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Article

The Speed of Convergence of the Threshold Estimator of Ruin Probability under the Tempered α-Stable Lévy Subordinator

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Abstract: In this paper, a nonparametric estimator of ruin probability is introduced in a spectrally negative Lévy process where the jump component is a tempered α-stable subordinator. Given a discrete record of high-frequency data, a threshold technique is proposed to estimate the mean of the jump size and use the Fourier transform and the Pollaczek–Khinchin formula to construct the estimator of ruin probability. The convergence rate of the integrated squared error for the estimator is studied.

Keywords: ruin probability; spectrally negative Lévy process; Fourier transform; high-frequency data

MSC: 62G20; 62M05

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1. Introduction

In actuarial science, the statistical inference of ruin probability has received much attention from scholars. Many works have been contributed to parametric and nonparametric estimation of ruin probability. See, for example, Croux and Veraverbeke [1], Frees [2], Mnatsakanov et al. [3], Pitts [4], Politis [5], and Veraverbeke [6]. In recent years, many nice results have been obtained by actuarial scholars, such as Huang et al. [7], Li et al. [8], You et al. [9], Zhang and Yang [10], Zhang and Yang [11], Zhang [12], and Zhang and Yang [13]. As an extension of ruin probability, the Gerber–Shiu function has been introduced and studied for its statistical properties. Interested readers can refer to Su and Yu [14,15], Yang et al. [16], Zhang and Su [17], Su et al. [18], Zhang and Su [19], Zhang [20], Shimizu [21], and Shimizu and Zhang [22], among others.

In Asmussen and Albrecher [23], an analytic (or probabilistic) approach was suggested, and it needs much more detailed information about the risk model, such as the claim size distribution. However, in practical situations, it is not easy to obtain the specific distribution information. Instead, one observes the surplus process at some discrete time points. Then, a statistical methodology can be directly used to estimate the claim size distribution with the observed data. In Zhang and Yang [10], a nonparametric estimator of ruin probability was proposed, based on the Pollaczek–Khinchin formula and the Fourier transform in a pure-jump Lévy risk model. This estimation approach was extended by Zhang and Yang [11] to a spectrally negative Lévy risk model. Subsequently, Shimizu and Zhang [22] estimated the Gerber–Shiu function for an insurance surplus process driven by a Lévy subordinator. In Zhang and Yang [10] and Comte and Genon-Catalot [24], they considered high-frequency sampling with n discrete time observations of step width $h_n > 0$ and derived asymptotics under the framework that $h_n \to 0$ and $nh_n \to \infty$.

In the present work, our interest is to estimate ruin probability for a spectrally negative Lévy risk model under the above framework. Assume that the surplus of the risk model can be observed at a sequence of discrete time points $\{t_k^n = kh_n, k = 0, 1, 2, 3, ...\}$ with $h_n \geq 0$ being the length of the sampling interval. Without observing the jump and diffusion parts of the risk model, it is challenging to estimate ruin probability, since it depends on both parts in a spectrally negative Lévy risk model. In Mancini [25–28] and Shimizu [29,30], they developed a threshold technique for identifying the times when jump sizes exceed a suitably defined threshold. Using the threshold technique and the Fourier transform, an estimator of ruin probability is constructed, and the convergence rate of its integrated squared error is obtained.

The remainder of this paper is organized as follows. In Section 2, the risk model, as well as some assumptions for the asymptotic theory are introduced. In Section 3, some estimators are suggested, based on the Fourier transform and the threshold technique. In Section 4, the convergence rate of our estimators is established. In Section 5, we conclude this paper.

2. Preliminaries

2.1. Risk Model and Some Assumptions

A spectrally negative Lévy process is specified by:

$$Y_t = ct + \sigma W_t - J_t, \quad t \geq 0, \tag{1}$$

where $c > 0$ is a parameter, $\sigma > 0$ represents the perturbation coefficient, W_t is a standard Brownian motion, and J_t is a subordinator. Suppose that W_t and J_t are independent of each other. Then, the characteristic exponent of Y_t is given by:

$$\psi_Y(s) = \frac{1}{t}\ln(\mathbf{E}[e^{isY_t}]) = ics - \frac{1}{2}\sigma^2 s^2 - \int_0^\infty (1 - e^{-isx})\nu(dx), \tag{2}$$

where ν is the Lévy measure on $(0, \infty)$. By Sato [31], it can be rewritten as:

$$J_t = L_t + M_t^*, \tag{3}$$

where L_t is the sum of jumps over $[0, t]$ with the jump size larger than one, and M_t^* is the sum of jumps over $[0, t]$ with the jump size less than one. Specifically, $L_t = \int_0^t \int_{x>1} x\mu(ds, dx) = \sum_{k=1}^{N_t} \gamma_k$, where μ is the Poisson random measure of J_t such that $E[\mu(ds, dx)] = \nu(dx)ds$, N_t is a Poisson process, and $\gamma_1, \gamma_2, \gamma_3, \ldots$ are i.i.d. random variables, that is L_t is a compound Poisson process representing the jumps of J_t with the jump size larger than one. Process M_t^* admits decomposition, $M_t^* = bt + M_t$, where $b = \int_{x\leq 1} x\nu(dx)$ and $M_t = \int_0^t \int_{x\leq 1} x\widetilde{\mu}(ds, dx)$ is a martingale with $\widetilde{\mu}(ds, dx) = \mu(ds, dx) - \nu(dx)ds$ being the compensated measure of $\mu(ds, dx)$. It is known that M_t is a square integrable martingale with infinite activity of the jump such that $E[M_t] = 0$ and $Var[M_t] = t\int_{x\leq 1} x^2\nu(dx) < \infty$. Suppose that γ_k, N_t, and M_t are independent of each other.

Let $u > 0$ be the initial surplus of an insurance company. Then, the surplus at time t can be modeled by:

$$U_t = u + Y_t = u + ct + \sigma W_t - J_t, \quad t \geq 0, \tag{4}$$

where c is the rate of the premium, σ represents the perturbation coefficient, J_t denotes the claim payments and other expenses in insurance businesses, and W_t is a perturbation.

2.2. Ruin Probability and Its Fourier Transform

The infinite-time horizon ruin probability $\Phi(u)$ is defined as:

$$\Phi(u) \quad = \quad \mathbf{P}\left(\inf_{0\leq t<\infty} U_t \leq 0 | U_0 = u\right).$$

By Equation (1) in Zhang and Yang [11], $\Phi(u)$ admits the following representation:

$$\Phi(u) = 1 - (1-\rho)\sum_{i=0}^{\infty} \rho^i (G^{(i+1)*} * H^{(i)*})(u), \quad u > 0, \tag{5}$$

where $\rho = \frac{\mu_1}{c}$, $\mu_1 = \int_0^\infty x\nu(dx)$, $H(x) = \frac{1}{\mu_1}\int_0^x \nu(y,\infty)dy$, and G is determined by the Fourier transform $\int_0^\infty e^{isx}dG(x) = c\{c - \frac{\sigma^2}{2}is\}^{-1}$. Setting $\Phi(u) \equiv 0$ for $u < 0$, Zhang and Yang [11] obtained the Fourier transform of $\Phi(u)$:

$$\mathcal{F}_\Phi(s) = \int_0^\infty e^{isu}\Phi(u)du = \frac{\frac{\sigma^2}{2}is + \frac{1}{is}\int_0^\infty (e^{isx}-1)\nu(dx) - \mu_1}{ics + \frac{\sigma^2}{2}s^2 - \int_0^\infty (e^{isx}-1)\nu(dx)} = \frac{ics(1-\rho) + \psi_Y(-s)}{-is\psi_Y(-s)}, \tag{6}$$

Once an estimator of $\mathcal{F}_\Phi(s)$ is available, $\Phi(u)$ can be estimated by the inverse Fourier transform.

3. Estimation of Ruin Probability

Suppose that a discrete sample $Y^n = \{Y_{t_i^n} | t_i^n = ih_n; i = 0,1,2...,n\}$ can be observed. Let $Z_i = Y_{t_i^n} - Y_{t_{i-1}^n}$, $h_n = t_i^n - t_{i-1}^n > 0$, $\lim_{n\to\infty} h_n = 0$, and $\lim_{n\to\infty} nh_n = \infty$. Our interest is to estimate $\Phi(u)$ by $Z_1, Z_2, ..., Z_n$ when Lévy measure ν and perturbation coefficient σ are unknown.

If one can estimate ρ and $\psi_Y(-s)$ in (6), then $\mathcal{F}_\Phi(s)$ can be estimated with the plug-in device. Inspired by Zhang and Yang [10,11] and You and Yin [32], we define the estimator of $\psi_Y(s)$:

$$\hat{\psi}_Y(s) = \frac{1}{h_n}(\frac{1}{n}\sum_{k=1}^n e^{isZ_k} - 1). \tag{7}$$

To estimate $\rho = \frac{\mu_1}{c}$, we need to estimate μ_1, the mean of J_1. Zhang and Yang [11] proposed to estimate μ_1, by:

$$\hat{\mu}_1^* = \frac{1}{nh_n}\sum_{k=1}^n (ch_n - Z_k). \tag{8}$$

Note that $ch_n - Z_k = (J_{t_k^n} - J_{t_{k-1}^n}) - \sigma(W_{t_k^n} - W_{t_{k-1}^n})$. Ideally, we hope that the estimator of μ_1 is $\frac{1}{nh_n}\sum_{k=1}^n (J_{t_k^n} - J_{t_{k-1}^n})$, but we cannot observe a discrete sample $J^n = \{J_{t_i^n} | t_i^n = ih_n; i = 0,1,2...,n\}$. To this end, we introduce a threshold technique. Motivated by Shimizu [29,30] and Mancini [26,27], we introduce the filter:

$$\mathcal{D}_k^n := \{\omega \in \Omega : (ch_n - Z_k) > r_n\}, \tag{9}$$

where $r_n > 0$ is a suitable threshold parameter dependent on n such that $\lim_{h_n\to 0} r_n = 0$. Let $\mathcal{C}_k^n := \{\omega \in \Omega : (ch_n - Z_k) \le r_n\}$ be the complement of $\mathcal{D}_k^n$. By (9), if $ch_n - Z_k > r_n$, we can detect the existence of a jump in an interval $(t_{k-1}^n, t_k^n]$, and then, we take $ch_n - Z_k$ as an approximation to $J_{t_k^n} - J_{t_{k-1}^n}$. This leads to a natural estimate of μ_1:

$$\hat{\mu}_1 = \frac{\sum_{k=1}^n (ch_n - Z_k)\mathbf{I}_{\mathcal{D}_k^n}}{nh_n}. \tag{10}$$

Then, ρ is estimated by:

$$\hat{\rho} = \frac{\sum_{k=1}^n (ch_n - Z_k)\mathbf{I}_{\mathcal{D}_k^n}}{cnh_n}. \tag{11}$$

Combining (6), (7), and (11) leads to our estimator:

$$\hat{\mathcal{F}}_\Phi(s) = \frac{ics(1-\hat{\rho}) + \hat{\psi}_Y(-s)}{-is\hat{\psi}_Y(-s)}. \tag{12}$$

Note that the above estimate has no definition at $s = 0$. When $s \to 0$, $\hat{\psi}_Y(-s) \to 0$, and thus, $\hat{\mathcal{F}}_\Phi(s)$ may behave erratically. Applying the inverse Fourier transform and removing a small neighborhood of $s = 0$, we propose to estimate $\Phi(u)$ by:

$$\hat{\Phi}(u) = \frac{1}{2\pi}\int_{m_n}^{M_n} e^{-ius}\hat{\mathcal{F}}_\Phi(s)ds + \frac{1}{2\pi}\int_{-M_n}^{-m_n} e^{-ius}\hat{\mathcal{F}}_\Phi(s)ds, \quad u > 0, \tag{13}$$

where m_n and M_n are positive threshold numbers such that $m_n \to 0$ and $M_n \to \infty$ as $n \to \infty$.

4. Asymptotic Properties of Estimators

In this section, the asymptotic properties of $\hat{\rho}$ and $\hat{\Phi}(u)$ are studied. For the ease of exposure, we first introduce some notations. For integer $k = 1, 2$, $\mu_k := \int_0^\infty x^k \nu(dx)$. For any two positive sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$, $x_n \lesssim y_n$ means that $x_n \le Cy_n$ for some constant C and large index n. For any function $f(x)$ with support $(0, \infty)$, define $\|f\|^2 = \int_0^\infty f^2(x)dx$. Let $\mu_L = \int_1^\infty x\nu(dx)$, $\mu_\gamma = E[\gamma_k]$, $\sigma_\gamma^2 = Var(\gamma_k)$, and $\sigma_M^2 = \int_0^1 x^2\nu(dx)$. Next, we make the following assumptions for our theoretical results:

Assumption 1. *The safety loading condition holds, i.e.,* $c - \mu_1 < \infty$.

Assumption 2. J_t *is the tempered* α*-stable subordinator.*

Assumption 3. $h_n = n^{-\kappa_1}$, $m_n = O(n^{-\kappa_2})$, $\alpha \in [0, 1)$, *and* $\theta \in (0, \frac{1}{2})$, *where* $\kappa_1, \kappa_2 > 0$, $\kappa_1 + 3\kappa_2 < 1$, $1 - \kappa_1(2 - 2\theta) + \kappa_2 < 0$, *and* $\kappa_2 - 2\theta\kappa_1(1 - \alpha) < 0$.

Assumption 1 guarantees that the ruin is not a certain event. Assumption 2 means that ν has a density of the form $\nu(x) = \frac{ze^{-\lambda x}}{x^{1+\alpha}}\mathbf{I}_{x>0}$, where $z > 0$, $\lambda > 0$, and $\alpha \in [0, 1)$. Assumption 2 implies that $\mu_1 < \infty$, $\mu_2 < \infty$,

$$\int_0^{r_n} x^k \nu(dx) \sim (r_n)^{k-\alpha}, \quad k = 1, 2, 3, 4, \tag{14}$$

$$\text{and} \quad \int_{r_n}^1 x\nu(dx) \sim (m + r_n^{1-\alpha}), \tag{15}$$

where m indicates a generic constant.

To establish the convergence rate of $\hat{\Phi}(u)$, we need to calibrate the estimation errors of $\hat{\rho}$. The following Theorem 1 gives the rate of convergence of $\hat{\rho}$.

Theorem 1. Let $r_n = h_n^\theta$ with $\theta \in (0, 1/2)$. Then, under Assumptions 1 and 2,

$$\hat{\rho} - \rho = \frac{1}{\sqrt{nh_n}} Q_{h_n} + O_P(h_n^{\theta(1-\alpha)} + nh_n^{2-2\theta}), \tag{16}$$

where $Q_{h_n} \xrightarrow{D} \mathcal{N}\Big(0, \frac{1}{c^2}(\mu_L\mu_\gamma + \frac{\mu_L\sigma_\gamma^2}{\mu_\gamma} + \sigma_M^2)\Big)$.

Proof. Let $\Delta_k J = (J_{t_k^n} - J_{t_{k-1}^n})$, $\Delta_k M = (M_{t_k^n} - M_{t_{k-1}^n})$, $\Delta_k L = (L_{t_k^n} - L_{t_{k-1}^n})$, $\Delta_k W = \sigma(W_{t_k^n} - W_{t_{k-1}^n})$, and $\Delta_k N = (N_{t_k^n} - N_{t_{k-1}^n})$. By (9) and (11), we have:

$$\begin{aligned}\hat{\rho} - \rho &= \frac{\sum_{k=1}^n (ch_n - Z_k)\mathbf{I}_{\mathcal{D}_k^n}}{cnh_n} - \frac{1}{c}\mu_1 \\ &= \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\Delta_k J - \Delta_k W)\mathbf{I}_{\{\Delta_k J - \Delta_k W > h_n^\theta\}}}{nh_n} - \mu_1\Big] \\ &= \frac{1}{c}\Big[\frac{\sum_{k=1}^n (bh_n + \Delta_k L - \Delta_k W + \Delta_k M)\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M > h_n^\theta\}}}{nh_n} - \mu_1\Big] \\ &= \sum_{j=1}^4 I_j, \end{aligned} \tag{17}$$

where:

$$I_1 = \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\Delta_k L - \Delta_k W)\mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta\}}}{nh_n} - \int_1^\infty x\nu(dx)\Big],$$

$$I_2 = \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\Delta_k L - \Delta_k W)}{nh_n}(\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M > h_n^\theta\}} - \mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta\}})\Big],$$

$$I_3 = \frac{1}{c}\Big[\frac{\sum_{k=1}^n (bh_n + \Delta_k M)}{nh_n} - \int_0^1 x\nu(dx)\Big],$$

$$I_4 = -\frac{1}{c}\frac{\sum_{k=1}^n (bh_n + \Delta_k M)}{nh_n}\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M \le h_n^\theta\}}.$$

In the following, we study each of I_1 to I_4.

(i) Note that:

$$\begin{aligned} I_1 &= \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\Delta_k L - \Delta_k W)\mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta, \Delta_k N = 1\}}}{nh_n} - \mu_L\Big] \\ &\quad + \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\Delta_k L - \Delta_k W)\mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta, \Delta_k N = 0\}}}{nh_n}\Big] \\ &\quad + \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\Delta_k L - \Delta_k W)\mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta, \Delta_k N \ge 2\}}}{nh_n}\Big] \\ &\equiv I_{11} + I_{12} + I_{13}. \end{aligned} \tag{18}$$

By Mancini [25], $\{\Delta_k L - \Delta_k W > h_n^\theta\}$ is equal to $\{\Delta_k N = 1\}$ almost surely for small h_n. Thus, for small h_n,

$$\begin{aligned} I_{11} &= \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\Delta_k L - \Delta_k W)\mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta, \Delta_k N = 1\}}}{nh_n} - \mu_L\Big] \\ &= \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\gamma_k - \Delta_k W)\mathbf{I}_{\{\Delta_k N = 1\}}}{nh_n} - \mu_L\Big]. \end{aligned} \tag{19}$$

Rewrite:

$$I_{11} = \frac{1}{\sqrt{nh_n}}\Big[\frac{\sum_{k=1}^n \{(\gamma_k - \Delta_k W)\mathbf{I}_{\{\Delta_k N = 1\}} - h_n\mu_L\}}{c\sqrt{nh_n}}\Big] \equiv \frac{Z_{h_n}}{\sqrt{nh_n}}. \tag{20}$$

Then, due to the independence of γ_k, N_t, and W_t, by the central limit theorem, we have $Z_{h_n} \xrightarrow{D} \mathcal{N}(0, \frac{1}{c^2}(\mu_L\mu_\gamma + \frac{\mu_L\sigma_\gamma^2}{\mu_\gamma}))$.

By (A.2)–(A.4) in You and Yin [32], we have:

$$I_{12} = \frac{1}{c}\Big[\frac{\sum_{k=1}^n (\Delta_k L - \Delta_k W)\mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta, \Delta_k N = 0\}}}{nh_n}\Big] = o_P(\frac{1}{\sqrt{nh_n}})$$

and:

$$I_{13} = \frac{1}{c}\Big[\frac{\sum_{k=1}^{n}(\Delta_k L - \Delta_k W)\mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta, \Delta_k N \geq 2\}}}{nh_n}\Big] = o_P(\frac{1}{\sqrt{nh_n}}).$$

Therefore, $I_1 = \frac{Z_{h_n}}{\sqrt{nh_n}} + o_P(\frac{1}{\sqrt{nh_n}})$.

(ii) Next, we show that $I_2 = o_P(h_n^{1-\alpha\theta}\sqrt{\log(\frac{1}{h_n})})$. Since $bh_n + \Delta_k M \geq 0$, we have $\{\Delta_k L - \Delta_k W > h_n^\theta\} \subseteq \{bh_n + \Delta_k M + \Delta_k L - \Delta_k W > h_n^\theta\}$. Thus,

$$\begin{aligned} I_2 &= \frac{1}{c}\Big[\frac{\sum_{k=1}^{n}(\Delta_k L - \Delta_k W)}{nh_n}(\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M > h_n^\theta\}} - \mathbf{I}_{\{\Delta_k L - \Delta_k W > h_n^\theta\}})\Big] \\ &= \frac{1}{c}\Big[\frac{\sum_{k=1}^{n}(\Delta_k L - \Delta_k W)}{nh_n}\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M > h_n^\theta, \Delta_k L - \Delta_k W \leq h_n^\theta\}}\Big]. \end{aligned} \tag{21}$$

For small h_n, $\{\Delta_k L - \Delta_k W \leq h_n^\theta\} = \{\Delta_k N = 0\}$ almost surely. As a result, we have:

$$I_2 = \frac{1}{c}\Big[\frac{\sum_{k=1}^{n}(-\Delta_k W)}{nh_n}\mathbf{I}_{\{\Delta_k M > h_n^\theta, \Delta_k L - \Delta_k W \leq h_n^\theta\}}\Big](1 + o_p(1)).$$

By Mancini [27], we obtain that:

$$\frac{1}{c}[\frac{\sum_{k=1}^{n}|\Delta_k W|}{nh_n}\mathbf{I}_{\{\Delta_k M > h_n^\theta\}}] = o_P(h_n^{1-\alpha\theta}\sqrt{\log(\frac{1}{h_n})}).$$

(iii) Applying the central limit theorem, we obtain:

$$I_3 = \frac{1}{c}[\frac{\sum_{k=1}^{n}(bh_n + \Delta_k M)}{nh_n} - \int_0^1 x\nu(dx)] \equiv \frac{T_{h_n}}{\sqrt{nh_n}}, \tag{22}$$

where $T_{h_n} \xrightarrow{D} \mathcal{N}(0, \frac{\sigma_M^2}{c^2})$.

(iv) Now, let us consider the last term of (17).

$$\begin{aligned} -I_4 &= \frac{1}{c}\frac{\sum_{k=1}^{n}(bh_n + \Delta_k M)}{nh_n}\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M \leq h_n^\theta\}} \\ &= \frac{1}{c}\frac{\sum_{k=1}^{n}(bh_n + \Delta_k M)}{nh_n}\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M \leq h_n^\theta, bh_n + \Delta_k M > 2h_n^\theta\}} \\ &\quad + \frac{1}{c}\frac{\sum_{k=1}^{n}(bh_n + \Delta_k M)}{nh_n}\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M \leq h_n^\theta, bh_n + \Delta_k M \leq 2h_n^\theta\}} \\ &\equiv B_1 + B_2. \end{aligned} \tag{23}$$

By (A.29)–(A.32) in You and Yin [32], we have

$$B_1 = \frac{1}{c}\frac{\sum_{k=1}^{n}(bh_n + \Delta_k M)}{nh_n}\mathbf{I}_{\{bh_n + \Delta_k L - \Delta_k W + \Delta_k M \leq h_n^\theta, bh_n + \Delta_k M > 2h_n^\theta\}} \leq O_p(nh_n^{1+(1-2\theta)}). \tag{24}$$

Next, we show that the second term of (23) is of order $O(r_n^{1-\alpha})$. In fact,

$$\begin{aligned} B_2 &= \frac{1}{c}\frac{\sum_{k=1}^n (bh_n + \Delta_k M)}{nh_n}\mathbf{I}_{\{bh_n+\Delta_k L-\Delta_k W+\Delta_k M\leq h_n^\theta, bh_n+\Delta_k M\leq 2h_n^\theta\}} \\ &\leq \frac{1}{c}\frac{\sum_{k=1}^n (bh_n + \Delta_k M)}{nh_n}\mathbf{I}_{\{bh_n+\Delta_k M\leq 2h_n^\theta\}} \\ &= \frac{1}{c}\frac{\sum_{k=1}^n (\int_{t_{k-1}^n}^{t_k^n}\int_0^{2h_n^\theta} x\mu(dx,dt))}{nh_n}\mathbf{I}_{\{bh_n+\Delta_k M\leq 2h_n^\theta\}} \\ &\leq \frac{1}{c}\frac{\sum_{k=1}^n (\int_{t_{k-1}^n}^{t_k^n}\int_0^{2h_n^\theta} x\mu(dx,dt))}{nh_n}. \end{aligned} \tag{25}$$

Using Assumption 2 and the law of large numbers, we establish that:

$$E[\frac{1}{c}\frac{\sum_{k=1}^n (\int_{t_{k-1}^n}^{t_k^n}\int_0^{2h_n^\theta} x\mu(dx,dt))}{nh_n}] = \frac{1}{c}\int_0^{2h_n^\theta} x\nu(dx) \sim h_n^{\theta(1-\alpha)}$$

and:

$$B_2 = O(h_n^{\theta(1-\alpha)}).$$

Thus, $I_4 = O_p(nh_n^{1+(1-2\theta)} + h_n^{\theta(1-\alpha)})$.

Finally, combining (i)–(iv) leads to $\hat{\rho} - \rho = \frac{1}{\sqrt{nh_n}}(Z_{h_n} + T_{h_n}) + O_p(nh_n^{1+(1-2\theta)} + h_n^{\theta(1-\alpha)})$. Note that Z_{h_n} and T_{h_n} are independent. It follows that the result of the theorem holds. □

The convergence rate of $\hat{\Phi}(u)$ depends on the choice of h_n, m_n, and M_n. The following theorem establishes the convergence rate of the integrated squared error of $\hat{\Phi}(u)$.

Theorem 2. Under Assumptions 1–3, as $n \to \infty$,

$$\|\hat{\Phi}(u) - \Phi(u)\|^2 = O_P\Big(\max\{n^{\kappa_1+3\kappa_2-1}, n^{\kappa_2-2\kappa_1\theta(1-\alpha)}, n^{1-\kappa_1(2-2\theta)+\kappa_2}, \frac{1}{M_n}\}\Big). \tag{26}$$

Proof. By (13), we have:

$$\|\hat{\Phi}(u) - \Phi(u)\|^2 = \int_0^\infty \left| \frac{1}{2\pi}\int_{m_n}^{M_n} e^{-ius}\hat{\mathcal{F}}_\Phi(s)ds + \frac{1}{2\pi}\int_{-M_n}^{-m_n} e^{-ius}\hat{\mathcal{F}}_\Phi(s)ds - \Phi(u)\right|^2 du. \tag{27}$$

Using Parseval's identity, we obtain that:

$$\begin{aligned} \|\hat{\Phi}(u) - \Phi(u)\|^2 &\lesssim \frac{1}{2\pi}\int_{m_n<|s|<M_n}\left|\frac{c(1-\hat{\rho})}{\hat{\psi}_Y(-s)} - \frac{c(1-\rho)}{\psi_Y(-s)}\right|^2 ds \\ &\quad + \frac{1}{2\pi}\int_{|s|\geq M_n}|\mathcal{F}_\Phi(s)|^2 ds + \frac{1}{2\pi}\int_{|s|\leq m_n}|\mathcal{F}_\Phi(s)|^2 ds \\ &= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3. \end{aligned} \tag{28}$$

Note that:

$$\begin{aligned} \mathrm{II}_1 &= \frac{1}{2\pi}\int_{m_n<|s|<M_n}\left|\frac{(c-\mu_1)[\hat{\psi}_Y(-s)-\psi_Y(-s)]+c\psi_Y(-s)[\hat{\rho}-\rho]}{\hat{\psi}_Y(-s)\psi_Y(-s)}\right|^2 ds \\ &\lesssim \int_{m_n<|s|<M_n}\left|\frac{(c-\mu_1)[\hat{\psi}_Y(-s)-\psi_Y(-s)]}{\hat{\psi}_Y(-s)\psi_Y(-s)}\right|^2 ds \\ &\quad + \int_{m_n<|s|<M_n}\left|\frac{c[\hat{\rho}-\rho]}{\hat{\psi}_Y(-s)}\right|^2 ds. \end{aligned} \tag{29}$$

Using the fact that $|\psi_Y(-s)| > s(c-\mu_1)$ and Theorem 4.1 in You and Cai [33], we establish that:

$$\int_{m_n<|s|<M_n}\left|\frac{(c-\mu_1)[\hat{\psi}_Y(-s)-\psi_Y(-s)]}{\hat{\psi}_Y(-s)\psi_Y(-s)}\right|^2 ds = O_P\left(\frac{1}{nh_n}m_n^{-3}\right) \tag{30}$$

and:

$$\int_{m_n<|s|<M_n}\left|\frac{c[\hat{\rho}-\rho]}{\hat{\psi}_Y(-s)}\right|^2 ds = O_P\left(\frac{1}{nh_n}\frac{1}{m_n}\right) + O_P\left((nh_n^{2-2\theta}+h_n^{2\theta(1-\alpha)})\frac{1}{m_n}\right). \tag{31}$$

By Lemma 1 in Zhang and Yang [11], we have $\mathrm{II}_2 = O(\frac{1}{M_n})$ and $\mathrm{II}_3 = O(m_n)$. Then, combining (30), (31), and Assumption 3 leads to:

$$\|\hat{\Phi}(u)-\Phi(u)\|^2 = O_P\Big(\max\{n^{\kappa_1+3\kappa_2-1}, n^{\kappa_2-2\kappa_1\theta(1-\alpha)}, n^{1-\kappa_1(2-2\theta)+\kappa_2}, \frac{1}{M_n}\}\Big).$$

□

5. Conclusions

In this paper, the threshold and Fourier transform (inversion) techniques were employed to construct a new estimator of ruin probability for the spectrally negative Lévy process. The convergence rate of the integrated squared error (ISE) of the estimator was obtained when the jump component was the tempered α-stable subordinator. This shows that the ISE of the estimated ruin probability function is well controlled. Further work includes, but is not limited to deriving the asymptotic distribution of the proposed estimator and making statistical inference for ruin probability, under the framework that the risk model is a spectrally negative Lévy process with dividend strategy and investment. Furthermore, statistical inference for the Gerber–Shiu function and the dividend function are worthy of study.

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