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Magic Square and Arrangement of Consecutive Integers That Avoids k-Term Arithmetic Progressions

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Abstract: In 1977, Davis et al. proposed a method to generate an arrangement of $[n] = \{1, 2, \ldots, n\}$ that avoids three-term monotone arithmetic progressions. Consequently, this arrangement avoids k-term monotone arithmetic progressions in $[n]$ for $k \geq 3$. Hence, we are interested in finding an arrangement of $[n]$ that avoids k-term monotone arithmetic progression, but allows $k-1$-term monotone arithmetic progression. In this paper, we propose a method to rearrange the rows of a magic square of order $2k-3$ and show that this arrangement does not contain a k-term monotone arithmetic progression. Consequently, we show that there exists an arrangement of n consecutive integers such that it does not contain a k-term monotone arithmetic progression, but it contains a $k-1$-term monotone arithmetic progression.

Keywords: magic square; arithmetic progression; permutations

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1. Introduction

A sequence $a_1, a_2, \ldots, a_n$ is said to have a k *-term monotone arithmetic progression* if there is a set of indices $\{i_1 < i_2 < \cdots < i_k\}$ such that the k-term subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ is either an increasing or a decreasing arithmetic progression.

Davis et al. [1] proposed a way to generate an arrangement of $[n] = \{1, 2, \ldots, n\}$ that avoids three-term monotone arithmetic progressions. An arrangement of $[n]$ is a sequence $a_1, a_2, \ldots, a_n$ such that $\{a_1, a_2, \ldots, a_n\} = [n]$. An arrangement of $[n]$ is also called a permutation of $[n]$.

Theorem 1 ([1])**.** *Let $n \geq 1$. There is a permutation of $[n]$ that does not contain a three-term monotone arithmetic progression.*

Let $\mathbb{Z}^+$ be the set of positive integers. Davis et al. [1] and Sidorenko [2] showed that there is no permutation of $\mathbb{Z}^+$ that avoids three-term monotone arithmetic progressions. However, they [1] showed that there exists permutations of $\mathbb{Z}^+$ that avoid five-term monotone arithmetic progression, implying the existence of permutations of the integers avoiding arithmetic progressions of length seven. Recently, Geneson [3] constructed a permutation of the integers avoiding arithmetic progressions of length six. Up to now, an intriguing question, whose answer is still unknown, is whether there exists a permutation of $\mathbb{Z}^+$ that avoids four-term monotone arithmetic progressions [4].

Let $\theta(n)$ denote the number of permutations of $[n]$ that contain no three-term monotone arithmetic progressions. Davis et al. [1] established that $2^{n-1} \leq \theta(n) \leq \lfloor \frac{n+1}{2} \rfloor ! \lceil \frac{n+1}{2} \rceil !$. These bounds were then improved by [5–7]. LeSaulnier and Vijay [5] also showed that any permutation of the positive integers must contain a three-term arithmetic progression with an odd common difference as a subsequence and constructed a permutation of the positive integers that does not contain any four-term arithmetic progression with an odd common difference. Geneson [3] also proved a lower bound of $\frac{1}{2}$ on the lower density of subsets of positive integers that can be permuted to avoid arithmetic progressions of length four, sharpening the lower bound of $\frac{1}{3}$ from [5].

As a consequence of Theorem 1, there exists an arrangement of $[n]$ that avoids k-term monotone arithmetic progression, where $k \geq 3$. However, up to now, there is no proposed arrangement of $[n]$ that avoids a k-term monotone arithmetic progression, but contains a $(k-1)$-term monotone arithmetic progression. In this paper, for $k \geq 3$, we show that the rows of a magic square of order $2k-3$ can be arranged in a way that the resulting arrangement does not contain a k-term monotone arithmetic progression, but it contains a $(k-1)$-term monotone arithmetic progression. Then, we apply the result to show that there exists an arrangement of n consecutive integers such that it does not contain a k-term monotone arithmetic progression, but it contains a $(k-1)$-term monotone arithmetic progression.

2. *K*-Term Monotone Arithmetic Progression

In this section, we prove that given any n consecutive integers with $n \geq k$, there is an arrangement that avoids k-term monotone arithmetic progressions, but contains a $(k-1)$-term monotone arithmetic progression.

2.1. Magic Square

In 1624 France, Claude Gaspard Bachet described the following "diamond method" for constructing odd ordered magic squares in his book *Problèmes Plaisants* [8].

Step 1: First, for $k \geq 3$, we arrange $[1,(2k-3)^2]$ in an $(2k-3) \times (2k-3)$ square. We extend a $(2k-3) \times (2k-3)$ square to form a diamond structure as in Figure 1. Then, we put the integers in order along descending diagonals into the square. For $k = 3$ and $k = 4$, Figures 1 and 2 illustrate the 3×3 and 5×5 extended squares, respectively.

		1		
	4		2	
7		5		3
	8		6	
		9		

Figure 1. A 3×3 square.

				1				
			6		2			
		11		7		3		
	16		12		8		4	
21		17		13		9		5
	22		18		14		10	
		23		19		15		
			24		20			
				25				

Figure 2. A 5×5 square.

Step 2: Then, move the numbers on the right leftwards in the same row. Similarly, move the numbers on the left rightwards in the same row. Furthermore, move the numbers on the top downwards in the same column, and move the numbers at the bottom upwards in the same column. See Figures 3 and 4 for $k = 3$ and 4. This gives a $(2k-3) \times (2k-3)$ magic square with magic sum $\frac{(2k-3)\left[(2k-3)^2+1\right]}{2}$.

4	9	2
3	5	7
8	1	6

Figure 3. A 3×3 magic square.

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Figure 4. A 5×5 magic square.

Let R_i be the i-th row of the magic square constructed this way, i.e.,

$$R_i = a_{i,1}, b_{i,1}, a_{i,2}, b_{i,2}, \ldots, a_{i,(k-2)}, b_{i,(k-2)}, a_{i,(k-1)}.$$

Note that for the first row R_1, we have:

$$\begin{aligned} a_{1,j} &= 1 + (k-2)(2k-3) - (j-1)(2k-4), 1 \le j \le k-1; \\ b_{1,j} &= k + (2k-4)(2k-3) - (j-1)(2k-4), 1 \le j \le k-2. \end{aligned} \tag{1}$$

For the last row R_{2k-3}, we have:

$$\begin{aligned} a_{(2k-3),j} &= k - 1 + (2k-4)(2k-3) - (j-1)(2k-4), 1 \le j \le k-1; \\ b_{(2k-3),j} &= 1 + (k-3)(2k-3) - (j-1)(2k-4), 1 \le j \le k-2. \end{aligned} \tag{2}$$

For row R_{k-1}, if k is odd, then:

$$\begin{aligned} a_{(k-1),j} &= 1 + (2k-4)(2k-3) - \left(\frac{3k-1}{2} + j - 3\right)(2k-4), 1 \le j \le \frac{k-1}{2}; \\ a_{(k-1),j} &= 1 + (2k-4)(2k-3) - \left(j - \frac{k-1}{2} - 1\right)(2k-4), \frac{k-1}{2} + 1 \le j \le k-1; \\ b_{(k-1),j} &= 1 + (2k-4)(2k-3) - \left(\frac{k-1}{2} + j - 1\right)(2k-4), 1 \le j \le k-2. \end{aligned} \tag{3}$$

whereas if k is even, then:

$$\begin{aligned} a_{(k-1),j} &= 1 + (2k-4)(2k-3) - \left(\frac{k}{2} + j - 2\right)(2k-4), 1 \le j \le k-1; \\ b_{(k-1),j} &= 1 + (2k-4)(2k-3) - \left(\frac{3k}{2} + j - 3\right)(2k-4), 1 \le j \le \frac{k}{2} - 1; \\ b_{(k-1),j} &= 1 + (2k-4)(2k-3) - \left(j - \frac{k}{2}\right)(2k-4), \frac{k}{2} \le j \le k-2, \end{aligned} \tag{4}$$

For row R_i with $2 \le i \le k-2$, if i is odd, then:

$$\begin{aligned} a_{i,j} &= 1 + (k-3+i)(2k-3) - \left(\frac{i-1}{2} + j - 1\right)(2k-4), 1 \le j \le k-1; \\ b_{i,j} &= 1 + (k-3+i)(2k-3) - \left(\frac{i-1}{2} + k + j - 2\right)(2k-4), 1 \le j \le \frac{i-1}{2}; \\ b_{i,j} &= k - 1 + i + (2k-4)(2k-3) - \left(j - \frac{i-1}{2} - 1\right)(2k-4), \frac{i-1}{2} + 1 \le j \le k - 2 - \frac{i-1}{2}; \\ b_{i,j} &= 1 + (k-3+i)(2k-3) - \left(j - k + 1 + \frac{i-1}{2}\right)(2k-4), k - 1 - \frac{i-1}{2} \le j \le k-2, \end{aligned} \tag{5}$$

whereas if i is even, then:

$$\begin{aligned}
a_{i,j} &= 1 + (k-3+i)(2k-3) - \left(\frac{i}{2} + k + j - 3\right)(2k-4), 1 \le j \le \frac{i}{2};\\
a_{i,j} &= k - 1 + i + (2k-4)(2k-3) - \left(j - \frac{i}{2} - 1\right)(2k-4), \frac{i}{2} + 1 \le j \le k - 1 - \frac{i}{2};\\
a_{i,j} &= 1 + (k-3+i)(2k-3) - \left(j - k + \frac{i}{2}\right)(2k-4), k - \frac{i}{2} \le j \le k-1;\\
b_{i,j} &= 1 + (k-3+i)(2k-3) - \left(\frac{i}{2} + j - 1\right)(2k-4), 1 \le j \le k-2.
\end{aligned} \tag{6}$$

For row R_i with $k \le i \le 2k-4$, if i is odd, then:

$$\begin{aligned}
a_{i,j} &= 2 + i - k + (2k-4)(2k-3) - \left(k - 2 - \frac{1+i}{2} + j\right)(2k-4), 1 \le j \le k-1;\\
b_{i,j} &= 2 + i - k + (2k-4)(2k-3) - \left(2k - 3 - \frac{1+i}{2} + j\right)(2k-4), 1 \le j \le k - 1 - \frac{1+i}{2};\\
b_{i,j} &= 1 + (i-k)(2k-3) - \left(j - k + \frac{1+i}{2}\right)(2k-4), k - \frac{1+i}{2} \le j \le \frac{1+i}{2} - 1;\\
b_{i,j} &= 2 + i - k + (2k-4)(2k-3) - \left(j - \frac{1+i}{2}\right)(2k-4), \frac{1+i}{2} \le j \le k-2,
\end{aligned} \tag{7}$$

whereas if i is even, then:

$$\begin{aligned}
a_{i,j} &= 2 + i - k + (2k-4)(2k-3) - \left(2k - 4 - \frac{i}{2} + j\right)(2k-4), 1 \le j \le k - 1 - \frac{i}{2};\\
a_{i,j} &= 1 + (i-k)(2k-3) - \left(j - k + \frac{i}{2}\right)(2k-4), k - \frac{i}{2} \le j \le \frac{i}{2};\\
a_{i,j} &= 2 + i - k + (2k-4)(2k-3) - \left(j - \frac{i}{2} - 1\right)(2k-4), \frac{i}{2} + 1 \le j \le k-1;\\
b_{i,j} &= 2 + i - k + (2k-4)(2k-3) - \left(k - 2 - \frac{i}{2} + j\right)(2k-4), 1 \le j \le k-2.
\end{aligned} \tag{8}$$

2.2. Arrangement That Avoids K-Term Arithmetic Progressions

In this section, we form a sequence P, which is an arrangement of the rows of the $(2k-3) \times (2k-3)$ magic square from Section 2.1 that avoids k-term arithmetic progressions.

Theorem 2. *Suppose $k \ge 4$. Let R_i be the sequence of integers in the i-th row from left to right in the magic square formed by using the method in Section 2.1 where $1 \le i \le 2k-3$. Then, the sequence $P = R_1, R_2, \ldots, R_{k-2}, R_k, R_{k+1}, \ldots, R_{2k-4}, R_{2k-3}, R_{k-1}$ avoids k-term arithmetic progressions, but it has a $(k-1)$-term arithmetic progression, $k-1, k, \ldots, 2k-3$.*

Remark 1. *Note that Theorem 2 is not true for $k = 3$. In fact, the row $R_3 = 3, 5, 7$ has a three-term arithmetic progression.*

Proof. Note that for $k = 4$,

$$P = 11, 24, 7, 20, 3, 4, 12, 25, 8, 16, 10, 18, 1, 14, 22, 23, 6, 19, 2, 15, 17, 5, 13, 21, 9,$$

does not contain a 4-term arithmetic progression, but has a 3-term arithmetic progression $3, 4, 5$. Therefore, we may assume that $k \ge 5$.

By Equations (1), (5) and (6), for $1 \le i \le k-2$, every integer in the row R_i is either congruent with $k-2+i$ or $k-1+i \mod (2k-4)$. By Equations (2), (7) and (8), for $k \le i \le 2k-3$, every integer in the row R_i is either congruent with $i-k+1$ or $i-k+2$

mod $(2k-4)$. Lastly, by Equations (3) and (4), every integer in the row R_{k-1} is congruent with 1 mod $(2k-4)$. Note that the only integer congruent with 1 mod $(2k-4)$ in row R_{k-2} is $(2k-3)^2$, whereas the only integer congruent with 1 mod $(2k-4)$ in row R_k is one. Thus, all the integers congruent with 1 mod $(2k-4)$ in $[(2k-3)^2]$ appear in rows R_{k-2}, R_k, and R_{k-1}. For $r = 2, 3, \ldots, 2k-4$, all the integers congruent with r mod $(2k-4)$ in $[(2k-3)^2]$ appear in exactly two different rows.

Since $k+j-2$ is in R_j for $1 \le j \le k-1$ and $P = R_1, R_2, \ldots, R_{k-2}, R_k, R_{k+1}, \ldots, R_{2k-4}, R_{2k-3}, R_{k-1}$, the progression $k-1, k, \ldots, 2k-3$ is a $(k-1)$-term arithmetic progression in P. Now, we proceed to show that P does not have any k-term arithmetic progressions. We prove it by contradiction. Assume that there exists a k-term monotone arithmetic progression $T = \{T_1, T_2, \ldots, T_k\}$ in P. Then, there exists a nonzero integer d such that:

$$T_j = T_1 + (j-1)d,$$

for all $1 \le j \le k$. Note that $|d| \le \frac{(2k-3)^2-1}{k-1} = 4k-8$.

Since:

$$P = R_1, R_2, \ldots, R_{k-2}, R_k, R_{k+1}, \ldots, R_{2k-4}, R_{2k-3}, R_{k-1},$$

T_j should appear before T_{j+1} if we read the elements from left to right in P. Thus, if T_j is in R_i for some $i \in \{1, 2, \ldots, k-2, k, k+1, \ldots, 2k-3\}$, then T_{j+1} will be in $R_{i'}$ where $i' \ge i$ or $i' = k-1$, and if T_j is in R_{k-1}, then T_{j+1} will be in R_{k-1}. Furthermore, if T_j and T_{j+1} are both in R_i for some $1 \le i \le 2k-3$, then T_{j+1} will appear after T_j in the sequence P when we read from left to right.

Suppose $T_j = 1$ or $(2k-3)^2$ for some $2 \le j \le k-1$. If the former holds, then $d = T_{j+1} - T_j > 0$ and $d = T_j - T_{j-1} < 0$, a contradiction. If the latter holds, then $d = T_{j+1} - T_j < 0$ and $d = T_j - T_{j-1} > 0$, a contradiction. Hence, we may assume that $T_j \ne 1$ or $(2k-3)^2$ for all $2 \le j \le k-1$.

If two consecutive terms of T are in R_i for some $1 \le i \le 2k-3$, then $d \equiv \pm 1$ or 0 mod $(2k-4)$. Suppose no two consecutive terms of T are in R_i. This means that if T_j is in R_{i_j}, then T_{j+1} will be in $R_{i_{j+1}}$ where $i_{j+1} > i_j$ or $i_{j+1} = k-1$ and $i_j \in \{1, 2, \ldots, k-2, k, k+1, \ldots, 2k-3\}$. Since $\frac{2k-3}{2} = k - \frac{3}{2} < k$, either there exists $1 \le j_0 \le k-1$ such that T_{j_0} and T_{j_0+1} are in R_{i_0} and R_{i_0+1}, respectively, for some $i_0 \in \{1, 2, \ldots, k-3, k, k+1, \ldots, 2k-4\}$ or T_{j_0} and T_{j_0+1} are in R_{k-2} and R_k, respectively, or T_{k-1} and T_k are in R_{2k-3} and R_{k-1}, respectively. If such a j_0 exists, then $d \equiv 0, 1$ or 2 mod $(2k-4)$, otherwise, $d \equiv k-2$ or $k-1$ mod $(2k-4)$.

Case 1: Suppose $d \equiv k-2 \mod (2k-4)$. This means no two consecutive terms of T are in R_i and T_{k-1} and T_k are in R_{2k-3} and R_{k-1}, respectively. Furthermore, $T_k \equiv 1 \mod (2k-4)$ and $T_{k-1} \equiv k-1 \mod (2k-4)$. Note that $T_{k-2} \equiv 1 \mod (2k-4)$. Therefore, T_{k-2} is in R_{k-2}, R_k or R_{k-1}. If T_{k-2} is in R_{k-1}, then T_{k-1} will be in R_{k-1}, a contradiction. If T_{k-2} is in R_{k-2}, then $T_{k-2} = (2k-3)^2$, a contradiction. If T_{k-2} is in R_k, then $T_{k-2} = 1$, a contradiction.

Case 2: Suppose $d \equiv k-1 \mod (2k-4)$. This means no two consecutive terms of T are in R_i and T_{k-1} and T_k are in R_{2k-3} and R_{k-1}, respectively. Furthermore, $T_k \equiv 1 \mod (2k-4)$ and $T_{k-1} \equiv k-2 \mod (2k-4)$. Note that $T_{k-2} \equiv 2k-5 \mod (2k-4)$ and $T_{k-3} \equiv k-4 \mod (2k-4)$. If $k = 5$, then $T_{k-2} = T_3$ is in R_1 or R_2, whereas $T_{k-3} = T_2$ is in R_3 or R_5. This is not possible as T_2 should appear before T_3 in the sequence P. Suppose $k \ge 6$. Now, T_{k-2} is in R_{k-3} or R_{k-4}, whereas T_{k-3} is in R_{2k-6} or R_{2k-5}, again not possible.

Case 3: Suppose $d \equiv 2 \mod (2k-4)$. This means no two consecutive terms of T are in R_i and there exists $1 \le j_0 \le k-1$ such that T_{j_0} and T_{j_0+1} are in R_{i_0} and R_{i_0+1}, respectively, for some $i_0 \in \{1, 2, \ldots, k-3, k, k+1, \ldots, 2k-4\}$ or T_{j_0} and T_{j_0+1} are in R_{k-2} and R_k, respectively. Now, T_1 cannot be in R_{k-1}; otherwise, T_2 will be in R_{k-1}. Note that $T_{k-1} = T_1 + (k-2)d \equiv T_1 \mod (2k-4)$. Suppose T_1 is in R_t for some $1 \le t \le k-2$. Then,

$T_1 \equiv k-2+t$ or $k-1+t \mod (2k-4)$. If $T_1 \equiv 1 \mod (2k-4)$, then $T_1 = (2k-3)^2$. Since $T_{k-1} \neq 1$, it must be in R_{k-1}. Therefore, T_k is in R_{k-1}, a contradiction. Suppose $T_1 \not\equiv 1 \mod (2k-4)$. Since T_{k-1} must appear after T_1 in P, we have $T_1 \equiv k-1+t \mod (2k-4)$ and T_1 is in R_t for some $1 \leq t \leq k-3$ and T_{k-1} is in R_{t+1}. This implies that either T_{k-2} is in the same row as T_{k-1} or T_2 is in the same row as T_1, a contradiction. Similarly, we also cannot have T_1 in R_t for some $k \leq t \leq 2k-3$.

Case 4: Suppose $d \equiv 1 \mod (2k-4)$. Now, T_1 cannot be in R_{k-1}; otherwise, T_2 will be in R_{k-1} and $d \equiv T_2 - T_1 \equiv 0 \mod (2k-4)$. Suppose T_1 is in R_i for some $1 \leq i \leq k-2$. Then, T_1 is either congruent with $k-2+i$ or $k-1+i \mod (2k-4)$. Suppose $T_1 \equiv k-2+i \mod (2k-4)$. Then, $T_{k-i} \equiv k-2+i+(k-i-1) \equiv 1 \mod (2k-4)$. Since $T_{k-i} \neq 1$ and $(2k-3)^2$, it must be in R_{k-1}. Thus, T_{k-i+1} is also in R_{k-1} and $d \equiv T_{k-i+1} - T_{k-i} \equiv 0 \mod (2k-4)$, a contradiction. Suppose $T_1 \equiv k-1+i \mod (2k-4)$. Then, $T_{k-i-1} \equiv k-1+i+(k-i-2) \equiv 1 \mod (2k-4)$. If $i \leq k-3$, then T_{k-i-1} must be in R_{k-1}. Therefore, T_{k-i} is also in R_{k-1} and $d \equiv 0 \mod (2k-4)$, a contradiction. Suppose $i = k-2$. Then, $T_1 \equiv 1 \mod (2k-4)$ and it is in R_{k-2}. Therefore, $T_1 = (2k-3)^2$. Now, $T_k \equiv 1+(k-1) \equiv k \mod (2k-4)$. Since $k \geq 5$, T_k is in R_1 or R_2, which is not possible as T_1 is in R_{k-2}.

Suppose T_1 is in R_i for some $k \leq i \leq 2k-3$. Then, T_1 is either congruent with $i-k+1$ or $i-k+2 \mod (2k-4)$. Suppose $T_1 \equiv i-k+2 \mod (2k-4)$. Then, $T_{k-2} \equiv i-k+2+(k-3) \equiv i-1 \mod 2k-3$. If $i \geq k+1$, then T_{k-2} is in R_{i-k+1} or R_{i-k}, which is not possible. If $i = k$, then $T_{k-2} \equiv k-1 \mod (2k-4)$ and $T_{k-1} \equiv k \mod (2k-4)$. Therefore, T_{k-1} is in R_1 or R_2, again not possible. Suppose $T_1 \equiv i-k+1 \mod (2k-4)$. Then, $T_{k-1} \equiv i-k+1+(k-2) \equiv i-1 \mod (2k-4)$. If $i \geq k+1$, then T_{k-1} is in R_{i-k+1} or R_{i-k}, which is not possible. If $i = k$, then $T_{k-1} \equiv k-1 \mod (2k-4)$ and $T_k \equiv k \mod (2k-4)$. Therefore, T_k is in R_1 or R_2, again not possible.

Case 5: Suppose $d \equiv -1 \mod (2k-4)$.

If two consecutive terms of T, say T_j and T_{j+1} are in R_{k-1}, then $d = T_{j+1} - T_j \equiv 0 \mod (2k-4)$, a contradiction. Suppose T_j and T_{j+1} are in R_1. Then, $T_j \equiv k \mod (2k-4)$ and $T_{j+1} \equiv k-1 \mod (2k-4)$. By Equation (1), $T_j = b_{1,j_1}$ for some $1 \leq j_1 \leq k-2$ and $T_{j+1} = a_{1,j_2}$ for some $1 \leq j_2 \leq k-1$. Now,

$$\begin{aligned} d = T_{j+1} - T_j = a_{1,j_2} - b_{1,j_1} &\leq a_{1,1} - b_{1,(k-2)} \\ &= 1 + (k-2)(2k-3) - (k + (2k-4)(2k-3) - (k-3)(2k-4)) \\ &= 1 - 2(2k-3) = -4k+7. \end{aligned}$$

Therefore, $|d| \geq 4k-7 > 4k-8$, a contradiction.

Suppose T_j and T_{j+1} are in R_{2k-3}. Then, $T_j \equiv k-1 \mod (2k-4)$ and $T_{j+1} \equiv k-2 \mod (2k-4)$. By Equation (2), $T_j = a_{(2k-3),j_1}$ for some $1 \leq j_1 \leq k-1$ and $T_{j+1} = b_{(2k-3),j_2}$ for some $1 \leq j_2 \leq k-2$. Furthermore, $j_2 \geq j_1$. Now,

$$\begin{aligned} d = T_{j+1} - T_j &= b_{(2k-3),j_2} - b_{(2k-3),j_1} + b_{(2k-3),j_1} - a_{(2k-3),j_1} \\ &\leq b_{(2k-3),j_1} - a_{(2k-3),j_1} \\ &= 1 + (k-3)(2k-3) - (k-1+(2k-4)(2k-3)) \\ &= 2-k-(k-1)(2k-3) = 1-(k-1)(2k-2). \end{aligned}$$

Therefore, $|d| \geq (k-1)(2k-2) - 1 > 4k-8$, a contradiction. Hence, we may assume that no consecutive terms of T are in R_1, R_{2k-3} or R_{k-1}.

Suppose T_1 is in R_1. Then, T_1 is either congruent with $k-1$ or $k \mod (2k-4)$. Suppose $T_1 \equiv k \mod (2k-4)$. Then, $T_2 \equiv k-1 \mod (2k-4)$. Note that T_2 is not in R_1, for no two consecutive terms of T are in R_1. Therefore, T_2 is in R_{2k-3}. This means T_3 is in R_{k-1}, for no two consecutive terms of T are in R_{2k-3}. Therefore, T_4 must be in R_{k-1}, a contradiction as no two consecutive terms of T are in R_{k-1}.

Suppose $T_1 \equiv k-1 \mod (2k-4)$. Then, $T_{k-1} \equiv k-1+(k-2)(-1) \equiv 1 \mod (2k-4)$. Therefore, T_{k-1} is in R_{k-2}, R_k, or R_{k-1}. Note that T_{k-1} cannot be in R_{k-1}, otherwise T_k will be in R_{k-1}. If T_{k-1} is in R_{k-2}, then $T_{k-1} = (2k-3)^2$, a contradiction. If T_{k-1} is in R_k, then $T_{k-1} = 1$, a contradiction.

Suppose T_1 is in R_i for some $2 \le i \le k-2$. Then, T_1 is either congruent with $k-2+i$ or $k-1+i \mod (2k-4)$. Suppose $T_1 \equiv k-1+i \mod (2k-4)$. Then, $T_2 \equiv k-2+i \mod (2k-4)$ and $T_3 \equiv k-3+i \mod (2k-4)$. Since T_2 is not in R_{i-1}, it must be in R_i. If $i=2$, then T_3 must be in R_{2k-3}, for it cannot be in R_1. Since $T_4 \equiv k-2 \mod (2k-4)$, we must have T_4 in R_{2k-3} or R_{2k-4}. Both cases also cannot happen. If $i \ge 3$, then T_3 is in R_{i-1} or R_{i-2}, which is not possible. Suppose $T_1 \equiv k-2+i \mod (2k-4)$. Then, $T_2 \equiv k-3+i \mod (2k-4)$. If $i=2$, then T_2 must be in R_{2k-3}. Since T_3 is not in R_{2k-3}, it must be in R_{k-1}. However, then T_4 is also in R_{k-1}, a contradiction. If $i \ge 3$, then T_2 is in R_{i-1} or R_{i-2}, which is not possible.

Suppose T_1 is in R_i for some $k \le i \le 2k-3$. Then, T_1 is either congruent with $i-k+1$ or $i-k+2 \mod (2k-4)$. Suppose $T_1 \equiv i-k+2 \mod (2k-4)$. Then, $T_{i-k+2} \equiv i-k+2+(i-k+1)(-1) \equiv 1 \mod (2k-4)$. Therefore, T_{i-k+2} is in R_{k-2}, R_k, or R_{k-1}. Note that T_{i-k+2} cannot be in R_{k-1}; otherwise, T_{i-k+3} will be in R_{k-1}. If T_{i-k+2} is in R_{k-2}, then $T_{i-k+2} = (2k-3)^2$, a contradiction. If T_{i-k+2} is in R_k, then $T_{i-k+2} = 1$, a contradiction.

Case 6: Suppose $d \equiv 0 \mod (2k-4)$.

Then, $T_j \equiv T_1 \mod (2k-4)$ for all j and $d = \pm(2k-4)$ or $d = \pm(4k-8)$. If $d = 4k-8$, then $T_k = T_1 + (k-1)d \ge 1 + (k-1)(4k-8) = (2k-3)^2$. Therefore, we must have $T_1 = 1$ and $T_k = (2k-3)^2$. Therefore, T_1 is in row R_k, whereas T_k is in R_{k-2}, which is not possible. If $d = -(4k-8)$, then $T_k = T_1 + (k-1)d \le (2k-3)^2 - (k-1)(4k-8) = 1$. Therefore, we must have $T_1 = (2k-3)^2$ and $T_k = 1$. Now, $T_j \equiv T_1 \equiv 1 \mod (2k-4)$ for $2 \le j \le k-1$ imply that $T_j \in R_{k-1}$ for all $2 \le j \le k-1$. This is also not possible as T_2 should appear after T_1, but before T_k in P. Hence, we may assume that $d = \pm(2k-4)$.

Case 6.1: Suppose $T_1 \equiv 1 \mod (2k-4)$. Then, T_j is either in R_{k-2}, R_k or R_{k-1} for all j. Note that $T_j \ne 1$ and $(2k-3)^2$ for all $2 \le j \le k$. Therefore, we may assume that T_j is in R_{k-1} for all $2 \le j \le k$.

Case 6.1.1: Suppose k is odd. Let

$$A_1 = \left\{ s \in [k] \ : \ T_s = a_{(k-1),j} \text{ for some } 1 \le j \le \frac{k-1}{2} \right\};$$
$$A_2 = \left\{ s \in [k] \ : \ T_s = a_{(k-1),j} \text{ for some } \frac{k-1}{2} + 1 \le j \le k-1 \right\};$$
$$A_3 = \left\{ s \in [k] \ : \ T_s = b_{(k-1),j} \text{ for some } 1 \le j \le k-2 \right\}.$$

Note that A_1, A_2, and A_3 are disjoint. By Equation (3), $|A_1| + |A_2| + |A_3| \ge k-1 \ge 4$. Therefore, one of the A_t where $t \in \{1,2,3\}$ must contain at least two elements. Thus, $d = -(2k-4)$.

Suppose $A_1 \ne \varnothing$. If $k \notin A_1$, then $T_{s_1} = a_{(k-1),\frac{k-1}{2}}$ for some $1 \le s_1 \le k-1$. Now,

$$\begin{aligned} T_{s_1} &= 1 + (2k-4)(2k-3) - \left(\frac{3k-1}{2} + \frac{k-1}{2} - 3 \right)(2k-4) \\ &= 1 + (2k-4) \end{aligned}$$

and $T_{s_1+1} = 1 + (2k-4) - (2k-4) = 1$. Therefore, T_{s_1+1} is in R_k. This is not possible. Thus, if $A_1 \ne \varnothing$, then $k \in A_1$.

Suppose $A_2 \neq \varnothing$. If $k \notin A_2$, then $T_{s_2} = a_{(k-1),k-1}$ for some $1 \leq s_2 \leq k-1$. Since $a_{(k-1),k-1}$ is the last term in R_{k-1}, T_{s_2+1} is not in R_{k-1}, a contradiction. Thus, if $A_2 \neq \varnothing$, then $k \in A_2$.

Suppose $A_3 \neq \varnothing$. If $k \notin A_3$, then $T_{s_3} = b_{(k-1),k-2}$ for some $1 \leq s_3 \leq k-1$. Now, $a_{(k-1),k-1}$ is the last term in R_{k-1}. Therefore, $T_{s_3+1} = a_{(k-1),k-1}$ and:

$$\begin{aligned} d = T_{s_3+1} - T_{s_3} &= \left(\frac{k-1}{2} + k - 2 - 1\right)(2k-4) - \left(k - 1 - \frac{k-1}{2} - 1\right)(2k-4) \\ &= (k-2)(2k-4), \end{aligned}$$

a contradiction. Thus, if $A_3 \neq \varnothing$, then $k \in A_3$.

Hence, $A_{t_0} \neq \varnothing$ for exactly one $t_0 \in [3]$. Now, $|A_1| \leq \frac{k-1}{2} < k-1$, $|A_2| \leq \frac{k-1}{2} < k-1$, and $|A_3| \leq k-2 < k-1$. Thus, $k-1 \leq |A_1| + |A_2| + |A_3| = |A_{t_0}| < k-1$, a contradiction.

Case 6.1.2: Suppose k is even. Let:

$$\begin{aligned} A_1 &= \left\{s \in [k] \ : \ T_s = a_{(k-1),j} \ \text{ for some } 1 \leq j \leq k-1\right\}; \\ A_2 &= \left\{s \in [k] \ : \ T_s = b_{(k-1),j} \ \text{ for some } 1 \leq j \leq \frac{k}{2} - 1\right\}; \\ A_3 &= \left\{s \in [k] \ : \ T_s = b_{(k-1),j} \ \text{ for some } \frac{k}{2} \leq j \leq k-2\right\}. \end{aligned}$$

Note that A_1, A_2, and A_3 are disjoint. By Equation (4), $|A_1| + |A_2| + |A_3| \geq k-1 \geq 4$. Therefore, one of the A_t where $t \in \{1,2,3\}$ must contain at least two elements. Thus, $d = -(2k-4)$.

Suppose $A_1 \neq \varnothing$. If $k \notin A_1$, then $T_{s_1} = a_{(k-1),k-1}$ for some $1 \leq s_1 \leq k-1$. Since $a_{(k-1),k-1}$ is the last term in R_{k-1}, T_{s_1+1} is not in R_{k-1}, a contradiction. Thus, if $A_1 \neq \varnothing$, then $k \in A_1$.

Suppose $A_2 \neq \varnothing$. If $k \notin A_2$, then $T_{s_2} = b_{(k-1),\frac{k}{2}-1}$ for some $1 \leq s_2 \leq k-1$. Now,

$$\begin{aligned} T_{s_2} &= 1 + (2k-4)(2k-3) - \left(\frac{3k}{2} + \frac{k}{2} - 1 - 3\right)(2k-4) \\ &= 1 + (2k-4) \end{aligned}$$

and $T_{s_2+1} = 1 + (2k-4) - (2k-4) = 1$. Therefore, T_{s_2+1} is in R_k. This is not possible. Thus, if $A_2 \neq \varnothing$, then $k \in A_2$.

Suppose $A_3 \neq \varnothing$. If $k \notin A_3$, $T_{s_3} = b_{(k-1),k-2}$ for some $1 \leq s_3 \leq k-1$. Now, $a_{(k-1),k-1}$ is the last term in R_{k-1}. Therefore, $T_{s_3+1} = a_{(k-1),k-1}$ and:

$$\begin{aligned} d = T_{s_3+1} - T_{s_3} &= \left(k - 2 - \frac{k}{2}\right)(2k-4) - \left(\frac{k}{2} + k - 1 - 2\right)(2k-4) \\ &= -(k-1)(2k-4), \end{aligned}$$

a contradiction. Thus, if $A_3 \neq \varnothing$, then $k \in A_3$.

Hence, $A_{t_0} \neq \varnothing$ for exactly one $t_0 \in [3]$. Now, $|A_1| \leq k-1$, $|A_2| \leq \frac{k}{2} - 1 < k-1$, and $|A_3| \leq \frac{k}{2} - 1 < k-1$. Thus, $t_0 = 1$ and $T_2 = a_{(k-1),1}$. This means T_1 can only be in R_{k-2} or R_k. Since $d < 0$, T_1 must be in R_k, that is $T_1 = (2k-3)^2$. Now,

$$\begin{aligned} d = T_2 - T_1 &= 1 + (2k-4)(2k-3) - \left(\frac{k}{2} + 1 - 2\right)(2k-4) - (2k-3)^2 \\ &= -\left(\frac{k}{2}\right)(2k-4), \end{aligned}$$

a contradiction.

Case 6.2: Suppose $T_1 \equiv k-1 \mod (2k-4)$. Then T_j is either in R_1 or R_{2k-3} for all j. Since there are at most $k-1$ integers in R_{2k-3} that are congruent with $k-1 \mod (2k-4)$, we must have T_1 in R_1. Suppose both T_1 and T_2 are in R_1. By Equation (1), we must have $d = -(2k-4)$ and $T_1 = a_{1,l}$ and $T_2 = a_{1,l+1}$ for some $1 \leq l \leq k-2$. Since there are at most $k-1$ integers in R_1 that are congruent with $k-1 \mod (2k-4)$, we must have T_k in R_{2k-3}. The largest integer congruent with $k-1 \mod (2k-4)$ in R_1 is $a_{1,1} = 1+(k-2)(2k-3)$, and the smallest integer congruent with $k-1 \mod (2k-4)$ in R_{2k-3} is $a_{(2k-3),k-1} = k-1+(2k-4)(2k-3)-(k-2)(2k-4) = (k-1)(2k-3)$. Note that $a_{1,1} - a_{(2k-3),k-1} = -(2k-4)$. Let s be the smallest integer such that T_s is in R_1 and T_{s+1} is in R_{2k-3}. Since both T_1 and T_2 are in R_1, $T_s \neq a_{1,1}$. Therefore,

$$\begin{aligned} 2k-4 = T_s - T_{s+1} &= (T_s - a_{1,1}) + a_{1,1} - a_{(2k-3),k-1} + (a_{(2k-3),k-1} - T_{s+1}) \\ &\leq 0 - (2k-4) + 0 < 0, \end{aligned}$$

a contradiction. Suppose T_1 is in R_1, but T_2 is in R_{2k-3}. Then, we must have $T_j = a_{(2k-3),j-1}$ for all $2 \leq j \leq k$. By Equation (2), $d = T_3 - T_2 = a_{(2k-3),2} - a_{(2k-3),1} = -(2k-4)$. On the other hand,

$$\begin{aligned} 2k-4 = T_1 - T_2 &\leq a_{1,1} - a_{(2k-3),1} \\ &= 1+(k-2)(2k-3) - (k-1+(2k-4)(2k-3)) \\ &= -(k-2)(2k-2) < 0, \end{aligned}$$

a contradiction.

Case 6.3: Suppose $T_1 \equiv r \mod (2k-4)$ where $k \leq r \leq 2k-4$. Then, for all j, T_j is either in R_{r-k+1} or R_{r-k+2}.

Case 6.3.1: Suppose $r-k+1$ is odd. Let:

$$\begin{aligned} A_1 &= \left\{ s \in [k] \ : \ T_s = b_{r-k+1,j} \text{ for some } \frac{r-k}{2}+1 \leq j \leq k-2-\frac{r-k}{2} \right\}; \\ A_2 &= \left\{ s \in [k] \ : \ T_s = a_{r-k+2,j} \text{ for some } 1 \leq j \leq \frac{r-k+2}{2} \right\}; \\ A_3 &= \left\{ s \in [k] \ : \ T_s = a_{r-k+2,j} \text{ for some } k-\frac{r-k+2}{2} \leq j \leq k-1 \right\}; \\ A_4 &= \left\{ s \in [k] \ : \ T_s = b_{r-k+2,j} \text{ for some } 1 \leq j \leq k-2 \right\}. \end{aligned}$$

Note that A_1, A_2, A_3, and A_4 are disjoint. By Equations (5) and (6) (or Equations (1) and (6) when $r = k$), $|A_1| + |A_2| + |A_3| + |A_4| = k \geq 5$. Therefore, one of the A_t where $t \in \{1,2,3,4\}$ must contain at least two elements. Thus, $d = -(2k-4)$.

Suppose $A_4 \neq \varnothing$. If $k \notin A_4$, then $T_{s_4} = b_{r-k+2,k-2}$ for some $1 \leq s_4 \leq k-1$. Now, $a_{r-k+2,k-1}$ is the last term in R_{r-k+2}. Therefore, $T_{s_4+1} = a_{r-k+2,k-1}$ and:

$$\begin{aligned} d = T_{s_4+1} - T_{s_4} &= -\left(k-1-k+\frac{r-k+2}{2}\right)(2k-4) + \left(\frac{r-k+2}{2}+k-2-1\right)(2k-4) \\ &= (k-2)(2k-4) > 0, \end{aligned}$$

a contradiction. Thus, if $A_4 \neq \varnothing$, then $k \in A_4$. Now, if $1 \notin A_4$, then $T_{s_5} = b_{r-k+2,1}$ for some $2 \leq s_5 \leq k$. By Equation (6), $T_{s_5-1} = a_{r-k+2,k-1}$. This is not possible as T_{s_5-1} should appear before T_{s_5} in P. Thus, if $A_4 \neq \varnothing$, then one and k must be in A_4. This implies that $j \in A_4$ for all $1 \leq j \leq k$. This is not possible as $|A_4| \leq k-2$. Hence, $A_4 = \varnothing$ and $|A_1| + |A_2| + |A_3| = k$.

Suppose $A_2 \neq \varnothing$. If $k \notin A_2$, then $T_{s_2} = a_{r-k+2,\frac{r-k+2}{2}}$ for some $1 \leq s_2 \leq k-1$. Now,

$$\begin{aligned} T_{s_2} &= 1 + (k-3+r-k+2)(2k-3) - \left(\frac{r-k+2}{2} + k + \frac{r-k+2}{2} - 3\right)(2k-4) \\ &= r, \end{aligned}$$

and $T_{s_2+1} = r - (2k-4) \leq 0$, a contradiction. Thus, if $A_2 \neq \varnothing$, then $k \in A_2$.

Suppose $A_3 \neq \varnothing$. If $k \notin A_3$, then $T_{s_3} = a_{r-k+2,k-1}$ for some $1 \leq s_3 \leq k-1$. Since $a_{r-k+2,k-1}$ is the last term in R_{r-k+2}, T_{s_3+1} is not in R_{r-k+2}, a contradiction. Thus, if $A_3 \neq \varnothing$, then $k \in A_3$.

Hence, $A_{t_0} \neq \varnothing$ for at most one $t_0 \in \{2,3\}$. If $A_3 = \varnothing$, then:

$$k = |A_1| + |A_2| \leq k - (r-k+2) + \frac{r-k+2}{2} = k - \frac{r-k+2}{2} < k,$$

a contradiction. Similarly, if $A_2 = \varnothing$, then $k = |A_1| + |A_3| < k$, again a contradiction.

Case 6.3.2: Suppose $r-k+1$ is even. Let:

$$\begin{aligned} A_1 &= \left\{ s \in [k] \ : \ T_s = a_{r-k+1,j} \ \text{ for some } \frac{r-k+1}{2} + 1 \leq j \leq k-1-\frac{r-k+1}{2} \right\}; \\ A_2 &= \left\{ s \in [k] \ : \ T_s = a_{r-k+2,j} \ \text{ for some } 1 \leq j \leq k-1 \right\}; \\ A_3 &= \left\{ s \in [k] \ : \ T_s = b_{r-k+2,j} \ \text{ for some } 1 \leq j \leq \frac{r-k+1}{2} \right\}; \\ A_4 &= \left\{ s \in [k] \ : \ T_s = b_{r-k+2,j} \ \text{ for some } k-1-\frac{r-k+1}{2} \leq j \leq k-2 \right\}. \end{aligned}$$

Note that A_1, A_2, A_3, and A_4 are disjoint. By Equations (5) and (6), $|A_1| + |A_2| + |A_3| + |A_4| = k \geq 5$. Therefore, one of the A_t where $t \in \{1,2,3,4\}$ must contain at least two elements. Thus, $d = -(2k-4)$.

Suppose $A_2 \neq \varnothing$. If $k \notin A_2$, then $T_{s_2} = a_{r-k+2,k-1}$ for some $1 \leq s_2 \leq k-1$. Since $a_{r-k+2,k-1}$ is the last term in R_{r-k+2}, T_{s_2+1} is not in R_{r-k+2}, a contradiction. Thus, if $A_2 \neq \varnothing$, then $k \in A_2$. Now, if T_1 is not in A_2, then $T_{s_5} = a_{r-k+2,1}$ for some $2 \leq s_5 \leq k$. By Equation (5) $T_{s_5-1} = b_{r-k+2,k-2}$. This is not possible as T_{s_5-1} should appear before T_{s_5} in P. Thus, if $A_2 \neq \varnothing$, then one and k must be in A_2. This implies that $j \in A_2$ for all $1 \leq j \leq k$. This is not possible as $|A_2| \leq k-1$. Hence, $A_2 = \varnothing$ and $|A_1| + |A_3| + |A_4| = k$.

Suppose $A_3 \neq \varnothing$. If $k \notin A_3$, then $T_{s_3} = b_{r-k+2,\frac{r-k+1}{2}}$ for some $1 \leq s_3 \leq k-1$. Now,

$$\begin{aligned} T_{s_3} &= 1 + (k-3+r-k+2)(2k-3) - \left(\frac{r-k+1}{2} + k + \frac{r-k+1}{2} - 2\right)(2k-4) \\ &= r, \end{aligned}$$

and $T_{s_3+1} = r - (2k-4) \leq 0$, a contradiction. Thus, if $A_3 \neq \varnothing$, then $k \in A_3$.

Suppose $A_4 \neq \varnothing$. If $k \notin A_4$, then $T_{s_4} = b_{r-k+2,k-2}$ for some $1 \leq s_4 \leq k-1$. By Equation (5), $T_{s_4+1} = a_{r-k+2,1}$, which is not possible. Thus, if $A_4 \neq \varnothing$, then $k \in A_4$.

Hence, $A_{t_0} \neq \varnothing$ for at most one $t_0 \in \{3,4\}$. If $A_3 = \varnothing$, then:

$$k = |A_1| + |A_4| \leq k - 1 - (r-k+1) + \frac{r-k+1}{2} = k - 1 - \frac{r-k+1}{2} < k,$$

a contradiction. Similarly, if $A_4 = \varnothing$, then $k = |A_1| + |A_3| < k$, again a contradiction.

Case 6.4: Suppose $T_1 \equiv r \mod (2k-4)$ where $2 \leq r \leq k-2$. Then, T_j is either in R_{k-2+r} or R_{k-1+r} for all j.

Case 6.4.1: Suppose $k-2+r$ is odd. Let:

$$\begin{aligned} A_1 &= \left\{ s \in [k] \ : \ T_s = a_{k-2+r,j} \ \text{ for some } 1 \le j \le k-1 \right\}; \\ A_2 &= \left\{ s \in [k] \ : \ T_s = b_{k-2+r,j} \ \text{ for some } 1 \le j \le k-1-\frac{k-1+r}{2} \right\}; \\ A_3 &= \left\{ s \in [k] \ : \ T_s = b_{k-2+r,j} \ \text{ for some } \frac{k-1+r}{2} \le j \le k-2 \right\}; \\ A_4 &= \left\{ s \in [k] \ : \ T_s = a_{k-1+r,j} \ \text{ for some } k-\frac{k-1+r}{2} \le j \le \frac{k-1+r}{2} \right\}. \end{aligned}$$

Note that A_1, A_2, A_3, and A_4 are disjoint. By Equations (7) and (8), $|A_1| + |A_2| + |A_3| + |A_4| = k \ge 5$. Therefore, one of the A_t where $t \in \{1,2,3,4\}$ must contain at least two elements. Thus, $d = -(2k-4)$.

Suppose $A_1 \ne \varnothing$. If $k \notin A_1$, then $T_{s_1} = a_{k-2+r,k-1}$ for some $1 \le s_1 \le k-1$. Therefore, $T_{s_1+1} = b_{k-2+r,1}$, and this is not possible. Thus, if $A_1 \ne \varnothing$, then $k \in A_1$. Now, if $1 \notin A_1$, then $T_{s_5} = a_{k-2+r,1}$ for some $2 \le s_5 \le k$. Therefore, $T_{s_5-1} = b_{k-2+r,k-2}$, again not possible. Thus, if $A_1 \ne \varnothing$, then one and k must be in A_1. This implies that $j \in A_1$ for all $1 \le j \le k$. Therefore, $k = |A_1| \le k-1$, a contradiction. Hence, $A_1 = \varnothing$ and $|A_2| + |A_3| + |A_4| = k$.

Suppose $A_2 \ne \varnothing$. If $1 \notin A_2$, then $T_{s_2} = b_{k-2+r,1}$ for some $2 \le s_2 \le k$. Therefore, $T_{s_2-1} = a_{k-2+r,k-1}$, and this is not possible. Thus, if $A_2 \ne \varnothing$, then $1 \in A_2$.

Suppose $A_3 \ne \varnothing$. If $1 \notin A_3$, then $T_{s_3} = b_{k-2+r,\frac{k-1+r}{2}}$ for some $2 \le s_3 \le k$. Therefore,

$$\begin{aligned} T_{s_3} &= 2 + k - 2 + r - k + (2k-4)(2k-3) \\ &= r + (2k-4)(2k-3), \end{aligned}$$

and $T_{s_3-1} = r + (2k-4)(2k-2) > (2k-3)^2$, a contradiction. Thus, if $A_3 \ne \varnothing$, then $1 \in A_3$. Hence, $A_{t_0} \ne \varnothing$ for at most one $t_0 \in \{2,3\}$. If $A_3 = \varnothing$, then:

$$k = |A_2| + |A_4| \le k - 1 - \frac{k-1+r}{2} + r = \frac{k-1+r}{2} < k,$$

a contradiction. Similarly, if $A_2 = \varnothing$, then $k = |A_3| + |A_4| < k$, again a contradiction.

Case 6.4.2: Suppose $k-2+r$ is even. Let:

$$\begin{aligned} A_1 &= \left\{ s \in [k] \ : \ T_s = a_{k-2+r,j} \ \text{ for some } 1 \le j \le k-1-\frac{k-2+r}{2} \right\}; \\ A_2 &= \left\{ s \in [k] \ : \ T_s = a_{k-2+r,j} \ \text{ for some } \frac{k-2+r}{2}+1 \le j \le k-1 \right\}; \\ A_3 &= \left\{ s \in [k] \ : \ T_s = b_{k-2+r,j} \ \text{ for some } 1 \le j \le k-2 \right\}; \\ A_4 &= \left\{ s \in [k] \ : \ T_s = b_{k-1+r,j} \ \text{ for some } k-\frac{k+r}{2} \le j \le \frac{k+r}{2} - 1 \right\}. \end{aligned}$$

Note that A_1, A_2, A_3, and A_4 are disjoint. By Equations (7) and (8) (or Equations (2) and (6) when $r = k-2$), $|A_1| + |A_2| + |A_3| + |A_4| = k \ge 5$. Therefore, one of the A_t where $t \in \{1,2,3,4\}$ must contain at least two elements. Thus, $d = -(2k-4)$.

Suppose $A_3 \ne \varnothing$. If $k \notin A_3$, then $T_{s_3} = b_{k-2+r,k-2}$ for some $1 \le s_3 \le k-1$. Therefore, $T_{s_3+1} = a_{k-2+r,1}$, and this is not possible. Thus, if $A_3 \ne \varnothing$, then $k \in A_3$. Now, if $1 \notin A_3$, then $T_{s_5} = b_{k-2+r,1}$ for some $2 \le s_5 \le k$. Therefore, $T_{s_5-1} = a_{k-2+r,k-1}$, again not possible. Thus, if $A_3 \ne \varnothing$, then one and k must be in A_3. This implies that $j \in A_3$ for all $1 \le j \le k$. Therefore, $k = |A_3| \le k-2$, a contradiction. Hence, $A_3 = \varnothing$ and $|A_1| + |A_2| + |A_4| = k$.

Suppose $A_1 \ne \varnothing$. If $1 \notin A_1$, then $T_{s_1} = a_{k-2+r,1}$ for some $2 \le s_1 \le k$. Therefore, $T_{s_1-1} = b_{k-2+r,k-2}$, and this is not possible. Thus, if $A_1 \ne \varnothing$, then $1 \in A_1$.

Suppose $A_2 \neq \varnothing$. If $1 \notin A_2$, then $T_{s_2} = a_{k-2+r, \frac{k-2+r}{2}+1}$ for some $2 \leq s_2 \leq k$. Therefore,

$$\begin{aligned} T_{s_2} &= 2 + k - 2 + r - k + (2k-4)(2k-3) \\ &= r + (2k-4)(2k-3), \end{aligned}$$

and $T_{s_2-1} = r + (2k-4)(2k-2) > (2k-3)^2$, a contradiction. Thus, if $A_2 \neq \varnothing$, then $1 \in A_2$. Hence, $A_{t_0} \neq \varnothing$ for at most one $t_0 \in \{1,2\}$. On the other hand, if $A_1 = \varnothing$, then:

$$k = |A_2| + |A_4| \leq k - 1 - \frac{k-2+r}{2} + r = \frac{k-2+r}{2} + 1 < k,$$

a contradiction. Similarly, if $A_2 = \varnothing$, then $k = |A_1| + |A_4| < k$, again a contradiction.

Hence, P does not contain any k-term monotone arithmetic progressions. This completes the proof of the theorem. □

Corollary 1. *Let $k \geq 3$. For each integer n with $k \leq n \leq (2k-3)^2$, there is an arrangement of $[n]$ that avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression.*

Proof. For $k = 3$, the sequence $P = 1,9,5,3,7,2,8,4,6$ avoids three-term arithmetic progressions. Now, for $3 \leq n \leq 9$, remove $n+1, n+2, \ldots, 9$ from P, and denote the resulting sequence as P'. For instance, if $n = 8$, then $P' = 1,5,3,7,2,8,4,6$, and if $n = 5$, then $P' = 1,5,3,2,4$. Note that P' avoids three-term arithmetic progressions. Therefore, we may assume that $k \geq 4$.

Suppose $2k-3 \leq n \leq (2k-3)^2$. First, we show that if $n = (2k-3)^2$, then there is an arrangement of $[n]$ that avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression. By Theorem 2, such an arrangement exists. Let us denote the arrangement by P. Now, remove $n+1, n+2, \ldots, (2k-3)^2$ from P, and denote the resulting sequence as P'. Note that P' avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression, that is $k-1, k, \ldots, 2k-3$.

Suppose $k \leq n \leq 2k-3$. Let $Q = 2k-3, 2k-4, \ldots, k, 1, 2, \ldots, k-1$. Then, it avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression, that is $1, 2, \ldots, k-1$. Now, remove $n+1, n+2, \ldots, (2k-3)$ from Q, and denote the resulting sequence as Q'. Thus, Q' avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression, that is $1, 2, \ldots, k-1$. □

Lemma 1. *Let $b_n = sa_n + r$ for some real numbers r, s. Then, the sequence $a_1, a_2, \ldots, a_n$ contains a k-term monotone arithmetic progression if and only if $b_1, b_2, \ldots, b_n$ contains a k-term monotone arithmetic progression.*

Proof. It is sufficient to show that if $a_1, a_2, \ldots, a_n$ contains a k-term monotone arithmetic progression, then $b_1, b_2, \ldots, b_n$ contains a k-term monotone arithmetic progression. Let $i_1 < i_2 < \cdots < i_k$ be such that $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ is a k-term monotone arithmetic progression. Therefore,

$$a_{i_j} = a_{i_1} + (j-1)d$$

for some real number d. This implies that:

$$\begin{aligned} b_{i_j} &= sa_{i_j} + r = s(a_{i_1} + (j-1)d) + r \\ &= (sa_{i_1} + r) + (j-1)sd = b_{i_1} + (j-1)sd. \end{aligned}$$

Hence, $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$ is a k-term monotone arithmetic progression. □

Let $\lceil x \rceil$ be the smallest integer such that $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ and $\lfloor x \rfloor$ be the largest integer such that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

Theorem 3. *Let $k \geq 3$. For each integer n with $n \geq k$, there is an arrangement of $[n]$ that avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression.*

Proof. By Corollary 1, this theorem is true for $n \leq (2k-3)^2$. Now, we may assume $n > (2k-3)^2$. We also assume that the theorem is true for all n' such that $n' < n$. Let $[n] = A_1 \cup A_2$ where A_1 and A_2 are odd and even integers, respectively. Note that $|A_1| = \lceil \frac{n}{2} \rceil$ and $|A_2| = \lfloor \frac{n}{2} \rfloor$. Note that:

$$A_1 = \left\{2j-1 \ : \ 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil \right\};$$
$$A_2 = \left\{2j \ : \ 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Since $\lceil \frac{n}{2} \rceil < n$, by assumption, the theorem is true for $\lceil \frac{n}{2} \rceil$. Let $P_1 = a_1, a_2, \ldots, a_{\lceil \frac{n}{2} \rceil}$ be an arrangement of $[\lceil \frac{n}{2} \rceil]$ that avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression. Furthermore, let $P_2 = b_1, b_2, \ldots, b_{\lfloor \frac{n}{2} \rfloor}$ be an arrangement of $[\lfloor \frac{n}{2} \rfloor]$ that avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression. We claim that:

$$P = 2a_1 - 1, 2a_2 - 1, \ldots, 2a_{\lceil \frac{n}{2} \rceil} - 1, 2b_1, 2b_2, \ldots, 2b_{\lfloor \frac{n}{2} \rfloor}$$

is an arrangement that avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression. In fact, if $T_1, T_2, \ldots, T_k$ is a k-term monotone arithmetic progression in P, then $T_j = T_1 + (j-1)d$. Let:

$$A_1 = \left\{s \in [k] \ : \ T_s = 2a_j - 1 \ \text{ for some } 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil \right\};$$
$$A_2 = \left\{s \in [k] \ : \ T_s = 2b_j \ \text{ for some } 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Since $k \geq 4$, one of the A_t where $t \in \{1,2\}$ must contain at least two elements. Thus, d is even. If both A_1 and A_2 are nonempty, then there is a $1 \leq s \leq k-1$ such that $s \in A_1$ and $s+1 \in A_2$. Now, $d = T_{s+1} - T_s = 2b_{j_2} - (2a_{j_1} - 1) = 2(b_{j_2} - a_{j_1}) + 1$ is odd, a contradiction. Hence, $A_{t_0} \neq \varnothing$ for at most one $t_0 \in \{1,2\}$. If $A_1 \neq \varnothing$, then by Lemma 1, P_1 contains a k-term arithmetic progression, a contradiction. Similarly, if $A_2 \neq \varnothing$, then P_2 contains a k-term arithmetic progression. Hence, P avoids k-term arithmetic progressions, but contains a $(k-1)$-term arithmetic progression. □

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