mathematics

Article

New Results on the Aggregation of Norms

Tatiana Pedraza and Jesús Rodríguez-López *

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain; tapedraz@mat.upv.es
* Correspondence: jrlopez@mat.upv.es

Abstract: It is a natural question if a Cartesian product of objects produces an object of the same type. For example, it is well known that a countable Cartesian product of metrizable topological spaces is metrizable. Related to this question, Borsík and Doboš characterized those functions that allow obtaining a metric in the Cartesian product of metric spaces by means of the aggregation of the metrics of each factor space. This question was also studied for norms by Herburt and Moszyńska. This aggregation procedure can be modified in order to construct a metric or a norm on a certain set by means of a family of metrics or norms, respectively. In this paper, we characterize the functions that allow merging an arbitrary collection of (asymmetric) norms defined over a vector space into a single norm (*aggregation on sets*). We see that these functions are different from those that allow the construction of a norm in a Cartesian product (*aggregation on products*). Moreover, we study a related topological problem that was considered in the context of metric spaces by Borsík and Doboš. Concretely, we analyze under which conditions the aggregated norm is compatible with the product topology or the supremum topology in each case.

Keywords: norm; asymmetric norm; aggregation; product topology; supremum topology

MSC: 46B99; 46A99; 54B10

Citation: Pedraza, T.; Rodríguez- López, J. New Results on the Aggregation of Norms. *Mathematics* **2021**, *9*, 2291. https://doi.org/10.3390/math9182291

Academic Editor: António M. Lopes

Received: 21 July 2021
Accepted: 11 September 2021
Published: 17 September 2021

1. Introduction

Aggregation functions are a special kind of function that allow merging several numerical values into a single one [1,2]. The study of these functions has attracted much attention during the last few years due to their applicability in several areas where decision-making is important, such as probabilities [3], computer science [4], economics [5,6], etc. A paradigmatic example of an aggregation function is the arithmetic mean, but there are many more. Following [2], given $\mathbb{I}$, a nonempty real interval, an aggregation function in $\mathbb{I}^n$ is a function $F : \mathbb{I}^n \to \mathbb{I}$ such that:

- It is isotone, that is nondecreasing;
- It satisfies the boundary conditions:

$$\inf_{\boldsymbol{x}\in\mathbb{I}^n} F(\boldsymbol{x}) = \inf \mathbb{I} \qquad \text{and} \qquad \sup_{\boldsymbol{x}\in\mathbb{I}^n} F(\boldsymbol{x}) = \sup \mathbb{I}.$$

There exist many families of aggregation functions that originated from a wide variety of research fields. Usually, these functions are used to aggregate a finite quantity of values into a representative output. Nevertheless, they can also be used to produce a new topological structure of some type from a family of this topological structure. It is well known how to endow a Cartesian product of a countable family of metric spaces $\{(X_n, d_n) : n \in \mathbb{N}\}$ with a metric inducing the product topology ([7], Theorem 4.2.2). Following this idea, Doboš and his collaborators [8,9] studied when, given a function $f : [0,+\infty)^I \to [0,+\infty)$ and an arbitrary family $\{(X_i, d_i) : i \in I\}$ of metric spaces, the function $f \circ \widetilde{\boldsymbol{d}} : (\prod_{i\in I} X_i) \times (\prod_{i\in I} X_i) \to [0,+\infty)$ given by $f \circ \widetilde{\boldsymbol{d}}(\boldsymbol{x},\boldsymbol{y}) = f((d_i(\boldsymbol{x}_i,\boldsymbol{y}_i))_{i\in I})$ is a metric on the Cartesian product $\prod_{i\in I} X_i$. They called these functions *metric-preserving* *functions*. In this way, they proved that f is a metric-preserving function if and only if $f^{-1}(0) = \mathbf{0}$ and f preserves triangular triplets (see Definition 4).

The same problem has also been studied for quasi-metrics (metrics that do not satisfy the symmetry axiom) by Mayor and Valero [10] using the terminology *asymmetric distance function*. It was shown that f is a quasi-metric-preserving function if and only if $f^{-1}(0) = \mathbf{0}$ and f preserves asymmetric triangular triplets (see Definition 4).

A related study was performed by Pradera and Trillas in [11], who analyzed the problem of characterizing those functions $f : [0, +\infty)^k \to [0, +\infty)$ that allow combining a finite family of pseudometrics $\{d_i : i = 1, \dots, k\}$ defined over the same set X into a single pseudometric on X given by $f \circ \boldsymbol{d}(x, y) = f(d_1(x, y), \dots, d_k(x, y))$ for all $x, y \in X$. In this case, the function *aggregates pseudometrics*. Moreover, they proved ([11], Theorem 6) that these functions, when acting on a finite Cartesian product, are equivalent to the pseudometric-preserving functions in the sense of Doboš. We observe that this is no longer true if one considers metrics instead of pseudometrics since the projection is an example of a function that aggregates metrics, but is not metric preserving.

Recently, Mayor and Valero [12] continued the study of Pradera and Trillas by characterizing the functions that aggregate metrics.

As we can observe, the term aggregation has been used for combining several metrics on the same set to obtain a new one, as well as to produce a metric in the Cartesian product of different metric spaces. Here, we use the terminology of [13], "aggregation", and add "on products" or "on sets" to it, depending on the problem we are treating (see Definition 2).

Of course, metrics are not the only topological structure that can be aggregated using this technique. In [14], Herburt and Moszyńska considered when the function $\| \cdot \|_f : V_1 \times V_2 \to [0, +\infty)$ is a norm on $V_1 \times V_2$ where $(V_1, \| \cdot \|_1), (V_2, \| \cdot \|_2)$ are two normed vector spaces and $f : [0, +\infty)^2 \to [0, +\infty)$ is a function (see their characterization in Theorem 1). This problem is the same as that studied by Borsík and Doboš, but in the context of normed spaces. The corresponding problem for asymmetric norms was solved by Martín, Mayor, and Valero [15]. In both cases, they considered the characterization of those functions that merge a family of (asymmetric) norms in different vector spaces into a (asymmetric) norm in the Cartesian product of these vector spaces.

In this paper, we continue the study of the aggregation of (asymmetric) norms. We first gather together the results on this topic due to Herburt and Moszyńska [14] and Martín, Mayor, and Valero [15]. Although Herburt and Moszyńska established their results for only two normed vector spaces, we state them for an arbitrary family of normed vector spaces since they remain valid at this level of generality. Moreover, we correct an assertion of [15], which is that there exist norm aggregation functions on products that are not asymmetric norm aggregation functions on products (see Definition 2). As we expose, the two concepts are equal (Corollary 1). As we previously observed, Mayor and Valero [12] and Pradera and Trillas [11] characterized the functions that allow merging a family of (quasi-)metrics or pseudometrics on a fixed nonempty set into a single one, respectively. However, to our knowledge, no study has been performed for (asymmetric) norms. In this manner, one of the goals of this paper is to characterize those functions (see Definition 2) that allow aggregating an arbitrary family of (asymmetric) norms in a fixed vector space into a norm in this space (*aggregation on sets*). In addition, inspired by the results of Borsík and Doboš [8,9], we also introduce the topology. In this way, we consider if, when $\{(V_i, n_i) : i \in I\}$ is an arbitrary family of normed vector spaces and f is a norm aggregation function on products, the product topology on $\prod_{i \in I} V_i$ coincides with the topology generated by the aggregated norm $f \circ \widetilde{\boldsymbol{n}}$ (see Definition 5). A similar question is considered when we have several norms in the same vector space.

The structure of the paper is as follows. In Section 2, we recall some basic facts about (asymmetric) norms and sublinear functions. Section 3 is devoted to studying so-called (asymmetric) norm aggregation functions on products and on sets (see Definition 2). In this way, we summarize some of the results of [14,15], who proved that (asymmetric) norm aggregation functions on products are isotone norms on the semivector space

$[0,+\infty)^I$. We next achieve the first goal of this paper, which is the characterization of the (asymmetric) norm aggregation functions on sets. In this case, in contrast with the case on products, norm aggregation function on sets are different from asymmetric norm aggregation function on sets (Theorems 3 and 4). We will also show that the difference between the aggregation on products and on sets lies in the images of the tuples which have at least one coordinate is equal to 0.

The last section of the paper deals with the study of the topology generated by the aggregated norm obtained by means of a family of norms. As before, we consider two different points of view: on products and on sets (see Definition 5). We will show that norm aggregation functions on products defined over a finite product $[0,+\infty)^k$ always preserve the product topology (see Theorem 5). On the other hand, we establish necessary and sufficient conditions under which the topology generated by a norm obtained by aggregating a family of norms on a fixed vector space coincides with the supremum topology of the topologies generated by each norm (see Theorem 6).

2. Norms and Asymmetric Norms

Our basic reference for normed vector spaces and topological vector spaces is [16]. For asymmetric norms we refer the reader to [17].

Let I be a set of indices. We will denote the elements of $[0,+\infty)^I$ by boldface letters $\boldsymbol{a}$ and we will write $\boldsymbol{a}_i$ instead of $\boldsymbol{a}(i)$ for all $i \in I$. Furthermore, $[0,+\infty)^I$ becomes a partially ordered set endowed with the partial order $\preceq$ given by $\boldsymbol{a} \preceq \boldsymbol{b}$ if $\boldsymbol{a}_i \leq \boldsymbol{b}_i$ for all $i \in I$. We will denote by $\mathbf{0}$ the element of $[0,+\infty)^I$ given by $\mathbf{0}_i = 0$ for all $i \in I$.

A function $f : [0,+\infty)^I \to [0,+\infty)$ will be called isotone if $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ whenever $\boldsymbol{x} \preceq \boldsymbol{y}$.

Notice that $[0,+\infty)^I$ is not a vector space but it is a semivector (or semilinear) space, that is, a unitary semimodule over the commutative semiring $[0,+\infty)$ (see [18–20]). Since the range of a(n) (asymmetric) norm is $[0,+\infty)$ (a semivector space) rather than $\mathbb{R}$ (a vector space), we give the following definitions for semivector spaces.

Definition 1 (cf. [16])**.** *Let V be a semivector space over $[0,+\infty)$ and let $n : V \to \mathbb{R}$ be a function. Consider the following properties for all $x, y \in V$:*

(N1) $n(x+y) \leq n(x) + n(y)$; *(subadditive)*
(N2) $n(\lambda x) = \lambda n(x)$ *for all* $\lambda \geq 0$; *(positive homogeneous)*
(N3) $n(x) \geq 0$ *for all* $x \in V$;
(N4) $n(x) = 0$ *implies* $x = 0_V$.

Then n is said to be:

- *A* sublinear function *if n satisfies (N1) and (N2).*
- *A* positive sublinear function *if n satisfies (N1), (N2) and (N3).*
- *A* norm *(on a semivector space) if n satisfies (N1), (N2), (N3) and (N4) [18].*

If V is a real vector space, consider the following properties:

(N5) $n(x) = n(-x) = 0$ *if and only if* $x = 0_V$;
(N6) $n(\lambda x) = |\lambda| n(x)$ *for all* $\lambda \in \mathbb{R}$. *(homogeneous)*

Then n is said to be:

- *An* asymmetric norm *if n satisfies (N1), (N2), (N3) and (N5).*
- *A* norm *if n satisfies (N1), (N5) and (N6).*

Norms are well-known mathematical objects but asymmetric norms could be something more unusual. Nevertheless, they have been well studied and there exists a parallel study to that of normed vector spaces [17].

As we will see in the next section, positive sublinear functions play a fundamental role in the characterization of norm aggregation functions. Therefore it is important to have tools that allow determining and constructing this kind of function. We first notice that a positive sublinear function n on a semivector space V can be characterized, the same way when they are defined on a vector space, by means of its epigraph which is the set

$$\mathrm{epi}(n) = \{(x, \alpha) \in V \times \mathbb{R} : n(x) \leq \alpha\}.$$

To achieve this, recall that if C is a subset of V, then C is said to be:

- convex if
$$\lambda v + (1 - \lambda)w \in C \text{ whenever } v, w \in C, \lambda \in (0, 1).$$
- a cone if $\lambda v \in C$ whenever $v \in C$, $\lambda \geq 0$.

Consequently, a cone C is convex if and only if $v + w \in C$ whenever $v, w \in C$. Moreover, if C is a convex subset of V, a function $f : C \to \mathbb{R}$ is said to be convex if

$$f(\lambda v + (1 - \lambda)w) \leq \lambda f(v) + (1 - \lambda) f(w)$$

for all $v, w \in C, \lambda \in (0, 1)$.

The following result is well-known for vector spaces (cf. [21], (Exercise 3.19))

Lemma 1. *Let V be a semivector space and let $n : V \to \mathbb{R}$. The following statements are equivalent:*

(1) *n is a sublinear function;*
(2) $\mathrm{epi}(n)$ *is a convex cone;*
(3) *n is convex and positive homogeneous.*

Proof. We only prove (2)⇒ (3) since the other implications are easy. Let $x, y \in V$. Given $\lambda \in [0, 1]$ since $\mathrm{epi}(n)$ is a convex cone then $(\lambda x + (1 - \lambda)y, \lambda n(x) + (1 - \lambda)n(y)) \in \mathrm{epi}(n)$ so $n(\lambda x + (1 - \lambda)y) \leq \lambda n(x) + (1 - \lambda)n(y)$. Hence n is convex.

Let us check that n is positive homogeneous. Notice that for every $x \in V$, $(0 \cdot x, 0 \cdot n(x)) = (0_V, 0) \in \mathrm{epi}(n)$ since it is a convex cone so $n(0 \cdot x) = n(0_V) = 0$. Consider $\lambda > 0$. Since $(x, n(x)) \in \mathrm{epi}(n)$ which is a convex cone then $(\lambda x, \lambda n(x)) \in \mathrm{epi}(n)$, that is, $n(\lambda x) \leq \lambda n(x)$. Suppose, in order to obtain a contradiction, that $n(\lambda x) \neq \lambda n(x)$. Since $(\lambda x, n(\lambda x)) \in \mathrm{epi}(n)$ then $(x, \frac{1}{\lambda} n(\lambda x)) \in \mathrm{epi}(n)$ so $n(x) \leq \frac{1}{\lambda} n(\lambda x) < \frac{1}{\lambda} \lambda n(x) = n(x)$ which is a contradiction. Consequently, n is positive homogeneous. □

As we will see, the following lemma is useful for constructing positive sublinear functions on $[0, +\infty)^k$.

Lemma 2. *Let $f : [0, +\infty)^k \to [0, +\infty)$ and consider $S = \{\boldsymbol{x} \in [0, +\infty)^k : \|\boldsymbol{x}\|_1 = \boldsymbol{x}_1 + \ldots + \boldsymbol{x}_k = 1\}$. Then f is a norm on the semivector space $[0, +\infty)^k$ if and only if $f|_S$ is strictly positive, convex and*

$$f(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\|_1 f|_S\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_1}\right) & \text{if } \boldsymbol{x} \neq \boldsymbol{0} \\ 0 & \text{if } \boldsymbol{x} = \boldsymbol{0} \end{cases}$$

for all $\boldsymbol{x} \in [0, +\infty)^k$.

Proof. Suppose that f is a norm. It is obvious that $f|_S$ is strictly positive. Since f is convex and S is convex then $f|_S$ is also convex. Furthermore, since f is positive homogeneous then

$$\|\boldsymbol{x}\|_1 f|_S\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_1}\right) = f(\boldsymbol{x})$$

whenever $\boldsymbol{x} \neq 0$.

Conversely, we first notice that f is positive homogeneous since if $\boldsymbol{0} \neq \boldsymbol{x} \in [0, +\infty)^k$ and $\lambda > 0$ then

$$f(\lambda \boldsymbol{x}) = \|\lambda \boldsymbol{x}\|_1 f|_S\left(\frac{\lambda \boldsymbol{x}}{\|\lambda \boldsymbol{x}\|_1}\right) = \lambda \|\boldsymbol{x}\|_1 f|_S\left(\frac{\lambda \boldsymbol{x}}{\lambda \|\boldsymbol{x}\|_1}\right) = \lambda f(\boldsymbol{x}).$$

Since $f|_S$ is strictly positive it is also clear that the range of f is $[0,+\infty)$ and that $f(\boldsymbol{x}) = 0$ implies $\boldsymbol{x} = \mathbf{0}$.

We next check that f is subadditive. Let $\boldsymbol{x}, \boldsymbol{y} \in [0,+\infty)^I$. If one of $\boldsymbol{x}, \boldsymbol{y}$ is $\mathbf{0}$ then the conclusion is obvious. So suppose that both are different from $\mathbf{0}$. Define

$$\beta = \frac{\|\boldsymbol{x}\|_1}{\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1} \in]0,1[.$$

Since $f|_S$ is convex then

$$f\left(\beta \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_1} + (1-\beta)\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_1}\right) \le \beta f\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_1}\right) + (1-\beta) f\left(\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_1}\right),$$

and hence

$$f\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1} + \frac{\boldsymbol{y}}{\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1}\right) \le \frac{\|\boldsymbol{x}\|_1}{\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1} f\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_1}\right) + \frac{\|\boldsymbol{y}\|_1}{\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1} f\left(\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_1}\right).$$

Since f is positive homogeneous

$$\frac{1}{\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1} f(\boldsymbol{x}+\boldsymbol{y}) \le \frac{1}{\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1} f(\boldsymbol{x}) + \frac{1}{\|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1} f(\boldsymbol{y})$$
$$f(\boldsymbol{x}+\boldsymbol{y}) \le f(\boldsymbol{x}) + f(\boldsymbol{y})$$

so f is subadditive. $\square$

The above lemma allows constructing easily norms on $[0,+\infty)^2$ by means of strictly positive convex functions f defined on $[0,1]$. We only have to take into account the homeomorphism φ between the set $S = \{(x,y) \in [0,+\infty)^2 : x+y = 1\}$ and $[0,1]$ given by $\varphi(x,y) = x$. In this way, we can consider $f \circ \varphi : S \to [0,+\infty)$ and then extend it to $[0,+\infty)^2$ by the previous lemma. Let us see some examples.

Example 1.

- *Consider the function $f : [0,1] \to [0,+\infty)$ given by*

  $$f(x) = 1$$

  for all $x \in [0,1]$. Obviously, f is strictly positive and convex. By the above lemma, the function $F : [0,+\infty)^2 \to [0,+\infty)$ given by

  $$F(x,y) = \begin{cases} \|(x,y)\|_1 f\left(\frac{x}{x+y}\right) = x+y & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

  is a norm on $[0,+\infty)^2$ which is obviously the restriction of the ℓ_1 norm to $[0,+\infty)^2$.

- *Consider the function $f : [0,1] \to [0,+\infty)$ given by*

  $$f(x) = \sqrt{2x^2 - 2x + 1}$$

  for all $x \in [0,1]$. Clearly f is strictly positive and it is straightforward to check that f is convex. By the above lemma, the function $F : [0,+\infty)^2 \to [0,+\infty)$ given by

$$\begin{aligned} F(x,y) &= \|(x,y)\|_1 (f\circ\varphi)\left(\frac{(x,y)}{\|(x,y)\|_1}\right) = (x+y) f\left(\frac{x}{x+y}\right) \\ &= (x+y)\sqrt{\frac{2x^2}{(x+y)^2} - \frac{2x}{x+y} + 1} = (x+y)\sqrt{\frac{x^2+y^2}{(x+y)^2}} \\ &= \sqrt{x^2+y^2} \end{aligned}$$

if $(x,y) \neq (0,0)$ *and* $F(0,0) = 0$ *is a norm on* $[0,+\infty)^2$ *which is obviously the restriction of the Euclidean norm to* $[0,+\infty)^2$.

- *Consider the function* $f : [0,1] \to [0,+\infty)$ *given by*

$$f(x) = e^x$$

for all $x \in [0,1]$. *Since* f *is strictly positive and convex, by the above lemma the function* $F : [0,+\infty)^2 \to [0,+\infty)$ *given by*

$$F(x,y) = \begin{cases} \|(x,y)\|_1 f\left(\frac{x}{x+y}\right) = (x+y)e^{\frac{x}{x+y}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is a norm on $[0,+\infty)^2$.

3. Aggregation of Norms and Asymmetric Norms

We next formalize the problem of the aggregation of norms solved by Herburt and Moszyńska in [14] but for an arbitrary product of normed vector spaces. The functions that allow this aggregation will be called *norm aggregation functions on products*. Nevertheless, following [11,13,22,23] we set another way of aggregating norms (*aggregation on sets*). Moreover, we also consider the case when the function aggregates asymmetric norms.

Definition 2 (cf. [9,13–15,22])**.** *A function* $f : [0,+\infty)^I \to [0,+\infty)$ *is said to be:*

- *a(n)* (asymmetric) norm aggregation function on products *if whenever* $\{(V_i, n_i) : i \in I\}$ *is a family of (asymmetric) normed vector spaces then* $f \circ \widetilde{\boldsymbol{n}}$ *is a(n asymmetric) norm on* $\prod_{i\in I} V_i$ *where*

$$f \circ \widetilde{\boldsymbol{n}}(\boldsymbol{v}) = f((n_i(\boldsymbol{v}_i))_{i\in I})$$

for all $\boldsymbol{v} \in \prod_{i\in I} V_i$.

- *a(n)* (asymmetric) norm aggregation function on sets *if whenever* $\{(V, n_i) : i \in I\}$ *is a family of (asymmetric) normed vector spaces then* $f \circ \boldsymbol{n}$ *is a(n) (asymmetric) norm on* V *where*

$$f \circ \boldsymbol{n}(v) = f((n_i(v))_{i\in I})$$

for all $v \in V$.

Remark 1. *A similar definition to the above can be given for metric spaces. As we have already observed:*

- *Metric aggregation functions on products were characterized by Borsík and Doboš [8,9] using the terminology metric-preserving functions;*
- *Metric aggregation functions on sets were characterized by Mayor and Valero in [12], using the terminology metric aggregation function;*
- *Quasi-metric aggregation functions on products were characterized by Mayor and Valero [10] using the terminology asymmetric distance aggregation functions;*
- *Pseudometric aggregation functions on products and pseudometric aggregation functions on sets were characterized in [11] using the terminology pseudometric-preserving functions and pseudometric aggregation functions, respectively.*

Obviously, if $|I| = 1$ then norm aggregation functions on products and norm aggregation functions on sets coincide. In general, it is easy to see that every norm aggregation function on products is a norm aggregation function on sets. Nevertheless, the converse is not true in general as the next example shows.

Example 2. *Let I be a set of indices and fix $j \in J$. Then the jth projection $p_j : [0,+\infty)^I \to [0,+\infty)$ is an example of a norm aggregation function on sets which is not a norm aggregation function on products. In fact, if $\{(V, n_i) : i \in I\}$ is a family of normed vector spaces it is obvious that $p_j \circ \boldsymbol{n} = n_j$ which is a norm on V. Nevertheless, if you consider the family of normed vector spaces $\{(\mathbb{R}, n_i) : i \in I\}$ where n_i is the absolute value for every $i \in I$, then $p_j \circ \widetilde{\boldsymbol{n}}$ is not a norm on $\mathbb{R}^I$. In fact, considering $\boldsymbol{v} \in \mathbb{R}^I$ such that $\boldsymbol{v}_i = 1$ if $i \neq j$ and $\boldsymbol{v}_j = 0$, then $p_j \circ \widetilde{\boldsymbol{n}}(\boldsymbol{v}) = p_j(\boldsymbol{v}) = 0$ but $\boldsymbol{v} \neq \boldsymbol{0}$.*

As we have previously commented, Herburt and Moszyńska [14] characterized the norm aggregation functions on products for functions when $|I| = 2$. Their characterization makes use of the following concept [9], although they did not use this terminology.

Definition 3 ([9,10])**.** *A triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0,+\infty)^I)^3$ is called:*

- *a* triangular triplet *if for all $i \in I$*

$$\boldsymbol{a}_i \leq \boldsymbol{b}_i + \boldsymbol{c}_i, \boldsymbol{b}_i \leq \boldsymbol{a}_i + \boldsymbol{c}_i \text{ and } \boldsymbol{c}_i \leq \boldsymbol{a}_i + \boldsymbol{b}_i;$$

- *an* asymmetric triangular triplet *if for all $i \in I$*

$$\boldsymbol{a}_i \leq \boldsymbol{b}_i + \boldsymbol{c}_i.$$

Definition 4. *We say that a function $f : [0,+\infty)^I \to [0,+\infty)$ preserves (asymmetric) triangular triplets if $(f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}))$ is a(n) (asymmetric) triangular triplet whenever $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ so is, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [0,+\infty)^I$.*

The next result is the reformulation, using the terminology of this paper, of the characterization proved in [14] about the norm aggregations functions on products. As we have already observed, this characterization is also valid for every cardinality of I so we state it in this level of generality. Moreover, we add the statements (2) and (4) to this characterization which were implicitly provided in [14].

Theorem 1 ([14])**.** *Let $f : [0,+\infty)^I \to [0,+\infty)$ be a function. The following statements are equivalent:*

(1) f is a norm aggregation function on products;
(2) $((\mathbb{R}^2)^I, \|\cdot\|_f)$ is a normed space where $\|\boldsymbol{x}\|_f = f((\|\boldsymbol{x}_i\|)_{i\in I})$ and $\|\cdot\|$ is the Euclidean norm, for all $\boldsymbol{x} \in (\mathbb{R}^2)^I$;
(3) $f^{-1}(0) = \boldsymbol{0}$, f is positive homogeneous and it preserves triangular triplets;
(4) $f^{-1}(0) = \boldsymbol{0}$, f is an isotone sublinear function.

Proof. We only prove (2)⇒ (3) and (3)⇒ (4) since the other implications are easy modifications of those in [14].

(2)⇒ (3) Since $\|\cdot\|_f$ is a norm on $(\mathbb{R}^2)^I$ we have that $0 = \|\boldsymbol{0}_{(\mathbb{R}^2)^I}\|_f = f(\boldsymbol{0})$. Moreover, suppose that we can find $\boldsymbol{a} \in [0,+\infty)^I \setminus \{\boldsymbol{0}\}$ such that $f(\boldsymbol{a}) = 0$. Consider $\boldsymbol{v} \in (\mathbb{R}^2)^I$ such that $\boldsymbol{v}_i = (\boldsymbol{a}_i, 0)$ for all $i \in I$. Then $\|\boldsymbol{v}\|_f = f((\|(\boldsymbol{a}_i, 0)\|)_{i\in I}) = f((|\boldsymbol{a}_i|)_{i\in I}) = f(\boldsymbol{a}) = 0$ but $\boldsymbol{v} \neq \boldsymbol{0}_{(\mathbb{R}^2)^I}$, which contradicts that $\|\cdot\|_f$ is a norm on $(\mathbb{R}^2)^I$. Hence $f^{-1}(0) = \boldsymbol{0}$.

Moreover, given $\lambda \geq 0$ and $\boldsymbol{a} \in [0,+\infty)^I$, if we define $\boldsymbol{v}$ as above then

$$\|\lambda \boldsymbol{v}\|_f = \lambda\|\boldsymbol{v}\|_f$$
$$f((\|\lambda \boldsymbol{v}_i\|)_{i\in I}) = \lambda f((\|\boldsymbol{v}_i\|)_{i\in I})$$
$$f((\|(\lambda \boldsymbol{a}_i, 0)\|)_{i\in I}) = \lambda f((\|(\boldsymbol{a}_i, 0)\|)_{i\in I})$$
$$f((|\lambda \boldsymbol{a}_i|)_{i\in I}) = \lambda f((|\boldsymbol{a}_i|)_{i\in I})$$
$$f(\lambda \boldsymbol{a}) = \lambda f(\boldsymbol{a})$$

so f is positive homogeneous.

Finally, let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0, +\infty)^I)^3$ be a triangular triplet. By [9], (Chapter 2, Proposition 1), for each $i \in I$ we can find $\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i \in \mathbb{R}^2$ such that $\boldsymbol{a}_i = \|\boldsymbol{y}_i - \boldsymbol{x}_i\|, \boldsymbol{b}_i = \|\boldsymbol{z}_i - \boldsymbol{x}_i\|$ and $\boldsymbol{c}_i = \|\boldsymbol{y}_i - \boldsymbol{z}_i\|$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. Since $\|\cdot\|_f$ is a norm on $(\mathbb{R}^2)^I$ we have that

$$\begin{aligned}\|\boldsymbol{y} - \boldsymbol{x}\|_f = \|(\boldsymbol{y} - \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{x})\|_f &\leq \|\boldsymbol{y} - \boldsymbol{z}\|_f + \|\boldsymbol{z} - \boldsymbol{x}\|_f,\\ f((\|\boldsymbol{y}_i - \boldsymbol{x}_i\|)_{i\in I}) &\leq f((\|\boldsymbol{y}_i - \boldsymbol{z}_i\|)_{i\in I}) + f((\|\boldsymbol{z}_i - \boldsymbol{x}_i\|_{i\in I}),\\ f(\boldsymbol{a}) &\leq f(\boldsymbol{b}) + f(\boldsymbol{c}).\end{aligned}$$

In a similar way you can prove the other inequalities so f preserves triangular triplets.

(3)⇒ (4) Let $\boldsymbol{a}, \boldsymbol{b} \in [0, \infty)^I$ such that $\boldsymbol{a} \preceq \boldsymbol{b}$. Then it is clear that $(\boldsymbol{a}, \frac{1}{2}\boldsymbol{b}, \frac{1}{2}\boldsymbol{b})$ is a triangular triplet so $(f(\boldsymbol{a}), f(\frac{1}{2}\boldsymbol{b}), f(\frac{1}{2}\boldsymbol{b}))$ is also triangular. Hence and since f is positive homogeneous we obtain that

$$f(\boldsymbol{a}) \leq f(\tfrac{1}{2}\boldsymbol{b}) + f(\tfrac{1}{2}\boldsymbol{b}) = \tfrac{1}{2}f(\boldsymbol{b}) + \tfrac{1}{2}f(\boldsymbol{b}) = f(\boldsymbol{b})$$

so f is isotone.

Moreover, for arbitrary $\boldsymbol{a}, \boldsymbol{b} \in [0, \infty)^I$, we have that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} + \boldsymbol{b})$ is a triangular triplet so $f(\boldsymbol{a} + \boldsymbol{b}) \leq f(\boldsymbol{a}) + f(\boldsymbol{b})$, that is, f is subadditive. □

Later on, Martín, Mayor and Valero [15] characterized asymmetric norm aggregation functions on products as follows:

Theorem 2 ([15])**.** *Let $f : [0, +\infty)^I \to [0, +\infty)$ be a function. The following statements are equivalent:*

(1) f is an asymmetric norm aggregation function on products;
(2) $f^{-1}(0) = \mathbf{0}$, f is positive homogeneous and it preserves asymmetric triangular triplets;
(3) $f^{-1}(0) = \mathbf{0}$, f is an isotone sublinear function.

From the two above results we can obtain the following:

Corollary 1. *Let $f : [0, +\infty)^I \to [0, +\infty)$ be a function. The following statements are equivalent:*

(1) f is an asymmetric norm aggregation function on products;
(2) f is a norm aggregation function on products;
(3) $((\mathbb{R}^2)^I, \|\cdot\|_f)$ is a normed space where $\|\boldsymbol{x}\|_f = f((\|\boldsymbol{x}_i\|)_{i\in I})$ and $\|\cdot\|$ is the Euclidean norm, for all $\boldsymbol{x} \in (\mathbb{R}^2)^I$;
(4) $f^{-1}(0) = \mathbf{0}$, f is positive homogeneous and it preserves asymmetric triangular triplets;
(5) $f^{-1}(0) = \mathbf{0}$, f is positive homogeneous and it preserves triangular triplets;
(6) $f^{-1}(0) = \mathbf{0}$, f is an isotone sublinear function.

Remark 2. *We notice that in [15], the authors asserted that there exist norm aggregation functions on products which are not asymmetric norm aggregation functions on products and they provided an example which supposedly showed this. Unfortunately, it was wrong and these two concepts coincide (see the above corollary).*

Remark 3. *By the previous corollary we have that (asymmetric) norm aggregation functions on products are precisely the isotone norms on the semivector space* $[0,+\infty)^I$.

Remark 4. *Since a norm n on $\mathbb{R}^I$ is a sublinear function verifying that $n^{-1}(0) = \mathbf{0}$ it is natural to wonder if it is also isotone since, in this case, its restriction to $[0,+\infty)^I$ would be a norm aggregation function on products (and on sets). Nevertheless, not all the norms are isotone. For example, let us consider the vector space $\mathbb{R}^2$ and the sup norm $\|(x,y)\|_\infty = \max\{|x|,|y|\}$. Consider $R : \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation through $\frac{\pi}{3}$. Since R is an injective linear transformation the composition of R with the sup norm is also a norm on $\mathbb{R}^2$ given by*

$$n(x,y) = \left\| \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\infty = \max\left\{ \left| \frac{x-\sqrt{3}y}{2} \right|, \left| \frac{\sqrt{3}x+y}{2} \right| \right\}.$$

Nevertheless, the restriction of n to $[0,+\infty)^2$ is not isotone since

$$n(0,1) = \frac{\sqrt{3}}{2} \not\leq n(\tfrac{1}{10},1) = \frac{\sqrt{3}}{2} - \frac{1}{20}.$$

By the above results, n is not a norm aggregation function on products.

Example 3.

- *The function $f : [0,+\infty)^k \to [0,+\infty)$ given by $f(x_1,\ldots,x_k) = x_1 + \ldots + x_k$ is a(n) (asymmetric) norm aggregation function on products since it is clearly isotone, positive homogeneous, subadditive and $f^{-1}(0) = \{\mathbf{0}\}$.*
- *The function $f : [0,+\infty)^k \to [0,+\infty)$ given by $f(x_1,\ldots,x_k) = \max\{x_1,\ldots,x_k\}$ is a(n) (asymmetric) norm aggregation function on products since it is clearly isotone, positive homogeneous, subadditive and $f^{-1}(0) = \{\mathbf{0}\}$.*
- *The function $f : [0,+\infty)^2 \to [0,+\infty)$ given by*

$$f(x,y) = \begin{cases} (x+y)e^{\frac{x}{x+y}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is a(n) (asymmetric) norm aggregation function on products.

The following lemma provides a method for obtaining new norm aggregation functions on products.

Lemma 3. *Let $\{f_i : [0,+\infty)^{q_i} \to [0,+\infty) : i \in \{1,\ldots,k\}\}$ be a finite family of (asymmetric) norm aggregation functions on products. If $f : [0,+\infty)^k \to [0,+\infty)$ is a(n) (asymmetric) norm aggregation function on products then $f \circ (f_1,\ldots,f_k) : [0,+\infty)^{q_1+\ldots+q_k} \to [0,+\infty)$ is a(n) (asymmetric) norm aggregation function on products.*

Proof. It is straightforward. □

Example 4.

- *Since $f : [0,+\infty)^2 \to [0,+\infty), f_1 : [0,+\infty)^2 \to [0,+\infty), f_2 : [0,+\infty) \to [0,+\infty)$ given by*

$$f(x,y) = x+y$$
$$f_1(x,y) = \sqrt{x^2+y^2}$$
$$f_2(x) = x$$

for all $x, y \in [0, +\infty)$ are norm aggregation function on products then, by the previous lemma, the function $g = f \circ (f_1, f_2) : [0, +\infty)^3 \to [0, +\infty)$ given by

$$g(x, y, z) = \sqrt{x^2 + y^2} + z$$

for all $x, y, z \in [0, +\infty)$ is a norm aggregation function on products.

- *Since $f : [0, +\infty)^2 \to [0, +\infty), f_1 : [0, +\infty)^2 \to [0, +\infty), f_2 : [0, +\infty)^2 \to [0, +\infty)$ given by*

$$f(x, y) = \max\{x, y\}$$
$$f_1(x, y) = 2x + y$$
$$f_2(x, y) = x + 2y$$

for all $x, y \in [0, +\infty)$ are norm aggregation function on products then, by the previous lemma, the function $g = f \circ (f_1, f_2) : [0, +\infty)^4 \to [0, +\infty)$ given by

$$g(x, y, z, t) = \max\{2x + y, z + 2t\}$$

for all $x, y, z, t \in [0, +\infty)$ is a norm aggregation function on products.

Next we deal with the problem of characterizing those functions which aggregate norms or asymmetric norms on sets. Contrarily to the situation with the aggregation on products, the functions that aggregate norms on sets are different from the functions that aggregate asymmetric norms on sets (see Example 5).

We first give a characterization of norm aggregation functions on sets. Notice that the proof of the following theorem is not a simple adaptation of the proof of Theorem 1 due to the difference between $\widetilde{\boldsymbol{n}}$ and $\boldsymbol{n}$ (see Definition 2).

Theorem 3. *Let $f : [0, +\infty)^I \to [0, +\infty)$ be a function and let g be the restriction of f to $(0, +\infty)^I \cup \{\mathbf{0}\}$. The following statements are equivalent:*

(1) f is a norm aggregation function on sets;
(2) for every family of norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$, $(\mathbb{R}^2, f \circ \boldsymbol{n})$ is a normed space;
(3) $g^{-1}(0) = \mathbf{0}$ and g is an isotone sublinear function;
(4) $g^{-1}(0) = \mathbf{0}$, g is positive homogeneous and it preserves asymmetric triangular triplets;
(5) $g^{-1}(0) = \mathbf{0}$, g is positive homogeneous and it preserves triangular triplets.

Proof. (1) $\Rightarrow$ (2) is obvious.

Let us prove (2) $\Rightarrow$ (3). By taking the family of norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$ such that n_i is the Euclidean norm for all $i \in I$ we have that $g(\mathbf{0}) = f(\mathbf{0}) = f((n_i(\mathbf{0}))_{i \in I}) = f \circ \boldsymbol{n}(\mathbf{0}) = 0$ since $f \circ \boldsymbol{n}$ is a norm on $\mathbb{R}^2$. Furthermore, suppose that we can find $\boldsymbol{a} \in (0, +\infty)^I$ such that $g(\boldsymbol{a}) = f(\boldsymbol{a}) = 0$. Since $\boldsymbol{a}_i \neq 0$ for all $i \in I$ then $n_i(x, y) = \frac{\boldsymbol{a}_i}{\sqrt{2}} \sqrt{x^2 + y^2}$ is a norm on $\mathbb{R}^2$. Then $f \circ \boldsymbol{n}$ is a norm on $\mathbb{R}^2$ but

$$f \circ \boldsymbol{n}(1, 1) = f((n_i(1, 1))_{i \in I}) = f((\boldsymbol{a}_i)_{i \in I}) = f(\boldsymbol{a}) = 0$$

which contradicts that $f \circ \boldsymbol{n}$ is a norm. Hence $g^{-1}(0) = \mathbf{0}$.

Take now an arbitrary $\boldsymbol{a} \in (0, +\infty)^I$, $\lambda \geq 0$ and consider again n_i as above for all $i \in I$. Then

$$f \circ \boldsymbol{n}(\lambda(1, 1)) = \lambda(f \circ \boldsymbol{n})(1, 1)$$
$$f((n_i(\lambda(1, 1)))_{i \in I}) = \lambda f((n_i(1, 1))_{i \in I})$$
$$f(\lambda \boldsymbol{a}) = \lambda f(\boldsymbol{a})$$
$$g(\lambda \boldsymbol{a}) = \lambda g(\boldsymbol{a})$$

so g is positive homogeneous.

We next show that g is subadditive. Let $\boldsymbol{a}, \boldsymbol{b} \in (0, +\infty)^I \cup \{\mathbf{0}\}$. Since $g(\mathbf{0}) = 0$, if any of $\boldsymbol{a}, \boldsymbol{b}$ is $\mathbf{0}$ then it is obvious that $g(\boldsymbol{a} + \boldsymbol{b}) \leq g(\boldsymbol{a}) + g(\boldsymbol{b})$. So suppose that $\boldsymbol{a}, \boldsymbol{b} \in (0, +\infty)^I$. Consider now the family $\{n_i : i \in I\}$ of norms in $\mathbb{R}^2$ given by $n_i(x, y) = \boldsymbol{a}_i|x| + \boldsymbol{b}_i|y|$ for all $i \in I$. By assumption, $f \circ \boldsymbol{n}$ is a norm on $\mathbb{R}^2$ so

$$\begin{aligned} f \circ \boldsymbol{n}(1,1) &\leq f \circ \boldsymbol{n}(1,0) + f \circ \boldsymbol{n}(0,1) \\ f((n_i(1,1))_{i\in I}) &\leq f((n_i(1,0))_{i\in I}) + f((n_i(0,1))_{i\in I}) \\ g((n_i(1,1))_{i\in I}) &\leq g((n_i(1,0))_{i\in I}) + g((n_i(0,1))_{i\in I}) \\ g((\boldsymbol{a}_i + \boldsymbol{b}_i)_{i\in I}) &\leq g((\boldsymbol{a}_i)_{i\in I}) + g((\boldsymbol{b}_i)_{i\in I}) \\ g(\boldsymbol{a} + \boldsymbol{b}) &\leq g(\boldsymbol{a}) + g(\boldsymbol{b}). \end{aligned}$$

Hence g is subadditive.

Finally, we prove that g is isotone. Let $\boldsymbol{a}, \boldsymbol{b} \in (0, +\infty)^I \cup \{\mathbf{0}\}$ such that $\boldsymbol{a} \preceq \boldsymbol{b}$. If $\boldsymbol{a} = \mathbf{0}$ or $\boldsymbol{a} = \boldsymbol{b}$ it is clear that $g(\boldsymbol{a}) \leq g(\boldsymbol{b})$. Therefore, we suppose that $\boldsymbol{a} \neq \boldsymbol{b}$ and $\boldsymbol{a} \neq \mathbf{0}$. In particular $0 < \boldsymbol{a}_i \leq \boldsymbol{b}_i$ for all $i \in I$. Let $I_1 = \{i \in I : \boldsymbol{a}_i \neq \boldsymbol{b}_i\}$ and $I_2 = \{i \in I : \boldsymbol{a}_i = \boldsymbol{b}_i\}$. Obviously $I = I_1 \cup I_2$. For each $i \in I_1$, let us consider the norm N_i on $\mathbb{R}^2$ given by

$$N_i(x, y) = \frac{\boldsymbol{b}_i - \boldsymbol{a}_i}{\sqrt{2}}|x| + \frac{\boldsymbol{a}_i}{\sqrt{2}}|y|$$

for every $(x, y) \in \mathbb{R}^2$. If we consider $R : \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation through $\frac{\pi}{4}$, then $n_i := N_i \circ R$ is also a norm on $\mathbb{R}^2$ for every $i \in I_1$ since R is an injective linear transformation. Concretely,

$$n_i(x, y) = \frac{\boldsymbol{b}_i - \boldsymbol{a}_i}{\sqrt{2}}\left|\frac{x - y}{\sqrt{2}}\right| + \frac{\boldsymbol{a}_i}{\sqrt{2}}\left|\frac{x + y}{\sqrt{2}}\right|$$

for every $(x, y) \in \mathbb{R}^2$ and every $i \in I_1$.

On the other hand, for each $i \in I_2$ consider the norm n_i on $\mathbb{R}^2$ given by

$$n_i(x, y) = \frac{\boldsymbol{b}_i}{2}(|x| + |y|)$$

for all $(x, y) \in \mathbb{R}^2$.

Since $\{n_i : i \in I\}$ is a family of norms on $\mathbb{R}^2$ then $f \circ \boldsymbol{n}$ is also a norm on $\mathbb{R}^2$. Therefore

$$\begin{aligned} f \circ \boldsymbol{n}(1,1) &\leq f \circ \boldsymbol{n}(1,0) + f \circ \boldsymbol{n}(0,1) \\ f((n_i(1,1))_{i\in I}) &\leq f((n_i(1,0))_{i\in I}) + f((n_i(0,1))_{i\in I}) \\ g((n_i(1,1))_{i\in I}) &\leq g((n_i(1,0))_{i\in I}) + g((n_i(0,1))_{i\in I}) \\ g((\boldsymbol{a}_i)_{i\in I}) &\leq g(\left(\tfrac{\boldsymbol{b}_i}{2}\right)_{i\in I}) + g(\left(\tfrac{\boldsymbol{b}_i}{2}\right)_{i\in I}) \\ g(\boldsymbol{a}) &\leq g\left(\tfrac{\boldsymbol{b}}{2}\right) + g\left(\tfrac{\boldsymbol{b}}{2}\right) = \tfrac{1}{2}g(\boldsymbol{b}) + \tfrac{1}{2}g(\boldsymbol{b}) = g(\boldsymbol{b}), \end{aligned}$$

since g is positive homogeneous. Consequently, g is isotone.

(3) $\Rightarrow$ (4) We only need to prove that g preserves asymmetric triangular triplets. Let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ((0, +\infty)^I \cup \{\mathbf{0}\})^3$ such that $\boldsymbol{a} \preceq \boldsymbol{b} + \boldsymbol{c}$. Since g is isotone and subadditive we have that

$$g(\boldsymbol{a}) \leq g(\boldsymbol{b} + \boldsymbol{c}) \leq g(\boldsymbol{b}) + g(\boldsymbol{c})$$

so $(g(\boldsymbol{a}), g(\boldsymbol{b}), g(\boldsymbol{c}))$ is an asymmetric triangular triplet.

(4) $\Rightarrow$ (5) This is obvious.

(5) $\Rightarrow$ (1) Let V be a vector space and $\{n_i : i \in I\}$ a family of norms on V. Let us check that $f \circ \boldsymbol{n}$ is a norm on V. It is clear that $f \circ \boldsymbol{n}(0_V) = 0$.

Suppose now that there exists $v \in V \backslash \{0_V\}$ such that $f \circ \boldsymbol{n}(v) = 0$. Since $\boldsymbol{n}(v) \in (0, +\infty)^I$ then $f \circ \boldsymbol{n}(v) = g \circ \boldsymbol{n}(v) = 0$. By hypothesis $g^{-1}(0) = \mathbf{0}$ so $n_i(v) = 0$ for all $i \in I$, which is a contradiction.

Let $\lambda \in \mathbb{R}$ and $v \in V$. Since $\boldsymbol{n}(\lambda v) \in (0, +\infty)^I \cup \{\mathbf{0}\}$ and g is positive homogeneous then

$$f \circ \boldsymbol{n}(\lambda v) = g \circ \boldsymbol{n}(\lambda v) = g((n_i(\lambda v))_{i \in I}) = g(|\lambda|(n_i(v))_{i \in I}) = |\lambda| g((n_i(v))_{i \in I})) = |\lambda| f \circ \boldsymbol{n}(v).$$

Finally, let $v, w \in V$. It is clear that $\boldsymbol{n}(v), \boldsymbol{n}(w) \in (0, +\infty)^I \cup \{\mathbf{0}\}$ and $(\boldsymbol{n}(v+w), \boldsymbol{n}(v), \boldsymbol{n}(w))$ is a triangular triplet. By assumption $(g(\boldsymbol{n}(v+w)), g(\boldsymbol{n}(v)), g(\boldsymbol{n}(w)))$ is a triangular triplet so

$$f(\boldsymbol{n}(v+w)) = g(\boldsymbol{n}(v+w)) \leq g(\boldsymbol{n}(v)) + g(\boldsymbol{n}(w)) = f(\boldsymbol{n}(v)) + f(\boldsymbol{n}(w)).$$

Consequently, $f \circ \boldsymbol{n}$ is a norm on V. □

From Theorems 1 and 3 we immediately deduce the already observed fact that if f is a norm aggregation function on products then it is a norm aggregation function on sets. We can also notice that the difference between these two kind of functions lies in the images of the tuples which have at least one coordinate equal to 0 (as we have show in Example 2). Moreover, we emphasize that in contrast with the characterization of norm aggregation functions on products and asymmetric norm aggregation functions on products, which is the same, we can find norm aggregation function on sets which are not asymmetric norm aggregation function on sets as the next example shows.

Example 5. *Let us consider $f : [0, +\infty)^2 \to [0, +\infty)$ given by*

$$f(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ x + y & \text{otherwise} \end{cases}.$$

It is clear that f is a norm aggregation function on sets since it satisfies the conditions of Theorem 3. Nevertheless, it is not an asymmetric norm aggregation function on sets. In fact, consider on $\mathbb{R}$ two asymmetric norms given by

$$n_1(x) = \max\{x, 0\}$$
$$n_2(x) = \max\{-x, 0\}$$

for all $x \in \mathbb{R}$. Then $f \circ \boldsymbol{n}(1) = f(n_1(1), n_2(1)) = f(1, 0) = 0$ and $f \circ \boldsymbol{n}(-1) = f(n_1(-1), n_2(-1)) = f(0, 1) = 0$. Since $1 \neq 0$ then $f \circ \boldsymbol{n}$ is not an asymmetric norm on $\mathbb{R}$.

The next result establish a characterization of functions aggregating asymmetric norms on sets.

Theorem 4. *Let $f : [0, +\infty)^I \to [0, +\infty)$ be a function. The following statements are equivalent:*

(1) f is an asymmetric norm aggregation function on sets;
(2) for every family of asymmetric norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$, $(\mathbb{R}^2, f \circ \boldsymbol{n})$ is an asymmetric normed space;
(3) $f(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in f^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; f is an isotone sublinear function;
(4) $f(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in f^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; f is positive homogeneous and it preserves asymmetric triangular triplets;
(5) $f(\mathbf{0}) = 0$; if $\boldsymbol{a}, \boldsymbol{b} \in f^{-1}(0)$ then there exists $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$; f is positive homogeneous and it preserves triangular triplets.

Proof. (1) ⇒ (2) is clear.

(2) $\Rightarrow$ (3) Using that the value in $(0,0)$ of every asymmetric norm on $\mathbb{R}^2$ is 0, we immediately deduce that $f(\mathbf{0}) = 0$.

Let $\boldsymbol{a}, \boldsymbol{b} \in [0,+\infty)^I$ such that $f(\boldsymbol{a}) = f(\boldsymbol{b}) = 0$. Suppose that we cannot find $j \in I$ such that $\boldsymbol{a}_j = \boldsymbol{b}_j = 0$. For each $i \in I$, let us consider the asymmetric norm n_i on the vector space $\mathbb{R}^2$ given by

$$n_i(x,y) = \boldsymbol{a}_i \max\{x,y,0\} + \boldsymbol{b}_i \max\{-x,-y,0\}$$

for all $x, y \in \mathbb{R}$. By assumption $f \circ \boldsymbol{n}$ is an asymmetric norm on $\mathbb{R}^2$. However, $f \circ \boldsymbol{n}(1,1) = f((n_i(1,1))_{i\in I}) = f(\boldsymbol{a}) = 0$ and $f \circ \boldsymbol{n}(-1,-1) = f((n_i(-1,-1))_{i\in I}) = f(\boldsymbol{b}) = 0$ but $(1,1)$ is different from $(0,0)$ which contradicts that $f \circ \boldsymbol{n}$ is an asymmetric norm on $\mathbb{R}^2$.

We next check that f is positive homogeneous. Let $\boldsymbol{a} \in [0,+\infty)^I$ and $\lambda \geq 0$. Let $\boldsymbol{b} \in [0,+\infty)^I$ such that for every $i \in I$ $\boldsymbol{b}_i \neq 0$ if $\boldsymbol{a}_i = 0$. Considering the same family of asymmetric norms $\{n_i : i \in I\}$ on $\mathbb{R}^2$ as above we have that

$$\begin{aligned} f \circ \boldsymbol{n}(\lambda(1,1)) &= \lambda(f \circ \boldsymbol{n})(1,1) \\ f((n_i(\lambda(1,1)))_{i\in I}) &= \lambda f((n_i(1,1))_{i\in I}) \\ f(\lambda \boldsymbol{a}) &= \lambda f(\boldsymbol{a}) \end{aligned}$$

so f is positive homogeneous.

Let us prove that f is subadditive. Let $\boldsymbol{a}, \boldsymbol{b} \in [0,+\infty)^I$. Let $I_1 = \{i \in I : \boldsymbol{a}_i = \boldsymbol{b}_i = 0\}$, $I_2 = \{i \in I : \boldsymbol{a}_i = 0, \boldsymbol{b}_i \neq 0\}$, $I_3 = \{i \in I : \boldsymbol{a}_i \neq 0, \boldsymbol{b}_i = 0\}$ and $I_4 = \{i \in I : \boldsymbol{a}_i \neq 0, \boldsymbol{b}_i \neq 0\}$. Given $i \in I$ let us consider the asymmetric norm n_i on $\mathbb{R}^2$ given by

- $n_i(x,y) = \max\{-x,0\} + \max\{-y,0\}$ if $i \in I_1$;
- $n_i(x,y) = \boldsymbol{b}_i \max\{-x,y,0\}$ if $i \in I_2$;
- $n_i(x,y) = \boldsymbol{a}_i \max\{x,-y,0\}$ if $i \in I_3$;
- $n_i(x,y) = \boldsymbol{a}_i \max\{x,0\} + \boldsymbol{b}_i \max\{y,0\}$ if $i \in I_4$.

Since $\{n_i : i \in I\}$ is a family of asymmetric norms on $\mathbb{R}^2$, by assumption, $f \circ \boldsymbol{n}$ is an asymmetric norm on $\mathbb{R}^2$. Consequently

$$\begin{aligned} f \circ \boldsymbol{n}(1,1) &\leq f \circ \boldsymbol{n}(1,0) + f \circ \boldsymbol{n}(0,1) \\ f((n_i(1,1))_{i\in I}) &\leq f((n_i(1,0))_{i\in I}) + f((n_i(0,1))_{i\in I}) \\ f((\boldsymbol{a}_i + \boldsymbol{b}_i)_{i\in I}) &\leq f((\boldsymbol{a}_i)_{i\in I}) + f((\boldsymbol{b}_i)_{i\in I}) \\ f(\boldsymbol{a} + \boldsymbol{b}) &\leq f(\boldsymbol{a}) + f(\boldsymbol{b}) \end{aligned}$$

so f is subadditive.

For proving that f is isotone, pick up $\boldsymbol{a}, \boldsymbol{b} \in [0,+\infty)^I$ with $\boldsymbol{a} \preceq \boldsymbol{b}$. Let us define $I_1 = \{i \in I : 0 = \boldsymbol{a}_i = \boldsymbol{b}_i\}$, $I_2 = \{i \in I : 0 = \boldsymbol{a}_i < \boldsymbol{b}_i\}$, $I_3 = \{i \in I : 0 < \boldsymbol{a}_i = \boldsymbol{b}_i\}$ and $I_4 = \{i \in I : 0 < \boldsymbol{a}_i < \boldsymbol{b}_i\}$. For each $i \in I$, consider an asymmetric norm n_i on $\mathbb{R}^2$ defined as

- $n_i(x,y) = \max\{-x,0\} + \max\{-y,0\}$, if $i \in I_1$;
- $n_i(x,y) = \dfrac{\boldsymbol{b}_i}{2}(|-x+y| + \max\{-x-y,0\})$, if $i \in I_2$;
- $n_i(x,y) = \dfrac{\boldsymbol{b}_i}{2}(|x| + |y|)$, if $i \in I_3$;
- $n_i(x,y) = \dfrac{\boldsymbol{b}_i - \boldsymbol{a}_i}{\sqrt{2}}\left|\dfrac{x-y}{\sqrt{2}}\right| + \dfrac{\boldsymbol{a}_i}{\sqrt{2}}\left|\dfrac{x+y}{\sqrt{2}}\right|$, if $i \in I_4$.

Since $\{n_i : i \in I\}$ is a family of asymmetric norms on $\mathbb{R}^2$ then $f \circ \boldsymbol{n}$ is an asymmetric norm on $\mathbb{R}^2$ so

$$\begin{aligned} f \circ \boldsymbol{n}(1,1) &\leq f \circ \boldsymbol{n}(1,0) + f \circ \boldsymbol{n}(0,1) \\ f((n_i(1,1))_{i\in I}) &\leq f((n_i(1,0))_{i\in I}) + f((n_i(0,1))_{i\in I}) \\ f((\boldsymbol{a}_i)_{i\in I}) &\leq f(\left(\tfrac{\boldsymbol{b}_i}{2}\right)_{i\in I}) + f(\left(\tfrac{\boldsymbol{b}_i}{2}\right)_{i\in I}) \\ f(\boldsymbol{a}) &\leq f\left(\tfrac{\boldsymbol{b}}{2}\right) + f\left(\tfrac{\boldsymbol{b}}{2}\right) = \tfrac{1}{2}f(\boldsymbol{b}) + \tfrac{1}{2}f(\boldsymbol{b}) = f(\boldsymbol{b}). \end{aligned}$$

so f is isotone.

The proofs of (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are similar to the proofs of the same implications of Theorem 3.

(4) $\Rightarrow$ (1) can be proved with an easy adaptation of the proof of (5) $\Rightarrow$ (1) of Theorem 3, since for every family of asymmetric norms $\{n_i : i \in I\}$ on a vector space V we have that $(\boldsymbol{n}(v+w), \boldsymbol{n}(v), \boldsymbol{n}(w))$ is an asymmetric triangular triplet (but not necessarily a triangular triplet) for every $v, w \in V$. The only slight difference is with proving that if $\{n_i : i \in I\}$ is a family of asymmetric norms on a vector space V then $f \circ \boldsymbol{n}(v) = f \circ \boldsymbol{n}(-v) = 0$ if and only if $v = 0_V$. To prove this, suppose that $f \circ \boldsymbol{n}(v) = f \circ \boldsymbol{n}(-v) = 0$. By assumption, there exists $j \in I$ such that $n_j(v) = n_j(-v) = 0$ so $v = 0_V$ since n_j is an asymmetric norm on V.

Finally, to conclude the equivalence of all the statements of this theorem, we will prove (5) $\Rightarrow$ (4). Since f is positive homogeneous and it preserves triangular triplets we have that f is isotone and subadditive (see the proof of (3) $\Rightarrow$ (4) of Theorem 1). In particular, if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0,+\infty)^I)^3$ is an asymmetric triangular triplet, that is $\boldsymbol{a} \preceq \boldsymbol{b} + \boldsymbol{c}$, since f is isotone and subadditive we have that

$$f(\boldsymbol{a}) \leq f(\boldsymbol{b} + \boldsymbol{c}) \leq f(\boldsymbol{b}) + f(\boldsymbol{c})$$

so $(f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}))$ is an asymmetric triangular triplet. □

4. Strongly Aggregation of Norms

In this section, we address another related problem which was studied in [9] for metrics. Suppose that $f : [0,+\infty)^I \to [0,+\infty)$ is a norm aggregation function on products and $\{(V_i, n_i) : i \in I\}$ is an arbitrary family of normed vector spaces. Then $f \circ \widetilde{\boldsymbol{n}}$ is a norm on $\prod_{i\in I} V_i$. It is natural to wonder when the topology $\tau(f \circ \widetilde{\boldsymbol{n}})$ on $\prod_{i\in I} V_i$ generated by the norm $f \circ \widetilde{\boldsymbol{n}}$ is equal to the product topology $\Pi\tau(n_i)$ associated with the Cartesian product of the family of topological spaces $\{(V_i, \tau(n_i)) : i \in I\}$, where $\tau(n_i)$ is the topology induced by the norm n_i. Recall that $\tau(n_i)$ has as a base the family $\{B_{n_i}(v, \varepsilon) : v \in V_i, \varepsilon > 0\}$ where $B_{n_i}(v, \varepsilon) = \{w \in V_i : n_i(w - v) < \varepsilon\}$.

Following the terminology of [9] we introduce the following concepts.

Definition 5. *A function $f : [0,+\infty)^I \to [0,+\infty)$ is said to be* an strongly norm aggregation function on products *if it is a norm aggregation function on products and whenever $\{(V_i, n_i) : i \in I\}$ is a family of normed vector spaces then $\Pi\tau(n_i) = \tau(f \circ \widetilde{\boldsymbol{n}})$.*

As we have already observed, the above concept was first considered for families of metric spaces [9]. In this way, strongly metric aggregation functions on products have been characterized in [9] under the name strongly metric-preserving functions. Below, we describe which are the strongly norm aggregation functions on products.

We begin proving the following result which follows the proof of ([9], (Ch. 10, Lemma 1)).

Lemma 4. *Let $f : [0,+\infty)^I \to [0,+\infty)$ be a norm aggregation function on products. If $\{(V_i, n_i) : i \in I\}$ is a family of normed vector spaces then*

$$\Pi\tau(n_i) \subseteq \tau(f \circ \widetilde{\boldsymbol{n}}) \text{ on } \prod_{i\in I} V_i.$$

Proof. Let $\varepsilon > 0$, $j \in I$ and $\boldsymbol{v} \in \prod_{i\in I} V_i$. Consider the subbasic open set $p_j^{-1}(B_{n_j}(\boldsymbol{v}_j, \varepsilon))$ in the product topology $\Pi\tau(n_i)$, where p_j denotes the jth projection map. Define $\boldsymbol{\varepsilon} \in [0,+\infty)^I$

in such a way that $\boldsymbol{\varepsilon}_j = \varepsilon$ and $\boldsymbol{\varepsilon}_i = 0$ if $i \neq j$. Since $\boldsymbol{\varepsilon} \neq \mathbf{0}$ and f is a norm aggregation function on products then $f(\boldsymbol{\varepsilon}) \neq 0$ by Theorem 1. Let us show that $\boldsymbol{v} \in B_{f\circ\widetilde{\boldsymbol{n}}}(\boldsymbol{v}, f(\boldsymbol{\varepsilon})) \subseteq p_j^{-1}(B_{n_j}(\boldsymbol{v}_j, \varepsilon))$. If $\boldsymbol{w} \in B_{f\circ\widetilde{\boldsymbol{n}}}(\boldsymbol{v}, f(\boldsymbol{\varepsilon}))$ then $f((n_i(\boldsymbol{w}_i - \boldsymbol{v}_i)_{i\in I}) < f(\boldsymbol{\varepsilon})$. If $n_j(\boldsymbol{w}_j - \boldsymbol{v}_j) \geq \varepsilon$ then $\boldsymbol{\varepsilon} \preceq_I (n_i(\boldsymbol{w}_i - \boldsymbol{v}_i))_{i\in I}$. Since by Theorem 1 f is isotone then $f(\boldsymbol{\varepsilon}) \leq f((n_i(\boldsymbol{w}_i - \boldsymbol{v}_i))_{i\in I})$ which is a contradiction. Consequently $n_j(\boldsymbol{w}_j - \boldsymbol{v}_j) < \varepsilon$, that is, $\boldsymbol{w} \in p_j^{-1}(B_{n_j}(\boldsymbol{v}_j, \varepsilon))$. We conclude that $\Pi\tau(n_i) \subseteq \tau(f \circ \widetilde{\boldsymbol{n}})$. □

Recall that an infinite product of nontrivial normed spaces is not normable ([16], (Theorem 6.4.5)), that is, the product topology cannot be induced by a norm. In fact, a topological vector space V is normable if and only if it is Hausdorff and has a convex bounded neighborhood of 0_V ([16], (Theorem 6.2.1)). Therefore, there does not exist a strongly norm aggregation function on products $f : [0, +\infty)^I \to [0, +\infty)$ when the cardinal of I is infinite. So in order to prove the converse of Lemma 4 we can restrict ourselves to functions $f : [0, +\infty)^k \to [0, +\infty)$, where $k \in \mathbb{N}$. In this case we can prove that strongly norm aggregation functions on products and norm aggregation functions on products are equivalent concepts (compare with [9]).

Theorem 5. *A function $f : [0, +\infty)^k \to [0, +\infty)$ is a strongly norm aggregation function on products if and only if it is a norm aggregation function on products.*

Proof. Necessity is obvious.

For sufficiency, suppose that f is a norm aggregation function on products. By Lemma 4 we only have to show that given a family of normed vector spaces $\{(V_i, n_i) : i = 1, \ldots, k\}$ then $\tau(f \circ \widetilde{\boldsymbol{n}}) \subseteq \Pi\tau(n_i)$.

Let us consider the set $F = \{\boldsymbol{x} \in [0, +\infty)^k : \max\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \leq 1\}$. By Theorem 1 f is isotone so $f(\boldsymbol{x}) \leq f(\mathbf{1}) := \alpha$ for all $\boldsymbol{x} \in F$.

Let $\boldsymbol{v} \in \prod_{i=1}^k V_i$ and $\varepsilon > 0$. We next show that $\boldsymbol{v} \in \prod_{i=1}^k B_{n_i}(\boldsymbol{v}_i, \frac{\varepsilon}{2\alpha}) \subseteq B_{f\circ\widetilde{\boldsymbol{n}}}(\boldsymbol{v}, \varepsilon)$. Let $\boldsymbol{w} \in \prod_{i=1}^k B_{n_i}(\boldsymbol{v}_i, \frac{\varepsilon}{2\alpha})$, that is, $n_i(\boldsymbol{w}_i - \boldsymbol{v}_i) < \frac{\varepsilon}{2\alpha}$ for all $i \in \{1, \ldots, k\}$. Let $\boldsymbol{y} = (n_1(\boldsymbol{w}_1 - \boldsymbol{v}_1), \ldots, n_k(\boldsymbol{w}_k - \boldsymbol{v}_k))$. Hence $\frac{2\alpha}{\varepsilon}\boldsymbol{y} \in F$ and since by Theorem 1 f is positive homogeneous then $f(\frac{2\alpha}{\varepsilon}\boldsymbol{y}) = \frac{2\alpha}{\varepsilon} f(\boldsymbol{y}) \leq \alpha$. Consequently $f(\boldsymbol{y}) = f \circ \widetilde{\boldsymbol{n}}(\boldsymbol{w} - \boldsymbol{v}) \leq \frac{\varepsilon}{2} < \varepsilon$, that is, $\boldsymbol{w} \in B_{f\circ\widetilde{\boldsymbol{n}}}(\boldsymbol{v}, \varepsilon)$. Since $\prod_{i=1}^k B_{n_i}(\boldsymbol{v}_i, \frac{\varepsilon}{2\alpha})$ is open in the product topology we have that $\tau(f \circ \widetilde{\boldsymbol{n}}) \subseteq \Pi\tau(n_i)$, which finishes the proof. □

Corollary 2. *If $f, g : [0, +\infty)^k \to [0, +\infty)$ are two norm aggregation functions on products then for any family $\{(V_i, n_i) : i = 1, \ldots, k\}$ of normed vector spaces, $f \circ \widetilde{\boldsymbol{n}}$ and $g \circ \widetilde{\boldsymbol{n}}$ are equivalent norms on $\prod_{i=1}^k V_i$.*

Proof. Since f, g are norm aggregation functions on products then they are strongly norm aggregation functions on products so

$$\tau(f \circ \widetilde{\boldsymbol{n}}) = \Pi\tau(n_i) = \tau(g \circ \widetilde{\boldsymbol{n}}).$$

□

Next we treat the problem of characterizing the strongly norm aggregation functions on sets. In the first place we must give an appropriate definition. Let $f : [0, +\infty)^I \to [0, +\infty)$ be a norm aggregation function on sets and $\{n_i : i \in I\}$ be a family of norms on a vector space V. In order to obtain a similar result to Lemma 4 we first must think about the topology which makes here the role of the product topology. Notice that the product topology $\Pi\tau_i$ on the Cartesian product of a family of topological spaces $\{(X_i, \tau_i) : i \in I\}$ is the initial topology making all the projections $p_j : \prod_{i\in I} X_i \to X_j$ continuous. If all the sets X_i are equal to X, then X is homeomorphic to the diagonal subspace of the Cartesian product X^I given by $\Delta = \{\boldsymbol{x} \in X^I : \boldsymbol{x}_i = \boldsymbol{x}_j \text{ for all } i, j \in I\}$ (see [7], (Corollary 2.3.21)). Then the restriction of the product topology on X^I to the diagonal is homeomorphic to X

endowed with the supremum topology $\bigvee_{i\in i}\tau_i$ of the family of all topologies $\{\tau_i : i \in I\}$. This suggests the following definition:

Definition 6. *A function $f : [0,+\infty)^I \to [0,+\infty)$ is said to be* an strongly norm aggregation function on sets *if it is a norm aggregation function on sets and whenever $\{(V, n_i) : i \in I\}$ is a family of normed vector spaces then $\bigvee_{i\in I}\tau(n_i) = \tau(f\circ \boldsymbol{n})$.*

Lemma 5. *If $f : [0,+\infty)^I \to [0,+\infty)$ is a strongly norm aggregation function on products then it is a strongly norm aggregation function on sets.*

Proof. Suppose that f is a strongly norm aggregation function on products and let $\{(V, n_i) : i \in I\}$ be a family of normed vector spaces. Then $f\circ \boldsymbol{n}(v) = f\circ\widetilde{\boldsymbol{n}}((v)_{i\in I})$ for all $v \in V$. As we have previously observed $(V, \bigvee_{i\in I}\tau(n_i))$ is homeomorphic to $(\Delta, \prod_{i\in I}\tau(n_i))$ where Δ is the diagonal of V^I. Since f is a strongly norm aggregation function on products then $(\Delta, \prod_{i\in I}\tau(n_i)) = (\Delta, \tau(f\circ\widetilde{\boldsymbol{n}}))$ which is homeomorphic to $(V, \tau(f\circ\boldsymbol{n}))$. This shows that f is a strongly norm aggregation function on sets. □

The first natural question is if we can obtain a similar result to Lemma 4 for norm aggregation functions on sets and the supremum topology. The following example shows that this is no longer true.

Example 6. *Let us consider the projection over the first coordinate $p_1 : [0,+\infty)^2 \to [0,+\infty)$ which clearly is a norm aggregation function on sets (see Theorem 3).*

Consider the vector space $C([0,1])$ of all continuous real-valued functions on $[0,1]$ and the family $\{n_1, n_2\}$ of two norms on $C([0,1])$ given by

$$n_1(f) = \|f\|_1 = \int_0^1 |f(x)|dx,$$
$$n_2(f) = \|f\|_\infty = \sup_{x\in[0,1]} |f(x)|,$$

for all $f \in C([0,1])$. It is obvious that $n_1(f) \le n_2(f)$ for all $f \in C([0,1])$. In particular $\tau(n_1) \subseteq \tau(n_2)$. Nevertheless, we cannot find $k > 0$ such that $kn_2(f) \le n_1(f)$ for all $f \in C([0,1])$. In fact, given $k > 0$ consider the function

$$f_k(x) = \begin{cases} 0 & \text{if } 0 \le x \le \max\{0,1-k\} \\ \dfrac{(x-\max\{0,1-k\})}{k(1-\max\{0,1-k\})} & \text{if } \max\{0,1-k\} < x \le 1 \end{cases}$$

Then $k\cdot n_2(f_k) = k\cdot\frac{1}{k} = 1 \not\le n_1(f_k) = \frac{1}{2k}\min\{1,k\}$. Consequently, n_1 and n_2 are not equivalent.

Therefore, $\tau(n_1)\vee\tau(n_2) = \tau(n_2) \not\subseteq \tau(p_1\circ\boldsymbol{n}) = \tau(n_1)$, so p_1 is not an strongly norm aggregation function on sets.

Notice that in the previous example p_1 is continuous and $\tau(p_1\circ\boldsymbol{n}) \subseteq \tau(n_1)\vee\tau(n_2)$. This fact is not accidental as we next prove.

Proposition 1. *Let $f : [0,+\infty)^I \to [0,+\infty)$ be a norm aggregation function on sets. Given an arbitrary vector space V and a family of norms $\{n_i : i \in I\}$ on V then $\tau(f\circ\boldsymbol{n}) \subseteq \bigvee_{i\in I}\tau(n_i)$ on V if and only if $f|_{\mathrm{Im}(\boldsymbol{n})}$ is continuous at $\mathbf{0}$.*

Proof. Let V be a vector space and $\{n_i : i \in I\}$ be a family of norms on V.

For sufficiency, given $\varepsilon > 0$ by assumption we can find a finite subset J of I and $\delta > 0$ such that $\bigcap_{j\in J} B_{n_j}(0_V, \delta) \subseteq B_{f\circ\boldsymbol{n}}(0_V, \varepsilon)$. Let $\boldsymbol{x} \in \mathrm{Im}(\boldsymbol{n})$ such that $\boldsymbol{x}_j < \delta$ for all $j \in J$. Let $v \in V$ with $\boldsymbol{n}(v) = \boldsymbol{x}$. Then $v \in \bigcap_{j\in J} B_{n_j}(0_V, \delta) \subseteq B_{f\circ\boldsymbol{n}}(0_V, \varepsilon)$ so $f\circ\boldsymbol{n}(v - 0_V) = f(\boldsymbol{x}) = |f(\boldsymbol{x}) - f(\mathbf{0})| < \varepsilon$ so $f|_{\mathrm{Im}(\boldsymbol{n})}$ is continuous at $\mathbf{0}$.

Conversely, let $v \in V$ and $\varepsilon > 0$. Since $f\mid_{\mathrm{Im}(\boldsymbol{n})}$ is continuous at $\mathbf{0}$ we can find $\delta > 0$ and a finite subset J of I such that given $\boldsymbol{x} \in \mathrm{Im}(\boldsymbol{n})$ with $\boldsymbol{x}_j < \delta$ for all $j \in J$ then $f(\boldsymbol{x}) < \varepsilon$. We claim that $\bigcap_{j\in J} B_{n_j}(v,\delta) \subseteq B_{f\circ\boldsymbol{n}}(v,\varepsilon)$. If $w \in \bigcap_{j\in J} B_{n_j}(v,\delta)$ then $n_j(w-v) < \delta$ for all $j \in J$ so $f((n_i(w-v))_{i\in I}) < \varepsilon$, that is, $w \in B_{f\circ\boldsymbol{n}}(v,\varepsilon)$. □

Corollary 3. *Let $f : [0,+\infty)^I \to [0,+\infty)$ be a norm aggregation function on sets. If f is continuous at $\mathbf{0}$ then for every arbitrary vector space V and every family of norms $\{n_i : i \in I\}$ on V*

$$\tau(f\circ\boldsymbol{n}) \subseteq \bigvee_{i\in I} \tau(n_i) \text{ on } V.$$

Remark 5. *It is well-known that a sublinear function f defined on $\mathbb{R}^k$ is continuous at zero if and only if it is bounded on a neighborhood of zero ([24], (Proposition 2.1.6)). Hence, if f is also isotone then it is continuous at zero since it can be easily checked that it is bounded on a neighborhood of zero. The same idea works for a norm aggregation function on sets f defined on $[0,+\infty)^k$ since its restriction to $(0,+\infty)^k \cup \{\mathbf{0}\}$ is sublinear and isotone. Notice that for every family of norms $\{n_i : i \in \{1,\ldots,k\}\}$ on a vector space V we have that $\mathrm{Im}(\boldsymbol{n}) \subseteq (0,+\infty)^k \cup \{\mathbf{0}\}$. Consequently, if I is finite we have $\tau(f\circ\boldsymbol{n}) \subseteq \bigvee_{i\in I}\tau(n_i)$ for every norm aggregation function on sets f, as we next state.*

Corollary 4. *Let $f : [0,+\infty)^k \to [0,+\infty)$ be a norm aggregation function on sets. Then for every arbitrary vector space V and every family of norms $\{n_i : i \in \{1,\ldots,k\}\}$ on V*

$$\tau(f\circ\boldsymbol{n}) \subseteq \bigvee_{i=1}^{k} \tau(n_i) \text{ on } V.$$

Recall that a multifunction [25] between two nonempty sets X and Y is a function that assigns to each element of X a subset of Y. Multifunctions are usually denoted by $h : X \rightrightarrows Y$. In order to obtain a characterization of the reciprocal inclusion given in the previous proposition, we need the following concept.

Definition 7 ([25], (Definition 6.2.4))**.** *A multifunction $h : X \rightrightarrows Y$ between two topological spaces is said to be* upper semicontinuous *at $x \in X$ if for every open subset G of Y containing $h(x)$ we can find an open subset O of X containing x such that $h(o) \subseteq G$ for every $o \in O$.*

Proposition 2. *Let $f : [0,+\infty)^I \to [0,+\infty)$ be a norm aggregation function on sets. Given an arbitrary vector space V and a family of norms $\{n_i : i \in I\}$ on V then $\bigvee_{i\in I}\tau(n_i) \subseteq \tau(f\circ\boldsymbol{n})$ on V if and only if $\left(f\mid_{\mathrm{Im}(\boldsymbol{n})}\right)^{-1} : [0,+\infty) \rightrightarrows \mathrm{Im}(\boldsymbol{n})$ is an upper semicontinuous multifunction at 0.*

Proof. Let V be a vector space and $\{n_i : i \in I\}$ be a family of norms on V. We suppose that $V \neq \{0_V\}$ since in this case the result is trivial. Since $\mathrm{Im}(\boldsymbol{n}) \subseteq (0,+\infty)^I \cup \{\mathbf{0}\}$ then $\left(f\mid_{\mathrm{Im}(\boldsymbol{n})}\right)^{-1}(0) = \mathbf{0}$ by Theorem 3. Let us denote $\left(f\mid_{\mathrm{Im}(\boldsymbol{n})}\right)^{-1}$ by $f_{\boldsymbol{n}}^{-1}$. Notice that $f_{\boldsymbol{n}}^{-1}(a)$ is nonempty for every $a \in [0,+\infty)$. In fact, since f is a norm aggregation function on sets given $0_V \neq w \in V$ then by Theorem 3 $f\circ\boldsymbol{n}(w) = b \neq 0$. Moreover $f\mid_{(0,+\infty)^I\cup\{\mathbf{0}\}}$ is positively homogeneous so $f\circ\boldsymbol{n}(\frac{a}{b}w) = \frac{a}{b}(f\circ\boldsymbol{n})(w) = a$ and $\boldsymbol{n}(\frac{a}{b}w) \in f_{\boldsymbol{n}}^{-1}(a)$.

For proving the necessity, let $\varepsilon > 0$. Consider a finite subset J of I and the open set in the induced product topology on $\mathrm{Im}(\boldsymbol{n})$ containing $\mathbf{0}$ given by $G = (\prod_{i\in I} G_i) \cap \mathrm{Im}(\boldsymbol{n})$ where $G_i = [0,+\infty)$ if $i \notin J$ and $G_i = [0,\varepsilon)$ if $i \in J$. Since $\bigvee_{i\in I}\tau(n_i) \subseteq \tau(f\circ\boldsymbol{n})$ we can find $\delta > 0$ such that $B_{f\circ\boldsymbol{n}}(0_V,\delta) \subseteq \bigcap_{j\in J} B_{n_j}(0_V,\varepsilon)$. Let $a \in [0,\delta)$ and $\boldsymbol{x} \in f_{\boldsymbol{n}}^{-1}(a)$. Then there exists $v \in V$ such that $\boldsymbol{n}(v) = (n_i(v))_{i\in I} = \boldsymbol{x}$. Hence $f(\boldsymbol{x}) = f((n_i(v))_{i\in I}) = a < \delta$ so $v \in B_{f\circ\boldsymbol{n}}(0_V,\delta) \subseteq \bigcap_{j\in J} B_{n_j}(0_V,\varepsilon)$. Then $n_j(v) = \boldsymbol{x}_j < \varepsilon$ for all $j \in J$. Therefore, $\boldsymbol{x} \in G$ so $f_{\boldsymbol{n}}^{-1}(a) \subseteq G$. Consequently, $f_{\boldsymbol{n}}^{-1}$ is upper semicontinuous at 0.

Conversely, suppose that $f_{\boldsymbol{n}}^{-1}$ is an upper semicontinuous multifunction at 0. Let $\varepsilon > 0$, $v \in V$ and J be a finite subset of I. Let us consider the open set in $\bigvee_{i\in I} \tau(n_i)$ given by $\bigcap_{j\in J} B_{n_j}(v,\varepsilon)$. Since $f_{\boldsymbol{n}}^{-1}(0) = \{\mathbf{0}\}$ then $(\prod_{i\in I} G_i) \cap \mathrm{Im}(\boldsymbol{n})$ is an open set containing $f_{\boldsymbol{n}}^{-1}(0)$ where $G_i = [0,+\infty)$ if $i \notin I$ and $G_i = [0,\varepsilon)$ if $i \in J$. By hypothesis there exists $\delta > 0$ such that if $a \in [0,\delta)$ then $f_{\boldsymbol{n}}^{-1}(a) \subseteq (\prod_{i\in I} G_i) \cap \mathrm{Im}(\boldsymbol{n})$. We claim that $B_{f\circ\boldsymbol{n}}(v,\delta) \subseteq \bigcap_{j\in J} B_{n_j}(v,\varepsilon)$. In fact, if $w \in B_{f\circ\boldsymbol{n}}(v,\delta)$ then $f \circ \boldsymbol{n}(w-v) = f((n_i(w-v))_{i\in I}) < \delta$. Since $f_{\boldsymbol{n}}^{-1}(f((n_i(w-v))_{i\in I})) \subseteq (\prod_{i\in I} G_i) \cap \mathrm{Im}(\boldsymbol{n})$ then $(n_i(w-v))_{i\in I} \in (\prod_{i\in I} G_i) \cap \mathrm{Im}(\boldsymbol{n})$ so $n_j(w-v) < \varepsilon$ for all $j \in J$, that is, $w \in \bigcap_{j\in J} B_{n_j}(v,\varepsilon)$. $\square$

Corollary 5. *Let $f : [0,+\infty)^I \to [0,+\infty)$ be a norm aggregation function on sets such that $\left(f\mid_{\{\mathbf{0}\}\cup(0,\infty)^I}\right)^{-1} : [0,+\infty) \rightrightarrows \{\mathbf{0}\} \cup (0,+\infty)^I$ is upper semicontinuous at 0. Then $\bigvee_{i\in I} \tau(n_i) \subseteq \tau(f\circ\boldsymbol{n})$ on V for every vector space V and every family of norms $\{n_i : i \in I\}$ on V.*

As a consequence of the previous results, we can characterize the strongly norm aggregation function on sets as follows:

Theorem 6. *Let $f : [0,+\infty)^I \to [0,+\infty)$ be a norm aggregation function on sets. Then f is an strongly norm aggregation function on sets if and only if for every vector space V and every family of norms $\{n_i : i \in I\}$ on V:*

(1) $f\mid_{\mathrm{Im}(\boldsymbol{n})}$ is continuous at $\mathbf{0}$ and

(2) the multifunction $\left(f\mid_{\mathrm{Im}(\boldsymbol{n})}\right)^{-1} : [0,+\infty) \rightrightarrows \mathrm{Im}(\boldsymbol{n})$ is upper semicontinuous at 0.

Corollary 6. *Let $f : [0,+\infty)^I \to [0,+\infty)$ be a norm aggregation function on sets. If f is continuous at $\mathbf{0}$ and $\left(f\mid_{\{\mathbf{0}\}\cup(0,+\infty)^I}\right)^{-1}$ is upper semicontinuous at 0 then f is a strongly norm aggregation function on sets.*

Corollary 7. *Let $f : [0,+\infty)^k \to [0,+\infty)$ be a norm aggregation function on sets. If $\left(f\mid_{\{\mathbf{0}\}\cup(0,+\infty)^k}\right)^{-1}$ is upper semicontinuous at 0 then f is a strongly norm aggregation function on sets.*

Notice that we cannot remove the condition about $\left(f\mid_{\{\mathbf{0}\}\cup(0,+\infty)^k}\right)^{-1}$ in the previous result as the projection shows (see Example 6).

5. Conclusions

The aggregation of mathematical structures by means of an aggregation function admits two different but related approaches. On one hand, you can consider a family of sets endowed each one with a mathematical structure of the same type and try to construct on the Cartesian product of the ground spaces a mathematical structure of the considered type using the aggregation function (we use the terminology *aggregation on products* for this approach). The problem of characterizing those aggregation functions that allow making this process with (quasi-)metrics and (asymmetric) norms has been solved by some authors [8–10,14,15]. On the other hand, you can wonder about which properties must satisfy a function that allows merging a family of mathematical structures defined on a fixed set into a similar structure on the same set (we use the terminology *aggregation on sets* for this situation). This question has been settled for (quasi-)metrics in [11,12]. However, it seems that the same problem for (asymmetric) norms has not been considered previously in the literature. In this paper, we have solved it by characterizing the (asymmetric) norm aggregation functions on sets (Theorems 3 and 4). Moreover, we have shown that, in general, these functions are different from the (asymmetric) norm aggregation functions on products.

Furthermore, we have studied the topology generated by the norm obtained by means of the aggregation of a family of norms both in the case on products and on sets. In the former case we have analyzed when the topology coincides with the product topology (strongly norm aggregation function on products) and in the latter when the topology agrees with the supremum topology (strongly norm aggregation function on sets). Roughly speaking, we have shown that strongly norm aggregation functions on products coincide with norm aggregation functions on products. However this is no longer true in the case of the aggregation on sets (see Example 6). Nevertheless, we have provided a characterization of these functions (see Theorem 6).

A future research direction that we are actually working on is the extension of the results that we have obtained to the fuzzy context. In the literature, there already exist papers which tackle the problem of the aggregation of fuzzy structures (for example [13,22,23,26]) so this question could be an interesting research.

Author Contributions: Conceptualization, investigation, writing–original draft preparation, writing–review and editing, supervision, T.P. and J.R.-L.; funding acquisition, J.R.-L. Both authors have read and agreed to the published version of the manuscript.

Funding: J. Rodríguez-López acknowledges financial support from FEDER/Ministerio de Ciencia, Innovación y Universidades-Agencia Estatal de Investigación Proyecto PGC2018-095709-B-C21.

Acknowledgments: We kindly acknowledge the comments of all the reviewers of this paper which have contributed to improve it.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Beliakov, G.; Pradera, A.; Calvo, T. *Aggregation Functions: A Guide for Practitioners*; Studies in Fuzziness and Soft Computing; Springer: Berlin/Heidelberg, Germany, 2007.
2. Grabisch, M.; Marichal, J.-L.; Mesiar, R.; Pap, E. Aggregation functions. In *Encyclopedia of Mathematics and its Applications*; Cambridge University Press: Cambridge, UK, 2009; Volume 127.
3. Torra, V.; Narukawa, Y. *Modeling Decisions: Information Fusion and Aggregation Operators*; Springer: Berlin/Heidelberg, Germany, 2007.
4. Paternain, D.; Fernández, J.; Bustince, H.; Mesiar, R.; Beliakov, G. Construction of image reduction operators using averaging aggregation functions. *Fuzzy Sets Syst.* **2015**, *261*, 87–111. [CrossRef]
5. García-Lapresta, J.L.; Pereira, R.A.M. The dual decomposition of aggregation functions and its application in welfare economics. *Fuzzy Sets Syst.* **2015**, *281*, 188–197. [CrossRef]
6. Raventós-Pujol, A.; Campión, M.J.; Induráin, E. Arrow theorems in the fuzzy setting. *Iran J. Fuzzy Syst.* **2020**, *15*, 29–41.
7. Engelking, R. General topology. In *Sigma Series in Pure Mathematics*; Heldermann Verlag: Berlin, Germany, 1989; Volume 6.
8. Borsík, J.; Doboš, J. On a product of metric spaces. *Math. Slovaca* **1981**, *31*, 193–205.
9. Doboš, J. *Metric Preserving Functions*; Štroffek: Košice, Slovakia, 1998.
10. Mayor, G.; Valero, O. Aggregation of asymmetric distances in Computer Science. *Inf. Sci.* **2010**, *180*, 803–812. [CrossRef]
11. Pradera, A.; Trillas, E. A note on pseudometrics aggregation. *Int. J. Gen. Syst.* **2002**, *31*, 41–51.
12. Mayor, G.; Valero, O. Metric aggregation functions revisited. *Eur. J. Combin.* **2019**, *80*, 390–400. [CrossRef]
13. Pedraza, T.; Rodríguez-López, J.; Valero, O. Aggregation of fuzzy quasi-metrics. *Inf. Sci.* **2021**, accepted.
14. Herburt, I.; Moszyńska, M. On metric products. *Colloq. Math.* **1991**, *62*, 121–133. [CrossRef]
15. Martín, J.; Mayor, G.; Valero, O. On aggregation of normed structures. *Math. Comput. Model.* **2011**, *54*, 815–827. [CrossRef]
16. Narici, L.; Beckenstein, E. *Topological Vector Spaces*; CRC Press LLC: London, UK, 2010.
17. Cobzaş, S. *Functional Analysis in Asymmetric Normed Spaces*; Birkhäuser: Basel, Switzerland, 2013.
18. Huh, W. Some properties of pseudonormable semilinear spaces. *J. Korean Math. Soc.* **1974**, *11*, 77–85.
19. Prakash, P.; Sertel, M.R. Topological semivector spaces: Convexity and fixed point theory. *Semigroup Forum* **1974**, *9*, 117–138. [CrossRef]
20. Romaguera, S.; Sanchis, M. Completeness of hyperspaces on topological groups. *J. Pure Appl. Algebra* **2000**, *149*, 287–293. [CrossRef]
21. Rockafellar, R.T.; Wets, R.J.-B. *Variational Analysis, Comprehensive Studies in Mathematics*; Springer: Berlin/Heidelberg, Germany, 1998, Volume 317.
22. Mayor, G.; Recasens, J. Preserving T-Transitivity. In *Artificial Intelligence Reserach and Development*; IOS Press: Amsterdam, The Netherlands, 2016; Volume 288, pp. 79–87.
23. Pedraza, T.; Rodríguez-López, J. Aggregation of L-probabilistic quasi-uniformities. *Mathematics* **2020**, *8*, 1980. [CrossRef]
24. Lucchetti, R. Convexity and well-posed problems. In *CMS Books in Mathematics*; Springer: Berlin/Heidelberg, Germany, 2006.
25. Beer, G. *Topologies on Closed and Closed Convex Sets*; Kluwer Academic Publishers: London, UK, 1993; Volume 268.
26. Mesiar, R.; Kolesárová, A. On the fuzzy set theory and aggregation functions: History and some recente advances. *Iran. J. Fuzzy Syst.* **2018**, *15*, 1–12.