mathematics

MDPI

Article

The Pauli Problem for Gaussian Quantum States: Geometric Interpretation

Maurice A. de Gosson

Faculty of Mathematics (NuHAG), University of Vienna, 1090 Vienna, Austria; maurice.de.gosson@univie.ac.at

Abstract: We solve the Pauli tomography problem for Gaussian signals using the notion of Schur complement. We relate our results and method to a notion from convex geometry, polar duality. In our context polar duality can be seen as a sort of geometric Fourier transform and allows a geometric interpretation of the uncertainty principle and allows to apprehend the Pauli problem in a rather simple way.

Keywords: covriance matrix; polar duality; uncertainty principle; reconstruction problem

check for updates

Citation: de Gosson, M.A. The Pauli Problem for Gaussian Quantum States: Geometric Interpretation. *Mathematics* **2021**, *9*, 2578. https://doi.org/10.3390/math9202578

Academic Editor: Rami Ahmad El-Nabulsi

Received: 29 August 2021
Accepted: 3 October 2021
Published: 14 October 2021

1. The Pauli Problem and Quantum Tomography

The problem goes back to Pauli's question [1]:

The mathematical problem as to whether, for given probability densities $W(p)$ and $W(x)$, wave function ψ (...) is always uniquely determined, has still not been investigated in its generality.

The answer to Pauli's question is negative [2]; there is a general nonuniqueness of the solution (for a detailed discussion of the Pauli problem and its applications, see [3]). The problem can actually be formulated as from statistical quantum mechanics as follows: can we estimate the density matrix of the said state using repeated measurements on identical quantum systems? After having obtained measurements on these identical systems, can we make a statistical inference about their probability distributions (e.g., [4])? Such a procedure is an instance of quantum state tomography, and is practically implemented using a set of measurements of a so-called quorum of observables. It can be performed using various mathematical techniques, for instance the Radon–Wigner transform that we discussed in [5]; the latter has important applications in medical imaging [6]. For details and explicit constructions, see [7–14], and [15] by Man'ko and Man'ko.

Remark 1. *Everything in this paper extends mutatis mutandis to time-frequency analysis, replacing the notion of wave function by that of a signal. In this case, one takes $\hbar = 1/2\pi$ and replaces phase-space variables (x,p) with time-frequency variables (x,ω).*

2. A Simple Example

Let us discuss the Pauli problem on the simplest possible example, that of a Gaussian wave function in one spatial dimension. Assuming for simplicity, it is centered at the origin and is given by formula

$$\psi(x) = \left(\frac{1}{2\pi\sigma_{xx}}\right)^{1/4} e^{-\frac{x^2}{4\sigma_{xx}}} e^{\frac{i\sigma_{xp}}{2\hbar\sigma_{xx}}x^2} \tag{1}$$

where σ_{xx} is the variance in the position variable, and σ_{xp} the covariance in the position and momentum variables. Fourier transform

$$\widehat{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}px}\psi(x)dx$$

of the ψ is explicitly given by

$$\widehat{\psi}(p) = \left(\frac{1}{2\pi\sigma_{pp}}\right)^{1/4} e^{-\frac{p^2}{4\sigma_{pp}}} e^{-\frac{i\sigma_{xp}}{2\hbar\sigma_{pp}}p^2} \tag{2}$$

hence, the knowledge of σ_{xx} and of σ_{pp}, that is, of moduli $|\psi(x)|^2$ and $|\widehat{\psi}(p)|^2$, determines covariance σ_{xp} up to a sign because state ψ saturates the Robertson–Schrödinger inequality; so, we have

$$\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \tfrac{1}{4}\hbar^2 \tag{3}$$

This identity can be solved in σ_{xp} yielding $\sigma_{xp} = \pm(\sigma_{xx}\sigma_{pp} - \frac{1}{4}\hbar^2)^{1/2}$. The state and its Fourier transform are given by formulas

$$\psi_\pm(x) = \left(\frac{1}{2\pi\sigma_{xx}}\right)^{1/4} e^{-\frac{x^2}{4\sigma_{xx}}} e^{\pm\frac{i\sigma_{xp}}{2\hbar\sigma_{xx}}x^2} \tag{4}$$

and

$$\widehat{\psi}_\pm(p) = \left(\frac{1}{2\pi\sigma_{pp}}\right)^{1/4} e^{-\frac{p^2}{4\sigma_{pp}}} e^{\mp\frac{i\sigma_{xp}}{2\hbar\sigma_{pp}}p^2} . \tag{5}$$

Both functions ψ_+ and $\psi_- = \psi_+^*$ and their Fourier transforms $\widehat{\psi}_+$ and $\widehat{\psi}_-$ satisfy conditions $|\psi_+(x)|^2 = |\psi_-(x)|^2$ and $|\widehat{\psi}_+(p)|^2 = |\widehat{\psi}_-(p)|^2$ showing that the Pauli problem does not have a unique solution. In Corbett's [16] terminology ψ_+ and ψ_- are "Pauli partners". Let us now have a look at these things from the perspective of the Wigner transform

$$W\psi(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}py}\psi(x+\tfrac{1}{2}y)\psi^*(x-\tfrac{1}{2}y)dy$$

of Gaussian ψ. A straightforward calculation involving Gaussian integrals [17] yields, setting $z = (x,p)$, normal distribution

$$W\psi_\pm(z) = \frac{1}{2\pi\sqrt{\det\Sigma_\pm}} e^{-\frac{1}{2}\Sigma_\pm^{-1}z\cdot z} \tag{6}$$

where covariance matrix

$$\Sigma_\pm = \begin{pmatrix} \sigma_{xx} & \pm\sigma_{xp} \\ \pm\sigma_{px} & \sigma_{pp} \end{pmatrix}$$

has determinant $\det\Sigma_\pm = \frac{1}{4}\hbar^2$ in view of equality (3); hence,

$$W\psi_\pm(z) = \frac{1}{\pi\hbar} e^{-\frac{1}{2}\Sigma_\pm^{-1}z\cdot z} . \tag{7}$$

Associated covariance matrices are thus

$$\Omega_\pm = \{z : \frac{1}{2}\Sigma_\pm^{-1}z\cdot z \leq 1 .\}$$

3. Multivariate Case: Asking the Right Questions

We generalize the discussion to the multivariate case where the real variables x and p are replaced with real vectors $x = (x_1, ..., x_n)$, $p = (p_1, ..., p_n)$.

The Wigner function cannot be directly measured, but its marginal distributions can (they are classical probability densities). In analogy with Formula (6) we determine a (centered) Gaussian, ψ such that

$$W\psi(z) = \left(\frac{1}{2\pi}\right)^n \frac{1}{\sqrt{\det\Sigma}} e^{-\frac{1}{2}\Sigma^{-1}z\cdot z} \tag{8}$$

where $z = (x, p)$, and the covariance matrix is

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix} \ , \ \Sigma_{PX} = \Sigma_{XP}^T \, . \tag{9}$$

Here, the the n-dimensional Wigner transform $W\psi$ is defined by

$$W\psi(x,p) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}p\cdot y}\psi(x+\tfrac{1}{2}y)\psi^*(x-\tfrac{1}{2}y)d^n y \, .$$

The most straightforward way to determine this state is to use the properties of the Wigner transform itself. Let us start with the marginal properties [17]:

$$\int W\psi(x,p)d^n p = |\psi(x)|^2 \tag{10}$$

$$\int W\psi(x,p)d^n x = |\widehat{\psi}(p)|^2 \tag{11}$$

where the n-dimensional Fourier transform $\widehat{\psi}$ is given by

$$\widehat{\psi}(p) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \int e^{-\frac{i}{\hbar}px}\psi(x)d^n x \, .$$

These formulas hold as soon as both ψ and $\widehat{\psi}$ are in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ [17]. These quantities allow for determining matrices

$$\Sigma_{XX} = (\sigma_{x_j x_k})_{1\leq j,k\leq n} \text{ and } \Sigma_{PP} = (\sigma_{p_j p_k})_{1\leq j,k\leq n}$$

by usual formulas

$$\sigma_{x_j x_k} = \int x_j x_k |\psi(x)|^2 d^n x \ , \ \sigma_{p_j p_k} = \int p_j p_k |\widehat{\psi}(p)|^2 d^n p$$

and an elementary calculation of Gaussian integrals yields the values

$$|\psi(x)| = \left(\tfrac{1}{2\pi}\right)^{n/4} (\det \Sigma_{XX})^{-1/4} e^{-\frac{1}{4}\Sigma_{XX}^{-1}x\cdot x} \tag{12}$$

$$|\widehat{\psi}(p)| = \left(\tfrac{1}{2\pi}\right)^{n/4} (\det \Sigma_{PP})^{-1/4} e^{-\frac{1}{4}\Sigma_{PP}^{-1}p\cdot p} \, . \tag{13}$$

Here, we are exactly in the situation discussed by Pauli: $|\psi(x)|$ and $|\widehat{\psi}(p)|$ are what we can measure, so we can determine covariance blocks Σ_{XX} and Σ_{PP}, but not covariance Σ_{XP}: knowledge of the latter (and hence of $\Sigma_{PX} = \Sigma_{XP}^T$) is necessary to entirely determine state ψ. In the previous section, the problem was solved: in case $n = 1$, blocks Σ_{XX}, Σ_{PP}, and Σ_{XP} were scalars σ_{xx}, σ_{pp}, and σ_{xp}, and these are related by the uncertainty principle in the form of $\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \frac{1}{4}\hbar^2$ yielding two possible values $\sigma_{xp} = \pm(\sigma_{xx}\sigma_{pp} - \frac{1}{4}\hbar^2)^{1/2}$, and hence the two states (5). In the multidimensional, case we also have a simple (but not immediately obvious) formula connecting the blocks of the covariance matrix. The way out of this problem consists in using a general formula [17–19], which was initially proved by Bastiaans [20] in connection with first-order optics. Let X and Y be real $n \times n$ matrices, such that $X = X^T > 0$ and $Y = Y^T$, and set

$$\psi_{X,Y}(x) = \left(\tfrac{1}{\pi\hbar}\right)^{n/4} (\det X)^{1/4} e^{-\frac{1}{2\hbar}(X+iY)x\cdot x} \, . \tag{14}$$

This function is normalized to unity: $||\psi_{X,Y}||_{L^2} = 1$, and its Wigner transform is given by

$$W\psi_{X,Y}(z) = \left(\tfrac{1}{\pi\hbar}\right)^n e^{-\frac{1}{\hbar}Gz\cdot z} \tag{15}$$

where G is the symmetric matrix

$$G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix} . \tag{16}$$

A fundamental fact, which is related to the uncertainty principle, is that G is a symplectic matrix, i.e., it belongs to symplectic group $\mathrm{Sp}(n)$. Equivalently, since $G = G^T$,

$$G^T J G = GJG = J$$

where

$$J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}$$

the standard symplectic matrix. We have $G = S^T S$, where

$$S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \tag{17}$$

is clearly symplectic. Assuming that function ψ for which we are looking is a Gaussian, comparing Formulas (8) and (15) leads to identification

$$\Sigma = \frac{\hbar}{2} G^{-1} .$$

Since $GJG = J$ the inverse G^{-1} is $-JGJ$, explicit formula

$$G^{-1} = \begin{pmatrix} X^{-1} & -X^{-1}Y \\ -YX^{-1} & X + YX^{-1}Y, \end{pmatrix}$$

so that there remains to solve matrix equation

$$\begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} X^{-1} & -X^{-1}Y \\ -YX^{-1} & X + YX^{-1}Y \end{pmatrix} . \tag{18}$$

It immediately follows that we have $X = \frac{\hbar}{2}\Sigma_{XX}^{-1}$ and $Y = -\frac{2}{\hbar}\Sigma_{XP}\Sigma_{XX}^{-1}$, so the unknown Gaussian for which we were looking is

$$\psi(x) = \left(\tfrac{1}{2\pi}\right)^{n/4} (\det \Sigma_{XX})^{-1/4} \exp\left[-\left(\frac{1}{4}\Sigma_{XX}^{-1} + \frac{i}{2\hbar}\Sigma_{XP}\Sigma_{XX}^{-1}\right) x \cdot x\right], \tag{19}$$

which is the n-dimensional variant of (1), replacing σ_{xx} with Σ_{XX} and σ_{xp} with Σ_{XP}. This does not solve completely our problem, however, because we do not know matrix Σ_{XP}. The crucial step is to notice that, as a bonus, we obtained from (18) the matrix form of the saturated Robertson–Schrödinger equality, namely,

$$\Sigma_{PP}\Sigma_{XX} - \Sigma_{XP}^2 = \tfrac{1}{4}\hbar^2 I_{n\times n} . \tag{20}$$

From this formula we can deduce Σ_{XP}^2, and one finds two Pauli partners

$$\psi_\pm(x) = \left(\tfrac{1}{2\pi}\right)^{n/4} (\det \Sigma_{XX})^{-1/4} \exp\left[-\left(\frac{1}{4}\Sigma_{XX}^{-1} \pm \frac{i}{2\hbar}\Sigma_{XP}\Sigma_{XX}^{-1}\right) x \cdot x\right] \tag{21}$$

once a value of Σ_{XP} is determined (even if $\Sigma_{XP}^2 = 0$, we can have $\Sigma_{XP} \neq 0$). Here, we solved a so-called "phase retrieval problem" (see Klibanov et al. [21] for a good review of the topic): in view of Formula (12), we know that

$$\psi(x) = e^{i\Phi(x)} \left(\tfrac{1}{2\pi}\right)^{n/4} (\det \Sigma_{XX})^{-1/4} e^{-\frac{1}{4}\Sigma_{XX}^{-1} x\cdot x} \tag{22}$$

where Φ is an unknown real function of the position variable. We identified this phase here as being function

$$\Phi(x) = -\left(\frac{1}{2\hbar}\Sigma_{XP}\Sigma_{XX}^{-1}\right)x \cdot x \,.$$

4. Geometric Interlude

We introduce the notion of $\hbar$-polarity and duality; we see in the next section that this notion from convex geometry is quite unexpectedly related to the Pauli problem, of which it gives a limpid geometric interpretation. For a very detailed study of polarity, see Charalambos and Aliprantis [22]. In both sources, alternative competing definitions are also described; the one we use here is the most common and the best fitted to our needs.

Let X be a nonempty subset of n-dimensional configuration space $\mathbb{R}^n_x$; this may be, for instance, a set of position measurements performed on some physical system with n degrees of freedom. One defines the polar set of X as the set X^o of all points $p = (p_1, ..., p_n)$ in the momentum space $\mathbb{R}^n_p$, such that

$$px = p_1 x_1 + \cdots + p_n x_n \leq 1$$

for all points $x = (x_1, ..., x_n)$ in X. Similarly, if P is a subset of $\mathbb{R}^n_p$, one defines its polar P^o as the set of all x in $\mathbb{R}^n_x$, such that $px \leq 1$ for all p in P. We use a rescaled variant of the notion of polarity here, which we call $\hbar$ polarity. By definition, the $\hbar$-polar $X^\hbar$ of X is the set of all p, such that

$$px = p_1 x_1 + \cdots + p_n x_n \leq \hbar$$

for all points x in X. We have $X^\hbar = \hbar X^o$ and $P^\hbar = \hbar P^o$ likewise.

From now on, we assume for simplicity that X and P are convex bodies, i.e., they are convex, compact, and with a nonempty interior; we also assume that they are symmetric (i.e., $X = -X$), which implies, by convexity, that they contain 0 in their interior. Simple examples of such sets are balls and ellipsoids centered at the origin. Polar duals have the following remarkable properties:

- *Biduality*: $(X^\hbar)^\hbar = X$
- *Antimonotonicity:* $X \subset Y \Longrightarrow Y^\hbar \subset X^\hbar$
- *Scaling property*: $L \in GL(n, \mathbb{R}) \Longrightarrow (LX)^\hbar = (L^T)^{-1}X^\hbar$.

Let $\mathcal{B}^n_X(R)$ (resp. $\mathcal{B}^n_P(R)$) be the ball $\{x : |x| \leq R\}$ in $\mathbb{R}^n_x$ (resp. $\{p : |p| \leq R\}$ in $\mathbb{R}^n_p$). We have

$$\mathcal{B}^n_X(\sqrt{\hbar})^\hbar = \mathcal{B}^n_P(\sqrt{\hbar}) \tag{23}$$

and one can show that $\mathcal{B}^n_X(\sqrt{\hbar})$ is the only self $\hbar$-dual set in $\mathbb{R}^n_x$. Let us extend this to the case of ellipsoids. An ellipsoid in $\mathbb{R}^n_x$ centered at the origin (which is just an ordinary plane ellipse when $n = 1$) can always be viewed as the image of ball $\mathcal{B}^n_X(\sqrt{\hbar})$ by some invertible linear transformation L, in which case, it is given by inequality

$$L^{-1}x \cdot L^{-1}x = (LL^T)^{-1}x \cdot x \leq \hbar \,.$$

Conversely, if A is a positive definite symmetric matrix, inequality $Ax \cdot x \leq \hbar$ always defines an ellipsoid, since it is equivalent to the above inequality, taking for L inverse square root $A^{-1/2}$ of A. It immediately follows from the scaling property that the $\hbar$-polar of the ellipsoid is obtained by inverting the matrix of the ellipsoid:

$$X : Ax^2 \leq \hbar \Longleftrightarrow X^\hbar : A^{-1}p \cdot p \leq \hbar \tag{24}$$

(that we have an equivalence follows from biduality property $(X^\hbar)^\hbar = X$).

5. The Pauli Problem and Polar Duality

Let us return to the Wigner transform of Gaussian states; using Formula (15), we can explicitly calculate $W\psi_{\pm}$, and one finds

$$W\psi_{\pm}(z) = (\pi\hbar)^{-n} e^{-\frac{1}{2}\Sigma_{\pm}^{-1} z \cdot z}$$

where covariance matrices $\Sigma_{\pm}$ are given by

$$\Sigma_{\pm} = \begin{pmatrix} \Sigma_{XX} & \pm\Sigma_{XP} \\ \pm\Sigma_{PX} & \Sigma_{PP} \end{pmatrix},$$

with $\Sigma_{PX} = \Sigma_{XP}^T$. Two ellipsoids $\Omega_{\pm}$ centered at the origin correspond to $\Sigma_{\pm}$. Let us determine orthogonal projections $\Omega_{X,\pm}$ and $\Omega_{P\pm}$ of $\Omega_{\pm}$ on the position and momentum spaces $\mathbb{R}_x^n$ and $\mathbb{R}_p^n$.

5.1. Case $n = 1$

We begin with case $n = 1$, and projections are line segments. Here, $\Sigma_{XX} = \sigma_{xx}$, $\Sigma_{PP} = \sigma_{pp}$, and $\Sigma_{XP} = \Sigma_{PX} = \sigma_{xp}$ and covariance ellipses $\Omega_{\pm}$ are defined by

$$\frac{\sigma_{pp}}{2D} x^2 \mp \frac{\sigma_{xp}}{D} px + \frac{\sigma_{xx}}{2D} p^2 \leq 1 \tag{25}$$

where $D = \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \frac{1}{4}\hbar^2$ (cf. Formula (3)). Orthogonal projections $\Omega_{X,\pm}$ and $\Omega_{P\pm}$ of $\Omega_{\pm}$ on the x and p axes are the same:

$$\Omega_X = [-\sqrt{2\sigma_{xx}}, \sqrt{2\sigma_{xx}}] \ , \ \ \Omega_P = [-\sqrt{2\sigma_{pp}}, \sqrt{2\sigma_{pp}}] \,. \tag{26}$$

Let $\Omega_X^\hbar$ be the polar dual of Ω_X: it is the set of all numbers p, such that $px \leq \hbar$ for $-\sqrt{2\sigma_{xx}} \leq x \leq \sqrt{2\sigma_{xx}}$ and is thus the interval

$$\Omega_X^\hbar = [-\hbar/\sqrt{2\sigma_{xx}}, \hbar/\sqrt{2\sigma_{xx}}] \,.$$

Since $\sigma_{xx}\sigma_{pp} \geq \frac{1}{2}\hbar$, we have inclusion

$$\Omega_X^\hbar \subset \Omega_P \tag{27}$$

and this inclusion reduces to equality $\Omega_X^\hbar = \Omega_P$ if and only if the Heisenberg inequality is saturated, i.e., $\sigma_{xx}\sigma_{pp} = \frac{1}{4}\hbar^2$, which is equivalent to $\sigma_{xp} = 0$.

5.2. General Case

We have similar properties in arbitrary dimension n. To study this case, we first must find the orthogonal projections of covariance ellipsoid Ω on the position and momentum spaces. Ellipsoid Ω is given by equation $Mz \cdot z \leq \hbar$ where $M = \frac{\hbar}{2}\Sigma^{-1}$ is symmetric and positive definite ($M > 0$). Writing M in block form

$$M = \begin{pmatrix} M_{XX} & M_{XP} \\ M_{PX} & M_{PP} \end{pmatrix}$$

where $M_{XX} = M_{XX}^T$, $M_{PP} = M_{PP}^T$, and $M_{XP} = M_{XP}^T$ are $n \times n$ matrices; since $M > 0$, we also have $M_{XX} > 0$ and $M_{PP} > 0$. Then, the projections of Ω on $\mathbb{R}_x^n$ and $\mathbb{R}_p^n$ are ellipsoids given by, respectively [23],

$$\Omega_X : (M/M_{PP})x \cdot x \leq \hbar\} \ , \ \ \Omega_P = (M/M_{XX})p \cdot p \leq \hbar \tag{28}$$

where symmetric matrices

$$M/M_{PP} = M_{XX} - M_{XP}M_{PP}^{-1}M_{PX} \tag{29}$$

$$M/M_{XX} = M_{PP} - M_{PX}M_{XX}^{-1}M_{XP} \tag{30}$$

are Schur complements in M of M_{PP} and M_{XX}; we have $M/M_{PP} > 0$ and $M/M_{XX} > 0$ so that Ω_X and Ω_P are nondegenerate (see Zhang's treatise [24] for a detailed study of the Schur complement). To prove that inclusion $\Omega_X^\hbar \subset \Omega_P$ holds, we must show that cf. implication (24)) that

$$(M/M_{PP})(M/M_{XX}) \leq I_{n\times n}\ , \tag{31}$$

that is, that the eigenvalues of $(M/M_{PP})(M/M_{XX})$ must be smaller than 1. To prove this, we use the following essential remark: we showed above that matrix $M = \frac{\hbar}{2}\Sigma^{-1}$ is symplectic; therefore, its entries obey some constraints. Considering that M is also symmetric, these constraints are

$$M_{XX}M_{PP} - M_{XP}^2 = I_{n\times n} \tag{32}$$

$$M_{XX}M_{PX} = M_{XP}M_{XX} \tag{33}$$

$$M_{PX}M_{PP} = M_{PP}M_{XP}\ . \tag{34}$$

Using Identities (33) and (34), it follows that Schur complements (29) and (30) can be rewritten as

$$\begin{aligned} M/M_{PP} &= M_{XX} - M_{PP}^{-1}M_{PX}^2 \\ &= M_{PP}^{-1}(M_{PP}M_{XX} - M_{PX}^2) \\ &= M_{PP}^{-1} \end{aligned}$$

the last equality by using the transpose of Identity (32). Similarly,

$$M/M_{XX} = M_{PP} - M_{XX}^{-1}M_{XP}^2 = M_{XX}^{-1}$$

So, summarizing, Schur complements are given by

$$M/M_{PP} = M_{PP}^{-1}\ \ ,\ M/M_{XX} = M_{XX}^{-1}\ . \tag{35}$$

It follows that

$$(M/M_{PP})(M/M_{XX}) = M_{PP}^{-1}M_{XX}^{-1} = (M_{XX}M_{PP})^{-1}\ .$$

We show that $(M/M_{PP})(M/M_{XX}) \leq I_{n\times n}$; equivalently, $M_{XX}M_{PP} \geq I_{n\times n}$. Now, since $M = \frac{\hbar}{2}\Sigma^{-1}$ is symplectic, so is matrix

$$M^{-1} = \frac{2}{\hbar}\Sigma = \begin{pmatrix} \frac{2}{\hbar}\Sigma_{XX} & \frac{2}{\hbar}\Sigma_{XP} \\ \frac{2}{\hbar}\Sigma_{PX} & \frac{2}{\hbar}\Sigma_{PP}; \end{pmatrix}$$

hence, reinverting,

$$M = \begin{pmatrix} \frac{2}{\hbar}\Sigma_{PP} & -\frac{2}{\hbar}\Sigma_{PX} \\ -\frac{2}{\hbar}\Sigma_{XP} & \frac{2}{\hbar}\Sigma_{XX} \end{pmatrix} \tag{36}$$

so that $M_{XX}M_{PP} = \frac{4}{\hbar^2}\Sigma_{PP}\Sigma_{XX}$. In view of the generalized RSUP (20), we have

$$\Sigma_{PP}\Sigma_{XX} - \Sigma_{XP}^2 = \tfrac{1}{4}\hbar^2 I_{n\times n} \tag{37}$$

hence

$$M_{XX}M_{PP} = I_{n\times n} + \frac{4}{\hbar^2}\Sigma_{XP}^2 \tag{38}$$

and we are finished, provided that we can prove that $\Sigma_{XP}^2 \geq 0$ (which is obvious if $n = 1$), or, which amounts to the same $M_{XP}^2 \geq 0$. For this, since $M_{XX}M_{PX} = M_{XP}M_{XX}$ (Formula (33)), we have

$$M_{XP} = M_{XX}M_{PX}M_{XX}^{-1}; \tag{39}$$

hence, M_{XP} and M_{PX} have the same eigenvalues; since $M_{PX} = M_{XP}^T$, these eigenvalues must be real, and those of M_{XP}^2 must be ≥ 0.

For completeness, we still need to discuss what happens when $\Omega_X^\hbar = \Omega_P$. In view of Formulas (28) and Equivalence (24), this means that (31) reduces to equality

$$(M/M_{PP})(M/M_{XX}) = I_{n\times n}$$

that is, by (35), $M_{XX}M_{PP} = I_{n\times n}$. Taking (38) into account, we must thus have $M_{XP}^2 = 0$, which does not imply that $M_{XP} = 0$. We are in the presence of states (21) in this case, saturating the Heisenberg inequalities.

6. Discussion and Outlook

Our discussion of polar duality suggests that a quantum system localized in the position representation in a set X cannot be localized in the momentum representation in a set smaller than that of its polar dual $X^\hbar$. The notion of polar duality thus appears informally as a generalization of the uncertainty principle of quantum mechanics, as expressed in terms of variances and covariances (see [23]). The idea of such generalizations is not new, and can already be found in the work of Uffink and Hilgevoord [25,26]; see Butterfield's discussion in [27]. It would certainly be interesting to explore the connection between convex geometry and quantum mechanics, but very little work has been conducted so far.

Funding: This work was financed by grant P 33447 of Austrian Research Agency FWF. Open Access Funding by the Austrian Science Fund (FWF).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work has been financed by the Grant P 33447 of the Austrian Research Agency FWF. Open Access Funding by the Austrian Science Fund (FWF).

Conflicts of Interest: The author declares no conflict of interest.

References

1. Pauli, W. *General Principles of Quantum Mechanics*; Springer Science & Business Media: Berlin, Germany, 2012; [Original Title: *Prinzipien der Quantentheorie*, Handbuch der Physik, v.5.1, 1958].
2. Esposito, G.; Marmo, G.; Miele, G.; Sudarshan, G. *Advanced Concepts in Quantum Mechanics*; Cambridge University Press: Cambridge, UK, 2015.
3. Ibort, A.; Man'ko, V.I.; Marmo, G.; Simoni, A.; Ventriglia, F. An introduction to the tomographic picture of quantum mechanics. *Phys. Scr.* **2009**, *79*, 065013. [CrossRef]
4. Wang, Y.; Xu, C. Density matrix estimation in quantum homodyne tomography. *Stat. Sin.* **2015**, *25*, 953–973. [CrossRef]
5. de Gosson, M. Quantum Harmonic Analysis of the Density Matrix. *Quanta* **2018**, *7*, 74–110. [CrossRef]
6. Wood, J.C.; Barry, D.T. Tomographic time-frequency analysis and its application toward time-varying filtering and ada ptive kernel design for multicomponent linear-FM signals. *IEEE Trans. Signal Process.* **1994**, *42*, 2094–2104. [CrossRef]
7. D'Ariano, G.M. Universal quantum observables. *Phys. Lett. A* **2002**, *300*, 1–6. [CrossRef]
8. D'Ariano, G.M.; Macchiavello, C.; Paris, M.G.A. Detection of the density matrix through optical homodyne tomography without filtered back projection. *Phys. Rev. A* **1994**, *50*, 4298–4303. [CrossRef] [PubMed]
9. Bužek, V.; Adam, G.; Drobný, G. Reconstruction of Wigner Functions on Different Observation Levels. *Ann. Phys.* **1996**, *245*, 37–97. [CrossRef]
10. Lvovsky, A.I.; Raymer, M.G. Continuous-variable optical quantum-state tomography. *Rev. Mod. Phys.* **2009**, *8*, 299–332. [CrossRef]
11. Leonhardt, U.; Paul, H. Realistic optical homodyne measurements and quasiprobability distributions. *Phys. Rev. A* **1993**, *48*, 4598. [CrossRef] [PubMed]

12. Mancini, S.; Man'ko, V.I.; Tombesi, P. Symplectic tomography as classical approach to quantum systems. *Phys. Lett. A* **1996**, *213*, 1–6. [CrossRef]
13. Thekkadath, G.S.; Giner, L.; Chalich, Y.; Horton, M.J.; Banker, J.; Lundeen, J.S. Direct Measurement of the Density Matrix of a Quantum System. *Phys. Rev. Lett.* **2016**, *117*, 120401. [CrossRef] [PubMed]
14. Vogel, K.; Risken, H. Determination of quasiprobability distributions in terms of probability distributions for the rotated quadrature phase. *Phys. Rev. A* **1989**, *40*, 2847–2849. [CrossRef] [PubMed]
15. Man'ko, O.; Man'ko, V.I. Quantum states in probability representation and tomography. *J. Russ. Laser Res.* **1997**, *18*, 407–444. [CrossRef]
16. Corbett, J.V. The Pauli problem, state reconstruction and quantum-real numbers. *Rep. Math. Phys.* **2006**, *57*, 53–68. [CrossRef]
17. de Gosson, M. *The Wigner Transform*; Series Advanced Texts in Mathematics; World Scientific: Singapore, 2017.
18. de Gosson, M. *Symplectic Geometry and Quantum Mechanics*; Birkhäuser: Basel, Switzerland, 2006.
19. Littlejohn, R.G. The semiclassical evolution of wave packets. *Phys. Reps.* **1986**, *138*, 193–291. [CrossRef]
20. Bastiaans, M.J. Wigner distribution function and its application to first-order optics. *J. Opt. Soc. Am.* **1979**, *69*, 1710. [CrossRef]
21. Klibanov, M.V.; Sacks, P.E.; Tikhonravov, A.V. The phase retrieval problem. *Inverse Probl.* **1995**, *11*, 1–28. [CrossRef]
22. Charalambos, D.; Aliprantis, B. *Infinite Dimensional Analysis: A Hitchhiker's Guide*; Springer: Berlin/Heidelberg, Germany, 2013.
23. de Gosson, M. Quantum Polar Duality and the Symplectic Camel: A New Geometric Approach to Quantization. *Found. Phys.* **2021**, *51*, 60. [CrossRef]
24. Zhang, F. *The Schur Complement and Its Applications*; Springer: Berlin, Germany, 2005.
25. Hilgevoord, J. The standard deviation is not an adequate measure of quantum uncertainty. *Am. J. Phys.* **2002**, *70*, 983. [CrossRef]
26. Hilgevoord, J.; Uffink, J.B.M. Uncertainty Principle and Uncertainty Relations. *Found. Phys.* **1985**, *15*, 925.
27. Butterfield, J. On Time in Quantum Physics. In *A Companion to the Philosophy of Time*; John Wiley and Sons: Singapore, 2013; pp. 220–241.