Caliber and Chain Conditions in Soft Topologies

José Carlos R. Alcantud [1],*[ID]**, Tareq M. Al-shami** [2][ID] **and A. A. Azzam** [3,4][ID]

1   BORDA Research Unit and Multidisciplinary Institute of Enterprise (IME), University of Salamanca,
    E37007 Salamanca, Spain
2   Department of Mathematics, Sana'a University, Sana'a P.O. Box 1247, Yemen; t.alshami@su.edu.ye
3   Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University,
    Alkharj 11942, Saudi Arabia; aa.azzam@psau.edu.sa
4   Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt
*   Correspondence: jcr@usal.es; Tel.: +34-923-294-666

Abstract: In this paper, we contribute to the growing literature on soft topology. Its theoretical underpinning merges point-set or classical topology with the characteristics of soft sets (a model for the representation of uncertain knowledge initiated in 1999). We introduce two types of axioms that generalize suitable concepts of soft separability. They are respectively concerned with calibers and chain conditions. We investigate explicit procedures for the construction of non-trivial soft topological spaces that satisfy these new axioms. Then we explore the role of cardinality in their study, and the relationships among these and other properties. Our results bring to light a fruitful field for future research in soft topology.

Keywords: soft topology; topology; countable chain condition; caliber; separability

1. Introduction

Topology and decision-making have a long joint tradition since the modern statement of the classical Weierstrass Extreme Value Theorem. This result combines two topological notions, namely, the continuity of a real-valued function and the compactness of the domain (both with respect to a given topology). The success of this technique was amplified by its adoption in fields like mathematical economics. A most celebrated version for the maximization of binary relations is the Bergstrom–Walker theorem [1,2], stating that continuous acyclic relations defined on compact topological spaces possess maximal elements. These sufficient conditions are in fact necessary [3].

Beyond the crisp setting, which is concerned with precise information, fuzzy set theory and other models of uncertain knowledge have given rise to new approaches to decision making since Zadeh's pioneering work. Particularly, topological concepts have been merged with fuzzy set theory since Chang [4] introduced fuzzy topology in 1968. This concept has been subsequently reformulated in renowned works like Goguen [5] and Lowen [6]. Topology has been combined with other models of vague knowledge too. Likewise, in 2011 Shabir and Naz [7] launched soft topology, which amounts to its hybridization with soft set theory [8]. Since then, many contributions have clarified the properties of soft topologies, whose performance has often been contrasted with the inspirational ideas from topology. For example, Das and Samanta [9], Nazmul and Samanta [10], and Zorlutuna et al. [11] soon produced the notion of soft point which is helpful in the analysis of the properties of soft interior points and soft neighborhood systems. As separation axioms are classical elements for the investigation of topologies, their structures have been exported to soft topology since the seminal [7]. Thus for example, El-Shafei et al. [12], Alshami and El-Shafei [13], El-Shafei and Al-shami [14], Hussain and Ahmad [15], Min [16], Terepeta [17] and Al-shami et al. [18] have investigated separation axioms in soft topology. Al-shami [19] disclosed some relationships among different types of soft separation axioms and amended some alleged results given in previous studies. Soft connectedness is the

subject of Hussain [20]. Aygünoğlu and Aygün [21] and Zorlutuna et al. [11] promoted the study of soft compactness, and Al-shami et al. [22] have recently defined novel classes of soft compact and Lindelöf spaces. Soft compactness and decision making are related by the properties of softarisons, in a result that reproduces the Weierstrass extreme value theorem in the framework of soft set theory [23]. Kočinac et al. [24] investigated Menger spaces in the context of soft sets. Separability axioms are another family of classical properties for topological research. It comes as no surprise that they are also becoming one of the important elements in the development of soft topology. They are being used to design well-behaved soft topologies in various forms [25].

Some interesting relations of soft topology with set-theoretical topological structures have been put forward recently [26]. In addition, other structures related to the combination of topology with the spirit of soft set theory have been proposed. Öztürk [27] designed bipolar soft topological spaces with the inspiration of bipolar soft sets [28]. Fuzzy soft topology has been studied since its introduction by Tanay and Kandemir [29] (v., Roy and Samanta [30] and Simsekler and Yuksel [31] among other references). Riaz and Tehrim [32] have put forward bipolar fuzzy soft topologies, Riaz et al. [33] have introduced hesitant fuzzy soft topology, and Riaz et al. [34] proposed N-soft topology which builds on N-soft sets (Fatimah et al. [35]). Both papers have presented some applications to decision-making. Surely other hybridizations are yet to come, for example, topology with bipolar N-soft set theory (Kamacı and Petchimuthu [36]) or spherical fuzzy sets (Mahmood et al. [37]).

Topological separability is not a universal concept. In the investigation of topological spaces that fail to satisfy this specific property, one can still check whether more general properties suffice to produce the required behavior. And the same argument applies to the notions of soft separability. With this motivation, this paper proposes some concepts that extend soft separability from very different perspectives, and then it studies their fundamental properties which make them valuable and worthy of further investigation. These new concepts are inspired by the notions of caliber of a topology, and the countable chain condition. We prove that they start a fruitful field of research with plenty of interactions among themselves and with respect to other soft topological properties.

This research paper consists of five sections. Section 2 contains preliminary concepts from topology, soft set theory and soft topology. Section 3 puts forward two novel soft topological concepts, namely, calibers and chain conditions. Examples defined by explicit constructions ensure that these notions are meaningful. Section 4 proves our main results which concern the fundamental properties of the new axioms, inclusive of the fact that they are topological properties. We also investigate their relationships with other topological axioms. The goal of Section 6 is to conclude our paper with a precise graphical summary of our main findings.

2. Preliminaries

Throughout this article X denotes a nonempty set (often called the universe of discourse), and E denotes a set of characteristics or attributes. For each set U, we denote by $\mathcal{P}(U)$ the set formed by all the subsets of U, that is, the set of parts of the set U.

Section 2.1 reviews some inspirational ideas from topology. Then in Section 2.2, we briefly review some basic elements of soft set theory and soft topology.

Sometimes we shall need to distinguish clearly between topologies and soft topologies, which are not topologies on the same set ([26] discusses this issue in more depth). In this case, we shall refer to a standard topology by the term "crisp topology".

2.1. Elements of (Crisp) Topology

A topology Σ on the nonempty set X is a collection of subsets of X, called the open sets of Σ or the Σ-open sets, such that (1) Σ contains both $\varnothing$ and X, (2) the union of members of Σ (or open sets) belongs to Σ (or is an open set), and (3) the intersection of a finite number of members of Σ (or open sets) belongs to Σ (or is an open set). In this section we recall

some fundamental topological concepts. For other notions in this field, we recommend Munkres [38] and Willard [39].

Let us fix a topology Σ on X. Then a base for Σ is a collection of open sets such that each $A \in \Sigma$ is a union of members of the collection (cf., Willard ([39], Definition 5.1)).

Topologies with countable bases are called second countable. Second countable topologies must be separable, that is, they must contain a countable and dense subset ([39], Problem 5F). The next axioms are designed to weaken topological separability (hence second countability) in various forms. We shall be inspired by these properties and their performance in order to introduce novel axioms for soft topological spaces in Section 3.

1. We say that Σ satisfies the countable chain condition, or CCC for short, if every disjoint family of Σ-open sets is countable. This family of pairwise disjoint open sets is also called a cellular open family [40]. The countable chain condition is often called Suslin's condition;

2. The topological space (X, Σ) has caliber $\aleph$, a cardinal number, if whenever $\mathcal{F}$ is a family of Σ-open sets with cardinality $\aleph$, a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ with cardinality $\aleph$ exists, such that $\cap \{A | A \in \mathcal{F}'\} \neq \varnothing$.

Separability implies caliber $\aleph_1$, which in turn implies CCC (cf., Willard ([39], Problem 16C)). Therefore separability implies CCC; but the converse is not true (cf., Steen and Seebach ([41], p. 22)).

2.2. Elements of Soft Set Theory and Soft Topology

Soft sets on the set X consist of a pair (F, E), where E is composed of all the attributes that are needed to characterize the elements of X, and F is a mapping $F : E \longrightarrow \mathcal{P}(X)$. Thus according to [8], a soft set over X is a parameterized family of subsets of X. In mathematical terms, a soft set on X is a multi-function (also called correspondence, point-to-set mapping, or multi-valued mapping) F from the set of characteristics E to X. In this context, the set of all soft sets on X will be expressed as $SS_E(X)$; when the set of attributes E is common knowledge, we write $SS(X)$ instead.

When $e \in E$, the set $F(e)$ is a subset of X that is sometimes denoted as $(F, E)(e)$ for higher accuracy.

Suppose that (F, E) is a soft set on X such that $F(a)$ is finite (resp., countable) for each $a \in E$. Then we say that (F, E) is a finite (resp., a countable) soft set on X [9,10].

A soft set (F, E) is frequently expressed as $\{(a, F(a)) : a \in E\}$.

Two extreme examples are the null and the absolute or full soft sets on X. The null soft set Φ satisfies $\Phi(a) = \varnothing$ for all $a \in E$. The absolute soft set $\tilde{X}$ satisfies $\tilde{X}(a) = X$ for each $a \in E$. Another special example of soft sets are soft points [9,11], which are soft sets (F, E) such that there are $y \in X$ and $a \in E$ with the property $F(a) = \{y\}$ and $F(a') = \varnothing$ when $a' \in E \setminus \{a\}$. Such soft point shall be denoted as $(\{y\}_a, E)$. The reader should be aware that in the related literature, the term soft point is different: in [17], it refers to (F, E) such that there is $y \in X$ with the property $F(a) = \{y\}$ for each $a \in E$.

Many set-theoretic operations have been exported to soft set theory. For example, union, intersection, and inclusion are defined for $SS_E(X)$ as follows [42]: for any $(F_1, E), (F_2, E) \in SS_E(X)$,

(1) $(F_1, E) \sqcup (F_2, E)$ is the soft set $(F_3, E) \in SS_E(X)$ such that $F_3(a) = F_1(a) \cup F_2(a)$ for each $a \in E$.

This can also be written as $((F_1, E) \sqcup (F_2, E))(a) = F_1(a) \cup F_2(a)$ when $a \in E$.

(2) $(F_1, E) \sqcap (F_2, E)$ is the soft set $(F_4, E) \in SS_E(X)$ such that $F_4(a) = F_1(a) \cap F_2(a)$ for each $a \in E$.

This can also be written as $((F_1, E) \sqcap (F_2, E))(a) = F_1(a) \cap F_2(a)$ when $a \in E$.

(3) $(F_1, E) \sqsubseteq (F_2, E)$ represents the property $F_1(a) \subseteq F_2(a)$ when $a \in E$.

Finally, each $(F, E) \in SS_E(X)$ has a complement, which is the soft set $(F, E)^c = (F^c, E) \in SS_E(X)$ such that $F^c(a) = X \setminus F(a)$ for each $a \in E$.

Remark 1. *The respective extensions of the concepts of union and intersection of two soft sets to arbitrary collections of soft sets are direct.*

A concept of soft equality derives from soft inclusions. Thus $(F_1, E) = (F_2, E)$ means $(F_1, E) \sqsubseteq (F_2, E)$ and $(F_2, E) \sqsubseteq (F_1, E)$. In other words, $F_1(a) = F_2(a)$ for all $a \in E$.

We say that $(G, E), (G', E) \in SS_E(X)$ are disjoint when $(G, E) \sqcap (G', E) = \Phi$. Clearly, disjoint non-empty soft sets are different (i.e., they are not soft equal).

The reader should be aware that different types of unions, intersections, and inclusions have been defined in the literature [43,44].

We are in a position to define soft topology on X (when the set of relevant characteristics is E). It consists of a family $\tau \subseteq SS_E(X)$ that satisfies the properties in the following definition:

Definition 1. *[7,45] A soft topology τ on X is collection of soft sets on X, $\tau \subseteq SS_E(X)$, that satisfies:*

(1) *$\Phi, \tilde{X} \in \tau$;*

(2) *The union of soft sets from τ is a member of τ;*

(3) *The intersection of a finite number of soft sets from τ is a member of τ.*

Every member of τ is a soft open set, or τ-soft open set for better clarity.

When τ is a soft topology on X and E is the set of relevant characteristics, we sometimes write (X, τ, E), and we call this triple a soft topological space.

Example 1. *Some soft topologies on X have been defined by various scholars:*

(i) The indiscrete soft topology is $\tau_{id} = \{\Phi, \tilde{X}\}$, and the discrete soft topology is $\tau_d = SS_E(X)$ [7].

(ii) The cofinite soft topology was defined in [10] as

$$\tau_c = \left\{ (F, E) \in SS_E(X) : (F, E)^c, \text{ the complement of } (F, E), \text{ is a finite soft set} \right\}. \quad (1)$$

In addition to specific examples of soft topologies, two general procedures exist that generate soft topologies from crisp topologies. The first one is quite direct:

Definition 2. *[17,25] Let $\Sigma = \{\Sigma_a\}_{a \in E}$ be a collection of crisp topologies on X indexed by the set of attributes E. Then,*

$$\tau(\Sigma) = \left\{ \{(a, F(a)) : a \in E\} \in SS_E(X) \text{ such that } F(a) \in \Sigma_a \text{ for each } a \in E \right\} \quad (2)$$

defines a soft topology on X that is called the soft topology on X generated by Σ.

We can also write $\tau(\Sigma) = \tau(\Sigma)$ if $\Sigma_a = \Sigma_{a'} = \Sigma$ for each $a, a' \in E$.

Alcantud [25] applies the construction above to prove that $\tau_c = \tau(\Sigma_c)$, that is, the cofinite soft topology can be obtained from the cofinite crisp topology on X, by the process in Definition 2.

The second procedure that generates soft topologies from crisp topologies relies on the behavior of soft open bases. Let us summarize some preliminary facts.

Remark 2. *The soft topology $\tau(\Sigma)$ given in Definition 2 was studied by Al-shami and Kočinac [46] under the name of extended soft topology. There is an equivalence between extended and enriched soft topologies ([46], Theorem 2): the soft topology τ is enriched if it contains all soft sets (F, E) such that $F(e) \in \{\varnothing, X\}$ for each $e \in E$. An interchangeability property of the interior and closure operators between soft set and its components ([46], Theorem 3) is another important feature of*

these types of soft topologies. It has been used to study many properties in soft topological spaces using their counterparts in classical topological spaces and vice versa.

Quite naturally, soft bases are defined as collections of soft sets that generate soft topologies by the proccess of selecting the set composed of the arbitrary soft unions of their elements. So $\mathcal{B} \subseteq \tau$ is a soft base for the soft topology τ if we can express each $(F, E) \in \tau$ as a union of members of $\mathcal{B}$ [45]. Roy and Samanta produced the more operative notion of a soft open base. Any soft open base can be used to generate a soft topology ([47], Theorem 13) in such way that the soft open base is a soft base of the topology that it generates by this process ([47], Theorem 16).

Alcantud ([25], Proposition 1) proves that every base for a standard topology produces a soft open base by an explicit definition. Now a soft topology ensues from this soft open base ([47], Theorem 13). Thus any base for a crisp topology defines an associated soft topology.

Notice that the two constructions of soft topologies defined above are closely related. Suppose that we fix a base for a crisp topology. We can produce both a crisp topology and a soft open base from it. Then we can apply Definition 2 to this crisp topology, and we can also produce a soft topology from the soft open base. The conclusion is that the respective soft topologies that we obtain are the same ([25], Theorem 3).

Theoretical consequences follow from these constructions. For example, a soft topology is soft second-countable, or S2C for short, if a base for it exists that has a countable number of elements ([9], Definition 4.32). Then, [25], Section 4 proves that Definition 2 produces a S2C soft topology, provided that each Σ_e is second-countable and E is at most countable. Actually, a converse to this result holds true as well ([25], Corollary 2).

It is also possible to combine soft topologies in order to produce new soft topological spaces, for example, as a sum of soft topological spaces [48].

We now recall two related concepts defined in [25]. Both concern adapted forms of topological separability for soft topological spaces.

Definition 3. *[25] The soft topology τ on X is soft-points countably-dense, or SPCD for short, when a countable set of soft points exists that is soft τ-dense in X, and $\mathcal{F} \subseteq \mathcal{P}(SS_E(X))$ is soft τ-dense in X when for each $\Phi \neq (G, E) \in \tau$, there is $(F, E) \in \mathcal{F}$ such that $(F, E) \sqcap (G, E) \neq \Phi$.*

Definition 4. *[25] The soft topology τ on X is countably soft-set-dense, or CS^2D for short, when a countable soft set exists that is τ-soft dense in X. $(F, E) \in SS_E(X)$ is τ-soft dense in X when for every $\Phi \neq (G, E) \in \tau$, it must be the case that $(G, E) \sqcap (F, E) \neq \Phi$.*

There are arguments that avail of the fact that both SPCD and CS^2D are good candidates for capturing separability-like axioms in the framework of soft topologies [25]. In addition, S2C implies SPCD ([25], Proposition 1), and SPCD implies CS^2D ([25], Lemma 2).

Any soft topology on a countable universe of discourse X is trivially SPCD therefore CS^2D.

In order to define properties that are preserved by suitable transformations, we need some previous definitions.

Definition 5. *Ref. [49] Let us fix two mappings $h : X \to Y$ and $\varphi : E \to E$. Then the pair (h, φ) defines a soft mapping $h_\varphi : SS_E(X) \longrightarrow SS_E(Y)$ as follows: when $(G, E) \in SS_E(X)$,*

$$h_\varphi(G, E) = (h(G), E) \in SS_E(Y) \text{ is such that } (h(G))(e) = h\left(\bigcup_{\lambda \in \varphi^{-1}(e)} G(\lambda)\right) \text{ for each } e \in E.$$

Observe that, when φ is bijective, the expression defining h_φ reduces to $(h(G))(e) = h(G(\varphi^{-1}(e)))$ for each $e \in E$ and each $(G, E) \in SS_E(X)$.

Definition 6. *Ref. [49] Let $h_\varphi : SS_E(X) \to SS_E(Y)$ be a soft mapping associated with the pair (h, φ). Then the pre-image of $(F, E) \in SS_E(Y)$ is $h_\varphi^{-1}(F, E) = (h^{-1}(F), E) \in SS_E(X)$ such that $(h^{-1}(F))(e) = h^{-1}(F(\varphi(e)))$ for each $e \in E$.*

Definition 7. *Ref. [11] A soft map h_φ is surjective (resp., injective, bijective) when both h and φ are surjective (resp., injective, bijective).*

Definition 8. *Ref. [11] Suppose that τ and τ' are soft topologies on X and Y, respectively. Then the soft map $h_\varphi : SS_E(X) \to SS_E(Y)$ is:*

(i) *soft continuous, when $(F, E) \in \tau'$ implies $h_\varphi^{-1}(F, E) \in \tau$;*

(ii) *soft open when $h_\varphi(F, E) \in \tau'$ if $(F, E) \in \tau$;*

(iii) *a soft homeomorphism when it is both soft continuous, soft open and bijective.*

When a property is preserved by soft homeomorphisms, we say that it is a topological property.

3. New Concepts in Soft Topology: Calibers and Chain Conditions

In this section, we define several novel properties of a soft topology. Then we prove that the fundamental construction given in Definition 2 allows to produce non-trivial examples of soft topological spaces that satisfy these properties, therefore they are not superfluous in the study of soft topology.

Afterwards in Section 4, we show some relationships among these new axioms, and also with respect to axioms of soft separability (understood by either Definition 3 or Definition 4).

The results that we obtain are motivated by the development presented in Section 2.1, however it will soon be apparent that the analysis of the soft topological formulation is much richer than this inspirational crisp case.

3.1. Axioms in Soft Topology

The next definitions are directly inspired by the analogous concepts under the same name in crisp topology. We keep the terminology because the context suffices to distinguish between the crisp axioms (defined in Section 2.1) and the soft axioms (that we introduce in this section). Anyhow some authors may prefer to add the adjective 'soft' to the terms in the new Definitions 9 to 11.

Definition 9. *The soft topology τ on X has caliber $\aleph$, a cardinal number, if for each family $\mathcal{F} \subseteq \tau$ formed by $\aleph$ soft open sets, there is $\mathcal{F}' \subseteq \mathcal{F}$ formed by $\aleph$ soft open sets such that $\sqcap\{(G, E)|(G, E) \in \mathcal{F}'\} \neq \Phi$.*

The next definition reformulates the concept of caliber of a soft topology:

Definition 10. *The soft topology τ on X has caliber $\aleph$ if for each family $\{(G_i, E)\}_{i \in I}$ of pairwise different soft open sets and $|I| = \aleph$, there is $P \subseteq I$ with $|P| = \aleph$ such that the subfamily $\{(G_i, E)\}_{i \in P}$ satisfies $\sqcap_{j \in P}(G_j, E) \neq \Phi$.*

To see why they are equivalent properties of a soft topology, observe first that the formulation of Definition 9 clearly implies Definition 10. Conversely, if τ satisfies Definition 10 then the Axiom of Choice allows us to ensure that it also satisfies Definition 9. Let us fix a family $\mathcal{F} = \{(G_i, E)\}_{i \in I}$ of soft open sets with $\aleph$ members (which may be repeated). In order to avoid repetitions we apply the Axiom of Choice to select $\aleph$ different members from the family, therefore without loss of generality we can assume $|I| = \aleph$ and $(G_i, E) \neq (G_j, E)$ for each different $i, j \in I$. Now Definition 10 ensures the consequent in Definition 9.

The next section will show how the caliber of certain soft topological spaces can be found, provided that some properties are satisfied. Soft topological separability will be especially useful for that purpose.

We now define two properties in the same spirit. The next section will also prove that they are related to soft topological spaces with known calibers.

Definition 11. *The soft topology τ on X satisfies:*

1. The finite chain condition, FCC, if every family of disjoint soft open sets (i.e., such that $(G, E) \sqcap (G', E) = \Phi$ holds when $(G, E) \neq (G', E)$ are members of the family) is finite;

2. The countable chain condition, CCC, if every family of disjoint soft open sets is countable.

There are trivial examples of soft topologies with the properties above. The indiscrete topology, and more generally, all soft hyperconnected topologies [50] satisfy FCC because their only disjoint soft open sets are the null and absolute soft sets. Section 4.1 proves to what extent the cardinalities of X and E ensure either FCC or at least CCC of any soft topology on X.

3.2. Axioms in Soft Topology Derived from Topological Conditions

Definition 2 produces the soft topology generated by a family of crisp topologies (one for each attribute). It is often possible to prove that properties of the crisp topologies can be 'transmitted' to conveniently similar properties of the soft topology that they generate. In Section 2.2 we have recalled some examples of this advantage of the construction given by Definition 2.

This section proves that results in this vein hold true also in the cases of calibers and chain conditions. We shall deduce that if the soft topology that a family of crisp topologies generates has any of the properties introduced in Section 3.1, then these crisp topologies have the analogous behavior; and the converse is true when the cardinality of the set of attributes is not too high.

First we focus on calibers. A very neat statement explains what types of topologies can generate soft topologies with a given caliber:

Proposition 1. *Let $\Sigma = \{\Sigma_a\}_{a \in E}$ be a family of crisp topologies on X.*

If the soft topology $\tau(\Sigma) \subseteq SS_E(X)$ has caliber $\aleph$ then Σ_a has caliber $\aleph$, for each $a \in E$.

Proof. Let us fix $a \in E$. Suppose that $\mathcal{F} = \{A_i\}_{i \in I}$ is a family of Σ_a-open sets whose cardinality is $\aleph$. By the Axiom of Choice, we can assume $|I| = \aleph$ (informally: we delete all repeated elements in the family).

Let $\mathcal{F}' = \{(G_i, E)\}_{i \in I}$ be the family of pairwise different soft open sets in $\tau(\Sigma)$ defined as follows:

$$\text{for each } i \in I, \ G_i(a') = \begin{cases} A_i, & \text{if } a = a', \\ \varnothing, & \text{otherwise} \end{cases} \quad \text{for each } a' \in E.$$

Then, $|\mathcal{F}'| = \aleph$ and we can appeal to Definition 10. Therefore there exists $P \subseteq I$ with $|P| = \aleph$ such that the subfamily $\{(G_i, E)\}_{i \in P}$ satisfies $\sqcap_{j \in P}(G_j, E) \neq \Phi$. Due to the construction of $\mathcal{F}'$, this fact warrants $\cap\{G_i(a) | i \in P\} \neq \varnothing$, and the thesis follows.　□

We proceed to investigate to what extent a converse of Proposition 1 can be stated. The sufficient conditions that we prove necessitate the finiteness of the set of attributes. Under this condition, our next result produces non-trivial examples of soft topological spaces that have caliber $\aleph_0$, and then Proposition 3 does the same with caliber $\aleph_1$:

Proposition 2. *Suppose that E is finite. Suppose further that $\Sigma = \{\Sigma_e\}_{e \in E}$ is a family of crisp topologies on X, each having caliber $\aleph_0$. Then the soft topology $\tau(\Sigma) \subseteq SS_E(X)$ has caliber $\aleph_0$.*

Proof. We appeal to Definition 10. Let $\mathcal{F} = \{(G_i, E)\}_{i \in I}$ be a family of pairwise different soft open sets in $\tau(\Sigma)$, with $|I| = \aleph_0$.

By construction, for each $e \in E$, $\Gamma_e = \{G_i(e) | i \in I\}$ is a family of Σ_e-open sets. It must be the case that, for at least one $e' \in E$, the corresponding family $\{G_i(e') | i \in I\}$ is countably infinite (i.e., it has countably many different members). Otherwise, the fact that E is finite would imply the contradiction that $\mathcal{F}$ is finite too: notice that $|\mathcal{F}| \leqslant \Pi_{e \in E} |\Gamma_e|$ because the elements in $\mathcal{F}$ are pairwise different.

We apply the assumption that $\Sigma_{e'}$ has caliber $\aleph_0$ in order to ensure the existence of $P \subseteq I$ such that $|P| = \aleph_0$ and $\cap \{G_i(e') | i \in P\} \neq \varnothing$.

Now the thesis follows easily because $\sqcap \{(G_i, E) | i \in P\} \neq \Phi$. $\square$

The proof of Proposition 2 can be replicated so as to prove the next variation, which allows us to design non-trivial soft topological spaces with caliber $\aleph_1$:

Proposition 3. *Suppose that E is finite. Suppose further that $\mathbf{\Sigma} = \{\Sigma_e\}_{e \in E}$ is a family of crisp topologies on X, each having caliber $\aleph_1$. Then the soft topology $\tau(\mathbf{\Sigma}) \subseteq SS_E(X)$ has caliber $\aleph_1$.*

Remark 3. *Notice that we cannot extend the conclusion of Proposition 3 to infinite sets of attributes, because the countably infinite product of countable sets may not be countable. Thus, in the argument above, we cannot assure that one of the $\Gamma_{e'}$ families has $\aleph_1$ members when E is infinite.*

Now we turn our attention to the countable chain condition. We can prove the following result with two targets. It ensures that the fact that the soft topology generated by a family of crisp topologies is CCC necessarily entails that each of these topologies is CCC. Furthermore, it provides an explicit construction of non-trivial soft topological spaces that satisfy CCC.

Proposition 4. *Suppose that $\mathbf{\Sigma} = \{\Sigma_e\}_{e \in E}$ is a family of crisp topologies on X. The following properties hold true:*
1. If $\tau(\mathbf{\Sigma})$ is CCC, then Σ_e is CCC for each $e \in E$;
2. The soft topology $\tau(\mathbf{\Sigma}) \subseteq SS_E(X)$ satisfies CCC, provided that Σ_e is CCC for each $e \in E$ and E is countable.

Proof. We prove the first claim by contradiction. Suppose that, for some $e' \in E$, the topology $\Sigma_{e'}$ is not CCC. Then there is an uncountable family $\{A_i\}_{i \in I}$ of $\Sigma_{e'}$-open sets that are pairwise disjoint. We can define the following uncountable family $\{(G_i, E)\}_{i \in I}$ of $\tau(\mathbf{\Sigma})$-soft open sets that are pairwise disjoint:

$$\text{for each } i \in I, \ G_i(e) = \begin{cases} A_i & e = e', \\ \varnothing & e \neq e' \end{cases} \text{ for each } e \in E.$$

This contradiction proves the claim.

In order to prove the second claim, let $\mathcal{F} = \{(G_i, E)\}_{i \in I}$ be a family of disjoint soft open sets in $\tau(\mathbf{\Sigma})$. We proceed to prove that $\mathcal{F}$ must be countable. If $\Phi \in \mathcal{F}$ we remove it from $\mathcal{F}$.

For each $e \in E$, the collection $\{G_i(e) | i \in I, G_i(e) \neq \varnothing\}$ must be countable: notice that $G_i(e) \in \Sigma_e$ for each $i \in I$, the sets in the collection are disjoint because so is $\mathcal{F}$, and Σ_e is CCC. Hence there is $I(e) \subseteq I$ which is countable, such that $G_i(e) \neq \varnothing$ if and only if $i \in I(e)$.

Define the set $J = \bigcup_{i \in E} \{(i, e) | i \in I(e)\}$. Then J is the union of countably many countable sets, since E is countable and for each fixed $e \in E$, the set $\{(i, e) | i \in I(e)\}$ has the same cardinality as $I(e)$. Therefore J is countable (cf., Willard ([39], 1.14 b))).

We appeal to the Axiom of Choice in order to select, for each $i \in I$, an $e_i \in E$ with $G_i(e_i) \neq \varnothing$. By construction, $i \in I(e_i)$.

Consider the mapping

$$\phi : \mathcal{F} \longrightarrow J$$
$$(G_i, E) \longmapsto \phi(G_i, E) = (i, e_i).$$

Clearly, ϕ is injective because $\phi(G_i, E) = \phi(G_j, E)$ means that $(i, e_i) = (j, e_j)$, therefore $i = j$ and $(G_i, E) = (G_j, E)$.

Now the thesis $|\mathcal{F}| \leqslant \aleph_0$ follows easily: $|\mathcal{F}| \leqslant |J|$ because ϕ is injective (cf., ([39], 1.12)), and $|J| \leqslant \aleph_0$ because J is countable. $\square$

4. Main Results

Our target in this Section is to investigate the properties of soft topologies introduced in Section 3. To begin with, we focus on cardinality and its impact on FCC and CCC in the next subsection. Afterwards, we shall investigate more sophisticated sufficient conditions for the new properties to hold true. We also prove new relationships among soft topological properties. Finally, we formalize the fact that calibers (of a fixed cardinality), FCC and CCC are topological properties. These results will arise from a more general analysis of which types of mappings preserve these properties.

4.1. The Role of Cardinality in FCC and CCC

We proceed to argue that very simple cardinality conditions ensure the properties in Definition 11. The next Proposition 5 proves that if X and E are finite, resp. countable, then any soft topology on X satisfies FCC, resp. CCC. Examples 2 and 3 below assure that the assumptions in these two claims are tight, that is, we cannot dispense with any of them in the statement of Proposition 5. Afterwards we shall prove by examples that neither of these sufficient conditions are not necessary (cf., Example 4 below).

Proposition 5. *If X and E are finite, resp. countable, then any soft topology on X satisfies FCC, resp. CCC.*

Proof. The finite case is trivial, since $SS_E(X)$ is finite therefore $\mathcal{P}(SS_E(X))$ is also finite.

Now suppose that X and E are countable. We proceed to prove that every family of disjoint soft sets on X is also countable, which in particular proves the claim that any soft topology on X satisfies CCC.

Let $\mathcal{F} = \{(G_i, E)\}_{i \in I}$ be a family of disjoint soft sets on X with attributes E. We do not lose generality if we assume $\Phi \notin \mathcal{F}$: if $(G_i, E) = \Phi$ for some (and only one) $i \in I$, we remove it from $\mathcal{F}$. For each $(G_i, E) \in \mathcal{F}$, let us fix a pair $(x_i, e_i) \in X \times E$ such that $x_i \in G_i(e_i)$. This can be done by the Axiom of Choice, because $(G_i, E) \neq \Phi$. Consider the mapping

$$\varphi : \mathcal{F} \longrightarrow X \times E$$
$$(G_i, E) \longmapsto \varphi(G_i, E) = (x_i, e_i).$$

Then, we claim that φ is injective. To this purpose, observe that $\varphi(G_i, E) = \varphi(G_j, E)$ means that $(x_i, e_i) = (x_j, e_j)$ therefore $x_i = x_j \in G_i(e_i) = G_j(e_i)$ by construction. However, (G_i, E) and (G_j, E) are disjoint when $i \neq j$, therefore $i = j$ or equivalently, $(G_i, E) = (G_j, E)$ as desired.

Now the thesis follows easily because $X \times E$ is countable, being the product of two countable sets (cf., Willard ([39], 1.14 c))). The reason is that because φ is injective, $|\mathcal{F}| \leqslant |X \times E| \leqslant \aleph_0$ (cf., ([39], 1.12)). $\square$

Example 2. *Neither of the two assumptions in the statement "when X and E are finite, FCC holds true" is redundant. Observe that:*

1. τ_d does not satisfy FCC when X is finite but E is infinite.

Suppose $X = \{x\}$ and $E = \{e_1, \ldots, e_n, \ldots\}$. The infinite family of τ_d-soft open sets $\{(G_i, E)\}_{i \in \mathbb{N}}$ with

$$\text{for each } i \in \mathbb{N} : G_i(e_j) = \begin{cases} \{x\} & i = j, \\ \varnothing & i \neq j \end{cases} \text{ for each } e_j \in E$$

is formed by disjoint τ_d-soft open sets;

 2. *τ_d does not satisfy FCC when E is finite but X is infinite.*

 Suppose $E = \{e\}$ and $X = \{x_1, \ldots, x_n, \ldots\}$. The infinite family of τ_d-soft open sets $\{(G_i, E)\}_{i \in \mathbb{N}}$ defined by $G_i(e) = \{x_i\}$ for each $i \in \mathbb{N}$ is formed by disjoint τ_d-soft open sets.

Example 3. *Neither of the two assumptions in the statement "when X and E are countable, CCC holds true" is redundant. Observe that:*

 1. *τ_d does not satisfy CCC when X is countable but E is uncountable.*

 Suppose $X = \{x\}$ and $E = \{e_i\}_{i \in \mathbb{R}}$. The uncountable family of τ_d-soft open sets $\{(G_i, E)\}_{i \in \mathbb{R}}$ with

$$\text{for each } i \in \mathbb{R}: \ G_i(e_j) = \begin{cases} \{x\} & i = j, \\ \varnothing & i \neq j \end{cases} \text{ for each } e_j \in E$$

is formed by disjoint τ_d-soft open sets;

 2. *τ_d does not satisfy CCC when E is finite but X is uncountable.*

 Suppose $E = \{e\}$ and $X = \{x_i\}_{i \in \mathbb{R}}$. The uncountable family of τ_d-soft open sets $\{(G_i, E)\}_{i \in \mathbb{R}}$ defined by $G_i(e) = \{x_i\}$ for each $i \in \mathbb{R}$ is formed by disjoint τ_d-soft open sets.

Section 4.2 sets forth some relationships among known properties of soft topological spaces and the structures developed above. We also identify classes of spaces that satisfy the new properties stated in this section. As a consequence, we shall deduce that the set of conditions that Proposition 7 provides for FCC, resp. CCC, are not necessary.

Armed with these tools, in Section 4.3 we shall produce new non-trivial examples of soft topological spaces that satisfy CCC.

4.2. Some Relationships among Properties

The proof of the next result is as simple as the crisp analogous:

Proposition 6. *Suppose that the soft topology τ on X has caliber $\aleph_0$ (resp., caliber $\aleph_1$). Then τ satisfies FCC (resp., CCC).*

Proof. We argue for the first statement, the second being a routine modification of it.

Suppose by contradiction that there exists an infinite family of pairwise disjoint soft open sets. By the Axiom of Choice, we select a countably infinite subfamily $\{(G_i, E)\}_{i \in I}$ of non-empty, pairwise disjoint soft sets. Thus $|I| = \aleph_0$. Because τ has caliber $\aleph_0$, and disjoint non-empty soft sets are different, there is a subfamily $\{(G_i, E)\}_{i \in P}$ with $|P| = \aleph_0$, $P \subseteq I$, that satisfies $\sqcap_{j \in P}(G_j, E) \neq \Phi$ in agreement with Definition 10. This is a contradiction because the soft sets $\{(G_i, E)\}_{i \in P}$ are pairwise disjoint.

Consider now the second claim, which states that if τ has caliber $\aleph_1$ then τ satisfies CCC. Suppose by contradiction that there exists an uncountably infinite family of pairwise disjoint soft open sets. By the Axiom of Choice, we can select an infinite subfamily $\{(G_i, E)\}_{i \in I}$ of non-empty, pairwise disjoint soft sets whose cardinality is the first uncountable cardinal. Thus $|I| = \aleph_1$. Because τ has caliber $\aleph_1$, and disjoint non-empty soft sets are different, there is a subfamily $\{(G_i, E)\}_{i \in P}$ with $|P| = \aleph_1$, $P \subseteq I$, that satisfies $\sqcap_{j \in P}(G_j, E) \neq \Phi$ in agreement with Definition 10. This is a contradiction because the soft sets $\{(G_i, E)\}_{i \in P}$ are pairwise disjoint. $\square$

Remark 4. *Under the Continuum Hypothesis—which is independent from the Zermelo-Fraenkel axioms together with the axiom of choice—$\aleph_1$ coincides with $2^{\aleph_0}$, which is the cardinality of the continuum (i.e., of $\mathbb{R}$). Nevertheless, the argument that proves Proposition 6 is independent of whether we assume the Continuum Hypothesis or not. In other words, whether $\aleph_1 = |\mathbb{R}|$ or $\aleph_1 < |\mathbb{R}|$ is irrelevant to our purposes.*

We now prove two propositions that show how properties related to the idea of "soft separability" permit us to identify the caliber of certain soft topological spaces.

Proposition 7. *Suppose that E is finite. Suppose further that for the soft topology τ on X, there exists a finite $(F, E) \in SS_E(X)$ that is τ-soft dense. Then τ has caliber $\aleph_0$.*

Proof. The argument relies on cardinalities.

Let us fix a family $\{(G_i, E)\}_{i \in I}$ of pairwise different soft open sets with $|I| = \aleph_0$. Without loss of generality we can also assume that Φ is not one of the $\aleph_0$ members of the family (if it is, we remove Φ and we still have a family of pairwise different soft open sets with $\aleph_0$ members).

Since (F, E) is τ-soft dense, for each $i \in I$ there is $e(i) \in E$ such that $F(e(i)) \cap G_i(e(i)) \neq \varnothing$ because $(G_i, E) \neq \Phi$.

Consider the mapping $f : I \longrightarrow E$ such that $f(i) = e(i)$ for each $i \in I$. Because $I = \cup_{e \in E} f^{-1}(e)$, $|I| = \aleph_0$ and E is finite, by cardinality we can assure the existence of $e \in E$ such that $|f^{-1}(e)| = \aleph_0$. Let us define $J = f^{-1}(e) \subseteq I$, then $|J| = \aleph_0$.

Because $F(e)$ is finite and $F(e) \cap G_i(e) \neq \varnothing$ for each $i \in J$ by construction, a similar cardinality argument guarantees that, for at least one $u \in F(e)$, it must be the case that $P = \{j \in J | u \in G_j(e)\}$ satisfies $|P| = \aleph_0$.

Notice that we have assured $u \in (\cap_{j \in P} G_j(e)) \cap F(e)$, thus proving $\sqcap_{j \in P}(G_j, E) \neq \Phi$. □

The gist of the argument in the proof of Proposition 7 is that a finite union of finite sets is finite. It is also true that the countable union of countable sets is a countable set. Therefore a routine modification of the proof of Proposition 7 produces the next statement:

Proposition 8. *Suppose that E is countable. Suppose further that the soft topology τ on X is CS^2D. Then τ has caliber $\aleph_1$.*

The next two Corollaries follow easily from the statements above in this section. Propositions 6 and 7 produce Corollary 1, whereas Propositions 6 and 8 produce Corollary 2:

Corollary 1. *Suppose that E is finite. Suppose further that for the soft topology τ on X, there exists a finite $(F, E) \in SS_E(X)$ that is τ-soft dense. Then τ satisfies FCC.*

Corollary 2. *Suppose that E is countable. Suppose further that for the soft topology τ on X is CS^2D. Then τ satisfies CCC.*

It is now obvious that SPCD implies CCC when E is countable.

We take advantage of the relationships stated above to prove that although Proposition 7 gives a set of conditions that are sufficient for FCC, resp. CCC, they are by no means necessary: soft topological spaces that satisfy FCC (hence CCC) can be designed with arbitrarily large sets of options and attributes.

Example 4. *Let us fix sets non-empty X and E, and an arbitrary element $x \in X$. Let Σ_x be the particular point topology on X, that is, the open sets of Σ_x are $\varnothing$ and any subset of X that contains x.*

Then, we claim that $\tau(\Sigma_x)$, the soft topology on X generated by Σ_x according to Definition 2, satisfies the FCC thus the CCC, irrespective of the size of X or E.

We just need to observe that the finite soft set (F_x, E) defined by $F_x(a) = \{x\}$ for each $a \in E$ is $\tau(\Sigma_x)$-soft dense (cf., Definition 4). Indeed, when $(G, E) \neq \Phi$ and $(G, E) \in \tau(\Sigma_x)$, there exists $a \in E$ with the property $\varnothing \neq G(a) \in \Sigma_x$; therefore, $x \in G(a) \cap F_x(a)$. Now Corollary 1 applies.

4.3. A Construction of Soft Topologies That Satisfy CCC

So far, we have ensured that certain properties of soft topologies, or related to cardinalities, guarantee some performance in terms of calibers or chain conditions. Now we shall be more specific because we are fully aware that precise examples nourish the development of both topology and soft topology.

Our next result shows that the behavior of the soft topology defined in Definition 2 allows us to provide a concrete process that generates soft topologies that satisfy CCC:

Corollary 3. *Suppose that E is countable. Suppose further that $\mathbf{\Sigma} = \{\Sigma_a\}_{a \in E}$ is a family of crisp topologies on X, and $\Sigma_{a'}$ is separable for some $a' \in E$. Then $\tau(\mathbf{\Sigma})$ satisfies CCC.*

Proof. The argument uses ([25], Proposition 2), which ensures that $\tau(\mathbf{\Sigma})$ satisfies SPCD, therefore CS^2D. The conclusion now follows from Corollary 2. $\square$

Example 4 proves that FCC and CCC are not mere conditions of cardinality. Arbitrarily large universes of discourses and sets and attributes may be endowed with soft topologies that satisfy even FCC. We proceed to specialize the statement in Corollary 3 in order to design further examples of CCC soft topological spaces with an uncountably infinite universe of discourse X:

Example 5. *Assume that E is countable. Let $X = \mathbb{R}^{\mathbb{R}}$ be the set all functions from $\mathbb{R}$ to itself, whose cardinality is $2^{\mathfrak{c}}$ and $\mathfrak{c}$ is the cardinality of $\mathbb{R}$ (hence $\mathfrak{c} \geqslant \aleph_1$). This uncountable set endowed with the product topology Σ_p of the Euclidean topologies is separable, being a product of a continuum of Hausdorff separable spaces (cf., Willard ([39], Theorem 16.4c))).*
  Then $\tau(\Sigma_p) \subseteq SS_E(X)$ satisfies CCC by Corollary 3.

4.4. Preservation of Properties by Mappings

To conclude this Section, we put forward some results on the preservation of properties when we transform a soft topological space by certain types of mappings. The next two propositions in particular prove that possessing a given caliber, FCC, and CCC are topological properties.

Proposition 9. *The property of having a caliber number is preserved under bijective soft continuous mappings. In particular, having caliber $\aleph$ is a topological property for each $\aleph$.*

Proof. Suppose that $h_\varphi : (X, \tau, E) \to (Y, \theta, E)$ is bijective soft continuous, and also that τ has caliber $\aleph$. In order to prove that θ has caliber $\aleph$, let $\{(G_i, E)\}_{i \in I}$ be a family of pairwise different soft open sets in θ and $|I| = \aleph$. Since h_φ is soft continuous, then $\{h_\varphi^{-1}(G_i, E)\}_{i \in I}$ is a family of soft open sets in τ. It is easy to prove that they are pairwise different because h_φ is bijective. As τ has caliber $\aleph$, there is $P \subseteq I$ with $|P| = \aleph$ such that the subfamily $\{h_\varphi^{-1}(G_i, E)\}_{i \in P}$ satisfies $\sqcap_{j \in P} h_\varphi^{-1}(G_j, E) \neq \Phi$. Now, it is straightforward to check $\sqcap_{j \in P}(G_j, E) \neq \Phi$ because h_φ is bijective. Therefore, we have proven that $\{(G_j, E)\}_{j \in P}$ is a family of pairwise different soft open sets in θ as desired. $\square$

Proposition 10. *The finite (resp., countable) chain condition is preserved under bijective soft continuous mappings. In particular, they are topological properties.*

Proof. Suppose that $\{(G_i, E)\}_{i \in I} \subseteq SS_E(Y)$ satisfy $\sqcap_{i \in I}(G_i, E) = \Phi$. It is easy to prove that, for any bijective soft mapping $h_\varphi : (X, \tau, E) \to (Y, \theta, E)$, the family $\{h_\varphi^{-1}(G_i, E)\}_{i \in I} \subseteq SS_E(X)$ has the same cardinality as $\{(G_i, E)\}_{i \in I}$. The fact that $\sqcap_{i \in I} h_\varphi^{-1}(G_i, E) = \Phi$ holds true too, as mentioned in the proof of Proposition 9.
  Thus, when h_φ is a bijective soft continuous mapping and $\{(G_i, E)\}_{i \in I} \subseteq \theta$ is a family of disjoint soft open sets, the family $\{h_\varphi^{-1}(G_i, E)\}_{i \in I} \subseteq \tau$ must be finite (resp. countable) when τ satisfies the finite (resp., countable) chain condition. This proves the claim because $\{(G_i, E)\}_{i \in I}$ has the same cardinality as $\{h_\varphi^{-1}(G_i, E)\}_{i \in I}$. $\square$

5. Conclusions

This paper introduces the finite and countable chain conditions for soft topological spaces. We abbreviate them with FCC and CCC, respectively. They are designed to

generalize the idea of soft separability while keeping a good theoretical performance. In passing, our results insist on the view that CS^2D and SPCD are important concepts in the investigation of separability issues in soft topology.

Considerations about cardinalities are quite relevant, although they do not exhaust the analysis of FCC and CCC. The reason is that soft topological spaces exist that satisfy FCC hence CCC, without restrictions to the sets of options and attributes (cf., Example 4).

Whenever cardinalities fail to ensure the required properties, calibers are a more advanced tool for the investigation of FCC and CCC. Calibers are a novel concept in the realm of soft topology. This paper proves that soft topological spaces with calibers $\aleph_1$ or smaller satisfy CCC; if the caliber is $\aleph_0$ then FCC holds true.

Explicit examples of the new concepts are presented in order to nurture their analysis and applications.

Let us summarize our main findings. Consider a soft topology τ.

1. If τ on X has caliber $\aleph_0$ (resp., caliber $\aleph_1$) then τ satisfies FCC (resp., CCC): cf., Proposition 6;

2. Suppose that E is finite.

   (2.1) Then when X is finite, FCC holds true trivially.

   (2.2) If this is not the case, but still there is a finite τ-soft dense set, then the caliber of τ is $\aleph_0$ and in particular, FCC holds true as well.

   (2.3) Observe that statement (2.1) is a particular case of (2.2), since a finite X ensures that the absolute soft set $\tilde{X}$ is a finite τ-soft dense set;

3. Suppose instead that E is countable.

   (3.1) Then when X is countable, CCC holds true: cf., Proposition 5.

   (3.2) If this is not the case, but still there is a countable τ-soft dense set (i.e., τ is CS^2D), then the caliber of τ is $\aleph_1$ (cf., Proposition 8) and in particular, CCC holds true as well.

   (3.3) Again, statement (3.1) is a particular case of (3.2): notice that a countable X ensures that the collection of all soft points is soft τ-dense in X and of course it is countable, therefore τ satisfies SPCD and CS^2D.

Now consider the important example of τ, a soft topology that has been generated by $\mathbf{\Sigma} = \{\Sigma_e\}_{e \in E}$, a family of crisp topologies. This fact is expressed as $\tau = \tau(\mathbf{\Sigma})$;

4. If τ has caliber $\aleph$ then Σ_e has caliber $\aleph$, for each $e \in E$: cf., Proposition 1;

5. Suppose that E is finite. Then, when each Σ_e has caliber $\aleph_0$, τ has caliber $\aleph_0$ (v., Proposition 2) and, when each Σ_e has caliber $\aleph_1$, τ has caliber $\aleph_1$ (v., Proposition 3);

6. If τ satisfies CCC then Σ_e satisfies CCC, for each $e \in E$ (v., part 1 of Proposition 4);

7. Suppose that E is countable. Then when each Σ_e satisfies CCC, τ satisfies CCC (v., part 2 of Proposition 4).
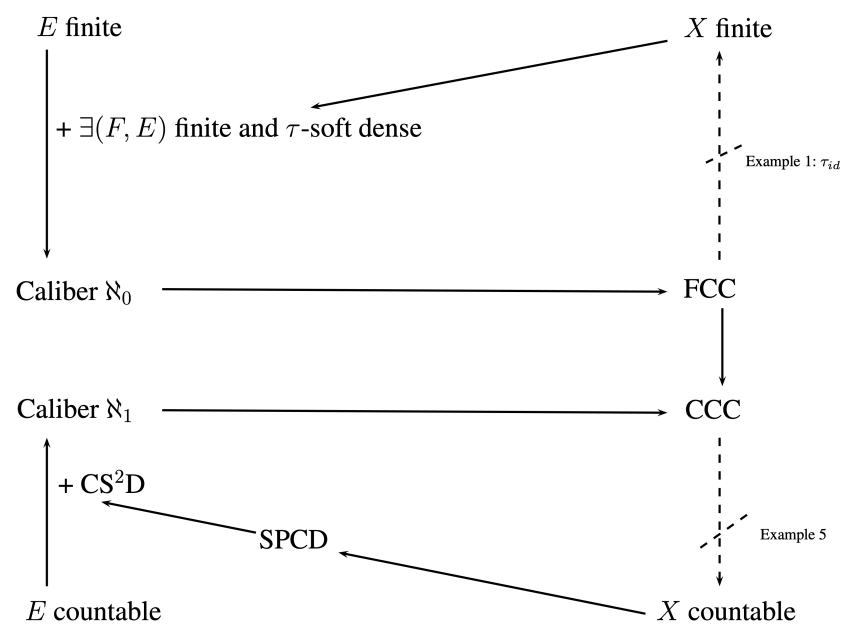
Figure 1 summarizes the results described above.

Figure 1. Summary of results about relationships among properties, and the role of cardinalities.

References

1. Bergstrom, T.C. Maximal elements of acyclic relations on compact sets. *J. Econ. Theory* **1975**, *10*, 403–404. [CrossRef]
2. Walker, M. On the existence of maximal elements. *J. Econ. Theory* **1977**, *16*, 470–474. [CrossRef]
3. Alcantud, J.C.R. Characterization of the existence of maximal elements of acyclic relations. *Economic Theory* **2002**, *19*, 407–416. [CrossRef]
4. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, *24*, 182–190. [CrossRef]
5. Goguen, J.A. Fuzzy Tychonoff theorem. *J. Math. Anal. Appl.* **1973**, *43*, 734–742. [CrossRef]
6. Lowen, R. Fuzzy topological spaces and fuzzy compactness. *J. Math. Anal. Appl.* **1976**, *56*, 621–633. [CrossRef]
7. Shabir, M.; Naz, M. On soft topological spaces. *Comput. Math. Appl.* **2011**, *61*, 1786–1799. [CrossRef]
8. Molodtsov, D. Soft set theory—First results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]
9. Das, S.; Samanta, S. Soft metric. *Ann. Fuzzy Math. Informatics* **2013**, *6*, 77–94.
10. Nazmul, S.; Samanta, S. Some properties of soft topologies and group soft topologies. *Ann. Fuzzy Math. Informatics* **2014**, *8*, 645–661.
11. Zorlutuna, I.; Akdag, M.; Min, W.K.; Atmaca, S. Remarks on soft topological spaces. *Ann. Fuzzy Math. Informatics* **2012**, *3*, 171–185.
12. El-Shafei, M.E.; Abo-Elhamayel, M.; Al-shami, T.M. Partial soft separation axioms and soft compact spaces. *Filomat* **2018**, *32*, 4755–4771. [CrossRef]
13. Al-shami, T.M.; El-Shafei, M.E. Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone. *Soft Comput.* **2020**, *24*, 5377–5387. [CrossRef]
14. El-Shafei, M.E.; Al-shami, T.M. Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem. *Comput. Appl. Math.* **2020**, *39*, 138. [CrossRef]
15. Hussain, S.; Ahmad, B. Some properties of soft topological spaces. *Comput. Math. Appl.* **2011**, *62*, 4058–4067. [CrossRef]
16. Min, W.K. A note on soft topological spaces. *Comput. Math. Appl.* **2011**, *61*, 3524–3528. [CrossRef]
17. Terepeta, M. On separating axioms and similarity of soft topological spaces. *Soft Comput.* **2019**, *23*, 1049–1057. [CrossRef]
18. Al-shami, T.M.; El-Shafei, M.E.; Abo-Elhamayel, M. On soft topological ordered spaces. *J. King Saud Univ.-Sci.* **2019**, *31*, 556–566.
19. Al-shami, T.M. Comments on some results related to soft separation axioms. *Afr. Mat.* **2020**, *31*, 1105–1119. [CrossRef]

20. Hussain, S. A note on soft connectedness. *J. Egypt. Math. Soc.* **2005**, *23*, 6–11. [CrossRef]
21. Aygünoğlu, A.; Aygün, H. Some notes on soft topological spaces. *Neural Comput. Appl.* **2012**, *21*, 113–119. [CrossRef]
22. Al-shami, T.M.; El-Shafei, M.E.; Abo-Elhamayel, M. Almost soft compact and approximately soft Lindelöf spaces. *J. Taibah Univ. Sci.* **2018**, *12*, 620–630. [CrossRef]
23. Alcantud, J.C.R. Softarisons: Theory and practice. *Neural Comput. Appl.* **2021**, forthcoming. [CrossRef]
24. Kočinac, L.D.R.; Al-shami, T.M.; Çetkin, V. Selection principles in the context of soft sets: Menger spaces. *Soft Comput.* **2021**, forthcoming. Available online: https://link.springer.com/article/10.1007/s00500-021-06069-6 (accessed on 21 September 2021).
25. Alcantud, J.C.R. Soft open bases and a novel construction of soft topologies from bases for topologies. *Mathematics* **2020**, *8*, 672. [CrossRef]
26. Alcantud, J.C.R. An operational characterization of soft topologies by crisp topologies. *Mathematics* **2020**,*9*, 1656. [CrossRef]
27. Öztürk, T.Y. On bipolar soft topological spaces. *J. New Theory* **2018**, *20*, 64–75.
28. Mahmood, T. A novel approach towards bipolar soft sets and their applications. *J. Math.* **2020**, 4690808. Available online: https://www.hindawi.com/journals/jmath/2020/4690808/ (accessed on 21 September 2021).
29. Tanay, B.; Kandemir, M.B. Topological structure of fuzzy soft sets. *Comput. Math. Appl.* **2011**, *61*, 2952–2957. [CrossRef]
30. Roy, S.; Samanta, T.K. A note on fuzzy soft topological spaces. *Ann. Fuzzy Math. Inform.* **2012**, *3*, 305–311.
31. Simsekler, T.; Yuksel, S. Fuzzy soft topological spaces. *Ann. Fuzzy Math. Inform.* **2013**, *5*, 87–96.
32. Riaz, M.; Tehrim, S.T. On bipolar fuzzy soft topology with decision-making. *Soft Comput.* **2020**, *24*, 18259–18272. [CrossRef]
33. Riaz, M.; Davvaz, B.; Fakhar, A.; Firdous, A. Hesitant fuzzy soft topology and its applications to multi-attribute group decision-making. *Soft Comput.* **2020**, *24*, 16269–16289. [CrossRef]
34. Riaz, M.; Çağman, N.; Zareef, I.; Aslam, M. *N*-soft topology and its applications to multi-criteria group decision making. *J. Intell. Fuzzy Syst.* **2019**, *36*, 6521–6536. [CrossRef]
35. Fatimah, F.; Rosadi, D.; Hakim, R.B.F.; Alcantud, J.C.R. *N*-soft sets and their decision making algorithms. *Soft Comput.* **2018**, *22*, 3829–3842. [CrossRef]
36. Kamacı, H.; Petchimuthu, S. Bipolar *N*-soft set theory with applications. *Soft Comput.* **2020**, *24*, 16727–16743. [CrossRef]
37. Mahmood, T.; Ullah, K.; Khan, Q.; Jan N. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Comput. Appl.* **2019**, *31*, 7041–7053. [CrossRef]
38. Munkres, J. Topology. In *Featured Titles for Topology*; Prentice Hall, Incorporated: Hoboken, NJ, USA, 2000.
39. Willard, S. General Topology. In *Addison-Wesley Series in Mathematics*; Dover Publications: Mignola, NY, USA, 2004.
40. Bal, P.; Kočinac, L.D.R. On selectively star-ccc spaces. *Topol. Its Appl.* **2020**, *281*, 107184. [CrossRef]
41. Steen, L.A.; Seebach, J.A. *Counterexamples in Topology*; Springer Science+Business Media: Berlin/Heidelberg, Germany, 1970.
42. Maji, P.; Biswas, R.; Roy, A. Soft set theory. *Comput. Math. Appl.* **2003**, *45*, 555–562. [CrossRef]
43. Al-shami, T.M. Investigation and corrigendum to some results related to *g*-soft equality and *gf*-soft equality relations. *Filomat* **2019**, *33* , 3375–3383. [CrossRef]
44. Al-shami, T.M.; El-Shafei, M.E. *T*-soft equality relation. *Turk. J. Math.* **2020**, *44* , 1427–1441. [CrossRef]
45. Çağman, N.; Karataş, S.; Enginoglu, S. Soft topology. *Comput. Math. Appl.* **2011**, *62*, 351–358 [CrossRef]
46. Al-shami, T.M.; Kočinac, L.D.R. The equivalence between the enriched and extended soft topologies. *Appl. Comput. Math.* **2019**, *18*, 149–162.
47. Roy, S.; Samanta, T.K. A note on a soft topological space. *Punjab Univ. J. Math.***2014**, *46*, 19–24
48. Al-shami, T.M.; Kočinac, L.D.R.; Asaad, B.A. Sum of soft topological spaces. *Mathematics* **2020**, *8*, 990. [CrossRef]
49. Kharal, A.; Ahmed, B. Mappings on soft classes. *New Math. Nat. Comput.* **2011**, *7*, 471–481. [CrossRef]
50. Kandil, A.; Tantawy, O.A.E.; El-Sheikh, S.A.; Abd, El-latif, A.M. Soft connectedness via soft ideals. *J. New Results Sci.* **2014**, *3*, 90–108.