

Article

# On Well-Posedness of Some Constrained Variational Problems

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**Abstract:** By considering the new forms of the notions of lower semicontinuity, pseudomonotonicity, hemicontinuity and monotonicity of the considered scalar multiple integral functional, in this paper we study the well-posedness of a new class of variational problems with variational inequality constraints. More specifically, by defining the set of approximating solutions for the class of variational problems under study, we establish several results on well-posedness.

**Keywords:** constrained variational problem; well-posedness; multiple integral functional

## 1. Introduction

The concept of well-posedness is a very useful mathematical tool in the study of optimization problems. Thus, beginning with the work of Tykhonov [1], many types of well-posedness associated with variational problems have been introduced (Levitin–Polyak well-posedness [2–5],  $\alpha$ -well-posedness [6,7], extended well-posedness [8–16],  $L$ -well-posedness [17]). Additionally, this mathematical tool can be used to study some related problems: variational inequality problems [18–20], complementary problems [21], equilibrium problems [22,23], fixed point problems [24], hemivariational inequality problems [25], Nash equilibrium problems [26], and so on. The well-posedness of generalized variational inequalities and the corresponding optimization problems have been analyzed by Jayswal and Shalini [27]. Moreover, an interesting and important extension of variational inequality problem is the multidimensional variational inequality problem and the associated multi-time optimization problems (see [28–33]). Recently, Treanță [30] investigated the well-posed isoperimetric-type constrained variational control problems. For other different but connected ideas, the reader is directed to Dridi and Djebabla [34] and Jana [35].

In this paper, motivated and inspired by the above research papers, we study the well-posedness property for new constrained variational problems, implying second-order multiple integral functionals and partial derivatives. In this regard, we formulate new forms of monotonicity, lower semicontinuity, hemicontinuity, and pseudomonotonicity for the considered multiple integral-type functional. Further, we introduce the set of approximating solutions for the constrained optimization problem under study and establish several theorems on well-posedness. The previous research works in this scientific area did not take into account the new form of the notions mentioned above. In essence, the results derived here can be considered as dynamic generalizations of the corresponding static results already existing in the literature. In this paper, the framework is based on function spaces of infinite-dimension and multiple integral-type functionals. This element is completely new for the well-posed optimization problems.

The present paper is structured as follows: In Section 2, we formulate the problem under study and introduce the new forms of monotonicity, lower semicontinuity, hemicontinuity, and pseudomonotonicity for the considered multiple integral-type functional. Additionally, an auxiliary lemma is provided. In Section 3, we study the well-posedness for the considered constrained variational problem. More precisely, we prove that well-posedness is equivalent with the existence and uniqueness of a solution in the aforesaid problem. Finally, Section 4 concludes the paper and provides further developments.



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### 2. Preliminaries and Problem Formulation

In this paper, we consider the following notations and mathematical tools: denote by  $K$  a compact domain in  $\mathbb{R}^m$  and consider the point  $K \ni \zeta = (\zeta^\alpha), \alpha = \overline{1, m}$ ; let  $\mathcal{E}$  denote the space of state functions of  $C^4$ -class  $s : K \rightarrow \mathbb{R}^n$  and  $s_\alpha := \frac{\partial s}{\partial \zeta^\alpha}, s_{\beta\gamma} := \frac{\partial^2 s}{\partial \zeta^\beta \partial \zeta^\gamma}$  denote the partial speed and partial acceleration, respectively; consider  $E \subseteq \mathcal{E}$  as a nonempty, closed and convex subset, with  $s|_{\partial K} = \text{given}$ , equipped with the inner product

$$\langle s, z \rangle = \int_K [s(\zeta) \cdot z(\zeta)] d\zeta = \int_K \left[ \sum_{i=1}^n s^i(\zeta) z^i(\zeta) \right] d\zeta, \quad \forall s, z \in \mathcal{E}$$

and the induced norm, where  $d\zeta = d\zeta^1 \dots d\zeta^m$  is the element of volume on  $\mathbb{R}^m$ .

Let  $J^2(\mathbb{R}^m, \mathbb{R}^n)$  be the second-order jet bundle for  $\mathbb{R}^m$  and  $\mathbb{R}^n$ . By using the real-valued continuously differentiable function  $f : J^2(\mathbb{R}^m, \mathbb{R}^n) \rightarrow \mathbb{R}$ , we define the multiple integral-type functional:

$$F : \mathcal{E} \rightarrow \mathbb{R}, \quad F(s) = \int_K f(\zeta, s(\zeta), s_\alpha(\zeta), s_{\beta\gamma}(\zeta)) d\zeta.$$

By using the above mathematical framework, we formulate the constrained variational problem (in short, CVP)  $((\pi_s(\zeta)) := (\zeta, s(\zeta), s_\alpha(\zeta), s_{\beta\gamma}(\zeta)))$ :

$$\begin{aligned} \text{(CVP)} \quad & \text{Minimize } \int_K f(\pi_s(\zeta)) d\zeta \\ & \text{subject to } s \in \Omega, \end{aligned}$$

where  $\Omega$  stands for the set of solutions for the variational inequality problem (in short, VIP): find  $s \in E$  such that

$$\begin{aligned} \text{(VIP)} \quad & \int_K \left[ \frac{\partial f}{\partial s}(\pi_s(\zeta))(z(\zeta) - s(\zeta)) + \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) D_\alpha(z(\zeta) - s(\zeta)) \right. \\ & \left. + \frac{1}{n(\beta, \gamma)} \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) D_{\beta\gamma}^2(z(\zeta) - s(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E, \end{aligned}$$

where  $D_{\beta\gamma}^2 := D_\beta(D_\gamma)$ , and  $n(\beta, \gamma)$  represents the multi-index notation (Saunders [36], Treanță [33]).

More precisely, the set of all feasible solutions of (VIP) is defined as

$$\begin{aligned} \Omega = \left\{ s \in E : \int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_s(\zeta)) + D_\alpha(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) \right. \right. \\ \left. \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E \right\}. \end{aligned}$$

**Definition 1.** The functional  $F(s) = \int_K f(\pi_s(\zeta)) d\zeta$  is monotone on  $E$  if the inequality holds:

$$\begin{aligned} & \int_K \left[ (s(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s}(\pi_s(\zeta)) - \frac{\partial f}{\partial s}(\pi_z(\zeta)) \right) \right. \\ & \left. + D_\alpha(s(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) - \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right) \right. \\ & \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(s(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) - \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right) \right] d\zeta \geq 0, \end{aligned}$$

for  $\forall s, z \in E$ .

**Definition 2.** The functional  $F(s) = \int_K f(\pi_s(\zeta))d\zeta$  is pseudomonotone on  $E$  if the implication holds:

$$\begin{aligned} & \int_K \left[ (s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right. \\ & \quad \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq 0 \\ \Rightarrow & \int_K \left[ (s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s}(\pi_s(\zeta)) + D_\alpha(s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) \right. \\ & \quad \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) \right] d\zeta \geq 0, \end{aligned}$$

for  $\forall s, z \in E$ .

**Example 1.** Consider  $m = 2, n = 1$ , and  $K = [0, 3]^2$ . Additionally, we define

$$f(\pi_s(\zeta)) = 2 \sin s(\zeta) + s(\zeta)e^{s(\zeta)}.$$

The functional  $F(s) = \int_K f(\pi_s(\zeta))d\zeta$  is pseudomonotone on  $E = C^4(K, [-1, 1])$ ,

$$\begin{aligned} & \int_K \left[ (s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right. \\ & \quad \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \\ & = \int_K \left[ (s(\zeta) - z(\zeta))(2 \cos z(\zeta) + e^{z(\zeta)} + z(\zeta)e^{z(\zeta)}) \right] d\zeta \geq 0 \\ & \quad \forall s, z \in E \\ \Rightarrow & \int_K \left[ (s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s}(\pi_s(\zeta)) + D_\alpha(s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) \right. \\ & \quad \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(s(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) \right] d\zeta \\ & = \int_K \left[ (s(\zeta) - z(\zeta))(2 \cos s(\zeta) + e^{s(\zeta)} + s(\zeta)e^{s(\zeta)}) \right] d\zeta \geq 0 \\ & \quad \forall s, z \in E. \end{aligned}$$

By direct computation, we obtain

$$\begin{aligned} & \int_K \left[ (s(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s}(\pi_s(\zeta)) - \frac{\partial f}{\partial s}(\pi_z(\zeta)) \right) \right. \\ & \quad \left. + D_\alpha(s(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) - \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right) \right. \\ & \quad \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(s(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) - \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right) \right] d\zeta \\ & = \int_K \left[ (s(\zeta) - z(\zeta)) [2(\cos s(\zeta) - \cos z(\zeta)) + s(\zeta)e^{s(\zeta)} + e^{s(\zeta)} - z(\zeta)e^{z(\zeta)} - e^{z(\zeta)}] \right] d\zeta \not\geq 0, \\ & \quad \forall s, z \in E, \end{aligned}$$

which implies that the functional  $F(s) = \int_K f(\pi_s(\zeta))d\zeta$  is not monotone on  $E$  (in the sense of Definition 1).

By considering the work of Usman and Khan [37], we provide the following definition.

**Definition 3.** The functional  $F(s) = \int_K f(\pi_s(\zeta))d\zeta$  is hemicontinuous on  $E$  if the application

$$\lambda \rightarrow \left\langle s(\zeta) - z(\zeta), \frac{\delta F}{\delta s_\lambda}(\zeta) \right\rangle, \quad 0 \leq \lambda \leq 1$$

is continuous at  $0^+$ , for  $\forall s, z \in E$ , where

$$\frac{\delta F}{\delta s_\lambda}(\zeta) := \frac{\partial f}{\partial s}(\pi_{s_\lambda}(\zeta)) - D_\alpha \frac{\partial f}{\partial s_\alpha}(\pi_{s_\lambda}(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2 \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_\lambda}(\zeta)) \in E,$$

$$s_\lambda := \lambda s + (1 - \lambda)z.$$

**Lemma 1.** Consider the functional  $F(s) = \int_K f(\pi_s(\zeta))d\zeta$  as hemicontinuous and pseudomonotone on  $E$ . Then, the function  $s \in E$  solves (VIP) if and only if it solves the variational inequality

$$\int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2 (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E.$$

**Proof.** Firstly, let us consider that the function  $s \in E$  solves (VIP). In consequence, it follows

$$\int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_s(\zeta)) + D_\alpha (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2 (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E.$$

By using the pseudomonotonicity property of  $F(s) = \int_K f(\pi_s(\zeta))d\zeta$ , the previous inequality involves

$$\int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2 (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E.$$

Conversely, assume that

$$\int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2 (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E.$$

For  $z \in E$  and  $\lambda \in (0, 1]$ , we define

$$z_\lambda = (1 - \lambda)s + \lambda z \in E.$$

Therefore, the above inequality can be rewritten as follows

$$\int_K \left[ (z_\lambda(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_{z_\lambda}(\zeta)) + D_\alpha(z_\lambda(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{z_\lambda}(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z_\lambda(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{z_\lambda}(\zeta)) \right] d\zeta \geq 0, \quad z \in E.$$

By considering  $\lambda \rightarrow 0$  (and the hemicontinuity property of  $F(s) = \int_K f(\pi_s(\zeta)) d\zeta$ ), it results that

$$\int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_s(\zeta)) + D_\alpha(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E,$$

which shows that  $s$  is solution for (VIP). The proof of this lemma is now complete.  $\square$

**Definition 4.** The functional  $F(s) = \int_K f(\pi_s(\zeta)) d\zeta$  is lower semicontinuous at  $s_0 \in E$  if

$$\int_K f(\pi_{s_0}(\zeta)) d\zeta \leq \liminf_{s \rightarrow s_0} \int_K f(\pi_s(\zeta)) d\zeta.$$

### 3. Well-Posedness Associated with (CVP)

In this section, we analyze the well-posedness property for the constrained variational problem (CVP). To this aim, we provide the following mathematical tools.

Let us denote by  $\mathcal{S}$  the set of all solutions for (CVP), that is,

$$\mathcal{S} = \left\{ s \in E \mid \int_K f(\pi_s(\zeta)) d\zeta \leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta)) d\zeta \text{ and } \int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_s(\zeta)) + D_\alpha(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) \right] d\zeta \geq 0, \forall z \in E \right\}.$$

Additionally, for  $\theta, \vartheta \geq 0$ , we define the set of approximating solutions for (CVP) as

$$\mathcal{S}(\theta, \vartheta) = \left\{ s \in E \mid \int_K f(\pi_s(\zeta)) d\zeta \leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta)) d\zeta + \theta \text{ and } \int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_s(\zeta)) + D_\alpha(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) \right] d\zeta + \vartheta \geq 0, \forall z \in E \right\}.$$

**Remark 1.** For  $(\theta, \vartheta) = (0, 0)$ , we have  $\mathcal{S} = \mathcal{S}(\theta, \vartheta)$  and, for  $(\theta, \vartheta) > (0, 0)$ , we obtain  $\mathcal{S} \subseteq \mathcal{S}(\theta, \vartheta)$ .

**Definition 5.** If there exists a sequence of positive real numbers  $\vartheta_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that the following inequalities

$$\limsup_{n \rightarrow \infty} \int_K f(\pi_{s_n}(\zeta)) d\zeta \leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta)) d\zeta$$

and

$$\int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_{s_n}(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s_n}(\zeta)) \right] d\zeta \geq 0,$$

$$+ \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_n}(\zeta)) \Big] d\zeta + \vartheta_n \geq 0, \quad \forall z \in E$$

are fulfilled, then the sequence  $\{s_n\}$  is called an approximating sequence of (CVP).

**Definition 6.** The problem (CVP) is called well-posed if:

- (i) It has a unique solution  $s_0$ ;
- (ii) Each approximating sequence of (CVP) will converge to this unique solution  $s_0$ .

Further, the symbol "diam B" stands for the diameter of B. Moreover, it is defined by

$$\text{diam } B = \sup_{x,y \in B} \|x - y\|.$$

**Theorem 1.** Consider the functional  $F(s) = \int_K f(\pi_s(\zeta)) d\zeta$  as lower semicontinuous, hemicontinuous and monotone on E. Then, the problem (CVP) is well-posed if and only if

$$\mathcal{S}(\theta, \vartheta) \neq \emptyset, \forall \theta, \vartheta > 0 \text{ and } \text{diam } \mathcal{S}(\theta, \vartheta) \rightarrow 0 \text{ as } (\theta, \vartheta) \rightarrow (0, 0).$$

**Proof.** Let us consider the case that (CVP) is well-posed. Therefore, it admits a unique solution  $\bar{s} \in \mathcal{S}$ . Since  $\mathcal{S} \subseteq \mathcal{S}(\theta, \vartheta)$ ,  $\forall \theta, \vartheta > 0$ , we obtain  $\mathcal{S}(\theta, \vartheta) \neq \emptyset, \forall \theta, \vartheta > 0$ . Contrary to the result, let us suppose that  $\text{diam } \mathcal{S}(\theta, \vartheta) \not\rightarrow 0$  as  $(\theta, \vartheta) \rightarrow (0, 0)$ . Then, there exists  $r > 0$ , a positive integer  $m$ ,  $\theta_n, \vartheta_n > 0$  with  $\theta_n, \vartheta_n \rightarrow 0$ , and  $s_n, s'_n \in \mathcal{S}(\theta_n, \vartheta_n)$  such that

$$\|s_n - s'_n\| > r, \quad \forall n \geq m. \tag{1}$$

Since  $s_n, s'_n \in \mathcal{S}(\theta_n, \vartheta_n)$ , we obtain

$$\begin{aligned} \int_K f(\pi_{s_n}(\zeta)) d\zeta &\leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta)) d\zeta + \theta_n, \\ \int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_{s_n}(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s_n}(\zeta)) \right. \\ &\left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_n}(\zeta)) \right] d\zeta + \vartheta_n \geq 0, \quad \forall z \in E \end{aligned}$$

and

$$\begin{aligned} \int_K f(\pi_{s'_n}(\zeta)) d\zeta &\leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta)) d\zeta + \theta_n, \\ \int_K \left[ (z(\zeta) - s'_n(\zeta)) \frac{\partial f}{\partial s}(\pi_{s'_n}(\zeta)) + D_\alpha(z(\zeta) - s'_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s'_n}(\zeta)) \right. \\ &\left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s'_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s'_n}(\zeta)) \right] d\zeta + \vartheta_n \geq 0, \quad \forall z \in E. \end{aligned}$$

It results that  $\{s_n\}$  and  $\{s'_n\}$  are approximating sequences of (CVP) which tend to  $\bar{s}$  (the problem (CVP) is well-posed, by hypothesis). By direct computation, it follows that

$$\begin{aligned} \|s_n - s'_n\| &= \|s_n - \bar{s} + \bar{s} - s'_n\| \\ &\leq \|s_n - \bar{s}\| + \|\bar{s} - s'_n\| \leq \vartheta, \end{aligned}$$

which contradicts (1) for some  $\vartheta = r$ . In consequence,  $\text{diam } \mathcal{S}(\theta, \vartheta) \rightarrow 0$  as  $(\theta, \vartheta) \rightarrow (0, 0)$ .

Conversely, let us consider that  $\{s_n\}$  is an approximating sequence of (CVP). Then there exists a sequence of positive real numbers  $\vartheta_n \rightarrow 0$  as  $n \rightarrow \infty$  such that the inequalities

$$\limsup_{n \rightarrow \infty} \int_K f(\pi_{s_n}(\zeta)) d\zeta \leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta)) d\zeta, \tag{2}$$

$$\int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_{s_n}(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s_n}(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_n}(\zeta)) \right] d\zeta + \vartheta_n \geq 0, \quad \forall z \in E \tag{3}$$

hold, including  $s_n \in \mathcal{S}(\theta_n, \vartheta_n)$ , for a sequence of positive real numbers  $\theta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\text{diam } \mathcal{S}(\theta_n, \vartheta_n) \rightarrow 0$  as  $(\theta_n, \vartheta_n) \rightarrow (0, 0)$ ,  $\{s_n\}$  is a Cauchy sequence which converges to some  $\bar{s} \in E$  as  $E$  is a closed set.

By hypothesis, the multiple integral functional  $\int_K f(\pi_s(\zeta)) d\zeta$  is monotone on  $E$ . Therefore, by Definition 1, for  $\bar{s}, z \in E$ , we have

$$\int_K \left[ (\bar{s}(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s}(\pi_{\bar{s}}(\zeta)) - \frac{\partial f}{\partial s}(\pi_z(\zeta)) \right) + D_\alpha(\bar{s}(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s_\alpha}(\pi_{\bar{s}}(\zeta)) - \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(\bar{s}(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{\bar{s}}(\zeta)) - \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right) \right] d\zeta \geq 0,$$

or, equivalently,

$$\int_K \left[ (\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s}(\pi_{\bar{s}}(\zeta)) + D_\alpha(\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{\bar{s}}(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{\bar{s}}(\zeta)) \right] d\zeta \geq \int_K \left[ (\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta. \tag{4}$$

Taking limit in inequality (3), we have

$$\int_K \left[ (\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s}(\pi_{\bar{s}}(\zeta)) + D_\alpha(\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{\bar{s}}(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(\bar{s}(\zeta) - z(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{\bar{s}}(\zeta)) \right] d\zeta \leq 0. \tag{5}$$

On combining (4) and (5), we obtain

$$\int_K \left[ (z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq 0.$$

Further, taking into account Lemma 1, it follows that

$$\int_K \left[ (z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s}(\pi_{\bar{s}}(\zeta)) + D_\alpha(z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{\bar{s}}(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{\bar{s}}(\zeta)) \right] d\zeta \geq 0, \tag{6}$$

which implies that  $\bar{s} \in \Omega$ .

Since the functional  $\int_K f(\pi_s(\zeta))d\zeta$  is lower semicontinuous, it results that

$$\int_K f(\pi_{\bar{s}}(\zeta))d\zeta \leq \liminf_{n \rightarrow \infty} \int_K f(\pi_{s_n}(\zeta))d\zeta \leq \limsup_{n \rightarrow \infty} \int_K f(\pi_{s_n}(\zeta))d\zeta.$$

By using (2), the above inequality reduces to

$$\int_K f(\pi_{\bar{s}}(\zeta))d\zeta \leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta))d\zeta. \tag{7}$$

Thus, from (6) and (7), we conclude that  $\bar{s}$  solves (CVP).

Now, let us prove that  $\bar{s}$  is the unique solution of (CVP). Suppose that  $s_1, s_2$  are two distinct solutions of (CVP). Then,

$$0 < \|s_1 - s_2\| \leq \text{diam } \mathcal{S}(\theta, \vartheta) \rightarrow 0 \text{ as } (\theta, \vartheta) \rightarrow (0, 0),$$

and the proof is complete.  $\square$

**Theorem 2.** Consider the functional  $F(s) = \int_K f(\pi_s(\zeta))d\zeta$  as lower semicontinuous, hemicontinuous and monotone on  $E$ . Then, the problem (CVP) is well-posed if and only if it has a unique solution.

**Proof.** Let us consider that (CVP) is well-posed. Thus, it possesses a unique solution  $s_0$ . Conversely, let us consider that (CVP) has a unique solution  $s_0$ , that is,

$$\begin{aligned} \int_K f(\pi_{s_0}(\zeta))d\zeta &\leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta))d\zeta, \\ \int_K \left[ (z(\zeta) - s_0(\zeta)) \frac{\partial f}{\partial s}(\pi_{s_0}(\zeta)) + D_\alpha(z(\zeta) - s_0(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s_0}(\zeta)) \right. \\ &\left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_0(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_0}(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E, \end{aligned} \tag{8}$$

but it is not well-posed. Therefore, by Definition 6, there exists an approximating sequence  $\{s_n\}$  of (CVP), which does not converge to  $s_0$ , such that the following inequalities

$$\limsup_{n \rightarrow \infty} \int_K f(\pi_{s_n}(\zeta))d\zeta \leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta))d\zeta$$

and

$$\begin{aligned} \int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_{s_n}(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s_n}(\zeta)) \right. \\ \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_n}(\zeta)) \right] d\zeta + \vartheta_n \geq 0, \quad \forall z \in E \end{aligned} \tag{9}$$

are fulfilled. Further, we proceed by contradiction to prove the boundedness of  $\{s_n\}$ . Contrary to the result, we suppose that  $\{s_n\}$  is not bounded; consequently,  $\|s_n\| \rightarrow +\infty$  as  $n \rightarrow +\infty$ . We define  $\delta_n = \frac{1}{\|s_n - s_0\|}$  and  $s_n = s_0 + \delta_n[s_n - s_0]$ . We observe that  $\{s_n\}$  is bounded in  $E$ . Therefore, if necessary, passing to a subsequence, we may consider that

$$s_n \rightarrow s \text{ weakly in } E \neq (s_0).$$

It is not difficult to see that  $s \neq s_0$  due to  $\|\delta_n[s_n - s_0]\| = 1$ , for all  $n \in \mathbb{N}$ . Since  $s_0$  is a solution of (CVP), the inequalities (8) are verified. By using Lemma 1, it follows that

$$\int_K f(\pi_{s_0}(\zeta))d\zeta \leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta))d\zeta,$$



$$\int_K \left[ (z(\zeta) - s_0(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - s_0(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_0(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E. \tag{10}$$

By considering the monotonicity property of the functional  $\int_K f(\pi_s(\zeta)) d\zeta$ , for  $s_n, z \in E$ , we obtain

$$\int_K \left[ (s_n(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s}(\pi_{s_n}(\zeta)) - \frac{\partial f}{\partial s}(\pi_z(\zeta)) \right) + D_\alpha(s_n(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s_\alpha}(\pi_{s_n}(\zeta)) - \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(s_n(\zeta) - z(\zeta)) \left( \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_n}(\zeta)) - \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right) \right] d\zeta \geq 0,$$

or, equivalently,

$$\begin{aligned} & \int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_{s_n}(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s_n}(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_n}(\zeta)) \right] d\zeta \\ & \leq \int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta. \end{aligned} \tag{11}$$

Combining with (9) and (11), we have

$$\int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq -\vartheta_n, \quad \forall z \in E.$$

Next, we can take  $n_0 \in \mathbb{N}$  be large enough such that  $\delta_n < 1$ , for all  $n \geq n_0$  (because of  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ ). Multiplying the above inequality and (10) by  $\delta_n > 0$  and  $1 - \delta_n > 0$ , respectively, we obtain

$$\int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq -\vartheta_n, \quad \forall z \in E, \forall n \geq n_0.$$

By using  $s_n \rightarrow s \neq s_0$  and  $s_n = s_0 + \mathbf{s}_n[s_n - s_0]$ , we obtain

$$\begin{aligned} & \int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \\ & = \lim_{n \rightarrow \infty} \int_K \left[ (z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_n(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \end{aligned}$$

$$\geq - \lim_{n \rightarrow \infty} \vartheta_n = 0, \quad \forall z \in E.$$

Taking into account Lemma 1 and by using the lower semicontinuity property, we obtain

$$\begin{aligned} \int_K f(\pi_{\bar{s}}(\zeta))d\zeta &\leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta))d\zeta, \\ \int_K \left[ (z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s}(\pi_s(\zeta)) + D_\alpha(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_s(\zeta)) \right. \\ &\left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_s(\zeta)) \right] d\zeta \geq 0, \quad \forall z \in E. \end{aligned} \tag{12}$$

This involves that  $s$  solves (CVP), contradiction with the uniqueness of  $s_0$ . Therefore,  $\{s_n\}$  is a bounded sequence having a convergent subsequence  $\{s_{n_k}\}$ , which converges to  $\bar{s} \in E$  as  $k \rightarrow \infty$ . Now, for  $s_{n_k}, z \in E$ , we obtain (see (11))

$$\begin{aligned} \int_K \left[ (z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s}(\pi_{s_{n_k}}(\zeta)) + D_\alpha(z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s_{n_k}}(\zeta)) \right. \\ \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_{n_k}}(\zeta)) \right] d\zeta \\ \leq \int_K \left[ (z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right. \\ \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta. \end{aligned} \tag{13}$$

Additionally, by (9), we can write

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_K \left[ (z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s}(\pi_{s_{n_k}}(\zeta)) + D_\alpha(z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{s_{n_k}}(\zeta)) \right. \\ \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{s_{n_k}}(\zeta)) \right] d\zeta \geq 0. \end{aligned} \tag{14}$$

By (13) and (14), we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_K \left[ (z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right. \\ \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - s_{n_k}(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq 0 \\ \Rightarrow \int_K \left[ (z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s}(\pi_z(\zeta)) + D_\alpha(z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_z(\zeta)) \right. \\ \left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_z(\zeta)) \right] d\zeta \geq 0. \end{aligned}$$

Using Lemma 1 and the lower semicontinuity property of the considered functional, we obtain

$$\begin{aligned} \int_K f(\pi_{\bar{s}}(\zeta))d\zeta &\leq \inf_{z \in \Omega} \int_K f(\pi_z(\zeta))d\zeta, \\ \int_K \left[ (z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s}(\pi_{\bar{s}}(\zeta)) + D_\alpha(z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s_\alpha}(\pi_{\bar{s}}(\zeta)) \right. \\ &\left. + \frac{1}{n(\beta, \gamma)} D_{\beta\gamma}^2(z(\zeta) - \bar{s}(\zeta)) \frac{\partial f}{\partial s_{\beta\gamma}}(\pi_{\bar{s}}(\zeta)) \right] d\zeta \geq 0, \end{aligned}$$

which shows that  $\bar{s}$  is a solution of (CVP). Hence,  $s_{n_k} \rightarrow \bar{s}$ , that is,  $s_{n_k} \rightarrow s_0$ , involving  $s_n \rightarrow s_0$  and the proof is complete.  $\square$

**Example 2.** We consider  $n = 1$  and  $K = [0, 2]^2 = [0, 2] \times [0, 2]$ . Let us minimize the mass of  $K$  having the density (that depends on the current point)  $f(\zeta, s(\zeta), s_\alpha(\zeta), s_{\beta\gamma}(\zeta)) = e^{s(\zeta)} - s(\zeta)$ , such that the following behavior (positivity property)

$$\iint_K (z(\zeta) - s(\zeta))(e^{s(\zeta)} - 1) d\zeta^1 d\zeta^2 \geq 0,$$

$$\forall z \in E = C^1(K, [-15, 15]), s|_{\partial K} = 0,$$

is satisfied.

To solve the previous practical problem, we consider the following constrained optimization problem:

$$(CVP1) \quad \text{Minimize} \quad \iint_K [e^{s(\zeta)} - s(\zeta)] d\zeta^1 d\zeta^2$$

subject to  $s \in \Omega$ ,

where  $\Omega$  is the solution set of the following inequality problem

$$\iint_K (z(\zeta) - s(\zeta))(e^{s(\zeta)} - 1) d\zeta^1 d\zeta^2 \geq 0,$$

$$\forall z \in E = C^1(K, [-15, 15]), s|_{\partial K} = 0.$$

Clearly,  $\mathcal{S} = \{0\}$  and the functional  $\int_K e^{s(\zeta)} - s(\zeta) d\zeta$  is hemicontinuous, monotone and lower semicontinuous on  $E$ . Thus, all the hypotheses of Theorem 2 hold and, in consequence, the problem (CVP1) is well-posed. Additionally,  $\mathcal{S}(\theta, \vartheta) = \{0\}$  and, therefore,  $\mathcal{S}(\theta, \vartheta) \neq \emptyset$  and  $\text{diam } \mathcal{S}(\theta, \vartheta) \rightarrow 0$  as  $(\theta, \vartheta) \rightarrow (0, 0)$ . In conclusion, by Theorem 1, the variational problem (CVP1) is well-posed.

#### 4. Conclusions

In this paper, we have studied the well-posedness property of new constrained variational problems governed by second-order partial derivatives. More precisely, by using the concepts of lower semicontinuity, monotonicity, hemicontinuity and pseudomonotonicity of considered multiple integral-type functional, we have proved that the well-posedness property of the problem under study is described in terms of existence and uniqueness of solution.

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