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Hermite–Hadamard–Mercer-Type Inequalities for Harmonically Convex Mappings

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Abstract: In this paper, we prove Hermite–Hadamard–Mercer inequalities, which is a new version of the Hermite–Hadamard inequalities for harmonically convex functions. We also prove Hermite–Hadamard–Mercer-type inequalities for functions whose first derivatives in absolute value are harmonically convex. Finally, we discuss how special means can be used to address newly discovered inequalities.

Keywords: Hermite–Hadamard–Mercer inequality; Jensen inequality; harmonically convex function

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1. Introduction

In the literature, the well-known Jensen inequality [1] states that if f is a convex function on an interval and contains $x_1, x_2, \ldots, x_n$, then:

$$f\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j f(x_j), \tag{1}$$

where $\lambda_j \geq 0, j = 0, 1, \ldots, n$ and $\sum_{j=1}^{n} \lambda_j = 1$.

In the theory of convex functions, the Hermite–Hadamard inequality is very important. It was independently discovered by C. Hermite and J. Hadamard (see also [2,3] (p. 137)):

$$f\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} f(x)dx \leq \frac{f(\kappa_1) + f(\kappa_2)}{2}, \tag{2}$$

where $f : I \to \mathbb{R}$ is a convex function over I, and $\kappa_1, \kappa_2 \in I$, with $\kappa_1 < \kappa_2$. In the case of concave mappings, the above inequality is satisfied in reverse order.

The following variant of the Jensen inequality, known as the Jensen–Mercer, was demonstrated by Mercer [4]:

Theorem 1. *If f is a convex function on $[a, b]$, then the following inequality is true:*

$$f\left(a + b - \sum_{j=1}^{n} \lambda_j x_j\right) \leq f(a) + f(b) - \sum_{j=1}^{n} \lambda_j f(x_j), \tag{3}$$

where $\sum_{j=1}^{n} \lambda_j = 1$, $x_j \in [a,b]$ *and* $\lambda_j \in [0,1]$.

In [5], the idea of the Jensen–Mercer inequality was used by Kian and Moslehian, and the following Hermite–Hadamard–Mercer inequality was demonstrated:

$$\begin{aligned} f\left(\kappa_1+\kappa_2-\frac{x+y}{2}\right) &\leq f(\kappa_1)+f(\kappa_2)-\frac{1}{y-x}\int_x^y f(\tau)d\tau \\ &\leq f(\kappa_1)+f(\kappa_2)-f\left(\frac{x+y}{2}\right) \end{aligned} \tag{4}$$

$$\begin{aligned} f\left(\kappa_1+\kappa_2-\frac{x+y}{2}\right) &\leq \frac{1}{y-x}\int_x^y f(\kappa_1+\kappa_2-\tau)d\tau \\ &\leq \frac{f(\kappa_1+\kappa_2-x)+f(\kappa_1+\kappa_2-y)}{2} \\ &\leq f(\kappa_1)+f(\kappa_2)-\frac{f(x)+f(y)}{2}, \end{aligned} \tag{5}$$

where f is a convex function on $[\kappa_1,\kappa_2]$. For some recent studies linked to the Jensen–Mercer inequality, one can consult [6,7].

2. Harmonic Convexity and Related Inequalities

In this section, we will study the concepts of harmonically convex functions and the integral inequalities associated with them.

Definition 1. *A mapping such as* $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *[8] is called harmonically convex if the following inequality holds for all* $x, y \in I$ *and* $\tau \in [0,1]$:

$$f\left(\frac{1}{\frac{\tau}{y}+\frac{1-\tau}{x}}\right) \leq \tau f(y)+(1-\tau)f(x). \tag{6}$$

When the inequality (6) is reversed, f *is described as harmonically concave.*

Dragomir recently proved the following Jensen-type inequality for harmonically convex functions:

Theorem 2 (Jensen inequality)**.** *If* f *is a harmonically convex function on an interval containing* $x_1, x_2, \ldots, x_n$*, then the following inequality is true [9]:*

$$f\left(\frac{1}{\sum_{i=1}^{n}\frac{\lambda_i}{x_i}}\right) \leq \sum_{i=1}^{n}\lambda_i f(x_i), \tag{7}$$

where $\lambda_j \geq 0, j = 0,1,\ldots,n$ *and* $\sum_{j=1}^{n}\lambda_j = 1$.

In [8], İşçan established the Hermite–Hadamard type of inequalities for harmonically convex functions as follows:

Theorem 3. *For a harmonically convex mapping* $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *with* $\kappa_1, \kappa_2 \in I$ *and* $\kappa_1 < \kappa_2$*, the following inequality holds:*

$$f\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \leq \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{f(x)}{x^2}dx \leq \frac{f(\kappa_1)+f(\kappa_2)}{2}. \tag{8}$$

Furthermore, to find right-hand-side estimates of inequality (8), İşçan proved the following lemma:

Lemma 1. *For a differentiable mapping $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ on I° with $\kappa_1, \kappa_2 \in I$ and $\kappa_1 < \kappa_2$, the following equality holds:*

$$\begin{aligned} &\frac{f(\kappa_1)+f(\kappa_2)}{2} - \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{f(x)}{x^2}dx \\ = \;& \frac{\kappa_1\kappa_2(\kappa_2-\kappa_1)}{2}\int_0^1 \frac{1-2\tau}{(\tau\kappa_2+(1-\tau)\kappa_1)^2} f'\left(\frac{\kappa_1\kappa_2}{\tau\kappa_2+(1-\tau)\kappa_1}\right)d\tau. \end{aligned} \tag{9}$$

For the estimates of the right-hand side of the inequality (8), one can consult [8].

The Jensen–Mercer inequality is a new variant of Jensen's inequality (7) for harmonically convex functions, as proved by Baloch:

Theorem 4 (Jensen–Mercer inequality)**.** *For harmonically convex functions on $[a, b]$, the following inequality is true [10]:*

$$f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\sum_{i=1}^{n}\frac{\lambda_i}{x_i}}\right) \le f(a)+f(b)-\sum_{i=1}^{n}\lambda_i f(x_i), \tag{10}$$

where $\sum_{i=1}^{n}\lambda_i = 1$, $x_i \in [a, b]$ and $\lambda_i \in [0, 1]$.

For some recent inequalities via harmonically convex functions, one can consult [11–15].

The authors used the inequality (10) to prove the following Hermite–Hadamard–Mercer type of inequalities in [10,16]:

Theorem 5. *For a harmonically convex mapping $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ with $\kappa_1, \kappa_2 \in I$ and $\kappa_1 < \kappa_2$, the following inequality holds [10]:*

$$f\left(\frac{1}{\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2}-\frac{2xy}{x+y}}\right) \le f(\kappa_1)+f(\kappa_2)-\frac{xy}{y-x}\int_x^y\frac{f(\tau)}{\tau^2}d\tau \le f(\kappa_1)+f(\kappa_2)-f\left(\frac{2xy}{x+y}\right) \tag{11}$$

for $x, y \in [\kappa_1, \kappa_2]$.

Theorem 6. *For a harmonically convex mapping $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ with $\kappa_1, \kappa_2 \in I$ and $\kappa_1 < \kappa_2$, the following inequality holds:*

$$\begin{aligned} f\left(\frac{1}{\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2}-\frac{1}{2}\left(\frac{x+y}{xy}\right)}\right) &\le \frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}\frac{f(\tau)}{\tau^2}d\tau \\ &\le f(\kappa_1)+f(\kappa_2)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{12}$$

for $x, y \in [\kappa_1, \kappa_2]$.

Inspired by these ongoing studies, we will then establish modified versions of inequalities (11) and (12) for harmonically convex functions because we discovered some flaws in the proof of inequality (11). We will also prove some new Hermite–Hadamard–Mercer-type inequalities for differentiable harmonically convex functions.

3. Main Results

For harmonically convex functions and differentiable harmonically convex functions, we will prove Hermite–Hadamard–Mercer-type inequalities in this section.

Theorem 7. *For a harmonically convex mapping $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ with $\kappa_1, \kappa_2 \in I$ and $\kappa_1 < \kappa_2$, the following inequality holds:*

$$\begin{aligned} f\left(\frac{1}{\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2} - \frac{1}{2}\left(\frac{x+y}{xy}\right)}\right) &\leq \frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}} \frac{f(\tau)}{\tau^2}d\tau \\ &\leq \frac{f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right) + f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)}{2} \\ &\leq f(\kappa_1) + f(\kappa_2) - \frac{f(x)+f(y)}{2} \end{aligned} \tag{13}$$

for $x, y \in [\kappa_1, \kappa_2]$.

Proof. Since the given mapping f is harmonically convex, we have

$$f\left(\frac{1}{\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2} - \frac{1}{2}\left(\frac{x_1+y_1}{x_1y_1}\right)}\right) \leq \frac{1}{2}\left[f\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y_1}}\right) + f\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x_1}}\right)\right]. \tag{14}$$

By setting

$$\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x_1} = \tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right) + (1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)$$

and

$$\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y_1} = (1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right) + \tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right),$$

inequality (14) becomes

$$\begin{aligned} f\left(\frac{1}{\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2} - \frac{1}{2}\left(\frac{x+y}{xy}\right)}\right) &\leq \frac{1}{2}\left[f\left(\frac{1}{\tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right) + (1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)}\right)\right. \\ &\quad \left. + f\left(\frac{1}{(1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right) + \tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)}\right)\right]. \end{aligned} \tag{15}$$

Integrating inequality (15) with respect to τ over an interval $[0, 1]$, we have

$$\begin{aligned} f\left(\frac{1}{\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2} - \frac{1}{2}\left(\frac{x+y}{xy}\right)}\right) &\leq \frac{1}{2}\left[\int_0^1 f\left(\frac{1}{\tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right) + (1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)}\right)d\tau\right. \\ &\quad \left. + \int_0^1 f\left(\frac{1}{(1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right) + \tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)}\right)d\tau\right]. \end{aligned} \tag{16}$$

Thus, we obtain the first inequality of (13) because each integral on the right side of (16) is equal to $\frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}} \frac{f(\tau)}{\tau^2}d\tau$. To prove the second inequality in (13) through the harmonic convexity function of f, we have the following:

$$\begin{aligned} & f\left(\frac{1}{\tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)+(1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)}\right) \\ \leq\ & \tau f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+(1-\tau)f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right) \end{aligned} \tag{17}$$

and

$$\begin{aligned} & f\left(\frac{1}{(1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)+\tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)}\right) \\ \leq\ & (1-\tau)f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+\tau f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right). \end{aligned} \tag{18}$$

By adding (17) and (18) and using inequality (10), we have

$$\begin{aligned} & f\left(\frac{1}{\tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)+(1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)}\right) \\ & +f\left(\frac{1}{(1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)+\tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)}\right) \\ \leq\ & f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)\leq 2[f(\kappa_1)+f(\kappa_2)]-[f(x)+f(y)]. \end{aligned} \tag{19}$$

Integrating inequality (19) with respect to τ over an interval $[0,1]$, we have

$$\begin{aligned} & \int_0^1 f\left(\frac{1}{\tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)+(1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)}\right)d\tau \\ & +\int_0^1 f\left(\frac{1}{(1-\tau)\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)+\tau\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)}\right)d\tau \\ \leq\ & f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)\leq 2[f(\kappa_1)+f(\kappa_2)]-[f(x)+f(y)]. \end{aligned} \tag{20}$$

Hence, we obtain the last inequality of (13). □

Remark 1. *In Theorem 7, if we set $x=\kappa_1$ and $y=\kappa_2$, then inequality (13) is reduced to inequality (8).*

The simple lemma below is needed to discover some new Hermite–Hadamard–Mercer-type inequalities for functions whose first derivatives are harmonically convex.

Lemma 2. *For a differentiable mapping $f: I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ on I° with $\kappa_1,\kappa_2\in I$ and $\kappa_1<\kappa_2$, the following equality holds:*

$$\begin{aligned} & \frac{f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)}{2}-\frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}}\frac{f(\tau)}{\tau^2}d\tau \\ =\ & \frac{y-x}{2xy}\int_0^1\frac{1-2\tau}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}f'\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)}\right)d\tau \end{aligned} \tag{21}$$

for $x,y\in[\kappa_1,\kappa_2]$.

Proof. Using the basic rules of integration, we have

$$\begin{aligned} & \int_0^1 \frac{1-2\tau}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2} f'\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)}\right) d\tau \\ = & -\frac{xy(1-2\tau)}{y-x} f\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)}\right)\Bigg|_0^1 - \frac{2xy}{y-x}\int_0^1 f\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)}\right) d\tau \\ = & \frac{xy}{y-x}\left[f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)\right] \\ & -\frac{2x^2y^2}{(y-x)^2}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}} \frac{f(\tau)}{\tau^2} d\tau. \end{aligned} \tag{22}$$

Thus, we obtain the resultant equality (21) by multiplying the equality (22) with $\frac{y-x}{2xy}$. □

Remark 2. *In Lemma 2, if we put $x = \kappa_1$ and $y = \kappa_2$, then equality (21) becomes equality (9).*

Now, for the sake of brevity, we shall use the following notations:

$$L = \frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x} \text{ and } M = \frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}.$$

Theorem 8. *The conditions of Lemma 2 are assumed to be true. The following inequality holds if the mapping $|f'|^q$, $q \geq 1$ is harmonically convex on I:*

$$\begin{aligned} & \left|\frac{f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)}{2} - \frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}} \frac{f(\tau)}{\tau^2} d\tau\right| \\ \leq & \frac{y-x}{2xy}\Lambda_1^{1-\frac{1}{q}}\left(\Lambda_1\left[\left|f'(\kappa_1)\right|^q+\left|f'(\kappa_2)\right|^q\right]-\left[\Lambda_2\left|f'(y)\right|^q+\Lambda_3\left|f'(x)\right|^q\right]\right)^{\frac{1}{q}}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \Lambda_1(L,M) &= \frac{1}{LM}-\frac{2}{(L-M)^2}\ln\left(\frac{(L+M)^2}{4LM}\right), \\ \Lambda_2(L,M) &= -\frac{1}{M(M-L)}+\frac{3L+M}{(M-L)^3}\ln\left(\frac{(L+M)^2}{4LM}\right) \end{aligned}$$

and

$$\Lambda_3(L,M) = \Lambda_1(L,M)-\Lambda_2(L,M).$$

Proof. We can deduce from Lemma 2 and the power mean inequality that:

$$\begin{aligned} & \left|\frac{f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)}{2} - \frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}} \frac{f(\tau)}{\tau^2} d\tau\right| \\ \leq & \frac{y-x}{2xy}\int_0^1 \left|\frac{1-2\tau}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}\right| \left|f'\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)}\right)\right| d\tau \\ \leq & \frac{y-x}{2xy}\left(\int_0^1 \left|\frac{1-2\tau}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}\right| d\tau\right)^{1-\frac{1}{q}} \end{aligned} \tag{24}$$

$$\times\left(\int_0^1\left|\frac{1-2\tau}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}\right|\left|f'\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)}\right)\right|^q d\tau\right)^{\frac{1}{q}}.$$

By inequality (10), we have the following:

$$\begin{aligned}
&\left|\frac{f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)}{2}-\frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}}\frac{f(\tau)}{\tau^2}d\tau\right| \\
\leq\ & \frac{y-x}{2xy}\left(\int_0^1\left|\frac{1-2\tau}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}\right|d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1-2\tau}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}\right|\right. \\
&\left.\times\left[\left|f'(\kappa_1)\right|^q+\left|f'(\kappa_2)\right|^q-\tau\left|f'(y)\right|^q-(1-\tau)\left|f'(x)\right|^q\right]d\tau\right)^{\frac{1}{q}} \\
=\ & \frac{y-x}{2xy}\Lambda_1^{1-\frac{1}{q}}\left(\Lambda_1\left[\left|f'(\kappa_1)\right|^q+\left|f'(\kappa_2)\right|^q\right]-\left[\Lambda_2\left|f'(y)\right|^q+\Lambda_3\left|f'(x)\right|^q\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

It is simple to verify this:

$$\begin{aligned}
\int_0^1\frac{|1-2\tau|}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}d\tau &= \int_0^1\frac{|1-2\tau|}{(\tau M+(1-\tau)L)^2}d\tau \\
&= \frac{1}{LM}-\frac{2}{(L-M)^2}\ln\left(\frac{(L+M)^2}{4LM}\right), \\
\int_0^1\frac{\tau|1-2\tau|}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}d\tau &= \int_0^1\frac{\tau|1-2\tau|}{(\tau M+(1-\tau)L)^2}d\tau \\
&= -\frac{1}{M(M-L)}+\frac{3L+M}{(M-L)^3}\ln\left(\frac{(L+M)^2}{4LM}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\frac{(1-\tau)|1-2\tau|}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}d\tau \\
=\ & \int_0^1\frac{|1-2\tau|}{(\tau M+(1-\tau)L)^2}d\tau-\int_0^1\frac{\tau|1-2\tau|}{(\tau M+(1-\tau)L)^2}d\tau \\
=\ & \frac{1}{LM}-\frac{2}{(L-M)^2}\ln\left(\frac{(L+M)^2}{4LM}\right)-\left(-\frac{1}{M(M-L)}+\frac{3L+M}{(M-L)^3}\ln\left(\frac{(L+M)^2}{4LM}\right)\right).
\end{aligned}$$

□

Remark 3. *In Theorem 8, if we set $x=\kappa_1$ and $y=\kappa_2$, then Theorem 8 becomes ([8], Theorem 2.6).*

Theorem 9. *The conditions of Lemma 2 are assumed to be true. The following inequality holds if the mapping $|f'|^q$, $q>1$ is harmonically convex on I:*

$$\begin{aligned}
&\left|\frac{f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)}{2}-\frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}}\frac{f(\tau)}{\tau^2}d\tau\right| \\
\leq\ & \frac{y-x}{2xy}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\Pi_1\left[\left|f'(\kappa_1)\right|^q+\left|f'(\kappa_2)\right|^q\right]-\left[\Pi_2\left|f'(y)\right|^q+\Pi_3\left|f'(x)\right|^q\right]\right)^{\frac{1}{q}},
\end{aligned}\tag{25}$$

where

$$\begin{aligned}\Pi_1 &= \frac{M^{1-2q}-L^{1-2q}}{(2q-1)(L-M)},\\ \Pi_2 &= \frac{L^{2-2q}+M^{1-2q}[(M-L)(1-2q)-L]}{2(M-L)^2(1-q)(1-2q)}\end{aligned}$$

and

$$\Pi_2 = \frac{M^{2-2q}+L^{1-2q}[(M-L)(1-2q)+M]}{2(M-L)^2(1-q)(1-2q)}.$$

Proof. We can deduce from Lemma 2 and Hölder's inequality that:

$$\begin{aligned}&\left|\frac{f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)}{2}-\frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}}\frac{f(\tau)}{\tau^2}d\tau\right| \qquad (26)\\ \leq\ &\frac{y-x}{2xy}\int_0^1\left|\frac{1-2\tau}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^2}\right|\left|f'\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)}\right)\right|d\tau\\ \leq\ &\frac{y-x}{2xy}\left(\int_0^1|1-2\tau|^p d\tau\right)^{\frac{1}{p}}\\ &\times\left(\int_0^1\frac{1}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^{2q}}\left|f'\left(\frac{1}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)}\right)\right|^q d\tau\right)^{\frac{1}{q}}.\end{aligned}$$

By inequality (10), we have the following:

$$\begin{aligned}&\left|\frac{f\left(\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}\right)+f\left(\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}\right)}{2}-\frac{xy}{y-x}\int_{\frac{\kappa_1\kappa_2 y}{\kappa_2 y+\kappa_1 y-\kappa_1\kappa_2}}^{\frac{\kappa_1\kappa_2 x}{\kappa_2 x+\kappa_1 x-\kappa_1\kappa_2}}\frac{f(\tau)}{\tau^2}d\tau\right|\\ \leq\ &\frac{y-x}{2xy}\left(\int_0^1|1-2\tau|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^{2q}}\right.\\ &\left.\times\left[\left|f'(\kappa_1)\right|^q+\left|f'(\kappa_2)\right|^q-\tau\left|f'(y)\right|^q-(1-\tau)\left|f'(x)\right|^q\right]d\tau\right)^{\frac{1}{q}}\\ =\ &\frac{y-x}{2xy}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\Pi_1\left[\left|f'(\kappa_1)\right|^q+\left|f'(\kappa_2)\right|^q\right]-\left[\Pi_2\left|f'(y)\right|^q+\Pi_3\left|f'(x)\right|^q\right]\right)^{\frac{1}{q}}.\end{aligned}$$

It is simple to verify this:

$$\begin{aligned}\int_0^1\frac{1}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^{2q}}d\tau &= \frac{M^{1-2q}-L^{1-2q}}{(2q-1)(L-M)},\\ \int_0^1\frac{\tau|1-2\tau|}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^{2q}}d\tau &= \frac{L^{2-2q}+M^{1-2q}[(M-L)(1-2q)-L]}{2(M-L)^2(1-q)(1-2q)}\end{aligned}$$

and

$$\int_0^1\frac{(1-\tau)|1-2\tau|}{\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\left(\frac{\tau}{y}+\frac{1-\tau}{x}\right)\right)^{2q}}d\tau = \frac{M^{2-2q}+L^{1-2q}[(M-L)(1-2q)+M]}{2(M-L)^2(1-q)(1-2q)}.$$

□

Remark 4. *In Theorem 9, if we choose $x = \kappa_1$ and $y = \kappa_2$, then Theorem 9 is reduced to ([8], Theorem 2.7).*

4. Application to Special Means

For arbitrary positive numbers κ_1, κ_2 $(\kappa_1 \neq \kappa_2)$, we consider the means as follows:

1. The arithmetic mean
$$\mathcal{A} = \mathcal{A}(\kappa_1, \kappa_2) = \frac{\kappa_1 + \kappa_2}{2};$$
2. The geometric mean
$$\mathcal{G} = \mathcal{G}(\kappa_1, \kappa_2) = \sqrt{\kappa_1 \kappa_2};$$
3. The harmonic mean
$$\mathcal{H} = \mathcal{H}(\kappa_1, \kappa_2) = \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2};$$
4. The logarithmic mean
$$\mathcal{L} = \mathcal{L}(\kappa_1, \kappa_2) = \frac{\kappa_2 - \kappa_1}{\ln \kappa_2 - \ln \kappa_1};$$
5. The generalized logarithmic mean
$$\mathcal{L}_p = \mathcal{L}_p(\kappa_1, \kappa_2) = \left[\frac{\kappa_2^{p+1} - \kappa_1^{p+1}}{(\kappa_2 - \kappa_1)(p+1)} \right]^{\frac{1}{p}}, \ p \in \mathbb{R} \backslash \{-1, 0\};$$
6. The identric mean
$$\mathcal{I} = \mathcal{I}(\kappa_1, \kappa_2) = \begin{cases} \frac{1}{e} \left(\frac{\kappa_2^{\kappa_2}}{\kappa_1^{\kappa_1}} \right)^{\frac{1}{\kappa_2 - \kappa_1}}, & \text{if } \kappa_1 \neq \kappa_2, \ \kappa_1, \kappa_2 > 0. \\ \kappa_1, & \text{if } \kappa_1 = \kappa_2, \end{cases}$$

These means are often employed in numerical approximations and other fields. However, the following straightforward relationship has been stated in the literature:

$$\mathcal{H} \leq \mathcal{G} \leq \mathcal{L} \leq \mathcal{I} \leq \mathcal{A}.$$

Proposition 1. *For $\kappa_1, \kappa_2 \in (0, \infty)$, the following inequality is true:*

$$\begin{aligned} \frac{1}{2\mathcal{H}^{-1}(\kappa_1, \kappa_2) - \mathcal{H}^{-1}(x, y)} &\leq \frac{1}{\mathcal{L}\left(\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2} - \frac{1}{y} \right)^{-1}, \left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2} - \frac{1}{x} \right)^{-1} \right)} \\ &\leq \mathcal{A}\left(\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2} - \frac{1}{y} \right)^{-1}, \left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2} - \frac{1}{x} \right)^{-1} \right) \\ &\leq 2\mathcal{A}(\kappa_1, \kappa_2) - \mathcal{A}(x, y). \end{aligned}$$

Proof. Inequality (13) in Theorem 7 for the mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$ leads to this conclusion. □

Proposition 2. *For $\kappa_1, \kappa_2 \in (0, \infty)$, the following inequality is true:*

$$\frac{1}{\left(2\mathcal{H}^{-1}(\kappa_1, \kappa_2) - \mathcal{H}^{-1}(x, y)\right)^2} \leq \mathcal{G}^{-1}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2} - \frac{1}{y}, \frac{1}{\kappa_1} + \frac{1}{\kappa_2} - \frac{1}{x} \right)$$

$$\leq \mathcal{A}\left(\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)^{-2},\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)^{-2}\right)$$
$$\leq 2\mathcal{A}\left(\kappa_1^2,\kappa_2^2\right)-\mathcal{A}\left(x^2,y^2\right).$$

Proof. Inequality (13) in Theorem 7 for the mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^2$ leads to this conclusion. □

Proposition 3. *For $\kappa_1,\kappa_2\in(0,\infty)$ and $p\in(-1,\infty)\backslash\{0\}$, the following inequality is true:*

$$\frac{1}{\left(2\mathcal{H}^{-1}(\kappa_1,\kappa_2)-\mathcal{H}^{-1}(x,y)\right)^{p+2}} \leq \mathcal{L}_p^p\left(\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)^{-1},\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)^{-1}\right)$$
$$\leq \mathcal{A}\left(\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{y}\right)^{-(p+2)},\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}-\frac{1}{x}\right)^{-(p+2)}\right)$$
$$\leq 2\mathcal{A}\left(\kappa_1^{p+2},\kappa_2^{p+2}\right)-\mathcal{A}\left(x^{p+2},y^{p+2}\right).$$

Proof. Inequality (13) in Theorem 7 for the mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^{p+2}$ leads to this conclusion. □

5. Conclusions

In this paper, we proved some new Hermite–Hadamard–Mercer inequalities for harmonically convex functions and differentiable harmonically convex functions. It was also demonstrated that the results of this paper generalize the findings of İşcan in [8]. It is an interesting and challenging problem, and researchers may be able to obtain similar inequalities for various fractional operators in their future work.

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