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A Compendium of Infinite Group Theory: Part 1—Countable Recognizability

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Abstract: Countably recognizable group classes were introduced by Reinhold Baer and provide a very ingenious way to study large groups through the properties of their countable subgroups. This is the reason we have chosen the countable recognizability to start this series of survey papers on infinite group theory.

1. Introduction

In 1950, Sergei N. Černikov [1] proved that a group G is hypercentral if and only if for each sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of G there exists a positive integer k such that $[g, x_1, \ldots, x_k] = 1$ for all $g \in G$. It follows that a group is hypercentral if and only if all its countable subgroups are hypercentral. A few years later, Reinhold Baer [2] obtained a corresponding characterization of the class of hyperabelian groups: a group G is hyperabelian if and only if for all sequences of elements $x_1, x_2, \ldots, x_n, \ldots$ and $y_1, y_2, \ldots, y_n, \ldots$ such that $[x_i, y_i, x_i] = x_{i+1}$, there is a positive integer k such that $x_k = 1$ (and hence $x_m = 1$ for every $m \geq k$). Also in this case, this result implies that the property of being hyperabelian can be detected from the behaviour of countable subgroups. This kind of results is motivated by the fact that certain relevant group classes, such as the above ones, cannot be characterized "in the small" (*im kleinen*, following Baer [3]), i.e., in terms of finitely generated subgroups. On the other hand, there are also important group classes that can be locally described. For instance, it is obvious that a group is abelian if and only if all its 2-generator subgroups are abelian, and for each positive integer c a similar local characterization can be given for nilpotent groups of class at most c, although it cannot be given for nilpotency in general. However, it is easy to see that nilpotent groups can be at least recognized from the behaviour of countable subgroups: in fact, Baer remarked that the union of any countable collection of *countably recognizable* group classes is likewise countably recognizable. Notice here that the celebrated local theorem of A.I. Mal'cev (see [4] Part 2, Section 8.2) provides a method to prove that many relevant group classes are local. All local classes are countably recognizable and in this survey we have chosen to focus on countably recognizable group classes that are not local.

The most obvious example of a group class which is not countably detectable is that of countable groups. A less trivial example is given by the class of free abelian groups, since the cartesian product of any infinite collection of infinite cyclic groups cannot be decomposed into a direct product of infinite cyclic groups, while all its countable subgroups are free abelian (see [5], Theorem 19.2). We mention also that Graham Higman proved that also free groups form a class which is not countably recognizable (see [6]).

In recent years, the increasing interest for group classes which are not defined by finiteness conditions led to a renewed attention towards countable recognizability as a

method to derive information on the structure of large groups by means of the behaviour of their small subgroups.

This paper aims to give an updated exposition on countable recognizability. It is supposed to be the first in a series of surveys dedicated to relevant topics in the theory of infinite groups, with the hope that it will be useful to young researchers approaching one of the most fascinating parts of mathematics: the *theory of groups*, of course.

2. General Properties

A group class $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group. Among the countably recognizable group classes there are of course the so-called local classes: a group class $\mathfrak{X}$ is *local* if it contains all groups in which every finite subset lies in some $\mathfrak{X}$-subgroup. The classes $\mathfrak{S}$ of soluble groups and $\mathfrak{N}$ of nilpotent groups are examples of countably recognizable group classes which are not local.

Lemma 1. (R. Baer [3], see also [7])

(a) *The intersection of any collection of countably recognizable group classes is countably recognizable.*
(b) *The union of countably many subgroups closed and countably recognizable group classes is countably recognizable.*

Notice that part (b) of the above statement cannot be extended to uncountable collections of countably recognizable group classes. In fact, if p is any prime number and σ is an ordinal, Baer proved that the class $\mathfrak{A}(\sigma)$ of all abelian p-groups of Ulm length at most σ is countably recognizable if and only if $\sigma < \aleph_1$ (see [3], Bemerkung 1.2); on the other hand, the group class

$$\bigcup_{\sigma < \aleph_1} \mathfrak{A}(\sigma)$$

cannot be countably recognizable since it is well known that there exist abelian p-groups of Ulm length $\aleph_1$. Recall here that the *Ulm length* of an abelian p-group A is the smallest ordinal σ such $A^{p^\sigma} = A^{p^{\sigma+1}}$.

Moreover, if we restrict to the union of countably many local group classes, it is possible to say something even on finite extensions. Recall that if $\mathfrak{X}$ is any group class, the symbol $\mathfrak{X}\mathfrak{F}$ denotes the class of all groups containing a normal $\mathfrak{X}$-subgroup of finite index.

Theorem 1 (FdG–MT [7], Theorem 3.2)**.** *Let* $(\mathfrak{X}_n)_{n \in \mathbb{N}}$ *be a collection of group classes which are local and subgroup closed, and let*

$$\mathfrak{X} = \bigcup_{n \in \mathbb{N}} \mathfrak{X}_n.$$

Then the class $\mathfrak{X}\mathfrak{F}$ *is countably recognizable.*

Corollary 1. *Let* $\mathfrak{X}$ *be a subgroup closed group class. Then the class* $(L\mathfrak{X})\mathfrak{F}$ *is countably recognizable.*

A crucial role in the proof of Theorem 1 is played by the following result due to Baer, for a proof of which we refer to [8], Proposition 1.K.2.

Lemma 2. *Let* $\mathfrak{X}$ *be a subgroup closed group class and let G be a group in which every finitely generated subgroup contains an $\mathfrak{X}$-subgroup of index at most k, where k is a fixed positive integer. Then G contains a subgroup of index at most k which is locally $\mathfrak{X}$.*

Our next result deals with the dual case of groups which are close to be in $\mathfrak{X}$ up to a finite section on the bottom; more precisely, it concerns with the class $\mathfrak{F}\mathfrak{X}$ of all groups G admitting a finite normal subgroup N such that G/N is in $\mathfrak{X}$.

Theorem 2 (FdG – MT [7], Theorem 3.6). *Let $\mathfrak{X}$ be a countably recognizable group class which contains all subgroups of direct products of finitely many $\mathfrak{X}$-groups. Then the class $\mathfrak{F}\mathfrak{X}$ is countably recognizable.*

Let $\mathfrak{X}$ be a group class. It is very easy to see that $\mathfrak{X}$ is countably recognizable if and only if the largest subgroup closed subclass $\mathfrak{X}^{\mathrm{S}}$ of $\mathfrak{X}$ is countably recognizable. Moreover, it has been proved in [9] that if $\mathfrak{X}$ is countably recognizable, then also its subclass $\mathfrak{X}^{\mathrm{Q}}$, consisting of all groups whose homomorphic images are in $\mathfrak{X}$, is countably recognizable. On the other hand, if we choose as $\mathfrak{X}$ the class of free abelian groups, then $\mathfrak{X}$ is not countably recognizable but $\mathfrak{X}^{\mathrm{Q}}$ is the class of trivial groups (and so it is obviously countably recognizable).

In contrast to the above remarks, it turns out that there are no connections between the countable recognizability of a group class $\mathfrak{X}$ and that of the classes $\mathrm{S}\mathfrak{X}$, consisting of all groups isomorphic to subgroups of $\mathfrak{X}$-groups, and $\mathrm{Q}\mathfrak{X}$, formed by all groups which are homomorphic images of $\mathfrak{X}$-groups. This is proved by the following examples.

- $\mathfrak{X}$ *countably recognizable* $\not\Rightarrow$ $\mathrm{S}\mathfrak{X}$ *countably recognizable*

Choose as $\mathfrak{X}$ the class of groups which are either trivial or free abelian of countably infinite rank

- $\mathrm{S}\mathfrak{X}$ *countably recognizable* $\not\Rightarrow$ $\mathfrak{X}$ *countably recognizable*

Let p and q be different prime numbers and choose as $\mathfrak{X}$ the class consisting of all countable p-groups and all groups that are direct product of a p-group and a group of order q

- $\mathfrak{X}$ *countably recognizable* $\not\Rightarrow$ $\mathrm{Q}\mathfrak{X}$ *countably recognizable*

Choose again as $\mathfrak{X}$ the class of groups which are either trivial or free abelian of countably infinite rank

- $\mathrm{Q}\mathfrak{X}$ *countably recognizable* $\not\Rightarrow$ $\mathfrak{X}$ *countably recognizable*

Choose as $\mathfrak{X}$ the class of all free groups

We now turn our attention to classes of groups that can be defined by the existence of certain ascending chains of subgroups. To this aim, we first look at the countable character of the existence of (non-trivial) normal subgroups with a given property.

Let $\mathcal{W}$ be a set of words on the alphabet

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{n+1}, b_{n+1}, \ldots$$

and let $\mathfrak{X}$ and $\mathfrak{Y}$ be group classes. We say that a normal subgroup N of a group G is $(\mathfrak{X}, \mathfrak{Y}, \mathcal{W})$-*embedded* in G if it satisfies the following conditions:

- N belongs to $\mathfrak{X}$;
- $G/C_G(N)$ belongs to $\mathfrak{Y}$;
- $w(x_1, \ldots; g_1, \ldots) = 1$ for any word $w \in \mathcal{W}$ and any choice of elements $x_1, \ldots$ in N and $g_1, \ldots$ in G.

Theorem 3 (R. Baer [3], Satz 3.1). *Let $\mathfrak{X}$ be a subgroup closed group class, $\mathfrak{Y}$ a quotient closed group class and $\mathcal{W}$ a set of words on the alphabet*

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{n+1}, b_{n+1}, \ldots$$

If $\mathfrak{X}$ and $\mathfrak{Y}$ are countably recognizable and G is a group such that every countable non-trivial subgroup X contains a non-trivial subgroup which is $(\mathfrak{X}, \mathfrak{Y}, \mathcal{W})$-embedded in X, then also G has a non-trivial $(\mathfrak{X}, \mathfrak{Y}, \mathcal{W})$-embedded subgroup.

The composition of Theorem 3 with the following easy remark (also due to Baer) provides several relevant countably recognizable group classes.

Lemma 3. *Let χ be a property pertaining to subgroups, and suppose that the class $\mathfrak{X}$ of all groups which either are trivial or contain a non-trivial χ-subgroup is countably recognizable. Then the class of all groups all whose homomorphic images belong to $\mathfrak{X}$ is countably recognizable.*

If $\mathfrak{X}$ is a group class, we say that a group G is *hyper-$\mathfrak{X}$* if it has an ascending normal series whose factors belong to $\mathfrak{X}$. For instance, if $\mathfrak{X}$ is chosen to be one of the classes $\mathfrak{A}$ of abelian groups, $\mathcal{C}$ of cyclic groups and $\mathfrak{F}$ of finite groups, this definition gives the important concepts of a *hyperabelian group*, a *hypercyclic group* and a *hyperfinite group*, respectively. It is well known that if the class $\mathfrak{X}$ is quotient closed, then G is hyper-$\mathfrak{X}$ if and only if every non-trivial homomorphic image of G has a non-trivial normal $\mathfrak{X}$-subgroup.

Corollary 2. *Let $\mathfrak{X}$ be a countably recognizable group class which is subgroup and quotient closed. Then the class of hyper-$\mathfrak{X}$ groups is countably recognizable.*

Of course, Corollary 2 proves in particular that hyperfinite groups form a countably recognizable class. On the other hand, K.K. Hickin and R.E. Phillips [10] constructed an uncountable locally finite p-group which is not a Specht group. Recall here that a group G is called a *Specht group* if it admits an ascending chain

$$\{1\} = G_0 < G_1 < \ldots G_\alpha < G_{\alpha+1} < \ldots G_\tau = G$$

such that G_α has finite index in $G_{\alpha+1}$ for all $\alpha < \tau$. Since any countable locally finite p-group admits an ascending series whose factors have order p, the above example shows that the class of Specht groups, as well as the classes $\acute{P}\mathfrak{F}$ and $\acute{P}\mathcal{C}$, is not countably recognizable (where $\acute{P}\mathfrak{X}$ denotes the class of groups admitting an ascending series with $\mathfrak{X}$-factors).

Our next result extends Corollary 2 to the class of groups containing a hyper-$\mathfrak{X}$ subgroup of finite index.

Theorem 4. *Let $\mathfrak{X}$ be a countably recognizable group class which is subgroup and quotient closed. If in any group the product of finitely many normal $\mathfrak{X}$-subgroups admits an ascending characteristic series with $\mathfrak{X}$-factors, then the class of all groups containing a hyper-$\mathfrak{X}$ subgroup of finite index is countably recognizable.*

Proof. Suppose the statement is false and let G be a counterexample, so all countable subgroups of G are (hyper-$\mathfrak{X}$)-by-finite, while G is not. Let X be any countable subgroup of G. Using the hypotheses on $\mathfrak{X}$, it is easy to see that X admits an ascending normal series

$$\{1\} = H_0 < H_1 < \ldots H_\alpha < H_{\alpha+1} < \ldots H_{\tau(X)} \leq X$$

such that $\rho(X) = |X : H_{\tau(X)}|$ is finite, $H_{\alpha+1}/H_\alpha$ is in $\mathfrak{X}$ for all ordinals α, and every normal hyper-$\mathfrak{X}$ subgroup of X is contained in $H_{\tau(X)}$. Assume that for each positive integer k there is a countable subgroup X_k of G such that $\rho(X_k) \geq k$. Then the subgroup generated by all X_k's is countable and cannot contain hyper-$\mathfrak{X}$ subgroups of finite index, which is of course a contradiction. Thus, there exists a countable subgroup M such that $\rho(M)$ is largest possible. If $\mathcal{M}$ is the set of all countable subgroups of G containing M, it follows that $\rho(M) = \rho(U)$ for every $U \in \mathcal{M}$ and so $H_{\tau(U)} \leq H_{\tau(V)}$ whenever U and V belong to $\mathcal{M}$. Put

$$H = \langle H_{\tau(U)} \mid U \in \mathcal{M} \rangle$$

and let L be any countable subgroup of H. For each $x \in L$ there is a countable subgroup $L_x \in \mathcal{M}$ such that $x \in H_{\tau(L_x)}$. Then $L \leq H_{\tau(W)}$, where $W = \langle L_x \mid x \in L \rangle$ and so L is hyper-$\mathfrak{X}$. Thus, H is hyper-$\mathfrak{X}$ by Corollary 2. On the other hand, G/H must be finite since all its countable subgroups are finite and hence G is (hyper-$\mathfrak{X}$)-by-finite. This contradiction completes the proof. $\square$

There are many relevant group classes which are not subgroup closed and to which of course Theorem 3 does not apply; this is for instance the case of the class of finitely generated metabelian groups. Baer overcame such difficulty by proving the following corresponding result. Notice here that any group class consisting only of finitely generated

groups is obviously countably recognizable, because every group which is not finitely generated must contain a countable subgroup which is not finitely generated.

Theorem 5 (R. Baer [3], Satz 3.2). *Let $\mathfrak{X}$ be a class consisting of finitely generated groups, $\mathfrak{Y}$ a subgroup closed countably recognizable group class and $\mathcal{W}$ a set of words on the alphabet*

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{n+1}, b_{n+1}, \ldots$$

If G is a group such that every countable non-trivial subgroup X contains a non-trivial subgroup which is $(\mathfrak{X}, \mathfrak{Y}, \mathcal{W})$-embedded in X, then also G has a non-trivial $(\mathfrak{X}, \mathfrak{Y}, \mathcal{W})$-embedded subgroup.

In analogy to Corollary 2, we note the following consequence of Theorem 5.

Corollary 3. *Let $\mathfrak{X}$ be a quotient closed group class consisting of finitely generated groups. Then the class of hyper-$\mathfrak{X}$ groups is countably recognizable.*

Other general theorems concerning the countable recognizability of classes of groups defined by ascending chains of subgroups can be found in Section 4 of [7] and Section 8.3 of [4], where it is proved in particular that if $\mathfrak{X}$ is any countably recognizable group class which is subgroup and quotient closed, then also the class of all groups admitting an ascending subnormal series with factors in $\mathfrak{X}$ is countably recognizable (see [4] Part 2, Theorem 8.36). Notice here that in such a case, this latter class is actually the smallest radical class containing $\mathfrak{X}$ (a group class is called *radical* if it is closed with respect to forming quotients, extensions and arbitrary products of normal subgroups).

Mal'cev proved that both the class of groups with abelian chief factors and the class of groups with central chief factors are local (see [4] Part 1, Theorem 5.27). Next theorem deals with conditions of this type.

Theorem 6 (R.E. Phillips [11,12]). *Let $\mathfrak{X}$ be a subgroup closed group class which is countably recognizable.*

(a)　*The class of all groups whose simple sections belong to $\mathfrak{X}$ is countably recognizable.*
(b)　*The class of all groups whose chief factors belong to $\mathfrak{X}$ is countably recognizable.*
(c)　*The class of all groups G such that X/M_X belongs to $\mathfrak{X}$, whenever M is a maximal subgroup of $X \leq G$, is countably recognizable.*

We point out that in part (a) of the above statement 'simple' may be replaced by 'characteristically simple' and similarly in part (b) 'chief factor' can be substituted by 'characteristic chief factor'. Notice also that, differently from Mal'cev's local theorem, the above result gives several countably recognizable group classes which are not local. For instance, the choice $\mathfrak{X} = \mathfrak{F}$ yields that the class of groups with finite chief factors, as well as that of groups whose maximal subgroups have finite index, is countably recognizable while they are not local, as shown by the consideration of the alternating group on a countably infinite set.

To prove the first part of Theorem 6, Phillips obtained a result of independent interest, showing in particular that the class of groups which are not (characteristically) simple is countably recognizable (see also [13], where a generalization of this result can be found).

Theorem 7. *Let G be a (characteristically) simple group and let X be any countable subgroup of G. Then there exists a countable (characteristically) simple subgroup of G containing X.*

It should also be mentioned that Phillips' proof of the second part of Theorem 6 depends on the following useful result, which rests ultimately on a method introduced by D.H. McLain in [14].

Lemma 4. *Let M be a minimal normal subgroup of a group G and let X be any countable subgroup of M. Then there exists a countable subgroup Y of G such that X embeds into a chief factor of Y.*

The last theorem of this section should be seen in connection with the above result.

Theorem 8. *Let $\mathfrak{X}$ be a subgroup closed and countably recognizable group class. Then the class of groups whose minimal normal subgroups are in $\mathfrak{X}$ is countably recognizable.*

Proof. Let G be a group admitting a minimal normal subgroup M which is not in $\mathfrak{X}$. Since $\mathfrak{X}$ is countably recognizable, M contains a countable subgroup X which is not in $\mathfrak{X}$. For each element x of X, choose a countable subgroup $Y_{1,x}$ of G containing x and such that $X \leq \langle x \rangle^{Y_{1,x}}$. Put

$$Y_1 = \langle Y_{1,x} \mid x \in X \rangle \quad \text{and} \quad X_1 = Y_1 \cap M.$$

Then X_1 is countable and the above argument can be iterated to define sequences of subgroups

$$Y_1, Y_2, \ldots, Y_n, \ldots \quad \text{and} \quad X \leq X_1 \leq X_2 \leq \ldots \leq X_n \leq \ldots$$

such that $X_n \leq \langle x \rangle^{Y_{n+1}}$ for each $x \in X_n$ and $X_n = \langle Y_1, \ldots, Y_n \rangle \cap M$, for all n. It is easy to see that

$$X_\omega = \bigcup_{n \in \mathbb{N}} X_n$$

is a minimal normal subgroup of the countable group

$$Y_\omega = \langle Y_n \mid n \in \mathbb{N} \rangle$$

and X_ω is not in $\mathfrak{X}$, because $\mathfrak{X}$ is subgroup closed. The statement is proved. $\square$

Corollary 4. *The class of all groups whose minimal normal subgroups are finite is countably recognizable.*

3. Nilpotency

This section deals with the countable detection of nilpotency and its generalizations. Of course, for each positive integer k, the group class $\mathfrak{N}_k$, made by all nilpotent groups of class at most k, is local and so also countably recognizable. Thus, the countably recognizability of $\mathfrak{N}$ follows at once from Lemma 1.

Theorem 9 (R. Baer [3], Bemerkung 1.3)**.** *The class $\mathfrak{N}$ of nilpotent groups is countably recognizable.*

Moreover, application of Corollary 1 yields that both the class $\mathfrak{N}\mathfrak{F}$ of all nilpotent-by-finite groups and the class $(L\mathfrak{N})\mathfrak{F}$ of all (locally nilpotent)-by-finite groups are countably recognizable. Notice that by Theorem 2 also the class $\mathfrak{F}\mathfrak{N}$ of all finite-by-nilpotent groups is countably recognizable.

Two relevant theorems of Baer and Philip Hall prove that for a group G there exists a positive integer h such that the subgroup $\gamma_h(G)$ is finite if and only if the index $|G : \zeta_k(G)|$ is finite for some integer $k \geq 0$ (see [4] Part 1, p. 113 and p. 117). Therefore, also the property of being finite over some term with finite ordinal type of the upper central series is countably recognizable. As a consequence of a general result, it turns out that the class of all groups G which are finite over $\zeta_k(G)$ for some fixed positive integer k is countably recognizable.

Let $\mathfrak{B}$ be a variety and let W be a set of words such that $\mathfrak{B}(W) = \mathfrak{B}$. Recall that a normal subgroup N of a group G is said to be $\mathfrak{B}$-*marginal* if

$$\theta(g_1, \ldots, g_{i-1}, g_i x, g_{i+1}, \ldots, g_n) = \theta(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n)$$

for each word $\theta \in W$ in n variables and for all elements $g_1, \ldots, g_n$ of G and x of N. Every group G contains a largest $\mathfrak{B}$-marginal subgroup, which is denoted by $W^*(G)$, and G belongs to $\mathfrak{B}$ if and only if $W^*(G) = G$.

Theorem 10 (FdG–MT [7], Theorem 3.8). *If $\mathfrak{B}$ is any variety, the class of groups containing a $\mathfrak{B}$-marginal subgroup of finite index is countably recognizable.*

If k is any positive integer and W_k is the set consisting of the single word $[x_1, \ldots, x_k]$, then $W_k^*(G) = \zeta_k(G)$ for any group G and hence Theorem 10 has the following consequence.

Corollary 5. *If k is any positive integer, the class of groups which are finite over the k-th term of their upper central series is countably recognizable.*

It has been recently proved that a group G is finite over its hypercentre $\bar{\zeta}(G)$ if and only if G contains a finite normal subgroup N such that G/N is hypercentral (see [15]). As the class of hypercentral groups is countably recognizable, it follows from Theorem 2 that also the class of groups which are finite over the hypercentre is countably recognizable (even if the upper central series stops only after infinitely many steps).

Recall that if n is any integer and G is a group, the set $\zeta(G; n)$ of all elements z of G such that $(zg)^n = z^n g^n$ and $(gz)^n = g^n z^n$ for all $g \in G$ is a subgroup, and G is called *n-abelian* if $\zeta(G; n) = G$, or equivalently if $(xy)^n = x^n y^n$ for all $x, y \in G$. Clearly, $\zeta(G; -1) = \zeta(G)$ and so G is (-1)-abelian if and only if it is abelian. The *upper n-central series* of G can now be defined in analogy to the ordinary upper central series and a group is *n-nilpotent* if its n-upper central series reaches G after finitely many steps. These concepts were introduced and studied by Baer (see [16,17]). It is not difficult to see that n-nilpotent groups of class at most k (with the obvious meaning) form a local class, so that *the class of n-nilpotent groups is countably recognizable*. Moreover, it follows in particular from Theorem 3 that if G is a non-trivial group and $\zeta(X; n) \neq \{1\}$ for every countable non-trivial subgroup X of G, then $\zeta(G; n) \neq \{1\}$. In particular, for $n = -1$, we observe that any non-trivial group whose non-trivial countable subgroups have non-trivial centre has a non-trivial centre itself. Consequently, we obtain the countable recognizability of the class of hypercentral groups, through methods which are different from those used by Černikov.

Let G be a group and let X be a subgroup of G. Recall that the *series of normal closures*

$$\{X^{G,n} \mid n \in \mathbb{N}o\}$$

of X in G is defined by putting $X^{G,0} = G$ and

$$X^{G,n+1} = X^{X^{G,n}}$$

for each non-negative integer n. In particular, $X^{G,1} = X^G$ and $X \leq X^{G,n}$ for all n. Notice that X is subnormal in G of defect at most k if and only if $X^{G,k} = X$. Moreover, if U is any subgroup of G, we put $X^{U,n} = X^{\langle X,U \rangle,n}$ for all $n \geq 0$.

Let P be a group of type p^∞ for some prime number p and let x be the automorphism of P defined by putting $a^x = a^{1+p}$ for all $a \in P$; then the semidirect product $G = \langle x \rangle \ltimes P$ is a locally nilpotent group, but $\langle x \rangle$ is not subnormal in G. On the other hand, it was proved by Baer in [3] that if X is a subgroup of a group G which is subnormal of defect at most k in $\langle X, U \rangle$ for each countable subgroup U of G, then X is subnormal in G with defect at most $k + 1$. Actually, the next result shows that the defect of the subnormal subgroup X is even bounded by k; according to Baer, this fact was already observed by E. Wirsing but never published.

Theorem 11 (FdG–MT [7], Theorem 2.4). *Let G be a group, and let X be a subgroup of G.*
(a) *If X is subnormal in $\langle X, U \rangle$ for each countable subgroup U of G, then X is subnormal in G.*

(b) *If k is a positive integer and X is subnormal in $\langle X, U \rangle$ of defect at most k for each countable subgroup U of G, then X is subnormal in G of defect at most k.*

Notice that the above theorem essentially depends on the following interesting result.

Lemma 5. *Let G be a group, and let X be a subgroup of G which is properly contained in $X^{G,n}$ for some positive integer n. Then there exists a countable subgroup U of G such that X is a proper subgroup of $X^{U,n}$.*

For every group G, let $\Xi(G)$ be a set of subgroups of G containing the identity subgroup $\{1\}$; the elements of $\Xi(G)$ are called Ξ-*subgroups* of G. We shall say that Ξ is an *embedding subgroup property* if the following conditions are satisfied:

(i) $\left(\Xi(G)\right)^{\varphi} = \Xi(G^{*})$ for every group isomorphism φ from G onto a group G^{*};

(ii) X belongs to $\Xi(Y)$, whenever $X \leq Y \leq G$ and $X \in \Xi(G)$.

An embedding property Ξ is called *absolute* if for each group G the set $\Xi(G)$ contains all subgroups which are isomorphic to a Ξ-subgroup of some group; in particular, if $\mathfrak{X}$ is any group class, the property for a subgroup to belong to $\mathfrak{X}$ is an absolute property. On the other hand, there are many relevant embedding properties (like for instance normality and subnormality) which are not absolute.

An embedding property Ξ is said to have *countable character* when a subgroup X of an arbitrary group G is a Ξ-subgroup if and only if Ξ holds for all countable subgroups of X. In particular, if Ξ is an embedding property of countable character, and X is a Ξ-subgroup of a group G, then all subgroups of X have the property Ξ. Notice also that if $\mathfrak{X}$ is a subgroup closed group class, the property for a subgroup to be in $\mathfrak{X}$ is an absolute property which has countable character if and only if $\mathfrak{X}$ is countably recognizable.

Next statement shows in particular that if Ξ is an embedding property of countable character, then the class of groups with a non-trivial normal Ξ-subgroup is countably recognizable.

Theorem 12 (FdG–MT [7], Lemmas 4.1 and 4.2)**.** *Let Ξ be an embedding property of countable character (and let k be a positive integer). Then the class of all groups which either are trivial or contain a non-trivial subnormal Ξ-subgroup (of defect at most k) is countably recognizable.*

The celebrated theorem of Hans Fitting on the nilpotency of the product of finitely many nilpotent normal subgroups yields that in any group G the *Fitting subgroup*, i.e., the subgroup generated by all nilpotent normal subgroups, is at least locally nilpotent, although easy examples show that the Fitting subgroup of an infinite group need not be nilpotent. Theorem 12 has the following consequence when $k = 1$ and Ξ is the property of being nilpotent.

Corollary 6. *Let G be a non-trivial group. If every countable non-trivial subgroup of G has a non-trivial Fitting subgroup, then the Fitting subgroup of G is not trivial.*

A group G is called a *Fitting group* if it coincides with its Fitting subgroup, or equivalently if G is generated by its nilpotent normal subgroups.

Theorem 13 (FdG–MT [7], Theorem 2.6)**.** *The class of Fitting groups is countably recognizable.*

The most natural generalization of Fitting's theorem has been obtained by K.A. Hirsch and B.I. Plotkin, who proved that the subgroup generated by any collection of ascendant locally nilpotent subgroups is likewise ascendant and locally nilpotent. Thus, any group G contains a largest locally nilpotent normal subgroup, the so-called *Hirsch-Plotkin radical* of G. Application of Theorem 12 for $k = 1$ and Ξ the property of being locally nilpotent yields the following result.

Corollary 7. *Let G be a non-trivial group. If every countable non-trivial subgroup of G has a non-trivial Hirsch-Plotkin radical, then the Hirsch-Plotkin radical of G is not trivial.*

Obviously, the class of cyclic groups is countably recognizable and hence Theorem 12 shows in particular that if every countable non-trivial subgroup of a group *G* contains a non-trivial cyclic subnormal subgroup, then the same property holds for *G* (see also [3], Anwendung 2.5). Since the subgroup generated by all cyclic subnormal subgroups of a group *G* is the so-called *Baer radical* of *G*, the above remark can be rephrased in the following way.

Corollary 8. *Let G be a non-trivial group. If every countable non-trivial subgroup of G has a non-trivial Baer radical, then the Baer radical of G is not trivial.*

A group *G* is called a *Baer group* if it coincides with its Baer radical, i.e., if *G* is generated by its abelian subnormal subgroups.

Theorem 14 (FdG–MT [7], Corollary 2.5). *The class of Baer groups is countably recognizable.*

In 1940, Baer [18] introduced and studied the class of groups in which every proper subgroup is properly contained in its normalizer; these groups are nowadays called *N-groups* and can be characterized as those groups in which all subgroups are ascendant, so that in particular they are locally nilpotent.

Theorem 15 (R. Baer [3], Satz 2.6). *The class of N-groups is countably recognizable.*

A relevant theorem of J.E. Roseblade shows that if all subgroups of a group *G* are subnormal of bounded defect, then *G* is nilpotent of bounded class (see [4] Part 2, Theorem 7.42). This result is far from being true if the hypothesis that the subgroups have bounded defect is omitted; in fact, H. Heineken and I.J. Mohamed constructed in [19] a countably infinite *p*-group with trivial centre in which every proper subgroup is subnormal and nilpotent. The structure of groups with only subnormal subgroups, the so-called N_1-*groups*, was later investigated by several researchers; in particular, W. Möhres, a student of Heineken, wrote some deep papers on the subject, culminating in the beautiful theorem stating that any N_1-group is at least soluble (see [20]). More recently, Howard Smith proved in [21] that torsion-free N_1-groups are even nilpotent, improving a previous result of Möhres (see [22]). Of course, N_1-groups form a proper subclass of the class of *N*-groups that can be as well recognized from the behaviour of countable subgroups.

Theorem 16 (FdG–MT [7], Theorem 2.7). *The class of N_1-groups is countably recognizable.*

The concept of Baer radical was generalized in 1959 by Karl Gruenberg [23], who proved that in any group *G* the elements generating a cyclic ascendant subgroup form a subgroup, which is now called the *Gruenberg radical* of *G*. A group *G* is said to be a *Gruenberg group* if it coincides with its Gruenberg radical, or equivalently if it is generated by abelian ascendant subgroups. Of course, groups with the *N*-property are Gruenberg groups and it follows from the Hirsch-Plotkin theorem that Gruenberg groups are locally nilpotent; it is also easy to see that any countable locally nilpotent group is Gruenberg. In contrast to Corollaries 7 and 8, it turns out that the Gruenberg radical cannot be countably detected, since M.I. Kargapolov [24] constructed a locally nilpotent uncountable group with no cyclic ascendant non-trivial subgroups. The same example proves that the class of Gruenberg groups is not countably recognizable and that in the statement of Theorem 12 subnormality cannot be replaced by ascendancy.

There are many further nilpotency conditions we did not deal with, being local ones, and we refer to [4,7] for an overview of most of them.

4. Solubility

In this section, we deal with solubility and its more relevant generalizations. Of course, the first statement provides the countable character of the class of soluble groups.

Proposition 1 (R. Baer [3], Bemerkung 1.3)**.** *The class $\mathfrak{S}$ of soluble groups is countably recognizable.*

Since for any positive integer k the class of soluble groups of derived length at most k is local, it follows from Theorem 1 that the class $\mathfrak{S}\mathfrak{F}$ of soluble-by-finite groups is countably recognizable. Moreover, application of Corollary 1 yields that also the class of (locally soluble)-by-finite groups can be countably detected, a fact that was already proved in [25], Lemma 3.5. Notice that even the class $\mathfrak{F}\mathfrak{S}$ of all finite-by-soluble groups is countably recognizable by Theorem 2.

The following statement is a special case of Corollary 2.

Theorem 17. *The class of hyperabelian is countably recognizable.*

It follows directly from Theorem 4 that hyperabelian-by-finite groups form a countably recognizable class, a result that was also proved in [26].

As with the case of hyperabelian groups, it turns out that the dual concept of a *hypoabelian group*, i.e., a group admitting a descending (normal) series with abelian factors, determines a countably recognizable group class. This is for instance a consequence of the fact that the class of groups which either are trivial or properly contain their commutator subgroup is countably recognizable (see [7], Lemma 4.10). Observe in this context that the class of perfect groups is local and so also countably recognizable.

Recall that a group is *subsoluble* if it admits an ascending series consisting of subnormal subgroups in which all factors are abelian. Thus, a group G is subsoluble if and only if every non-trivial homomorphic image of G has a non-trivial Baer radical and hence Corollary 8 has the following application.

Proposition 2. *The class of subsoluble groups is countably recognizable.*

A group G is called an SN^*-*group* if it has an ascending series with abelian factors and it is known that any group has a largest ascendant SN^*-subgroup (the so-called SN^*-*radical*), which is of course characteristic. It was proved by Gruenberg [23] that for a locally nilpotent group the SN^*-property and the property of being a Gruenberg group coincide. It follows now either from Kargapolov's example quoted in Section 3 (or also from the example of Hickin and Phillips after Corollary 2) that neither the class of SN^*-groups is countably recognizable nor the SN^*-radical is countably detectable.

Recall finally that a group G is *radical* if it has an ascending (normal) series with locally nilpotent factors, while G is called *generalized radical* if it admits an ascending (normal) series each of whose factors is either locally nilpotent or locally finite. Since the classes $L\mathfrak{N}$ and $L\mathfrak{N} \cup L\mathfrak{F}$ are local, it follows from Corollary 2 that radical groups, as well as generalized radical groups, form a countably recognizable group class.

There are many other solubility conditions we did not deal with, being local ones, and we refer to [4,7] for an overview of most of them. Here we just observe that the only problem concerning the countable recognizability of the main classes of generalized soluble groups which seems to be still open concerns the class SJ of groups admitting a subnormal series with arbitrary ordinal type and abelian factors (see Question A$_D$V-2B in *Adv. Group Theory Appl.* 2 (2016), p.127–128).

5. Chain Conditions

It is easy to see that both the class of groups satisfying the minimal condition and that of groups satisfying the maximal condition are countably recognizable. Since the class of abelian-by-finite groups is countably recognizable by Corollary 1, we have in

particular that Černikov groups form a countably recognizable class, a result that was first proved by Baer (see [27], Zusatz 2.5). It also follows from the above fact that any group class whose countable members are finitely generated is countably recognizable (see [3], Folgerung 1.5); in particular, polycyclic groups and polycyclic-by-finite groups form countably recognizable classes. On the other hand, a similar remark cannot be done for the minimal condition, since there exists an uncountable group satisfying the minimal condition (see [28]) and so the class of countable groups with the minimal condition is not countably recognizable.

A group G is said to satisfy the *weak minimal condition* on subgroups if it has no infinite descending chains of subgroups

$$X_1 > X_2 > \ldots > X_n > \ldots$$

such that the index $|X_n : X_{n+1}|$ is infinite for all n. The *weak maximal condition* on subgroups is defined replacing descending chains by ascending chains. It was independently proved by Baer [29] and Zaicev [30] that for soluble groups the weak minimal condition, the weak maximal condition and the property of being minimax are equivalent. It turns out that also the class of groups satisfying the weak minimal condition and that of groups satisfying the weak maximal condition on subgroups are countably recognizable. It follows in particular that the class of soluble minimax groups is countably recognizable, and it has recently been proved that the solubility assumption can be dropped out. Recall here that a group is said to be *minimax* if it admits a finite series each of whose factors satisfies either the minimal or the maximal condition on subgroups.

Theorem 18 (FdG–MT [31])**.** *The class of minimax groups is countably recognizable.*

The crucial point in the proof of the above result is the analysis of groups whose countable subgroups admit a fixed minimax type according to the following definition: if $\vee$ and $\wedge$ denote the minimal and the maximal condition on subgroups, respectively, a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ whose entries belong to the set $\{\vee, \wedge\}$ is called a *minimax type* for a minimax group G if G has a finite series

$$\{1\} = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

of length n such that the factor group G_i/G_{i-1} satisfies σ_i for each positive integer $i \leq n$.

Hyperminimax groups, i.e., groups admitting an ascending normal series with minimax factors, have been considered in [32], as a natural generalization of both minimax groups and *FC*-hypercentral groups (see the next section for the definition). In view of Corollary 2, the above theorem has the following consequence.

Corollary 9. *The class of hyperminimax groups is countably recognizable.*

The last part of this section is devoted to some final remarks about the countable character of chain conditions. For the sake of brevity, we discuss only classes of groups connected to the maximal and minimal conditions, but our remarks can be generalized to encompass essentially any chain condition. Let χ be any embedding subgroup property such that if H is a χ-subgroup of a group G and X is any subgroup of G, then $H \cap X$ is a χ-subgroup of X. A group is said to satisfy the *minimal* (*maximal*, respectively) condition on χ-subgroups if it does not admit any descending (ascending, respectively) chain of χ-subgroups. The usual proof that the class of groups satisfying the minimal (maximal) condition is countably recognizable can be slightly modified to prove that groups satisfying the minimal (maximal) condition on χ-subgroups also form a countably recognizable class (see [4], Part 2, Theorem 8.32). This is for instance the case of subnormality, so the class of groups satisfying the minimal (maximal) condition on subnormal subgroups is countably recognizable. The situation is a little more complicated if we look at groups satisfying the

minimal (maximal) condition on subgroups which do not possess the χ-property. Here, further assumptions on the property χ must be required; we prove the following result in the case of subnormality and the maximal condition, leaving the reader with the task of deriving the conditions on χ which are necessary to the proof.

Theorem 19. *Let G be a group whose countable subgroups satisfy the maximal condition on non-subnormal subgroups. Then G itself satisfies the maximal condition on non-subnormal subgroups.*

Proof. Suppose the statement is false and let

$$X_1 < X_2 < \ldots < X_n < X_{n+1} < \ldots$$

be an ascending chain of non-subnormal subgroups of G. For each positive integer i choose an element $x_i \in X_{i+1} \setminus X_i$ and a countable subgroup U_i of G such that $[U_i \cap X_i, {}_n U_i]$ is not contained in X_i for each $n \in \mathbb{N}$. Put $Y = \langle x_i, U_i \,|\, i \in \mathbb{N} \rangle$ and notice that

$$X_1 \cap Y < X_2 \cap Y < \ldots < X_n \cap Y < X_{n+1} \cap Y < \ldots$$

is an ascending chain of non-subnormal subgroups of Y, a contradiction. □

6. Conjugacy Classes

In this section, we deal with the countable detection of the most relevant classes of generalized *FC*-groups. Recall that a group G is an *FC-group* if it has finite conjugacy classes of elements, or equivalently if the centralizer of each element has finite index in G; moreover, an *FC*-group G is said to be a *BFC-group* if there is a bound on the sizes of conjugacy classes of its elements. A well-known result of B.H. Neumann shows that a group is *BFC* if and only if it has a finite commutator subgroup, and it is straightforward to note that *FC*-groups, as well as *BFC*-groups, form a countably recognizable class, although these classes are not local. Thus Corollary 1 cannot be applied to prove that the class $\mathfrak{F}\mathfrak{A}\mathfrak{F}$ of all finite-by-abelian-by-finite groups is countably recognizable. On the other hand, both $\mathfrak{F}\mathfrak{A}$ and $\mathfrak{F}\mathfrak{A}\mathfrak{F}$ can be obtained as union of a countable collection of local classes (see for instance [33–35]) and hence it is countably recognizable by Theorem 1. Notice here that similar remarks hold if the class $\mathfrak{A}$ is replaced by the class $\mathfrak{Q}$ consisting of all *quasihamiltonian groups*, i.e., groups in which $XY = YX$ for all subgroups X and Y, so that the classes $\mathfrak{F}\mathfrak{Q}$ and $\mathfrak{F}\mathfrak{Q}\mathfrak{F}$ are countably recognizable (see [36,37]).

More generally, the set $FC(G)$ of all elements of a group G with only finitely many conjugates is a characteristic subgroup of G, called the *FC-centre*. Clearly, G is an *FC*-group if and only if $FC(G) = G$. If in the statement of Theorem 3 we choose as $\mathfrak{X}$ the class of groups satisfying the maximal condition, as $\mathfrak{Y}$ the class of finite groups and $\mathcal{W} = \emptyset$, we obtain the following result.

Lemma 6. *Let G be a non-trivial group. If every countable non-trivial subgroup of G has a non-trivial FC-centre, then $FC(G) \neq \{1\}$.*

If G is a group, the *upper FC-central series* of G is the ascending characteristic series $\{FC_\alpha(G)\}_\alpha$ defined by setting $FC_0(G) = \{1\}$,

$$FC_{\alpha+1}(G)/FC_\alpha(G) = FC\big(G/FC_\alpha(G)\big)$$

for each ordinal α and

$$FC_\lambda(G) = \bigcup_{\alpha < \lambda} FC_\alpha(G)$$

if λ is a limit ordinal. The group G is said to be *FC-hypercentral* if $FC_\tau(G) = G$ for some ordinal τ. Clearly, a group G is *FC*-hypercentral if and only if every non-trivial

homomorphic image of G has a non-trivial *FC*-centre, and hence Corollary 6 has the following consequence.

Theorem 20. *The class of FC-hypercentral groups is countably recognizable.*

Notice that *hypercyclic groups* generalize supersoluble groups and form a very relevant subclass of the class of *FC*-hypercentral groups. As a consequence of Corollary 2, we have that the class of hypercyclic groups is countably recognizable.

In relation to Theorem 20, we observe that also groups admitting an ascending normal series whose factors are either central or finite form a countably recognizable class; this is a special case of Corollary 4.7 of [7].

If a group G coincides with some term with finite ordinal type of its upper *FC*-central series, then it is called *FC-nilpotent* and the *FC-nilpotency class* of G is the least non-negative integer k such that $FC_k(G) = G$. The class of *FC*-nilpotent groups strictly lies between the class of nilpotent-by-finite groups and that of *FC*-hypercentral groups.

Theorem 21 (FdG–MT [35], Theorem 3.2)**.** *For each positive integer k, the class of FC-nilpotent groups of class at most k is countably recognizable.*

The following statement follows from a combination of Theorem 21 and Lemma 1.

Corollary 10 (FdG–MT [35], Corollary 3.4)**.** *The class of FC-nilpotent groups is countably recognizable.*

We point out that the proof of Theorem 4 can be adapted to prove, for instance, the following result, which is analogous to Corollary 5.

Corollary 11. *If k is any positive integer, the class of groups which are finite over the k-th term of their upper FC-central series is countably recognizable.*

In the last part of this section, we deal with the countable character of certain group classes which generalize that of *FC*-groups, recalling first their definitions. Let $\mathfrak{X}$ be any group class. We denote by $M\mathfrak{X}$ the class of all groups in which every finite subset lies in a normal $\mathfrak{X}$-subgroup; so, the well-known Dietzmann's lemma actually states that $M\mathfrak{F}$ coincides with the class of periodic *FC*-groups and in particular $M\mathfrak{F}$ is countably recognizable. Moreover, we say that a group G is an $\mathfrak{X}C$-*group* (or that G has $\mathfrak{X}$-*conjugacy classes*) if $G/C_G(\langle g \rangle^G)$ belongs to $\mathfrak{X}$ for each element g of G. Then $\mathfrak{F}C$-groups are just groups with the *FC*-property. Next result shows that many classes of the form $M\mathfrak{X}$ and $\mathfrak{X}C$ are countably recognizable.

Theorem 22 (FdG–MT [35], Theorems 3.6 and 3.7)**.** *If $\mathfrak{X}$ is any subgroup closed and countably recognizable group class, then the classes $M\mathfrak{X}$ and $\mathfrak{X}C$ are countably recognizable.*

If we put $FC^0 = \mathfrak{F}$ and recursively $FC^k = (FC^{k-1})C$ for each positive integer k, we obtain an increasing sequence of classes of generalized *FC*-groups that were introduced in [38]. It follows by induction from Theorem 22 that the class FC^k is countably recognizable for each non-negative integer k.

For further information on the subject, we refer to [35], where it is proved in particular that the class $SD\mathfrak{F}$ of all groups which are isomorphic to subgroups of direct products of finite groups is not countably recognizable.

7. Residual Properties

This section deals with the countable character of certain relevant residual properties. Recall that for any group class $\mathfrak{X}$, the $\mathfrak{X}$-*residual* of a group G is the intersection of all normal subgroups N of G such that G/N belongs to $\mathfrak{X}$, and G is called *residually* $\mathfrak{X}$ if its

$\mathfrak{X}$-residual is trivial. In particular, a group is *residually finite* if the intersection of all its (normal) subgroups of finite index is trivial. Residually finite groups form a large class, containing in particular all free groups. It was proved by B.H. Neumann in [13] that the class of residually finite groups is countably recognizable. This result was widely extended in the following way.

Theorem 23 (FdG–MT [9], Theorem B). *Let $\mathfrak{X}$ be a subgroup closed and countably recognizable group class. Then the class of all groups whose finite residual belongs to $\mathfrak{X}$ is countably recognizable.*

Corollary 12 (FdG–MT [9], Corollary 3.1). *The class of all groups admitting a finite (normal) series with residually finite factors is countably recognizable.*

   In relation to Corollary 12, we also notice that groups admitting a descending series with residually finite factors form a countably recognizable class. In fact, it has been proved that the class of groups which either are trivial or contain a proper (normal) subgroup of finite index is countably recognizable (see [9], Theorem 3.2), so that the class of groups admitting a descending series with finite factors is countably recognizable and it is easy to show that a group has such a series if and only if it admits a descending (normal) series with residually finite factors. It is also easy to see that the class of groups which have no proper subgroups of finite index ($\mathfrak{F}$-*perfect groups*) is local and so countably recognizable. Finally, it seems to be an open question whether *hypofinite groups*, i.e., groups admitting a descending normal series with finite factors, are countably detectable or not.
   Neumann's theorem admits other types of generalization, such as for instance the following two results.

Theorem 24 (FdG–MT [9], Corollary 2.7). *Let $\mathfrak{X}$ be a subgroup closed class of finite groups. Then the class of residually $\mathfrak{X}$ groups is countably recognizable.*

Theorem 25 (FdG–MT [9], Corollary 2.8). *Let $\mathfrak{X}$ be a group class which is subgroup and quotient closed. If every $\mathfrak{X}$-group is residually finite, then the class of residually $\mathfrak{X}$ groups is countably recognizable.*

   Further results related to Neumann's theorem can be found in [9], where it is proved for instance that the class of profinite groups is not countably recognizable.
   The class of residually nilpotent groups was discovered to be countably recognizable by Phillips [12], and this is a special case of a result on collections of varieties that was obtained in [7].
   Let W be a set of words on a countably infinite alphabet. If G is any group, the *verbal subgroup* of G determined by W is defined as the subgroup $W(G)$ generated by all values of words in W on elements of G. Recall also that the *variety* determined by W is the class $\mathfrak{B}(W)$ consisting of all groups G such that each word in W reduces to the identity when the variables are replaced by arbitrary elements of G. Thus, a group G belongs to $\mathfrak{B}(W)$ if and only if $W(G) = \{1\}$. Clearly, every variety is S, Q, L and R-closed, and in particular is countably recognizable. Moreover, it is well known that a group class is a variety if and only if it is Q and R-closed (see for instance [4] Part 1, Theorem 1.13).

Theorem 26 (FdG–MT [7], Lemma 2.10). *Let $(\mathfrak{B}_n)_{n\in\mathbb{N}}$ be a countable collection of varieties of groups, and let*

$$\mathfrak{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n.$$

Then the class $R\mathfrak{B}$ of residually $\mathfrak{B}$-groups is countably recognizable.

Corollary 13. *The class of residually nilpotent groups and the class of residually soluble groups are countably recognizable.*

8. Further Properties

This section is devoted to discussing group classes that do not fit any of the previous topics and for which the status of art can be summarized in a few lines.

Complementation

A subgroup X of a group G is *complemented* in G if it admits a *complement*, i.e., if there is a subgroup Y of G such that $G = \langle X, Y \rangle$ and $X \cap Y = \{1\}$. A group G is called a *K-group* if all its subgroups are complemented in G, while G is a *C-group* if each subgroup X of G admits a complement Y such that $XY = YX$. It turns out that there exists a metabelian group G such that each countable subgroup of G is a C-group but G' does not admit any complement, so that the class of K-groups and the class of C-groups are not countably recognizable. This and other results on the local and countable character of group classes related to complementation properties can be found in [39].

f-Subnormality

A subgroup X of a group G is said to be *f-subnormal* if there is a finite chain

$$X = X_0 \leq X_1 \leq \ldots \leq X_k = G$$

of subgroups such that for $i = 1, 2, \ldots, k$ either X_{i-1} is normal in X_i or the index $|X_i : X_{i-1}|$ is finite. This concept was introduced by Phillips in [40] and later studied by various authors. As for subnormality, it turns out that many classes of groups which are connected to f-subnormality are countably recognizable. This is for instance the case of the class of groups in which all subgroups all f-subnormal (see [41]) and the group classes described in [42–44]. These results can be all obtained as consequences of a general theorem which, in turn, depends on a generalization of the ideas behind Mal'cev's local theorem (see [45]).

Finally, we remark that in [46] it is proved that there is an equivalence between certain chain conditions on subnormal subgroups and the corresponding restrictions on f-subnormal subgroups. In particular, we have that the class of groups satisfying the minimal (maximal) condition on f-subnormal subgroups is countably recognizable.

Linearity

A classical theorem of Mal'cev shows that for each positive integer n, the class of linear groups of degree n (even of a fixed characteristic) is local and so also countably recognizable (see for instance [47], Theorem 2.7). Then it follows from Lemma 1 that the class of all linear groups and that of linear groups of a fixed characteristic are countably recognizable. The case of finitary linear groups is much more complicated, since ultraproduct methods employed by Mal'cev do not work in general for finitary linear groups. Actually, an example of a periodic group which is not finitary linear although all its countable subgroups are such was given by F. Leinen and O. Puglisi in [48]; this example is based on a previous one due to Leinen ([49], Example 3.5). The problem seems to be still open in the case of primary groups, although B.A.F. Wehrfritz provided in [50] a partial positive solution in the case of hypercentral groups of central height at most ω. We also mention that J.I. Hall [51] proved that any simple group which is not finitary linear contains a countable subgroup that has no finitary linear representation. Further interesting results on this topic can be found in [48,52].

Pronormality

Recall that a subgroup X of a group G is called *pronormal* if X and X^g are conjugate in $\langle X, X^g \rangle$ for every $g \in G$. Of course, normal subgroups and maximal subgroups of arbitrary groups are pronormal, as well as Sylow subgroups of finite groups and Hall subgroups of finite soluble groups. Pronormality is strictly related to groups in which normality is a transitive relation (the so-called *T-groups*), for instance groups with only pronormal subgroups have the *T*-property. It is well known that the class of these groups is local and so also countably recognizable. Although the class of groups in which all

subgroups are pronormal is not local, it is at least countably recognizable. This and other facts connected to the countable character of pronormality and its generalizations can be found in [53,54]. For instance, it turns out that if X is a subgroup of a group G and all its countable subgroups are pronormal in G, then X itself is a pronormal subgroup.

Rank Conditions

Recall that a group G has *finite rank r* if every finitely generated subgroup of G can be generated by r elements and r it the smallest positive integer with such a property. Of course, for each positive integer r, the class of groups of rank at most r is local and so it follows from Lemma 1 that groups of finite rank form a countably recognizable class $\mathfrak{R}$. It was proved in [55] that for all choices k and r of positive integers, the class of groups G such that $\gamma_{k+1}(G)$ has rank at most r is local, and that a similar theorem holds for the class of groups G for which $G^{(k)}$ has rank at most r. From these results it follows that the classes $\mathfrak{R}\mathfrak{N}$ and $\mathfrak{R}\mathfrak{S}$ are countably recognizable.

In the same paper, an example is constructed to show that the class of groups which are extension of an abelian group by a group of rank 1 is not local (see [55], Theorem E), although it seems to be unknown if this class is at least countably recognizable. On the other hand, it is at least true that the classes $\mathfrak{N}\mathfrak{R}$ and $\mathfrak{S}\mathfrak{R}$ are countably recognizable (see [55], Theorems B and C). Further interesting results of this type can be found in [55,56], where in particular it is proved that the classes $\mathfrak{R}(L\mathfrak{S})$, $\mathfrak{R}(L\mathfrak{N})$ and $(L\mathfrak{N})\mathfrak{R}$ are countably recognizable.

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