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Existence of Absolutely Continuous Fundamental Matrix of Linear Fractional System with Distributed Delays

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Abstract: The goal of the present paper is to obtain sufficient conditions that guaranty the existence and uniqueness of an absolutely continuous fundamental matrix for a retarded linear fractional differential system with Caputo type derivatives and distributed delays. Some applications of the obtained result concerning the integral representation of the solutions are given too.

Keywords: Caputo fractional derivative; linear fractional system; distributed delay

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1. Introduction

It is well known that many mathematical models of real world phenomena can be described more accurately through fractional derivative formulation. For more details on fractional calculus theory and fractional differential equations, we recommend the monographs of Kilbas et al. [1] and Podlubny [2]. For distributed order fractional differential equations see Jiao at all [3] and for an application-oriented exposition Diethelm [4]. Impulsive differential and functional differential equations with fractional derivative and some applications are studied by Stamova and Stamov [5].

The theme of the integral representation (variation of constants formula) of the solutions of linear fractional differential equations and/or systems (ordinary or with delay) is an "evergreen" theme for research. This explains why a lot of papers are devoted to different aspects of this problem. For linear fractional ordinary differential equations and systems, we refer the works [1,2,6–10] and the references therein. Relatively, as far as we know, there are not many works devoted to the variation of constants formula for delayed linear fractional systems [11–15]. The case of neutral fractional systems is studied in [16–20].

The establishment of a fundamental matrix with appropriate properties (for example in [21] smoothness is obtained) is the basis for obtaining any integral representation and is a key tool in the study of different types of stability of linear and nonlinear disturbed systems (see [20]).

In the present work, we consider linear fractional systems with distributed delays and incommensurate order derivatives in the Caputo sense. The first goal of the work is to establish sufficient conditions for existence and uniqueness of a fundamental matrix $C(t,s)$, which is absolutely continuous in t on every compact subinterval of $\mathbb{R}$. The second one is to clarify the analytic properties in s, which are very similar to these in the integer case. As an application of the obtained results, some results concerning the integral representation of the solutions given in [15,18] are improved.

The paper is organized as follows. In Section 2 we recall the definitions of Riemann–Liouville and Caputo fractional derivatives with some of their properties. In the same section is the statement of the problem, as well as some necessary preliminary results used later. Section 3 is devoted to the existence and the uniqueness of the solutions of the Initial

Problem (IP) for linear fractional systems with distributed delays and incommensurate order derivatives in the Caputo sense and special type discontinuous initial function. In Section 4 the existence and uniqueness of an absolutely continuous fundamental matrix is proved und its analytical properties are studied. Using the obtained results, in Section 5 we establish new integral representations for the solutions of the studied systems.

2. Preliminaries and Problem Statement

For readers convenience, below we recall the definitions of Riemann–Liouville and Caputo fractional derivatives as well as some needed properties. For details and other properties we refer to [1–3].

Let $\alpha \in (0,1)$ be an arbitrary number and denote by $L_1^{loc}(\mathbb{R},\mathbb{R})$ the linear space of all locally Lebesgue integrable functions $f:\mathbb{R}\to\mathbb{R}$. Then for each $a\in\mathbb{R}$, each $t>a$ and $f\in L_1^{loc}(\mathbb{R},\mathbb{R})$ the definitions of the left-sided fractional integral operator, the left side Riemann–Liouville and Caputo fractional derivatives of order α and some properties are given below (see [1]):

$$(D_{a+}^{-\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s,$$

$$_{RL}D_{a+}^{\alpha}f(t) = \frac{\mathrm{d}}{\mathrm{d}t}(D_{a+}^{-(1-\alpha)}f(t));$$

$$_{C}D_{a+}^{\alpha}f(t) = {}_{RL}D_{a+}^{\alpha}[f(s)-f(a)](t);$$

$$_{C}D_{a+}^{\alpha}f(t) = {}_{RL}D_{a+}^{\alpha}f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha};$$

$$(a)\,{}_{C}D_{a+}^{\alpha}D_{a+}^{-\alpha}f(t) = (D_{a+}^{0}f)(t) = f(t);\,(b)\,D_{a+}^{-\alpha}\,{}_{C}D_{a+}^{\alpha}f(t) = f(t)-f(a)$$

If $f\in AC(\mathbb{R},\mathbb{R})$ then the next formula gives a direct definition of the Caputo left side derivative:

$$_{C}D_{a+}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}f'(s)\mathrm{d}s,$$

Everywhere below, the following notations will be used: $\mathbb{R}_+ = (0,\infty), \bar{\mathbb{R}}_+ = [0,\infty)$, $J_a = [a,\infty), a\in\mathbb{R}, J_{s+M} = [s,s+M], s\in J_a, M\in\mathbb{R}_+, \langle n\rangle = \{1,2,\ldots,n\}, \langle m\rangle_0 = \langle m\rangle\cup\{0\}$, $n,m\in\mathbb{N}, I,\Theta\in\mathbb{R}^{n\times n}$ denote the identity and zero matrix, respectively, $I^k, k\in\langle n\rangle$ denotes the k-th column of the identity matrix and $\mathbf{0}\in\mathbb{R}^n$ is the zero element.

For $Y(t) = (y^1(t),\ldots,y^n(t))^T : J_a\to\mathbb{R}^n, \beta = (\beta_1,\ldots,\beta_n), \beta_k\in[-1,1], k\in\langle n\rangle$ we will use the notations $I_\beta(Y(t)) = \mathrm{diag}((y_1(t))^{\beta_1},\ldots,(y_n(t))^{\beta_n})$, for $W(t) = \{w_{kj}(t)\}_{k,j=1}^n :$ $J_a\to\mathbb{R}^{n\times n}$ we denote $|W(t)| = \sum_{k,j=1}^{n}|w_{k,j}(t)|, t\in J_a$ and for simplicity we will use the notation $D_{a+}^{\alpha} = {}_{C}D_{a+}^{\alpha}$ for the left side Caputo fractional derivative.

Consider the homogeneous linear delayed system of incommensurate type and distributed delay in the following general form

$$D_{a+}^{\alpha}X(t) = \int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]X(t+\theta),\ t>a \tag{1}$$

or described in more detailed form

$$D_{a+}^{\alpha_k}x_k(t) = \sum_{j=1}^{n}\int_{-h}^{0}x_j(t+\theta)\mathrm{d}_\theta u_{kj}(t,\theta),\ \ k\in\langle n\rangle, t>a$$

and the corresponding nonhomogeneous one

$$D^{\alpha}_{a+}X(t) = \int_{-h}^{0} [\mathrm{d}_\theta U(t,\theta)]X(t+\theta) + F(t),\ t > a \tag{2}$$

where $h \in \mathbb{R}_+, X(t) = (x_1(t), \dots, x_n(t))^T, F(t) = (f_1(t), \dots, f_n(t))^T : J_a \to \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n), \alpha_k \in (0,1), \alpha_m = \min_{k\in\langle n\rangle} \alpha_k, D^{\alpha}_{a+} = \mathrm{diag}(D^{\alpha_1}_{a+}, \dots, D^{\alpha_n}_{a+})^T, U : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $U(t,\theta) = \{u_{kj}(t,\theta)\}^n_{k,j=1}$.

Definition 1. *A function $Z(t) = (z_1(t), \dots, z_n(t))^T : J \to \mathbb{R}^n$ is called piecewise absolutely continuous on some interval $J \subset \mathbb{R}$ (denoted $Z(t) \in PAC(J, \mathbb{R}^n)$), if it is with bounded variation (BV) in t on J, there is no singular term in the Lebesgue decomposition of $Z(t)$ and the set of discontinuity points of the function $Z(t)$ has no limit points in J.*

Definition 2. *With $\boldsymbol{C}^*_a, a \in \mathbb{R}$ we denote the Banach space of right continuous vector functions $\Phi(t) \in PAC([a-h,a], \mathbb{R}^n)$ with norm $||\Phi|| = \sup_{t\in[a-h,a]} |\Phi(t)| = \sup_{t\in[a-h,a]} \sum_{k=1}^{n} |\phi_k(t)| < \infty$ and the subspace of all absolutely continuous functions by $\boldsymbol{C}_a = AC([a-h,a], \mathbb{R}^n)$, i.e., $\boldsymbol{C}_a \subset \boldsymbol{C}^*_a$.*

The set of jump points of every initial vector function $\Phi \in \boldsymbol{C}^*_a$ we denote by S_Φ. We emphasize that the set $S_\Phi \cap K$ is finite for every compact interval $K \subset \mathbb{R}$ and the case $S_\Phi = \varnothing$ is not excluded.

For the system (1) or (2) introduce the following initial conditions:

$$X(t) = \Phi(t)\ \ (x_k(t) = \phi_k(t), k \in \langle n\rangle),\ \ t \in [a-h,a], \Phi \in \boldsymbol{C}^*_a \tag{3}$$

We say that for the kernel $U : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$ the conditions **(S)** hold, if the following conditions are fulfilled:

(S1) The functions $(t,\theta) \to U(t,\theta)$ are measurable in $(t,\theta) \in J_a \times \mathbb{R}$ and normalized so that for $t \in J_a, U(t,\theta) = 0$ when $\theta \in \bar{\mathbb{R}}_+$ and $U(t,\theta) = U(t,-h)$ for all $\theta \in (-\infty, -h]$, $h \in \mathbb{R}_+$ and $Var_{\theta\in[-h,0]} U(t,\cdot) < \infty$ for $t \in J_a$.

(S2) The Lebesgue decomposition of the kernel $U(t,\theta)$ for $t \in J_a$ and $\theta \in [-h,0]$ has the form:

$$U(t,\theta) = U_J(t,\theta) + U_{AC}(t,\theta) + U_S(t,\theta)$$

where $U_J(t,\theta) = \sum_{i=0}^{m} A^i(t)H(\theta + \sigma_i(t)), m \in \mathbb{N}, A^i(t) = \{a^i_{kj}(t)\}^n_{k,j=1} \in L^{loc}_1(J_a, \mathbb{R}^{n\times n})$ are locally bounded on J_a, $H(t)$ is the Heaviside function, the delays $\sigma_i(t) \in C(J_a, \bar{R}_+)$ are bounded with $\sigma_i = \sup_{t\in J_a} \sigma_i(t)$, $\max_{i\in\langle m\rangle_0} \sigma_i \le h$ and $A^i(t)H(\theta + \sigma_i(t))$ are continuous from left in θ on $(-\sigma_i, 0)$, $i \in \langle m\rangle, \sigma_0(t) \equiv 0$.

$U_{AC}(t,\theta) = \int_{-h}^{0} B(t,\theta)\mathrm{d}\theta, B(t,\theta) = \{b^j_k(t,\theta)\}^n_{k,j=1} \in L^{loc}_1(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ are locally bounded on J_a and $U_S(t,\theta) \in C(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$.

(S3) There exists a locally bounded function $z_u \in L^{loc}_1(J_a, \mathbb{R}_+)$ such that $Var_{[-h,0]} U(t,\cdot) \le z_u(t)$ for $t \in J_a$ and for every $t_* \in J_a$ the following relations hold: $\lim_{t\to t_*} \int_{-h}^{0} |U(t,\theta) - U(t_*,\theta)|\mathrm{d}\theta = 0$.

(S4) The sets $S^i_\Phi = \{t \in J_a \mid t - \sigma_i(t) \in S_\Phi\}$ for every $i \in \langle m\rangle$ do not have limit points.

Consider the following auxiliary system in matrix form

$$\begin{aligned} X(t) = \Phi(a) + I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\eta)F(\eta)\mathrm{d}\eta \\ + I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\eta) \int_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)]X(\eta+\theta)\mathrm{d}\eta,\ t > a \end{aligned} \tag{4}$$

where $I_{-1}(\Gamma(\alpha)) = \mathrm{diag}(\Gamma^{-1}(\alpha_1), \ldots, \Gamma^{-1}(\alpha_n))$, or for $k \in \langle n \rangle$ in more detailed form

$$\begin{aligned} x_k(t) =& \phi_k(a) + \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-\eta)^{\alpha_k - 1} f_k(\eta)\mathrm{d}\eta \\ &+ \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-\eta)^{\alpha_k-1} (\sum_{j=1}^n \int_{-h}^0 x_j(\eta+\theta)\mathrm{d}_\theta u_{kj}(\eta,\theta))\mathrm{d}\eta, t > a \end{aligned} \tag{5}$$

with the initial condition (3).

Definition 3. *The vector function $X(t) = (x_1(t), \ldots, x_n(t))^T$ is a solution of the IP (2) and (3) or IP (3) and (4) in $J_{a+M}(J_a), M \in \mathbb{R}_+$ if $X \in C([a, a+M], \mathbb{R}^n)(X \in C(J_a, \mathbb{R}^n))$ satisfies the system (2), respectively, (4) for all $t \in (a, a+M](t \in (a, \infty))$ and the initial condition (3) for each $t \in [a-h, a]$.*

In virtue of Lemma 3.3 in [15] every solution $X(t)$ of IP (2) and (3) is a solution of the IP (3) and (4) and vice versa.

We will need a slightly modified version of the Weissinger generalization of the Banach's fixed point theorem for complete metric spaces (see [22], Fixpunktsatz, p. 195).

Theorem 1. *Let Ω be a complete metric space with metric d_Ω and let the following conditions hold:*

1. *There exists a sequence $\gamma_q \geq 0, q \in \mathbb{N}$, with $\sum_{q=1}^{\infty} \gamma_q < \infty$.*
2. *The operator $T : \Omega \to \Omega$ satisfies for each $q \in \mathbb{N}$ and for arbitrary $x, y \in \Omega$ the inequality*

$$\mathrm{d}_\Omega(T^q x, T^q y) \leq \gamma_q d_\Omega(x, y)$$

Then T has a uniquely fixed point $x^* \in \Omega(x^* = Tx^*)$ and for every $x \in \Omega$ we have that $\lim_{q\to\infty} T^q x = x^*$.

Remark 1. *This modification of the Weissinger generalization of the Banachs fixed point is not new. It is used in [23] and in the case when Ω is a Banach space in [24]. It is clear that the original Weissinger proof is correct for the presented variant too, with elementary modifications.*

Let $\boldsymbol{B}$ be an arbitrary real Banach space.

Definition 4 ([25])**.** *The function $f(t) : \mathbb{R} \to \boldsymbol{B}$ is called a regulated function if it has one-sided (left and right) limits at every point $t \in \mathbb{R}$.*

Remark 2. *If $f(t) : J_K \to \boldsymbol{B}$, where $J_K \subset \mathbb{R}$ is an arbitrary compact interval and $f(t)$ is a regulated function, then it is assumed that in the left (right) end on the interval J_K the function $f(t)$ has only a right (left) limit.*

Theorem 2 ([25])**.** *Let $f(t) : \mathbb{R} \to \boldsymbol{B}$ be an arbitrary function.*

Then a necessary and sufficient condition for $f(t)$ to be a regulated function is that $f(t)$ must in every compact interval $J_K \subset \mathbb{R}$ be a limit of n uniformly convergent sequence of step-functions (i.e., with respect to the supremum norm $\|\cdot\|_\infty = \sup\limits_{t\in J_K} \|\cdot\|_{\boldsymbol{B}}$).

Theorem 3 ([26]). *Let $t_0 \in J_a$ and $\alpha > 0$ be arbitrary fixed numbers and the following conditions hold:*

1. *The functions $p(t), u(t) \in L^1_{loc}([t_0, T), \bar{\mathbb{R}}_+)$ for some $T \leq \infty$.*
2. *The function $g(t) \in C([t_0, T), [0, M])$ for some $M \in \mathbb{R}_+$ and is nondecreasing.*
3. *For every $t \in J_a$ is fulfilled $u(t) \leq p(t) + g(t) \int\limits_{t_0}^{t} (t-\eta)^{\alpha-1} u(\eta) \mathrm{d}\eta$.*

Then for $t \in J_a$ the following inequality holds

$$u(t) \leq p(t) + \int\limits_{t_0}^{t} [\sum_{q=1}^{\infty} \frac{(g(\eta)(\Gamma(\alpha))^q}{\Gamma(\alpha q)} (t-\eta)^{\alpha q - 1}] p(\eta) \mathrm{d}\eta.$$

Remark 3. *Note that the statement of Theorem 3 is proved in the partial case $t_0 = 0$, but with small modifications the proof will be correct for arbitrary $t_0 \in J_a$.*

3. Existence and Uniqueness of the Solutions

Let it be that for every $\Phi \in \boldsymbol{C}^*_a$ consider the corresponding linear space

$$E^\Phi = \{G : J_a \to \mathbb{R}^n \mid G|_{J_a} \in AC(J_a, \mathbb{R}^n), G(t) = \Phi(t), t \in [a-h, a]\}$$

For each $M \in \mathbb{R}_+$ define the set

$$E^\Phi_M = \{G_M : J_{a+M} \to \mathbb{R}^n \mid G_M = G|_{[a,a+M]}, G \in E^\Phi),$$

where $\Phi \in \boldsymbol{C}^*_a$ is arbitrary and define a metric function $\mathrm{d}_M : E^\Phi_M \times E^\Phi_M \to \bar{\mathbb{R}}_+$ with

$$\mathrm{d}_M(G_M, G^*_M) = \sum_{k=1}^{n} \sup_{t \in J_{a+M}} |g_k(t) - g^*_k(t)|$$

for each $G_M, G^*_M \in E^\Phi_M$.

Since $G_M(a) = \Phi(a) = G^*_M(a)$ then for every $M \in \mathbb{R}_+$, according to a well-known result we conclude that E^Φ_M is a complete metric space concerning the metric

$$\mathrm{d}^{Var}_M(G_M, G^*_M) = Var_{[a,a+M]}(G_M(t) - G^*_M(t)) = \sum_{k=1}^{n} Var_{[a,a+M]}(g_k(t) - g^*_k(t)).$$

It was a very strange for us that we could not find a result from which the statement of the next lemma directly follows.

Lemma 1. *For every $M \in \mathbb{R}_+$ the set E^Φ_M is a complete metric space concerning the metric d_M too.*

Proof. Let $M \in \mathbb{R}_+$ be an arbitrary fixed number and consider an arbitrary Cauchy sequence $\{G^l_M(t) = (g^l_1(t), \ldots, g^l_1(t))^T\}^\infty_{l=1} \subset E^\Phi_M$, i.e., $\lim\limits_{l,r\to\infty} \mathrm{d}_M(G^l_M(t), G^r_M(t)) = 0$. It is clear that there exists a vector valued function $G^0_M(t) = (g^0_1(t), \ldots, g^0_1(t))^T \in C(J_{a+M}, \mathbb{R}^n)$, $G^0_M(t) = \Phi(t), t \in [a-h, a]$ such that $\lim\limits_{l\to\infty} \mathrm{d}_M(G^l_M(t), G^0_M(t)) = 0$.

Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be arbitrary numbers. There exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $\delta = \delta(\varepsilon, k) > 0$ such that for every $n \geq n_0$ we have that $|G_n(t) - G_0(t)| < \frac{\varepsilon}{k}$ for each $t \in [a, a+M]$. Since $G_{n_0} \in AC([a, a+M])$ then there exists $\delta_* = \delta_*(\varepsilon, n_0) \in (0, \delta)$, such that for every finite sequence of pairwise disjoint subintervals $\{[a_j, b_j]\}_{j\in\langle k\rangle}$, $\bigcup\limits_{j\in\langle k\rangle} [a_j, b_j]$

$\subset [a, a+M]$ with $\sum\limits_{j=1}^{k}(b_j - a_j) < \delta_*$ the inequality $\sum\limits_{j=1}^{k}|G_{n_0}(a_j) - G_{n_0}(b_j)| < \varepsilon$ holds. Then whenever when $\sum\limits_{j=1}^{k}(b_j - a_j) < \delta_*$ we have that

$$\sum_{j=1}^{k}|G_0(a_j) - G_0(b_j)| = \sum_{j=1}^{k}|G_0(a_j) - G_0(b_j) + G_{n_0}(a_j) - G_{n_0}(a_j) + G_{n_0}(b_j) - G_{n_0}(b_j)|$$

$$\leq \sum_{j=1}^{k}|G_0(a_j) - G_{n_0}(a_j)| + \sum_{j=1}^{k}|G_{n_0}(b_j) - G_0(b_j)| + \sum_{j=1}^{k}|G_{n_0}(a_j) - G_{n_0}(b_j)| \leq 3\varepsilon$$

Thus $G_0 \in AC([a, a+M])$ for every $M \in \mathbb{R}_+$ and the statement is proved. □

Remark 4. *It must be noted that the statement of the next theorem cannot be obtained directly as a corollary from analogical results for the considered initial problems in [15,18,23].*

Theorem 4. *Let the following conditions hold:*

1. *Conditions (S) hold.*
2. *The function $\Phi \in \boldsymbol{C}_a^*$ is arbitrary.*

Then the IP (1) and (3) has a unique absolutely continuous solution in the interval $[a, a+M]$ for every $M \geq \max\{h, 1\}$.

Proof. In the proof of this theorem we will use the approach introduced in [23].

Let $\Phi \in \boldsymbol{C}_a^*$ and $M \geq \max\{h, 1\}$ be arbitrary. From (5) in the case $f_k(t) \equiv 0$ for $k \in \langle n \rangle$ we obtain the system

$$x_k(t) = \phi_k(a) + \frac{1}{\Gamma(\alpha_k)}\int_a^t (t-\eta)^{\alpha_k - 1}\Big(\sum_{j=1}^{n}\int_{-h}^{0} x_j(\eta+\theta)\mathrm{d}_\theta u_{kj}(\eta,\theta)\Big)\mathrm{d}\eta \tag{6}$$

For every $G(t) = (g_1(t), \ldots, g_n(t))^T \in E_M^\Phi$ define the operator $(\Re G)(t) = (\Re_1 g_1(t), \ldots, \Re_n g_n(t))^T$ via the operators $\Re_k g_k(t)$ for $t \in (a, a+M], k \in \langle n \rangle$ by

$$\Re_k g_k(t) = \phi_k(a) + \frac{1}{\Gamma(\alpha_k)}\int_a^t (t-s)^{\alpha_k - 1}\Big(\sum_{j=1}^{n}\int_{-h}^{0} g_j(s+\theta)\mathrm{d}_\theta u_k j(s,\theta)\Big)\mathrm{d}s, \tag{7}$$

and the following additional condition:

$$\Re_k g_k(t) = \phi_k(t), t \in [a-h, a] \tag{8}$$

Since Φ is PAC, the conditions (S) imply that for each $k, j \in \langle n \rangle$, the functions $t \to \int\limits_{-h}^{0} g_j(t+\theta)\mathrm{d}_\theta u_{k,j}(t,\theta)$ are at least piecewise continuous on the interval $[a, a+M]$ (see [27], Lemma 1). Then the right side of (7) is absolutely continuous on the interval $[a, a+M]$, which implies that the function $\Re_k g_k(t)$ is absolutely continuous on the same interval and since $\lim\limits_{t \to a+} \Re_k g_k(t) = \phi_k(a)$ for $k \in \langle n \rangle$, then $(\Re G) \in E_M^\Phi$. Thus the operator $\Re$ maps E_M^Φ into E_M^Φ.

Since according to Lemma 1 the space E_M^Φ is a full metric space concerning the metric d_M, it is enough to check that for the operator $\Re$ the conditions of Theorem 1 hold and then in virtue of Theorem 1 we will obtain that the operator $\Re$ has a unique fixed point in E_M^Φ.

We recall that the $\Gamma(z), z \in \mathbb{R}_+$, has a local minimum at $z_{min} \approx 1.46163$, where it attains the value $\Gamma(z_{min}) \approx 0.885603$. There exists $q_0 \in \mathbb{N}$ such that $q_0 + 1 > \alpha_m^{-1} \geq q_0$

and for every $q \in \langle q_0 + 1 \rangle$ we denote with $\alpha_q, \alpha_q \in \{\alpha_1, \dots, \alpha_n\}$ that number for which $\Gamma(1 + \alpha_q q) = \min_{k \in \langle n \rangle} \Gamma(1 + \alpha_k q)$.

Let denote $P = \max_{k\in\langle n\rangle}(\sum_{j=1}^{n} \sup_{s\in[a,a+M]} Var_{\theta\in[-h,0]} u_{kj}(s,\theta))$ and let $G, \bar{G} \in E_M^\Phi$ be arbitrary. Then from (7) for $k \in \langle n \rangle$ and every $t \in [a, a + M]$ we obtain

$$\begin{aligned}
|\Re_k g_k(t) - \Re_k \bar{g}_k(t)| &\le \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} (\sum_{j=1}^{n} | \int_{-h}^{0} (g_j(s+\theta) - \bar{g}_j(s+\theta)) \mathrm{d}u_{kj}(s,\theta)|) \mathrm{d}s \\
&\le \frac{(t-a)^{\alpha_k}}{\Gamma(1+\alpha_k)} \sum_{j=1}^{n} \sup_{s\in[a,a+M]} Var_{\theta\in[-h,0]} u_{kj}(s,\theta) \sup_{t\in[a,a+M]} |g_j(t) - \bar{g}_j(t)| \qquad (9) \\
&\le \frac{(t-a)^{\alpha_k} P}{\Gamma(1+\alpha_k)} \sum_{j=1}^{n} \sup_{t\in[a,a+M]} |g_j(t) - \bar{g}_j(t)| \le \frac{(t-a)^{\alpha_k} P}{\Gamma(1+\alpha_k)} \mathrm{d}_M(G, \bar{G})
\end{aligned}$$

Let assume that for some $q \in \mathbb{N}, k \in \langle n \rangle$ and for every $t \in [a, a + M]$ the inequality

$$|\Re_k^q g_k(t) - \Re_k^q \bar{g}_k(t)| \le \frac{(t-a)^{q\alpha_k} P^q}{\Gamma(1+q\alpha_k)} \mathrm{d}_M(G, \bar{G}) \qquad (10)$$

holds. Obviously from (9) it follows that the inequality (10) holds for each $k \in \langle n \rangle$ and every $t \in [a, a + M]$ at least for $q = 1$. Denoting for simplicity $\Re^q G(t) = Y(t) = (y(t), \dots, y_n(t))^T, \Re^q \bar{G}(t) = \bar{Y}(t) = (\bar{y}(t), \dots, \bar{y}_n(t))^T$, we obtain

$$|\Re_k^{q+1} g_k(t) - \Re_k^{q+1} \bar{g}_k(t)| = |\Re \Re_k^q g_k(t) - \Re \Re_k^q \bar{g}_k(t)| = |\Re y_k(t) - \Re \bar{y}_k(t)|. \qquad (11)$$

Let us assume that (10) holds for each $k \in \langle n \rangle$, every $t \in [a, a + M]$ and for some $q \in \mathbb{N}$. Then a similar way as in (9) from (10) and (11) we obtain

$$\begin{aligned}
|\Re_k y_k(t) - \Re \bar{y}_k(t)| &\le \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} \sum_{j=1}^{n} | \int_{-h}^{0} (y_j(s+\theta) - \bar{y}_j(s+\theta)) \mathrm{d}_\theta u_{kj}(s,\theta)| \mathrm{d}s \\
&\le \frac{P}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} \sum_{j=1}^{n} \sup_{\eta\in[a,a+M]} |y_j(\eta) - \bar{y}_j(\eta)| \mathrm{d}s \qquad (12) \\
&\le \frac{P P^q}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)} \mathrm{d}_M(G, \bar{G}) \int_a^t (t-s)^{\alpha_k - 1} (s-a)^{q\alpha_k} \mathrm{d}s
\end{aligned}$$

Substituting $s - a = z(t - a)$ in the integral in the right side of (12) and using the well-known relation between the gamma and beta functions we obtain

$$\begin{aligned}
|\Re_k y_k(t) - \Re \bar{y}_k(t)| &\le \frac{P^{q+1}(t-a)^{\alpha_k(q+1)}}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)} \mathrm{d}_M(G, \bar{G}) \int_0^1 (1-z)^{\alpha_k - 1} z^{q\alpha_k} \mathrm{d}z \\
&\le \frac{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)(P(t-a)^{\alpha_k})^{q+1}}{\Gamma(\alpha_k)\Gamma(1+q\alpha_k)\Gamma(1+(q+1)\alpha_k)} \mathrm{d}_M(G, \bar{G}) \\
&\le \frac{(P(t-a)^{\alpha_k})^{q+1}}{\Gamma(1+(q+1)\alpha_k)} \mathrm{d}_M(G, \bar{G}) \qquad (13) \\
&\le \frac{(PM^{\alpha_k})^{(q+1)}}{\Gamma(1+(q+1)\alpha_k)} \mathrm{d}_M(G, \bar{G}) \\
&\le \frac{(PM)^{(q+1)}}{\Gamma(1+(q+1)\alpha_k)} \mathrm{d}_M(G, \bar{G})
\end{aligned}$$

and hence (10) holds for every $q \in \mathbb{N}$, for each $k \in \langle n \rangle$ and every $t \in [a, a+M]$.

For $q \in \langle q_0 + 1 \rangle$ from (10) and (13) it follows that

$$\mathrm{d}_M(\Re^{q+1}G, \Re^{q+1}\bar{G}) \leq \frac{n(MP)^{(q+1)}}{\Gamma(1+\alpha_q(q+1))}\mathrm{d}_M(G, \bar{G}). \tag{14}$$

and denote $\gamma_q = \frac{n(MP)^q}{\Gamma(1+\alpha_q q)}$ for $q \in \langle q_0 + 1 \rangle$.

For all $q > q_0 + 1$ from (10) and (13) we obtain that

$$\mathrm{d}_M(\Re^{q+1}G, \Re^{q+1}\bar{G}) \leq \frac{n(MP)^{(q+1)}}{\Gamma(1+\alpha_m(q+1))}\mathrm{d}_M(G, \bar{G}). \tag{15}$$

and denote $\gamma_q = \frac{n(MP)^q}{\Gamma(1+\alpha_m q)}$ for $q > q_0 + 1$.

Consider the one parameter Mittag–Leffler function $E_{\alpha_m,1}(z) = \sum_{q=1}^{\infty} \frac{z^q}{\Gamma(1+\alpha_m q)}, z \in \bar{\mathbb{R}}_+$. It is simply to see that the series $\sum_{q=1}^{\infty} \frac{(MP)^q}{\Gamma(1+\alpha_m q)}$ is convergent because it is the considered Mittag–Leffler function evaluated at $z = MP$. Then we have that

$$\sum_{q=1}^{\infty} \gamma_q = n(\sum_{q=1}^{q_0} \frac{(MP)^q}{\Gamma(1+\alpha_q q)} + \sum_{q=q_0+1}^{\infty} \frac{(MP)^q}{\Gamma(1+\alpha_m q)}) < \infty$$

and then from Theorem 1 it follows that the IP (1) and (3) has a unique solution in $t \in [a, a+M]$. □

Corollary 1. *Let the conditions of Theorem 4 hold.*

Then the IP (1) and (3) has a unique absolutely continuous solution in the interval J_a.

Proof. Let $\Phi \in C_a^*$ be arbitrary and assume the contrary, that there exists $M^{max} < \infty$ such that the solution $X^{max}(t)$ in the interval $t \in [a, a+M^{max}]$ does not possess a prolongation. Let $M^* > M^{max}$ be an arbitrary number. In virtue of Theorem 4 the IP (1) and (3) has a unique absolutely continuous solution $X^*(t)$ in the interval $t \in [a, a+M^*]$. The solution $X^*(t)$ obviously is a prolongation of the solution $X^{max}(t)$, which contradicts of our assumption that $M^{max} < \infty$. □

For arbitrary fixed $s > a$ consider the following auxiliary system

$$D_{a+}^{\alpha} X(t) = \int_{-h}^{0} [\mathrm{d}_\theta U(t,\theta)] X(t+\theta) \tag{16}$$

with the following condition

$$X(t) = \tilde{\Phi}_{js}(t), \ \ t \in (-\infty, s]. \tag{17}$$

Corollary 2. *Let the following conditions hold:*

1. *Conditions (S) hold.*
2. *The function $\tilde{\Phi}_{js}$ has the form*

$$\tilde{\Phi}_{js}(t) = \begin{cases} I^j, & t = s, j \in \langle n \rangle \\ \mathbf{0}, & t < s \end{cases}$$

Then for each $s > a$ and arbitrary $j \in \langle n \rangle$ the problem (16) and (17) has a unique solution, which satisfies Equation (16) for $t > s$, the condition (17) for $t \leq s$ and is absolutely continuous in $(-\infty, s) \cup (s, \infty)$ with a first kind jump at $t = s$.

Proof. Let $s > a$ be an arbitrary fixed number, introduce the system

$$D^{\alpha}_{s+}X(t) = \int_{-h}^{0}[\mathrm{d}_{\theta}U(t,\theta)]X(t+\theta),\ t > s \tag{18}$$

and consider the IP (17) and (18). Since Theorem 4 is proved for arbitrary $a \in \mathbb{R}$ and $\tilde{\Phi}_{js}(t)$ for arbitrary $j \in \langle n \rangle$ is PAC on the interval $[s-h,s]$, then from Theorem 4 it follows that the IP (17) and (18) possess a unique solution $X_j(\cdot) \in AC(J_s,\mathbb{R}^n)$ and moreover, from (17) it follows also that $X_j(\cdot) \in AC([a,s) \cup J_s,\mathbb{R}^n)$. Then for every $k \in \langle n \rangle$ we have

$$\begin{aligned}
D^{\alpha_k}_{a+}x_{jk}(t) &= \frac{1}{\Gamma(1-\alpha_k)}\int_{a}^{t}(t-\eta)^{-\alpha_k}x'_{jk}(\eta)\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-\alpha_k)}\int_{a}^{s}(t-\eta)^{-\alpha_k}x'_{jk}(\eta)\mathrm{d}\eta + \frac{1}{\Gamma(1-\alpha_k)}\int_{s}^{t}(t-\eta)^{-\alpha_k}x'_{jk}(\eta)\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-\alpha_k)}\int_{s}^{t}(t-\eta)^{-\alpha_k}x'_{jk}(\eta)\mathrm{d}\eta = D^{\alpha_k}_{s+}x_{jk}(t)
\end{aligned}$$

and hence $X_j(t) = (x_{j1},\ldots,x_{jn})^T$ satisfies the Equation (16) for $t \in (s,\infty)$ and the condition (17) for $t \in (-\infty,s]$.

Let consider an IP with Equation (16) for $t > a$ and initial condition $X(t) = X_j(t) = \mathbf{0}$ for $t \in [a-h,a]$. Then obviously $X_j(t) = \mathbf{0}$ is its unique solution in $t \in [a,s)$. This completes the proof. □

4. Fundamental Matrix

Let $s \in J_a$ be an arbitrary fixed number and define the following matrix valued function $\bar{\Phi}(t,s) = (\bar{\varphi}_{kj}(t,s))^n_{k,j=1} : \mathbb{R} \times J_a \to \mathbb{R}^{n\times n}$ with

$$\bar{\Phi}(t,s) = \begin{cases} I, & t = s \\ \Theta, & t < s \end{cases}$$

and denote $\bar{\Phi}_j(t,s) = (\bar{\varphi}_{1j}(t,s),\ldots,\bar{\varphi}_{nj}(t,s))^T, j \in \langle n \rangle$.

For arbitrary fixed number $s \in J_a$ consider the following matrix IP

$$D^{\alpha}_{a+}C(t,s) = \int_{-h}^{0}[\mathrm{d}_{\theta}U(t,\theta)]C(t+\theta,s),\ t \in (s,\infty) \tag{19}$$

$$C(t,s) = \bar{\Phi}(t,s),\ t \in (-\infty,s]. \tag{20}$$

Definition 5. *The matrix valued function* $t \to C(t,s) = (C^1(t,s),\ldots,C^n(t,s)) = \{c^j_k(t,s)\}^n_{k,j=1}$, $s \in J_a$ *, is called a solution of the IP* (19) *and* (20) *in* J_s *if* $C(\cdot,s) : [s,\infty) \to \mathbb{R}^{n\times n}$ *is continuous for* $t \in [s,\infty)$ *and satisfies the matrix Equation* (19) *on* $t \in [s,\infty)$ *as well as the initial condition* (20) *too.*

Remark 5. *Practically in condition* (20) *we need only the values of* $\bar{\Phi}(\cdot,s)$ *for* $t \in [s-h,s]$, *but for convenience we define* $C(t,s) = \Theta$ *also for* $t \in (-\infty,s-h)$. *Then* $C(t,s)$ *is prolonged as continuous in t function on* $(-\infty,s)$.

Theorem 5. *Let the conditions (S) hold.*

Then for every initial point $s \in J_a$, *the matrix IP* (19) *and* (20) *has a unique absolutely continuous solution* $t \to C(t,s)$ *in the interval* J_s.

Proof. The statement of the Theorem follows immediately from Corollary 2. □

Definition 6. *The matrix $C(t,s)$, which is a solution of the IP (19) and (20), will be called fundamental (or Cauchy) matrix for the homogeneous system (1).*

Lemma 2. *Let the conditions (S) hold and the matrix valued function $t \to C(t,s)$ is the fundamental matrix of the system (1).*

Then for every $\bar{t} \in J_a$ the matrix function $C(t,\cdot) : [a,\bar{t}] \to \mathbb{R}^{n\times n}$ is locally bounded in s for $s \in [a,\bar{t}]$ and $t \in (-\infty,\bar{t}]$.

Proof. Let $\bar{t} \in J_a$ be an arbitrary fixed number, $s \in [a,\bar{t}]$ be arbitrary and consider the fundamental matrix $C(t,\cdot) : [a,\bar{t}] \to \mathbb{R}^{n\times n}$. According to Remark 5 for $s \in [a,\bar{t}]$ we have that $C(t,s) = \Theta$ for $t \in (-\infty,a)$ and $C(a,a) = I$.

Taking into account Theorem 5 and Corollary 2 it is easy to be seen, that the unique solution $C(t,s)$ of IP (19) and (20) is a solution of the equation

$$C(t,s) = \bar{\Phi}(s,s) + I_{-1}(\Gamma(\alpha)) \int_s^t I_{\alpha-1}(t-\eta)\Big(\int_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)]C(\eta+\theta,s)\Big)\mathrm{d}\eta. \tag{21}$$

Introduce for every $k,j \in \langle n\rangle$ the notations: $w_k^j(t,s) = \sup_{\xi\in[s,t]} |c_k^j(\xi,s)| = \sup_{\xi\in[a,t]} |c_k^j(\xi,s)|$ (since $c_k^j(\xi,s) = 0$ for $\xi < s$), $V_U^* = \sup_{t\in[a,\bar{t}]} Var_{\theta\in[-h,0]} U(t,\theta)$, $b = |I_{-1}(\Gamma(\alpha))|V_U^*$ and then from (21) it follows

$$|C(t,s)| \le |\bar{\Phi}(s,s)| + |I_{-1}(\Gamma(\alpha))| \, \Big|\int_s^t I_{\alpha-1}(t-\eta)\Big(\int_{-\sigma}^0 [\mathrm{d}_\theta U(\eta,\theta)]C(\eta+\theta,s)\Big)\mathrm{d}\eta\Big|$$

and hence for each $k,j \in \langle n\rangle$

$$\begin{aligned} w_k^j(t,s) &\le |\bar{\varphi}_{kj}(s,s)| + \frac{V_U^*}{\Gamma(\alpha_k)} \int_s^t (t-\eta)^{\alpha_k-1} w_k^j(\eta,s)\mathrm{d}\eta \\ &\le |\bar{\varphi}_{kj}(s,s)| + b\int_a^t (t-\eta)^{\alpha_k-1} w_k^j(\eta,s)\mathrm{d}\eta \end{aligned} \tag{22}$$

From Theorem 3 and (22) for $a \le t \le s \le \bar{t}$ we obtain the estimation

$$w_k^j(t,s) \le 1 + \int_a^t \Big(\sum_{q=1}^{\infty} \frac{(b\Gamma(\alpha_k))^q}{\Gamma(\alpha_k q)}(t-\eta)^{\alpha_k q-1}\Big)\mathrm{d}\eta \le 1 + \int_a^{\bar{t}} \Big(\sum_{q=1}^{\infty} \frac{(b\Gamma(\alpha_k))^q}{\Gamma(\alpha_k q)}(t-\eta)^{\alpha_k q-1}\Big)\mathrm{d}\eta$$

and thus $C(t,s)$ is locally bounded in s for $s \in [a,\bar{t}]$ and $t \in (-\infty,\bar{t}]$. □

Theorem 6. *Let the conditions (S) hold and the matrix valued function $t \to C(t,s)$ be the fundamental matrix of system (1).*

Then for every fixed $\bar{t} \in J_a$ the matrix function $C(t,\cdot) : [a,\bar{t}] \to \mathbb{R}^{n\times n}$ is continuous for $s \in [a,\bar{t}]$ when $s \neq t$, for $s = t$ possess first kind jumps and hence is Lebesgue integrable in s on $[a,\bar{t}]$ for each $t \in (-\infty,\bar{t}]$.

Proof. Let $\bar{t} \in J_a$ be an arbitrary fixed number, $s \in [a,\bar{t}]$ be arbitrary and consider the fundamental matrix $C(t,\cdot) : [a,\bar{t}] \to \mathbb{R}^{n\times n}$. According to Remark 5 for $s \in (a,\bar{t}]$ we have that $C(t,s) = \Theta$ for $t \in (-\infty,a]$ and $C(a,a) = I$. In virtue of Theorem 2 and Lemma 2 it is enough to prove that $C(t,s)$ has left and right limits for each $s \in (a,\bar{t})$ and there exist the limits $C(\bar{t},\bar{t}-0)$ and $C(a,a+0)$.

(i) Let $s^* \in [a,\bar{t}]$ be arbitrary and let $t < s^*$. Then it is simply to see that for every $s^* \in [a,\bar{t})$ $\lim_{s\to s^*+0} C(t,s) = \Theta = \lim_{s\to s^*-0} C(t,s)$. Note that for $s^* = \bar{t}$ we have that $\lim_{s\to\bar{t}-0} C(t,s) = \Theta$ holds.

(ii) Let $s^* \in [a,\bar{t}]$ be arbitrary and let $t = s^*$. Then $\lim_{s\to s^*+0} C(s^*,s) = \Theta$ and since $C(s^*,s^*) = I$, then we can conclude that $C(s^*,s^*+0)$ exists and hence $C(t,s)$ has jumps of first kind on the line $t = s^*$ for each $s^* \in [a,\bar{t})$.

(iii) Let $s^* \in [a,\bar{t}], t > s^*$. Then we have to consider two cases: either $s^* \in (a,\bar{t}], s \in (a,s^*)$ (left limit in s^*) or $s^* \in [a,\bar{t})$ with $s \in (s^*,t)$ (right limit in s^*).

(iii.a) For purposes of clarity we assume that $s^* \in (a,\bar{t}], s \in (a,s^*)$. According to Corollary 2 for each $j \in \langle n\rangle$, the IP (16) and (17) has unique solutions $C^j(t,s)$ and $C^j(t,s^*)$ for the initial functions $\bar{\Phi}_j(t,s) = (\bar{\varphi}_{1j}(t,s),\ldots,\bar{\varphi}_{nj}(t,s)(t,s))^T$ and $\bar{\Phi}_j(t,s^*) = (\bar{\varphi}_{1j}(t,s^*),\ldots,\bar{\varphi}_{nj}(t,s^*))^T$, respectively.
Then we obtain

$$C^j(t,s^*) = \bar{\Phi}_j(s^*,s^*) + I_{-1}(\Gamma(\alpha))\int_{s^*}^{t} I_{\alpha-1}(t-\eta)\Big(\int_{-h}^{0}[\mathrm{d}_\theta U(\eta,\theta)]C^j(\eta+\theta,s^*)\Big)\mathrm{d}\eta \tag{23}$$

$$C^j(t,s) = \bar{\Phi}_j(s,s) + I_{-1}(\Gamma(\alpha))\int_{s}^{t} I_{\alpha-1}(t-\eta)\Big(\int_{-h}^{0}[\mathrm{d}_\theta U(\eta,\theta)]C^j(\eta+\theta,s)\Big)\mathrm{d}\eta. \tag{24}$$

Taking into account that $\bar{\Phi}(s,s) = \bar{\Phi}(s^*,s^*) = I$ and subtracting both sides of (24) from the corresponding sides of (23) we obtain

$$\begin{aligned} C^j(t,s^*) - C^j(t,s) &= I_{-1}(\Gamma(\alpha))\int_{s}^{s^*} I_{\alpha-1}(t-\eta)\Big(\int_{-h}^{0}[\mathrm{d}_\theta U(\eta,\theta)]C^j(\eta+\theta,s)\Big)\mathrm{d}\eta \\ &+ I_{-1}(\Gamma(\alpha))\int_{s^*}^{t} I_{\alpha-1}(t-\eta)\Big(\int_{-h}^{0}[\mathrm{d}_\theta U(\eta,\theta)](C^j(\eta+\theta,s^*) - C^j(\eta+\theta,s))\Big)\mathrm{d}\eta. \end{aligned} \tag{25}$$

Let us denote $y_k^j(t,s,s^*) = \sup_{\xi\in[a,t]} |c_k^j(\xi,s^*) - c_k^j(\xi,s)|$ and $\bar{C}^j = \sup_{\xi,s\in[a,\bar{t}]} C^j(\xi,s)|$ ($\bar{C}^j$ exists according to Lemma 2) and $b = |I_{-1}(\Gamma(\alpha))|V_U^*$. Then for each $k,j \in \langle n\rangle$ and $s \in (a,s^*)$ from (25) we obtain that

$$\begin{aligned} y_k^j(t,s,s^*) &\le b\bar{C}^j\int_{s}^{s^*}(t-\eta)^{\alpha_k-1}\mathrm{d}\eta + bV_U^*\int_{s^*}^{t}(t-\eta)^{\alpha_k-1}y_k^j(\eta,s,s^*)\mathrm{d}\eta \\ &= b\bar{C}^j\frac{(t-s)^{\alpha_k}-(t-s^*)^{\alpha_k}}{\alpha_k} + bV_U^*\int_{s^*}^{t}(t-\eta)^{\alpha_k-1}y_k^j(\eta,s,s^*)\mathrm{d}\eta \end{aligned} \tag{26}$$

Define for $\eta \in [s^*,t]$ the function $p(\eta,s,s^*) = b\bar{C}^j\frac{(\eta-s)^{\alpha_k}-(\eta-s^*)^{\alpha_k}}{\alpha_k}$.

It is simple to see that the function p is monotonically decreasing in η for $\eta \in [s^*,t]$ when $s \in (a,s^*)$. Furthermore, for arbitrary fixed $\eta \in [s^*,t]$ we have that

$$\lim_{s\to s^*} p(\eta,s,s^*) = 0. \tag{27}$$

Then from (26) and Theorem 3 it follows that for each $j \in \langle n\rangle, k \in \langle n\rangle$ and $s \in (a, s^*)$ we obtain the following estimation:

$$\begin{aligned} y_k^j(t,s,s^*) &\le p(t,s,s^*) + \int_{s^*}^{t} [\sum_{q=1}^{\infty} \frac{(bV_U^*(\Gamma(\alpha_k))^q}{\Gamma(\alpha_k q)}(t-\eta)^{\alpha_k q-1}] p(\eta,s,s^*)\mathrm{d}\eta \\ &\le p(t,s,s^*) + p(s^*,s,s^*) \int_{s^*}^{t} [\sum_{q=1}^{\infty} \frac{(bV_U^*(\Gamma(\alpha_k))^q}{\Gamma(\alpha_k q)}(t-\eta)^{\alpha_k q-1}]\mathrm{d}\eta \end{aligned} \tag{28}$$

From (27) and (28) it follows that $\lim_{s\to s^*} y_k^j(\eta,s,s^*) = \lim_{s\to s^*} \sup_{\xi\in[a,t]} |c_k^j(\xi,s^*) - c_k^j(\xi,s)| = 0$ and hence $c_k^j(t,s^*-0)$ exists and $c_k^j(t,s^*-0) = c_k^j(t,s^*)$ for $t > s^*$.

(iii.b) The case when $t > s^*$ and $s \in (s^*, t)$ can be treated fully analogically to obtain that $c_k^j(t,s^*+0)$ exists and $c_k^j(t,s^*+0) = c_k^j(t,s^*)$.

□

Let $s^* \in [a-h, a]$ be an arbitrary fixed number and define the following matrix valued function $\Phi_{s^*}^*(t,s) = (\varphi_{kj}^*(t,s))_{k,j=1}^n : \mathbb{R} \times [a-h,a] \to \mathbb{R}^{n\times n}$ with

$$\Phi_{s^*}^*(t,s) = \begin{cases} I, & s^* \le s \le t \le a \\ \Theta, & t < s \text{ or } s < s^* \end{cases}$$

and consider following IP:

$$D_{a+}^{\alpha} T_{s^*}(t,s) = \int_{-h}^{0} [\mathrm{d}_\theta U(t,\theta)] T_{s^*}(t+\theta,s),\ t > a \tag{29}$$

$$T_{s^*}(t,s) = \Phi_{s^*}^*(t,s),\ t \in (-\infty, a]. \tag{30}$$

Theorem 7. *Let the conditions (S) hold and $\bar{t} \in J_a$ be arbitrary.*

Then the following statements hold:

1. *The matrix IP (29) and (30) has a unique absolutely continuous solution $T_{s^*}(t,s)$ in t for $t \in J_a$ for every $s^* \in [a-h, a]$.*
2. *The matrix function $T_{a-h}(t,\cdot) : [a,\bar{t}] \to \mathbb{R}^{n\times n}$ is continuous in s for each $s \in [a,\bar{t}]$ with $s \neq t$.*
3. *When $s = t$ with $t \ge a$, T_{a-h} possess first kind jumps and hence is Lebesgue integrable in s on $[a,\bar{t}]$.*

Proof. 1. Let $s^* \in [a-h, a]$ be an arbitrary fixed number. Then since $\Phi_{s^*}^*$ is PAC for each $s^* \in [a-h, a]$, then the statement of point 1 follows from Theorem 4.

2. Let $s \in [a,\bar{t}]$ with $s \neq t$. Then if $t < s$ we have that $T_{a-h}(t,s) = \mathbf{0}$. Consider the case $t > s$. Then the same way as in the proof of point (iii) of Theorem 6 we obtain that $T_{a-h}(t,s)$ is continuous in s when $s \neq t$.
3. Let $s = t$ with $t \ge a$. Then obviously $\lim_{t\to s+0} T_{a-h}(t,s) = I$ and $\lim_{t\to s-0} T_{a-h}(t,s) = \mathbf{0}$ and this completes the proof.

□

5. Applications

We will demonstrate that the obtained results concerning the fundamental matrix allow to improve the integral representation of the solution of the IP (2) and (3) and simplify the proofs.

As usual according the superposition principle we will seek a solution of IP (2) and (3) with initial condition $\Phi(t) \equiv 0, t \in [a-h, a]$ for the case when the function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded.

Let

$$X_F(t) = \int_a^t K(t,s)\mathrm{d}s, \tag{31}$$

where $K(t,s) = C(t,s)T(s), T(s) =_{RL} D_{a+}^{1-\alpha}F(s)$.

Theorem 8. *Let the following conditions hold.*

1. *The conditions (S) hold.*
2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded and $D_{a+}^{\alpha}F(t) \in L_1^{loc}(J_a, \mathbb{R}^n)$.*

Then the vector function $X_F(t)$ defined by equality (31) is a solution of IP (2) and (3) with initial condition $X_F(t) = \Phi(t) \equiv \mathbf{0}, t \in [a-h, a]$.

Proof. From (31) and Theorem 5 it follows that $X_F(t)$ is an absolutely continuous function in J_a. Then we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}X_F(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_a^s K(s,\eta)\mathrm{d}\eta = \int_a^s C_s(s,\eta)T(\eta)\mathrm{d}\eta + C(t,t)T(t) = \int_a^s C_s(s,\eta)T(\eta)\mathrm{d}\eta + T(t). \tag{32}$$

Taking into account (31) and (32), Lemma 2.5 in [1], Condition 2 of the Theorem and applying Fubini's theorem we obtain

$$\begin{aligned} D_{a+}^{\alpha}X_F(t) &= \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}(\int_a^s T(\eta)C_s(s,\eta))\mathrm{d}\eta)\mathrm{d}s + \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}T(s)\mathrm{d}s \\ &= \int_a^t T(\eta)(\frac{1}{\Gamma(1-\alpha)}(\int_\eta^t (t-s)^{-\alpha}C_s(s,\eta))\mathrm{d}s)\mathrm{d}\eta + D_{a+}^{\alpha-1}T(t) \\ &= \int_a^t T(\eta)(\frac{1}{\Gamma(1-\alpha)}(\int_a^t (t-s)^{-\alpha}C_s(s,\eta))\mathrm{d}s)\mathrm{d}\eta + D_{a+}^{\alpha-1}{}_{RL}D_{a+}^{1-\alpha}F(t) \\ &= \int_a^t T(\eta)_C D_{a+}^{\alpha}C(t,\eta)\mathrm{d}\eta + F(t) \end{aligned} \tag{33}$$

For the right side of (33) we have

$$\begin{aligned} \int_a^t {}_C D_{a+}^{\alpha}C(t,\eta)T(\eta)\mathrm{d}\eta &= \int_a^t T(\eta)[\sum_{i=0}^m \int_{-h}^0 [\mathrm{d}_\theta U^i(t,\theta)]C(t+\theta,\eta)]\mathrm{d}\eta \\ &= \sum_{i=0}^m \int_{-h}^0 [\mathrm{d}_\theta U^i(t,\theta)](\int_a^t C(t+\theta,\eta)T(\eta)\mathrm{d}\eta) = \sum_{i=1}^m \int_{-h}^0 [\mathrm{d}_\theta U^i(t,\theta)]X_F(t+\theta) \end{aligned} \tag{34}$$

and then the statement of the corollary follows from (33) and (34). □

Corollary 3. *Let the following conditions hold.*

1. *The conditions (S) hold.*
2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded and $F(a) = \mathbf{0}$.*

Then the vector function $X_F(t)$ defined by equality (31) is a solution of IP (2) and (3) with initial condition $X_F(t) = \Phi(t) \equiv \mathbf{0}, t \in [a-h, a]$.

Proof. For $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ with $F(a) = \mathbf{0}$ is fulfilled

$$D_{a+}^{\alpha-1} T(t) = D_{a+}^{\alpha-1} {}_{RL}D_{a+}^{1-\alpha} F(t) = D_{a+}^{\alpha-1} {}_{C}D_{a+}^{1-\alpha} F(t) = F(t) - F(a) = F(t).$$

Then the proof is the same as in Theorem 8. □

Let $T_{a-h}(t,s)$ be a solution of the IP (29) and (30) for $s \in [a-h, a]$. For arbitrary function $\Phi(t) \in BV([a-h,a], \mathbb{R}^n)$ define the following function:

$$X_{\Phi}(t) = \int_{a-h}^{a} T_{a-h}(t,s) \mathrm{d}_s \Phi(s) + T_{a-h}(t, a-h)\Phi(a-h), \; t \in J_a \tag{35}$$

Theorem 9. *Let the following conditions hold.*

1. *The conditions (S) hold.*
2. *The initial function $\Phi(t) \in BV([a-h,a], \mathbb{R}^n)$ is not a constant and has finitely many jumps.*

Then the function $X_{\Phi}(t)$ defined with (35) is the unique solution of IP (1) and (3).

Proof. Let $t \in [a-h, a]$ be an arbitrary fixed number and $s \in [a-h, a]$. Then in virtue of Theorem 6 for every $t \in [a-h, a]$ the matrix function $T_{a-h}(t, \cdot) : [a-h, a] \to \mathbb{R}^{n \times n}$ is continuous in s on $[a-h, a]$ for $s \neq t$ and when $t = s$ possess first kind jumps. Thus we have that for $s \in [t, a]$ from (30) it follows that $T_{a-h}(t,s) = \Theta$ and for $s \in [a-h, t]$ we have $T_{a-h}(t,s) = I$. Then $X_{\Phi}(t) = \Phi(t)$ and hence $X_{\Phi}(t)$ satisfies the initial condition (3).

Theorem 7 implies that the matrix valued function $T_{a-h}(t,s)$ is an absolutely continuous function for $t \in J_a$ and then the vector valued $X_{\Phi}(t)$ is defined by equality (35) too. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} X_{\Phi}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{a-h}^{a} T_{a-h}(t,s) \mathrm{d}_s \Phi(s) = \int_{a-h}^{a} \frac{\partial T_{a-h}(t,s)}{\partial t} \mathrm{d}_s \Phi(s)$$

and hence applying the Fubini's theorem we have that

$$\begin{aligned} {}_{C}D_{a+}^{\alpha} X_{\Phi}(t) &= {}_{C}D_{a+}^{\alpha} \int_{a-h}^{a} T_{a-h}(t,s) \mathrm{d}_s \Phi(s) = \int_{a}^{t} (t-\eta)^{-\alpha} \Big(\int_{a-h}^{a} \frac{\partial T_{a-h}(\eta,s)}{\partial \eta} \mathrm{d}_s \Phi(s) \Big) \mathrm{d}\eta \\ &= \int_{a-h}^{a} \Big(\int_{a}^{t} (t-\eta)^{-\alpha} \frac{\partial T_{a-h}(\eta,s)}{\partial \eta} \mathrm{d}_s \Phi(s) \Big) = \int_{a-h}^{a} {}_{C}D_{a+}^{\alpha} T_{a-h}(t,s) \mathrm{d}_s \Phi(s) \end{aligned} \tag{36}$$

In the right side of (1) for each $i \in \langle m \rangle$ applying the unsymmetric Fubini theorem [28] we obtain that

$$\begin{aligned} \int_{-h}^{0} [\mathrm{d}_\theta U(t,\theta)] X_{\Phi}(t+\theta) &= \int_{-h}^{0} [\mathrm{d}_\theta U(t,\theta)] \Big(\int_{a-h}^{a} T_{a-h}(t+\theta, s) \mathrm{d}_s \Phi(s) \Big) \\ &= \int_{a-h}^{a} \Big(\int_{h}^{0} [\mathrm{d}_\theta U(t,\theta)] (T_{a-h}(t+\theta, s) \mathrm{d}_s \Phi(s) \Big) \end{aligned} \tag{37}$$

From (36) and (37) it follows that

$$\int_{a-h}^{a} [{}_{C}D_{a+}^{\alpha} T_{a-h}(t,s) - \int_{-h}^{0} [\mathrm{d}_\theta U^i(t,\theta)] T_{a-h}(t+\theta, s)] \mathrm{d}_s \Phi(s) = 0$$

and hence $X_{\Phi}(t)$ satisfies (1) for $t > a$. □

Theorem 10. *Let the following conditions hold.*

1. *The conditions of Theorem 9 hold.*
2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded.*
3. *Either $F(0) = \mathbf{0}$ or $D_{a+}^{\alpha} F(t) \in L_1^{loc}(J_a, \mathbb{R}^n)$ holds.*

Then the function

$$X_f^{\Phi}(t) = X_{\Phi}(t) + X_f(t) = \int_a^t C(t,s)_{RL}D_{a+}^{1-\alpha}F(s)\mathrm{d}s + \int_{a-h}^{a} T_{a-h}(t,s)\mathrm{d}_s\Phi(s) + T_{a-h}(t,a-h)\Phi(a-h) \quad (38)$$

where $X_F(t)$ and $X_{\Phi}(t)$ are defined by (31) and (35), respectively, is the unique solution of IP (2) and (3).

Proof. The statement of the theorem follows immediately from the superposition principle, Corollary 3 and Theorems 8 and 9. □

Corollary 4. *Let the following conditions hold.*

1. *The conditions of Theorem 10 hold.*
2. *The Lebesgue decomposition of the function $\Phi(t) \in BV([a-h,a], \mathbb{R}^n)$ does not include a singular term.*

Then the function $X_F^{\Phi}(t)$ defined by (38) possesses the following integral representation:

$$\begin{aligned} X_f^{\Phi}(t) &= T_{a-h}(t,a)(\Phi_J(a+) - \Phi_J(a-)) + \sum_i T_{a-h}(t,s_i)((\Phi_J(s_i+) - \Phi_J(s_i-)) \\ &+ \int_{a-h}^{a} T_{a-h}(t,s)\Phi_A'(s)\mathrm{d}s + \int_a^t C(t,s)_{RL}D_{a+}^{1-\alpha}F(s)\mathrm{d}s + T_{a-h}(t,a-h)\Phi(a-h) \end{aligned} \quad (39)$$

where $\Phi(t) = \Phi_J(t) + \Phi_A(t)$ and $\Phi_J(t), \Phi_A(t)$ are the jump term and the absolutely continuous term, respectively, in its Lebesgue decomposition and the summation is taken over all jump points $s_i \in S^{\Phi}$.

Proof. The statement of the corollary follows immediately from Theorem 10. □

One of the important questions that arise when we use a fundamental matrix is what kind are its analytical properties concerning the variable s. It is well known that for the integer-order linear differential systems without delay, this problem is fully solved, i.e., both variables are symmetric. Generally speaking, this is not true even in the integer case for the delayed differential systems (the symmetry disappears for the non-autonomous systems and in some cases for the autonomous systems too). As far as we know there are no results in this direction for delayed non-autonomous fractional differential systems. The next result is a first attempt to establish some analytical properties of the fundamental matrix in the mentioned case concerning the variable s.

Theorem 11. *Let the conditions of Theorem 9 hold and the matrix valued function $t \to C(t,s)$ is the fundamental matrix of the system (1).*

Then for every fixed $\bar{a} \in J_a$ the matrix function $C(t,\cdot) : [a,\bar{a}] \to \mathbb{R}^{n\times n}$ for every fixed $t \in (-\infty, s) \cup (s, \infty)$ is absolutely continuous in s for every compact subinterval $s \in [a_1, a_2] \subset (a, \bar{a})$ with $t \notin [a_1, a_2]$.

Proof. Let $\bar{a} \in J_a$ be an arbitrary fixed number, $s \in [a, \bar{a}]$ be arbitrary and $t \in (-\infty, s) \cup (s, \infty)$.

(i) When $t \in (-\infty, s)$ then for the fundamental matrix $C(t,\cdot) : [a,\bar{a}] \to \mathbb{R}^{n\times n}$ according to Remark 5 for arbitrary $s \in (a, \bar{a})$ we have that $C(t,s) = \Theta$ and $C(s,s) = I$. Thus $C(t,s)$ has jumps of first kind for $t = s$ and $C(t,s)$ is absolutely continuous in s for $s \in (a, \bar{a})$.

(ii) Let $s \in (a, \bar{a})$ and $\bar{t} \in (\bar{a}, \infty)$ be an arbitrary fixed number. For purposes of clarity we assume that $s, s_* \in (a, \bar{a})$ with $s < s^*$. Then as in the proof of Theorem 6 we obtain that (25) holds for $t \in [a, \bar{a}]$ and introduce the same notations used there.

Since the function $p(\eta, s, s^*) = b\bar{C}^j \frac{(\eta - s)^{\alpha_k} - (\eta - s^*)^{\alpha_k}}{\alpha_k}$ is monotonically decreasing in η for $\eta \in [s^*, \bar{a}]$ then in a similar way as in the proof of Theorem 6 for each $k, j \in \langle n \rangle$ from (25) we obtain that

$$\begin{aligned} y_k^j(t, s, s^*) &\le b\bar{C}^j \int_s^{s^*} (t - \eta)^{\alpha_k - 1} \mathrm{d}\eta + bV_U^* \int_{s^*}^t (t - \eta)^{\alpha_k - 1} y_k^j(\eta, s, s^*) \mathrm{d}\eta \\ &= b\bar{C}^j \frac{(t - s)^{\alpha_k} - (t - s^*)^{\alpha_k}}{\alpha_k} + bV_U^* \int_{s^*}^t (t - \eta)^{\alpha_k - 1} y_k^j(\eta, s, s^*) \mathrm{d}\eta \\ &\le [(t - s^*)^{\alpha_k} - (t - s)^{\alpha_k}] \frac{b\bar{C}^j + \bar{C}^j bV_U^*}{\alpha_k} \\ &\le (s - s^*)^{\alpha_k} \frac{b\bar{C}^j + \bar{C}^j bV_U^*}{\alpha_k} \le (s - s^*)^{\alpha_m} \frac{b\bar{C}^j + \bar{C}^j bV_U^*}{\alpha_m} \end{aligned} \tag{40}$$

when $s^* - s \le 1$. Then we can conclude that $c_k^j(t, s)$ for each $k, j \in \langle n \rangle$ is α_m-Hoelder continuous in s in every closed subinterval $[a_1, a_2] \subset (a, \bar{a})$ with $a_2 - a_1 \le 1$. Thus $c_k^j(t, s)$ is absolutely continuous in s for $s \in (a, \bar{a})$.

(iii) Let $s \in (a, \bar{a}), \bar{t} \in (s, \bar{a}]$ and $s < s^* < \bar{t}$. Then as in the former case we conclude that (40) holds and then $c_k^j(t, s)$ for each $k, j \in \langle n \rangle$ is α_m-Hoelder continuous in s for every closed subinterval $[a_1, a_2] \subset (a, \bar{a})$ for which $a_2 - a_1 \le 1$. Thus $c_k^j(t, s)$ is absolutely continuous in s for every compact subinterval $[a_1, a_2] \subset (a, \bar{a})$.

□

Remark 6. *It is not difficult to see that under the conditions of Theorem 11 the statement of the theorem holds for $T_{a-h}(t, s)$ too.*

Corollary 5. *Let the following conditions hold.*

1. *The conditions of Theorem 9 hold.*
2. *The Lebesgue decomposition of the function $\Phi(t) \in BV([a - h, a], \mathbb{R}^n)$ does not include a singular term.*
3. *The delays $\sigma_i(t) \in C^1(J_a, \bar{\mathbb{R}}_+), \sigma_i'(t) < 1$ for $t \in J_a$ and $i \in \langle m \rangle$.*

Then the unique solution $X_\Phi(t)$ of IP (1) and (3) defined with (35) has the following representation:

$$X_\Phi(t) = T_{a-h}(t, a)\Phi_J(a+) - \int_{a-h}^{a} \frac{\partial T_{a-h}(t, s)}{\partial s} \Phi_A(s) \mathrm{d}s + T_{a-h}(t, a - h)\Phi(a - h) \tag{41}$$

Proof. From conditions 1 and 2 of the theorem it follows that in virtue of Corollary 4 the unique solution $X_\Phi(t)$ of IP (1) and (3) defined with (35) has the representation:

$$\begin{aligned} X_\Phi(t) &= T_{a-h}(t, a)(\Phi_J(a+) - \Phi_J(a-)) \\ &+ \sum_i T_{a-h}(t, s_i)(\Phi_J(s_i+) - \Phi_J(s_i-)) + \int_{a-h}^{a} T_{a-h}(t, s)\Phi_A'(s) \mathrm{d}s + T_{a-h}(t, a - h)\Phi(a - h) \end{aligned} \tag{42}$$

Let $\bar{a} \in J_a$ be an arbitrary fixed number and $[a_1, a_2] \subset (a, \bar{a})$ be an arbitrary compact subinterval. Condition 3 of the theorem implies that for arbitrary fixed number $t \in \mathbb{R}$ the set $S_t^\sigma = \{s \in [a_1, a_2] | s = t - \sigma_i(t), i \in \langle m \rangle_0\}$ is finite. Then according Theorem 11 for arbitrary fixed number $t \in \mathbb{R}$, $T_{a-h}(t,s)$ is absolutely continuous in s for every compact subinterval $[a_1, a_2] \subset (a, \bar{a})$. Then integrating by parts the integral in (42) we obtain:

$$\int_{a-h}^{a} T_{a-h}(t,s)\Phi'_A(s)\mathrm{d}s = \int_{a-h}^{a} T_{a-h}(t,s)\mathrm{d}\Phi_A(s)$$
$$= T_{a-h}(t,a)\Phi_J(a-) + \sum_i T_{a-h}(t,s_i)(\Phi_J(s_i-) - \Phi_J(s_i+)) - \int_{a-h}^{a} \frac{\partial T_{a-h}(t,s)}{\partial s}\Phi_A(s)\mathrm{d}s \tag{43}$$

Then the statement of the corollary follows from (42) and (43). □

6. Conclusions

In this article, first the existence and uniqueness of the solutions of an initial problem for linear differential systems with incommensurate order Caputo fractional derivatives and with piecewise absolutely continuous (PAC) initial function is proved.

Then we prove the existence and uniqueness of an absolutely continuous fundamental matrix $C(t,s)$, which has the following properties:

1. $C(t,s)$ is absolutely continuous in t for PAC initial functions;
2. $C(t,s)$ is absolutely continuous in s (with appropriate additional assumptions).

It must be noted that when the fundamental matrix is absolutely continuous in t and in s, the fundamental matrix has integrable derivatives in t and in s and this allows simpler and more applicable formulas to be obtained in the integral representations, as well as simpler and shorter proofs.

As far we know there are no other articles where such properties of the fundamental matrix $C(t,s)$ concerning the variable s for delayed non-autonomous fractional differential systems are obtained. A brief comparison with similar fundamental matrix studies shows that the same system was studied in [15], but there is proof of existence of a continuous fundamental matrix, which is only continuous in t for initial functions with bounded variation. Our result is more general than that obtained in [21] where the smoothness of the fundamental matrix is proven.

Finally, using the properties of the fundamental matrix thus obtained, integral representations are obtained in the paper for the particular solution of the inhomogeneous system with zero initial conditions and for the general solution of the homogeneous system.

A comparison, for example, with the integral representations obtained in [15] shows, that all the proofs are shorter and the obtained formula for the general solution of the homogeneous system is simpler and more applicable.

A general comparison with the analogous results for integer order derivatives shows that those obtained in the article results coincide with them at $\alpha = 1$, which means that they are a generalization of the classical ones.

We hope that the results obtained will be useful both for future research and generalizations from a mathematical point of view, as well as for modeling of real-world phenomena.

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Abbreviations

In the paper are used the following abbreviations:

AC	Absolutely Continuous
PAC	Piecewise Absolutely Continuous
BV	Bounded Variation
IP	Initial Problem

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