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Lebesgue Points of Besov and Triebel–Lizorkin Spaces with Generalized Smoothness

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Abstract: In this article, the authors study the Lebesgue point of functions from Hajłasz–Sobolev, Besov, and Triebel–Lizorkin spaces with generalized smoothness on doubling metric measure spaces and prove that the exceptional sets of their Lebesgue points have zero capacity via the capacities related to these spaces. In case these functions are not locally integrable, the authors also consider their generalized Lebesgue points defined via the γ-medians instead of the classical ball integral averages and establish the corresponding zero-capacity property of the exceptional sets.

Keywords: Hajłasz–Sobolev space; Hajłasz–Besov space; Hajłasz–Triebel–Lizorkin space; generalized smoothness; Lebesgue point; capacity

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1. Introduction

The study of function spaces on the Euclidean space $\mathbb{R}^n$ and its subsets with generalized smoothness started from the middle of the 1970s (see, for instance, [1–4]), and has found various applications in interpolations, embedding properties of function spaces [5–8], fractal analysis ([9], Chapters 18–23), and many other fields such as probability theory and stochastic processes [10,11]. Recall that, in [11], Farkas and Leopold studied the generalized Besov spaces $B^{(\sigma,N)}_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin spaces $F^{(\sigma,N)}_{p,q}(\mathbb{R}^n)$ for the full range of parameters, in which the smoothness, instead of the classical smoothness sequence $\{2^{js}\}_{j\geq 0}$, was given via a weight sequence $\sigma := \{\sigma_j\}_{j\geq 0}$ of positive numbers. Intensive investigations on generalized Besov and Triebel–Lizorkin spaces also exist in which smoothness is described by a parameter function; see, for instance [6,12–16]. In recent years, a lot of attention has been paid to Besov and Triebel–Lizorkin spaces on $\mathbb{R}^n$ with logarithmic smoothness; see, for instance [17–27].

Recently, using Hajłasz gradient sequences, the authors [28] introduced Hajłasz–Besov and Hajłasz–Triebel–Lizorkin spaces with generalized smoothness on a given metric space $\mathcal{X}$ with a doubling measure and, when $\mathcal{X} = \mathbb{R}^n$, proved their coincidence with the classical Besov and Triebel–Lizorkin spaces with generalized smoothness. Recall that the Hajłasz gradients were originally introduced by Hajłasz [29] and have been an important tool used to develop Sobolev spaces on metric measure spaces (see, for instance [30–34]). The fractional Hajłasz gradients were introduced independently by Hu [35] and Yang [36] in 2003. In 2011, Koskela et al. [37] introduced the notion of sequences of Hajłasz gradients and characterized Besov and Triebel–Lizorkin spaces via some pointwise inequalities involving these Hajłasz gradient sequences; as an application, this pointwise characterization has been used in [37] to show the invariance of quasi-conformal mappings on some Triebel–Lizorkin spaces.

It is well known, by the Lebesgue differentiation theorem, that almost every point is a Lebesgue point of a locally integrable function. Then, it is very natural to expect a smaller exceptional set when the function has higher regularity. In [38], Kinnunen and Latvala considered the Lebesgue point of functions in the Hajłasz–Sobolev space $M^{1,p}(\mathcal{X})$

on a given metric measure space $\mathcal{X}$ and proved that, when the measure doubles and $p \in (1, Q]$, a Hajłasz–Sobolev function has Lebesgue points outside a set of zero Hajłasz–Sobolev capacity, where Q represents the doubling dimension of $\mathcal{X}$. This result leads to a series of related work on many other function spaces such as fractional Hajłasz–Sobolev spaces [39], Orlicz–Sobolev spaces [40], as well as Hajłasz–Besov and Hajłasz–Triebel–Lizorkin spaces [41]. We also refer the reader to [42,43] for a related study on variable function spaces.

Inspired by these works, in this article, we study the Lebesgue point of functions from the Hajłasz–Sobolev space $M^{\phi,p}(\mathcal{X})$, the Hajłasz–Besov space $N^{\phi}_{p,q}(\mathcal{X})$, and the Hajłasz–Triebel–Lizorkin space $M^{\phi}_{p,q}(\mathcal{X})$ with generalized smoothness on a given doubling measure space $\mathcal{X}$, via measuring the related exceptional sets of Lebeguse points. Note that functions in the Hajłasz–Besov or Hajłasz–Triebel–Lizorkin spaces with generalized smoothness might fail to be locally integrable when their index p or q is close to zero. To overcome this obstacle, similar to [41,44,45], we also consider a class of generalized Lebesgue points, which are defined via the γ-medians introduced in [46,47], instead of the classical integrals. As the main results of this article, we prove that the exceptional sets of (generalized) Lebesgue points of functions from the above spaces have zero capacity, where those capacities are defined by related spaces. These results can apply to a wide class of function spaces due to the generality of the smoothness factor ϕ. In particular, the logarithmic Hajłasz–Sobolev space is an admissible function space for our main results.

The structure of this article is as follows.

In Section 2, we state some basic notions and assumptions on the smoothness function ϕ. We also introduce the inhomogeneous Hajłasz–Sobolev space $M^{\phi,p}(\mathcal{X})$, the inhomogeneous Hajłasz–Besov space $N^{\phi}_{p,q}(\mathcal{X})$, and the inhomogeneous Hajłasz–Triebel–Lizorkin space $M^{\phi}_{p,q}(\mathcal{X})$ with generalized smoothness and establish their coincidence with those classical Besov and Triebel–Lizorkin spaces with generalized smoothness when $\mathcal{X} = \mathbb{R}^n$.

Section 3 is devoted to studying the Lebesgue point of functions from $N^{\phi}_{p,q}(\mathcal{X})$ and $M^{\phi}_{p,q}(\mathcal{X})$ and, in particular, $M^{\phi,p}(\mathcal{X}) = M^{\phi}_{p,\infty}(\mathcal{X})$, via the capacities $\mathrm{Cap}_{N^{\phi}_{p,q}(\mathcal{X})}$ and $\mathrm{Cap}_{M^{\phi}_{p,q}(\mathcal{X})}$ related to the spaces $N^{\phi}_{p,q}(\mathcal{X})$ and $M^{\phi}_{p,q}(\mathcal{X})$, respectively. To this end, via establishing some Poincaré-type inequalities and estimates related to Hajłasz-type spaces with generalized smoothness, we first prove the convergence of discrete convolution approximations in $N^{\phi}_{p,q}(\mathcal{X})$ and $M^{\phi}_{p,q}(\mathcal{X})$ when $p,\ q < \infty$, and a dense subset in $M^{\phi,p}(\mathcal{X}) = M^{\phi}_{p,\infty}(\mathcal{X})$ exists when $p < \infty$, which consists of continuous functions. Recall that, when $s \in (0, 1]$ and $p \in (0, \infty)$, the class of all s-Hölder continuous functions is dense in the classical Hajłasz–Sobolev space $M^{s,p}(\mathcal{X})$ (see, for instance, ([48], Theorem 5.19)), which was proved via an extension argument together with the inequality

$$[d(x, y)]^s \le [d(x, z)]^s + [d(z, y)]^s$$

for any $x,\ y,\ z \in \mathcal{X}$. However, this inequality may not be true if one replaces $[d(\cdot, \cdot)]^s$ by $\phi(d(\cdot, \cdot))$ due to the generality of ϕ. To overcome the difficulties caused by this, we borrow the notion of the modulus of continuity and, for certain ϕ that satisfies such assumptions, find a dense subset of $M^{\phi,p}(\mathcal{X})$ consisting of generalized Lipschitz functions. Applying these dense properties, we obtain the boundedness of discrete maximal operators on these Hajłasz-type spaces and then a weak-type capacitary estimate for restricted maximal functions, which is further used to prove that the exceptional sets of Lebesgue points of functions from $M^{\phi,p}(\mathcal{X})$, $N^{\phi}_{p,q}(\mathcal{X})$, and $M^{\phi}_{p,q}(\mathcal{X})$ have zero $\mathrm{Cap}_{M^{\phi,p}(\mathcal{X})}$, $\mathrm{Cap}_{N^{\phi}_{p,q}(\mathcal{X})}$, and $\mathrm{Cap}_{M^{\phi}_{p,q}(\mathcal{X})}$ capacities, respectively.

In Section 4, we deal with the generalized Lebesgue point of functions from the spaces $M^{\phi,p}(\mathcal{X})$, $N^{\phi}_{p,q}(\mathcal{X})$, and $M^{\phi}_{p,q}(\mathcal{X})$, which are defined via the γ-medians instead of the classical ball integral averages. Following a procedure similar to that of Section 3, we also prove that the exceptional sets of generalized Lebesgue points of functions from $\mathcal{F}$ have zero $\mathrm{Cap}_{\mathcal{F}}$-capacity with

$$\mathcal{F} \in \left\{N^{\phi}_{p,q}(\mathcal{X}), M^{\phi}_{p,q}(\mathcal{X}), M^{\phi,p}(\mathcal{X})\right\}.$$

Finally, we compare the capacity $\mathrm{Cap}_{\mathcal{F}}$ with some Netrusov–Hausdorff contents and prove that they have the same null sets. This enables us to also use some Netrusov–Hausdorff contents to measure the exceptional set of Lebesgue points of functions from these Hajłasz-type spaces.

2. Hajłasz–Besov and Hajłasz–Triebel–Lizorkin Spaces with Generalized Smoothness

In this section, we recall some basic notation and notions as well as the definitions of the function spaces used in this article. Let $\mathbb{Z}$ be the collection of all integers, $\mathbb{N}$ be the collection of all positive integers, and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We write $A \lesssim B$ if there exists a positive constant C that is independent of the main parameters such that $A \le CB$ and write $A \sim B$ if $A \lesssim B \lesssim A$. We also denote by $C_{(a_1,a_2,\dots)}$ a positive constant depending on the parameters $a_1,\ a_2,\dots$.

A triple $(\mathcal{X}, d, \mu)$ is called a *metric measure space* if $\mathcal{X}$ is a non-empty set, d is a metric on $\mathcal{X}$, and μ is a regular Borel measure on $\mathcal{X}$ such that all of the balls defined by d have finite and positive measures. Recall that (see [48], [Convention 1.4]) a measure μ on $\mathcal{X}$ is called a *regular Borel measure* if open sets are μ-measurable and every set is contained in a Borel set with the same measure. Additionally, the measure μ is said to *double* if there exists a positive constant $C_\mu \in [1,\infty)$ such that, for any ball $B \subset \mathcal{X}$,

$$\mu(2B) \le C_\mu \mu(B).$$

Here and thereafter, for any $\lambda \in (0,\infty)$, λB denotes the ball with the same center as B but λ-times radius of B. The doubling property of μ implies that, for any ball $B \subset \mathcal{X}$ and any $\lambda \in [1,\infty)$,

$$\mu(\lambda B) \le C_\mu \lambda^D \mu(B), \tag{1}$$

where $D := \log_2 C_\mu$. Here and thereafter, we assume that C_μ is the smallest positive constant such that (1) holds true. Clearly, when $\mathcal{X} = \mathbb{R}^n$, $D = n$. Throughout this article, we always let$(\mathcal{X}, d, \mu)$ be a *metric space with a doubling measure* (for short, a *doubling metric measure space*). For any subset $E \subset \mathcal{X}$, we denote by $\mathbf{1}_E$ the characteristic function of E.

Let $L^0(\mathcal{X})$ be the collection of all measurable functions on $\mathcal{X}$ that are finite almost everywhere and $L^1_{\mathrm{loc}}(\mathcal{X})$ be the collection of all measurable functions on $\mathcal{X}$ satisfying that, for any $x_0 \in \mathcal{X}$, there exists an $r_0 \in (0,\infty)$ such that $f\mathbf{1}_{B(x_0,r_0)} \in L^1(\mathcal{X})$. For any $p,\ q \in (0,\infty]$, let $L^p(\mathcal{X}, l^q)$ and $l^q(\mathcal{X}, L^p)$ be, respectively, the collections of all sequences $\{u_k\}_{k\in\mathbb{Z}} \subset L^0(\mathcal{X})$ such that

$$\|\{u_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} := \left\|\left(\sum_{k\in\mathbb{Z}} |u_k|^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} < \infty$$

and

$$\|\{u_k\}_{k\in\mathbb{Z}}\|_{l^q(\mathcal{X},L^p)} := \left[\sum_{k\in\mathbb{Z}} \|u_k\|^q_{L^p(\mathcal{X})}\right]^{1/q} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

For any $u \in L^0(\mathcal{X})$ and $E \subset \mathcal{X}$ with $\mu(E) \in (0,\infty)$, let

$$u_E := \fint_E u\,d\mu := \frac{1}{\mu(E)} \int_E u\,d\mu := \frac{1}{\mu(E)} \int_E u(x)\,d\mu(x). \tag{2}$$

For any $L \in (0,\infty)$, a function f is said to be L*-Lipschitz* if it satisfies

$$\left|f(x) - f(y)\right| \le L\,d(x,y), \qquad \forall\, x,\ y \in \mathcal{X}.$$

For a Lipschitz function f, the smallest constant L satisfying the above inequality is called the *Lipschitz constant* of f and denoted by $\operatorname{Lip} f$.

We also frequently use the following inequality: if $q \in (0,1]$, then, for any $\{a_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$,

$$\left(\sum_{i\in\mathbb{Z}} |a_i|\right)^q \le \sum_{i\in\mathbb{Z}} |a_i|^q. \tag{3}$$

We now recall the definition and some basic properties of weight functions used to describe the smoothness of function spaces under consideration. We begin with a classical notion of admissible sequences; see, for instance [11,49].

Definition 1. *Let $E \in \{\mathbb{Z}, \mathbb{Z}_+\}$. A sequence of positive numbers, $\{\sigma_j\}_{j\in E}$, is said to be* admissible *if there exist two positive constants d_0 and d_1 such that, for any $j \in E$, $d_0\sigma_j \le \sigma_{j+1} \le d_1\sigma_j$.*

Several examples of admissible sequences can be found in [11], which illustrate the flexibility of this assumption.

Definition 2. *A continuous function $\phi : [0,\infty) \to [0,\infty)$ is said to be* of admissible growth *if $\{\phi(2^j)\}_{j\in\mathbb{Z}}$ is an admissible sequence and $\phi(t) \sim \phi(2^k)$ for any $k \in \mathbb{Z}$ and $t \in [2^k, 2^{k+1})$ with the positive equivalence constants independent of both t and k.*

We point out that, for any given admissible sequence $\sigma := \{\sigma_j\}_{j\in\mathbb{Z}}$, there exists a continuous function ϕ of admissible growth such that, for any $j \in \mathbb{Z}$, $\phi(2^{-j}) = 1/\sigma_j$. Indeed, the function

$$\phi_\sigma(t) := 2^{j+1}\left(\frac{1}{\sigma_j} - \frac{1}{\sigma_{j+1}}\right)(t - 2^{-j-1}) + \frac{1}{\sigma_{j+1}}, \quad \forall\, t \in [2^{-j-1}, 2^{-j}),\ \forall\, j \in \mathbb{Z}, \tag{4}$$

suits this job; see ([28] [Proposition 2.4]) or ([14] [Example 2.3]). Throughout this article, for any given admissible sequence $\sigma := \{\sigma_j\}_{j\in\mathbb{Z}}$, we *always* let ϕ_σ be as in (4).

For any given sequence $\sigma := \{\sigma_k\}_{k\in\mathbb{Z}}$ of positive numbers or any given function $\phi : [0,\infty) \to [0,\infty)$, let

$$\alpha_\sigma := \max\{\alpha_\sigma^-, \alpha_\sigma^+\} := \max\left\{\limsup_{k\to-\infty} \frac{\sigma_k}{\sigma_{k+1}},\ \limsup_{k\to\infty} \frac{\sigma_k}{\sigma_{k+1}}\right\},$$

$$\beta_\sigma := \max\{\beta_\sigma^-, \beta_\sigma^+\} := \max\left\{\limsup_{k\to-\infty} \frac{\sigma_{k+1}}{\sigma_k},\ \limsup_{k\to\infty} \frac{\sigma_{k+1}}{\sigma_k}\right\},$$

$$\alpha_\phi := \max\{\alpha_\phi^-, \alpha_\phi^+\} := \max\left\{\limsup_{k\to-\infty} \frac{\phi(2^k)}{\phi(2^{k+1})},\ \limsup_{k\to\infty} \frac{\phi(2^k)}{\phi(2^{k+1})}\right\},$$

and

$$\beta_\phi := \max\{\beta_\phi^-, \beta_\phi^+\} := \max\left\{\limsup_{k\to-\infty} \frac{\phi(2^{k+1})}{\phi(2^k)},\ \limsup_{k\to\infty} \frac{\phi(2^{k+1})}{\phi(2^k)}\right\}.$$

Since, for any $j \in \mathbb{Z}$, $\phi_\sigma(2^{-j}) = 1/\sigma_j$, then $\alpha_{\phi_\sigma}^- = \alpha_\sigma^+$, $\alpha_{\phi_\sigma}^+ = \alpha_\sigma^-$, $\beta_{\phi_\sigma}^- = \beta_\sigma^+$, and $\beta_{\phi_\sigma}^+ = \beta_\sigma^-$, which means that $\alpha_{\phi_\sigma} = \alpha_\sigma$ and $\beta_{\phi_\sigma} = \beta_\sigma$. By an obvious observation that $1/\alpha_\sigma^- \le \beta_\sigma^-$ and $1/\alpha_\sigma^+ \le \beta_\sigma^+$, it is also easy to show that $1/\alpha_\sigma \le \beta_\sigma$; furthermore, $\alpha_\phi < 1$ implies $\beta_\phi > 1$, and $\beta_\phi < 2$ implies $\alpha_\phi > 1/2$.

Observe that, if $\alpha_\phi^- \in (0,1)$ (resp., $\alpha_\phi^+ \in (0,1)$), then there exists a $\delta_1 \in (0,\infty)$ such that $\alpha_\phi^- + \delta_1 < 1$ (resp., $\alpha_\phi^+ + \delta_1 < 1$). Let K_0 be a given integer. By the definition of α_ϕ^- (resp., α_ϕ^+), we find that there exists an integer K_1 (resp., K_2) such that, for any $k \in (-\infty, \min\{K_1, K_0\}]$ (resp., $k \in [\max\{K_2, K_0\}, \infty)$),

$$\frac{\phi(2^k)}{\phi(2^{k+1})} < \alpha_\phi^- + \delta_1 \qquad \left(\text{resp., } \frac{\phi(2^k)}{\phi(2^{k+1})} < \alpha_\phi^- + \delta_1\right)$$

and hence, for any $i,\ j \in (-\infty, \min\{K_1, K_0\}]$ (resp., $i,\ j \in [\max\{K_2, K_0\}, \infty)$) with $i \le j$,

$$\frac{\phi(2^i)}{\phi(2^j)} < (\alpha_\phi^- + \delta_1)^{j-i} \quad \left(\text{resp., } \frac{\phi(2^i)}{\phi(2^j)} < (\alpha_\phi^+ + \delta_1)^{j-i}\right). \tag{5}$$

Since $\phi(2^k)/\phi(2^{k+1})$ is bounded on $[\min\{K_1, K_0\}, K_0]$ (resp., $k \in [K_0, \max\{K_2, K_0\}]$), then, from (5), we deduce that there exists a positive constant C, depending only on K_0, ϕ, and δ_1, such that, for any $i,\ j \in (-\infty, K_0] \cap \mathbb{Z}$ (resp., $i,\ j \in [K_0, \infty) \cap \mathbb{Z}$) with $i \le j$,

$$\frac{\phi(2^i)}{\phi(2^j)} \le C(\alpha_\phi^- + \delta_1)^{j-i} \quad \left(\text{resp., } \frac{\phi(2^i)}{\phi(2^j)} \le C(\alpha_\phi^+ + \delta_1)^{j-i}\right). \tag{6}$$

By this, we further obtain, for any $k_0 \in (-\infty, K_0] \cap \mathbb{Z}$ (resp., $k_0 \in [K_0, \infty) \cap \mathbb{Z}$) and $r \in (0, \infty]$,

$$\begin{aligned}\left\{\sum_{k \le k_0} \left[\phi(2^k)\right]^r\right\}^{1/r} &= \phi(2^{k_0}) \left\{\sum_{k \le k_0} \left[\frac{\phi(2^k)}{\phi(2^{k_0})}\right]^r\right\}^{1/r} \\ &\lesssim \phi(2^{k_0}) \left\{\sum_{k \le k_0} (\alpha_\phi^- + \delta_1)^{(k_0 - k)r}\right\}^{1/r} \lesssim \phi(2^{k_0})\end{aligned}$$

$$\left(\text{resp., } \left\{\sum_{k \ge k_0} \left[\frac{1}{\phi(2^k)}\right]^r\right\}^{1/r} \le \frac{1}{\phi(2^{k_0})} \left\{\sum_{k \ge k_0} (\alpha_\phi^+ + \delta_1)^{(k - k_0)r}\right\}^{1/r} \lesssim \frac{1}{\phi(2^{k_0})}\right),$$

where the implicit positive constants depend only on K_0, ϕ, and δ_1.

If $\beta_\phi^- \in (0, 2)$ (resp., $\beta_\phi^+ \in (0, 2)$), by an argument similar to the above, we conclude that there exist a $\delta_2 \in (0, \infty)$ such that $\beta_\phi^- + \delta_2 < 2$ (resp., $\beta_\phi^+ + \delta_2 < 2$) and a positive constant C, depending only on K_0, ϕ, and δ_2, such that, for any $i,\ j \in (-\infty, K_0] \cap \mathbb{Z}$ (resp., $i,\ j \in [K_0, \infty) \cap \mathbb{Z}$) with $i \le j$,

$$2^{i-j} \frac{\phi(2^j)}{\phi(2^i)} \le C\left(\frac{\beta_\phi^- + \delta_2}{2}\right)^{j-i} \quad \left(\text{resp., } 2^{i-j} \frac{\phi(2^j)}{\phi(2^i)} \le C\left(\frac{\beta_\phi^+ + \delta_2}{2}\right)^{j-i}\right). \tag{7}$$

Furthermore, for any $k_0 \in (-\infty, K_0] \cap \mathbb{Z}$ (resp., $k_0 \in [K_0, \infty) \cap \mathbb{Z}$) and $r \in (0, \infty]$, we have

$$\begin{aligned}\left\{\sum_{k \le k_0} \left[\frac{2^k}{\phi(2^k)}\right]^r\right\}^{1/r} &= \frac{2^{k_0}}{\phi(2^{k_0})} \left\{\sum_{k \le k_0} \left[2^{k - k_0} \frac{\phi(2^{k_0})}{\phi(2^k)}\right]^r\right\}^{1/r} \\ &\lesssim \frac{2^{k_0}}{\phi(2^{k_0})} \left\{\sum_{k \le k_0} \left(\frac{\beta_\phi^- + \delta_2}{2}\right)^{(k_0 - k)r}\right\}^{1/r} \lesssim \frac{2^{k_0}}{\phi(2^{k_0})}\end{aligned} \tag{8}$$

$$\left(\text{resp., } \left\{\sum_{k \ge k_0} \left[2^{-k} \phi(2^k)\right]^r\right\}^{1/r} \lesssim 2^{-k_0} \phi(2^{k_0}) \left\{\sum_{k \ge k_0} \left(\frac{\beta_\phi^+ + \delta_2}{2}\right)^{(k - k_0)r}\right\}^{1/r} \lesssim 2^{-k_0} \phi(2^{k_0})\right).$$

If $\alpha_\phi \in (0,1)$ (resp., $\beta_\phi \in (0,2)$), then $\alpha_\phi^- \in (0,1)$ and $\alpha_\phi^+ \in (0,1)$ (resp., $\beta_\phi^- \in (0,2)$ and $\beta_\phi^+ \in (0,2)$). Thus, by (6) and (7), we obtain, for any $i,\ j \in \mathbb{Z}$ with $i \le j$,

$$\frac{\phi(2^i)}{\phi(2^j)} \lesssim (\alpha_\phi + \delta_1)^{j-i} \quad \left(\text{resp., } 2^{i-j}\frac{\phi(2^j)}{\phi(2^i)} \lesssim \left(\frac{\beta_\phi + \delta_2}{2}\right)^{j-i}\right),$$

where δ_1 (resp., δ_2) is any given positive constant such that $\alpha_\phi + \delta_1 < 1$ (resp., $\beta_\phi + \delta_2 < 2$), and the implicit positive constants depend only on ϕ and δ_1 (resp., δ_2). By this, we conclude that, for any $r \in (0,\infty]$ and $k_0 \in \mathbb{Z}$,

$$\left\{\sum_{k \le k_0} \left[\phi(2^k)\right]^r\right\}^{1/r} \lesssim \phi(2^{k_0}) \quad \text{and} \quad \left\{\sum_{k \ge k_0} \left[\frac{1}{\phi(2^k)}\right]^r\right\}^{1/r} \lesssim \frac{1}{\phi(2^{k_0})} \tag{9}$$

$$\left(\text{resp., } \left\{\sum_{k \le k_0} \left[\frac{2^k}{\phi(2^k)}\right]^r\right\}^{1/r} \lesssim \frac{2^{k_0}}{\phi(2^{k_0})} \quad \text{and} \quad \left\{\sum_{k \ge k_0} \left[2^{-k}\phi(2^k)\right]\right\}^{1/r} \lesssim 2^{-k_0}\phi(2^{k_0})\right). \tag{10}$$

Here, the implicit positive constants depend only on ϕ.

The following lemma is just ([28] [Lemma 2.5]).

Lemma 1. *Let $\phi : [0,\infty) \to [0,\infty)$ satisfy $\alpha_\phi \in (0,1)$, $\varepsilon \in (0, -\log_2 \alpha_\phi)$, and $\delta \in (\log_2 \beta_\phi, \infty)$. Then,*

(i) *there exist positive constants C_1 and C_2, depending on ϕ, such that, for any $k \in \mathbb{Z}$,*

$$\sum_{j \ge k} \frac{2^{j\varepsilon}}{\phi(2^j)} \le C_1 \frac{2^{k\varepsilon}}{\phi(2^k)} \quad \textit{and} \quad \sum_{j \le k} 2^{-j\varepsilon}\,\phi(2^j) \le C_2\, 2^{-k\varepsilon}\phi(2^k);$$

(ii) *there exist positive constants c_1 and c_2, depending on ϕ, such that, for any $i,\ j \in \mathbb{Z}$ with $i \le j$,*

$$2^{(j-i)\varepsilon}\,\frac{\phi(2^i)}{\phi(2^j)} \le c_1 \quad \textit{and} \quad 2^{(i-j)\delta}\,\frac{\phi(2^j)}{\phi(2^i)} \le c_2.$$

We recall another widely used notion (see, for instance, [50], Section 2.2.1) to describe the smoothness function as follows.

Definition 3. *A function $f : [0,\infty) \to [0,\infty)$ is said to be* almost increasing *(resp.,* decreasing*) if there exists a positive constant $C \in [1,\infty)$ such that, for any $t_1,\ t_2 \in [0,\infty)$ with $t_1 \le t_2$ (resp., $t_1 \ge t_2$), $f(t_1) \le C f(t_2)$.*

Throughout this article, for simplicity, we *always denote* by $\mathcal{A}$ the class of all continuous and almost increasing functions $\phi : [0,\infty) \to [0,\infty)$ satisfying that $\phi(0) = 0$, $\phi(1) = 1$, and $\{\phi(2^j)\}_{j\in\mathbb{Z}}$ is admissible.

Let $\mathcal{A}_\infty$ be the set of all functions $\phi \in \mathcal{A}$ satisfying that the function $\widetilde{\phi}$, defined by setting, for any $t \in [0,\infty)$, $\widetilde{\phi}(t) := \phi(t)/t$, almost decreases.

For any $r \in (0,\infty)$, let $\mathcal{A}_r$ be the set of all functions $\phi \in \mathcal{A}_\infty$ satisfying that ϕ is of admissible growth and that there exist a $k_0 \in \mathbb{Z}$ and two positive constants X_{k_0} and Y_{k_0}, depending on k_0 and r, such that

$$\left\{\sum_{j \ge k_0} \left[\phi(2^j)\right]^{-r}\right\}^{1/r} \le X_{k_0} \quad \text{and} \quad \left\{\sum_{j \ge k_0} 2^{-jr}\left[\phi(2^{-j})\right]^{-r}\right\}^{1/r} \le Y_{k_0}. \tag{11}$$

We claim that if, for some $k_0 \in \mathbb{Z}$, there exist positive constants X_{k_0} and Y_{k_0} such that (11) holds true, then, for any $k \in \mathbb{Z}$, there exist positive constants X_k and Y_k, depending on k and r, such that (11) holds true with k_0 replaced by k. Indeed, this claim is trivial when $k \geq k_0$, while when $k < k_0$, it easily follows from the fact that $\sum_{j=k}^{k_0-1}[\phi(2^j)]^{-r}$ and $\sum_{j=k}^{k_0-1} 2^{-jr}[\phi(2^{-j})]^{-r}$ are always finite. This proves the above claim.

Clearly, by (3), $\mathcal{A}_{r_1} \subset \mathcal{A}_{r_2} \subset \mathcal{A}_\infty$ for any $r_1,\ r_2 \in (0,\infty)$ with $r_1 \leq r_2$. For instance, for any $b \in (0,\infty)$ and $r \in (1/b,\infty]$, the function

$$\phi(t) := \begin{cases} [\log_2(1+t)]^b, & t \in (0,1), \\ (1+\log_2 t)^b, & t \in [1,\infty) \end{cases} \tag{12}$$

belongs to $\mathcal{A}_r$.

If ϕ is of admissible growth, then $\alpha_\phi \in (0,1)$ implies $\phi \in \mathcal{A}$; furthermore, $\alpha_\phi \in (0,1)$, together with $\beta_\phi^- \in (0,2)$, implies that, for any $r \in (0,\infty]$, $\phi \in \mathcal{A}_r$. In view of these, we let $\mathcal{A}_0$ be the class of all functions ϕ satisfying that $\alpha_\phi \in (0,1)$, $\beta_\phi^- \in (0,2)$, and ϕ is of admissible growth.

Now, we state the notions of generalized Hajłasz gradients and the related Hajłasz-type spaces with respect to the smoothness function $\phi \in \mathcal{A}$.

Definition 4. *Let $\phi \in \mathcal{A}$ and $u \in L^0(\mathcal{X})$.*

(i) *A nonnegative measurable function g is called a ϕ-*Hajłasz gradient *of u if there exists a set $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that, for any $x,\ y \in \mathcal{X} \setminus E$,*

$$|u(x) - u(y)| \leq \phi(d(x,y))[g(x) + g(y)]. \tag{13}$$

Denote by $\mathcal{D}^\phi(u)$ the collection of all ϕ-Hajłasz gradients of u.

(ii) *A sequence of nonnegative measurable functions, $\vec{g} := \{g_k\}_{k\in\mathbb{Z}}$, is called a ϕ-*Hajłasz gradient sequence *of u if, for any $k \in \mathbb{Z}$, there exists a set $E_k \subset \mathcal{X}$ with $\mu(E_k) = 0$ such that, for any $x,\ y \in \mathcal{X} \setminus E_k$ with $2^{-k-1} \leq d(x,y) < 2^{-k}$,*

$$|u(x) - u(y)| \leq \phi(d(x,y))[g_k(x) + g_k(y)].$$

Denote by $\mathbb{D}^\phi(u)$ the collection of all ϕ-Hajłasz gradient sequences of u.

The following are basic properties of these generalized gradients, which can be proved by an argument similar to those about classical Hajłasz gradients (see, for instance, ([51] [Lemma 2.4]), ([38] [Lemma 2.6]), ([41] [Lemmas 2.3 and 2.4]), and ([45][Lemmas 4 and 5])); we omit the details.

Lemma 2. (i) *Let $u,\ v \in L^0(\mathcal{X})$, $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(u)$, and $\{h_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(v)$. Then,*

$$\{\max(g_k,h_k)\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(\max\{u,v\}) \quad \textit{and} \quad \{\max(g_k,h_k)\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(\min\{u,v\}).$$

(ii) *Let $\{u_i\}_{i\in\mathbb{N}} \subset L^0(\mathcal{X})$ and, for any $i \in \mathbb{N}$, let $\{g_k^{(i)}\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(u_i)$. Let $u := \sup_{i\in\mathbb{N}} u_i$ and $\{g_k\}_{k\in\mathbb{Z}} := \{\sup_{i\in\mathbb{N}} g_k^{(i)}\}_{k\in\mathbb{Z}}$. If $u \in L^0(\mathcal{X})$, then $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(u)$.*

Using these generalized gradients, we introduced the following homogeneous ϕ-Hajłasz–Triebel–Lizorkin and ϕ-Hajłasz–Besov spaces in [28].

Definition 5. *Let $\phi \in \mathcal{A}$ and $p,\ q \in (0,\infty]$.*

(i) *The homogeneous ϕ-Hajłasz–Triebel–Lizorkin space $\dot{M}^\phi_{p,q}(\mathcal{X})$ is defined to be the set of all $u \in L^0(\mathcal{X})$ such that*

$$\|u\|_{\dot{M}^\phi_{p,q}(\mathcal{X})} := \inf_{\vec{g}\in\mathbb{D}^\phi(u)} \|\vec{g}\|_{L^p(\mathcal{X},l^q)} < \infty$$

when $p \in (0,\infty)$ *and* $q \in (0,\infty]$, *or* $p = q = \infty$, *and*

$$\|u\|_{\dot{M}^{\phi}_{\infty,q}(\mathcal{X})} := \inf_{\vec{g}\in\mathbb{D}^{\phi}(u)} \sup_{k\in\mathbb{Z}} \sup_{x\in\mathcal{X}} \left\{ \sum_{j\geq k} \fint_{B(x,2^{-k})} [g_j(y)]^q \, d\mu(y) \right\}^{\frac{1}{q}} < \infty$$

when $p = \infty$ *and* $q \in (0,\infty)$.

(ii) *The homogeneous* ϕ*-Hajłasz–Besov space* $\dot{N}^{\phi}_{p,q}(\mathcal{X})$ *is defined to be the set of all* $u \in L^0(\mathcal{X})$ *such that*

$$\|u\|_{\dot{N}^{\phi}_{p,q}(\mathcal{X})} := \inf_{\vec{g}\in\mathbb{D}^{\phi}(u)} \|\vec{g}\|_{l^q(\mathcal{X},L^p)} < \infty.$$

In [28], we proved that, when $\mathcal{X} = \mathbb{R}^n$, for any given admissible sequence $\sigma := \{\sigma_j\}_{j\in\mathbb{Z}}$ with $\alpha_\sigma \in (0,1)$ and $\beta_\sigma \in (0,2)$, $\dot{M}^{\phi_\sigma}_{p,q}(\mathbb{R}^n) = \dot{F}^{\sigma}_{p,q}(\mathbb{R}^n)$ for any given $p,\ q \in (n/[n-\log_2 \alpha_\sigma],\infty]$, and $\dot{N}^{\phi_\sigma}_{p,q}(\mathbb{R}^n) = \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ for any given $p \in (n/[n-\log_2 \alpha_\sigma],\infty]$ and $q \in (0,\infty]$, where $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\sigma}_{p,q}(\mathbb{R}^n)$ are, respectively, the classical generalized Besov and Triebel–Lizorkin spaces in which smoothness is described by an admissible sequence σ (see Definition 7 below). In this sense, the spaces $\dot{M}^{\phi}_{p,q}(\mathcal{X})$ and $\dot{N}^{\phi}_{p,q}(\mathcal{X})$ serve as natural generalizations of classical Besov and Triebel–Lizorkin spaces with generalized smoothness on metric measure spaces.

In this article, we also consider the inhomogeneous version of the above spaces.

Definition 6. *Let* $\phi \in \mathcal{A}$ *and* $p,\ q \in (0,\infty]$.

(i) *The inhomogeneous* ϕ*-Hajłasz–Triebel–Lizorkin space* $M^{\phi}_{p,q}(\mathcal{X})$ *is defined as the set* $L^p(\mathcal{X}) \cap \dot{M}^{\phi}_{p,q}(\mathcal{X})$. *Moreover, for any* $u \in M^{\phi}_{p,q}(\mathcal{X})$, *let*

$$\|u\|_{M^{\phi}_{p,q}(\mathcal{X})} := \|u\|_{L^p(\mathcal{X})} + \|u\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})}.$$

(ii) *The inhomogeneous* ϕ*-Hajłasz–Besov space* $N^{\phi}_{p,q}(\mathcal{X})$ *is defined as the set* $L^p(\mathcal{X}) \cap \dot{N}^{\phi}_{p,q}(\mathcal{X})$. *Moreover, for any* $u \in M^{\phi}_{p,q}(\mathcal{X})$, *let*

$$\|u\|_{M^{\phi}_{p,q}(\mathcal{X})} := \|u\|_{L^p(\mathcal{X})} + \|u\|_{\dot{N}^{\phi}_{p,q}(\mathcal{X})}.$$

Remark 1. (i) *Recall that, for any given* $p \in (0,\infty]$, $\dot{M}^{\phi}_{p,\infty}(\mathcal{X}) = \dot{M}^{\phi,p}(\mathcal{X})$ *(see [28], [Remark 3.4(i)]), where* $\dot{M}^{\phi,p}(\mathcal{X})$ *denotes the homogeneous Hajłasz–Sobolev space with respect to* ϕ, *which consists of all* $u \in L^0(\mathcal{X})$ *such that*

$$\|u\|_{\dot{M}^{\phi,p}(\mathcal{X})} := \inf_{g\in\mathcal{D}^{\phi}(u)} \|g\|_{L^p(\mathcal{X})} < \infty.$$

Consequently, if the inhomogeneous Hajłasz–Sobolev space $M^{\phi,p}(\mathcal{X})$ *is defined as the set* $L^p(\mathcal{X}) \cap \dot{M}^{\phi,p}(\mathcal{X})$, *then* $M^{\phi}_{p,\infty}(\mathcal{X}) = M^{\phi,p}(\mathcal{X})$. *In particular, when* ϕ *is as in (12), the related spaces are called the* logarithmic Hajłasz–Sobolev spaces.

(ii) *Let* $\phi \in \mathcal{A}$, $k_0 \in \mathbb{Z}$, *and* $u \in L^0(\mathcal{X})$. *Let* $\mathbb{D}^{\phi}_{k_0}(u)$ *be the set of all sequences* $\vec{h} := \{h_k\}_{k\in\mathbb{Z}}$, *defined by setting* $h_k := \widetilde{h}_k$ *when* $k \geq k_0$ *and* $h_k \equiv 0$ *when* $k < k_0$, *where* $\widetilde{h} := \{\widetilde{h}_k\}_{k\in\mathbb{Z}}$ *is a* ϕ*-Hajłasz gradient sequence of* u. *Naturally,* $\mathcal{D}^{\phi}_{k_0}(u)$ *denotes the set of all functions* g *such that, for almost every* $x,\ y \in \mathcal{X}$ *with* $d(x,y) < 2^{-k_0}$, *(13) holds true. Then, for any given* $p \in (0,\infty]$, $q = \infty$, *and* $\phi \in \mathcal{A}$ *or for any given* $p \in (0,\infty]$, $q \in (0,\infty)$, *and* $\phi \in \mathcal{A}$ *with* $\alpha^+_\phi \in (0,1)$,

$$|||u|||_{N^{\phi}_{p,q}(\mathcal{X})} := \|u\|_{L^p(\mathcal{X})} + \inf_{\vec{h}\in\mathbb{D}^{\phi}_{k_0}(u)} \|\vec{h}\|_{l^q(\mathcal{X},L^p)}, \quad \forall\, u \in N^{\phi}_{p,q}(\mathcal{X}),$$

is an equivalent quasi-norm of $N^{\phi}_{p,q}(\mathcal{X})$ with the positive equivalence constants depending on k_0. Indeed, for any $u \in L^0(\mathcal{X})$, $|||u|||_{N^{\phi}_{p,q}(\mathcal{X})} \le \|u\|_{N^{\phi}_{p,q}(\mathcal{X})}$ obviously holds true. Conversely, let $q \in (0,\infty)$ and $u \in L^0(\mathcal{X})$. Notice that, for any $k \in \mathbb{Z}$ and $x, y \in \mathcal{X}$,

$$\left|u(x) - u(y)\right| \le \phi(2^{-k})\left[\frac{|u(x)|}{\phi(2^{-k})} + \frac{|u(y)|}{\phi(2^{-k})}\right].$$

Then, $\{\frac{|u|}{\phi(2^{-k})}\}_{k\in\mathbb{Z}}$ is a ϕ-Hajłasz gradient sequence of u modulo some uniform constant, which implies that, for any $\vec{h} := \{h_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^{\phi}_{k_0}(u)$, the sequence $\vec{g} := \{g_k\}_{k\in\mathbb{Z}}$, defined by setting, for any $k \ge k_0$, $g_k := h_k$ and, for any $k < k_0$, $g_k := \frac{|u|}{\phi(2^{-k})}$ is an element of $\mathbb{D}^{\phi}_{k_0}(u)$. By $\alpha^{+}_{\phi} \in (0,1)$, we can choose a $\delta_1 \in (0,\infty)$ such that $\alpha^{+}_{\phi} + \delta_1 < 1$. Then, there exists a $K \in \mathbb{Z}$ such that, for any integer $k \le K$, $\phi(2^{-k})/\phi(2^{-k+1}) < a^{+}_{\phi} + \delta_1$. Notice that $\phi(2^{-k})/\phi(2^{-k+1})$ is bounded when $k \in [K, k_0]$. We then have

$$\begin{aligned}\sum_{k\le k_0}[\phi(2^{-k})]^{-q} &= [\phi(2^{-k_0})]^{-q}\sum_{k\le k_0}\left[\frac{\phi(2^{-k-1})}{\phi(2^{-k})}\frac{\phi(2^{-k-2})}{\phi(2^{-k-1})}\cdots\frac{\phi(2^{-k_0})}{\phi(2^{-k_0+1})}\right]^{q}\\ &\lesssim \sum_{k\le k_0}(\alpha^{+}_{\phi}+\delta_1)^{(k_0-k)q} \lesssim 1,\end{aligned}$$

where the implicit positive constants depend only on ϕ, q, and k_0. This implies that

$$\|u\|_{N^{\phi}_{p,q}(\mathcal{X})} \le \|\vec{g}\|_{l^q(\mathcal{X},L^p)} + \|u\|_{L^p(\mathcal{X})} \lesssim \|\vec{h}\|_{l^q(\mathcal{X},L^p)} + \|u\|_{L^p(\mathcal{X})} \lesssim |||u|||_{N^{\phi}_{p,q}(\mathcal{X})}.$$

The proof for the case $q = \infty$ is similar, and we omit the details here.

Similarly, for any $\phi \in \mathcal{A}$ with $\alpha^{+}_{\phi} \in (0,1)$, $p \in (0,\infty]$, and $q \in (0,\infty)$ or any $\phi \in \mathcal{A}$ with $p \in (0,\infty]$ and $q = \infty$, $|||u|||_{M^{\phi}_{p,q}(\mathcal{X})}$, defined by replacing $\vec{g} \in \mathbb{D}^{\phi}(u)$ in $\|u\|_{M^{\phi}_{p,q}(\mathcal{X})}$ by $\vec{h} \in \mathbb{D}^{\phi}_{k_0}(u)$, is also an equivalent quasi-norm of $M^{\phi}_{p,q}(\mathcal{X})$.

As was mentioned above, the spaces $\dot{M}^{\phi_\sigma}_{p,q}(\mathbb{R}^n)$ and $\dot{N}^{\phi_\sigma}_{p,q}(\mathbb{R}^n)$ coincide, respectively, with the Triebel–Lizorkin space $\dot{F}^{\sigma}_{p,q}(\mathbb{R}^n)$ and the Besov space $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ with generalized smoothness; see [28]. It is natural to expect to obtain their inhomogeneous counterparts. To this end, we let $\mathcal{S}(\mathbb{R}^n)$ be the collection of all Schwartz functions on $\mathbb{R}^n$, in which the topology is determined by a family of norms, $\{\|\cdot\|_{\mathcal{S}_{k,m}(\mathbb{R}^n)}\}_{k,m\in\mathbb{Z}_+}$, where, for any k, $m \in \mathbb{Z}_+$ and any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\varphi\|_{\mathcal{S}_{k,m}(\mathbb{R}^n)} := \sup_{\alpha\in\mathbb{Z}^n_+,|\alpha|\le k}\ \sup_{x\in\mathbb{R}^n}(1+|x|)^m\left|\partial^\alpha\varphi(x)\right|$$

with $\alpha := (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n_+$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$. Additionally, let $\mathcal{S}'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology. Define

$$\mathcal{S}_\infty(\mathbb{R}^n) := \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) :\ \int_{\mathbb{R}^n}\varphi(x)\,x^\gamma\,dx = 0 \text{ for all multi-indices } \gamma \in \mathbb{Z}^n_+\right\},$$

and let $\mathcal{S}'_\infty(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}'_\infty(\mathbb{R}^n)$ equipped with the weak-$*$ topology. For any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, we use $\widehat{f}$ to denote its Fourier transform in the sense of $\mathcal{S}'_\infty(\mathbb{R}^n)$; in particular, for any $f \in L^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)\,e^{-2\pi i x\cdot\xi}\,dx$. For any $t \in (0,\infty)$ and $x \in \mathbb{R}^n$, let $\varphi_t(x) := t^{-n}\varphi(x/t)$.

Definition 7. *Let $\sigma := \{\sigma_j\}_{j\in\mathbb{Z}}$ be an admissible sequence. Let $p, q \in (0,\infty]$ and $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ be such that*

$$\operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}, \ \text{and} \ |\widehat{\varphi}(\xi)| \ge C_1 \ \text{if} \ 3/5 \le |\xi| \le 5/3$$

and

$$\operatorname{supp}\widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}, \ \text{and} \ |\widehat{\Phi}(\xi)| \ge C_2 \ \text{if} \ |\xi| \le 5/3,$$

where C_1, C_2 are two positive constants.

(i) *The* homogeneous Triebel–Lizorkin space $\dot{F}^{\sigma}_{p,q}(\mathbb{R}^n)$ with generalized smoothness *is defined as the set of all $u \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|u\|_{\dot{F}^{\sigma}_{p,q}(\mathbb{R}^n)} < \infty$, where, when $p < \infty$,*

$$\|u\|_{\dot{F}^{\sigma}_{p,q}(\mathbb{R}^n)} := \|\{\sigma_k\,\varphi_{2^{-k}} * u\}_{k\in\mathbb{Z}}\|_{L^p(\mathbb{R}^n,l^q)} := \left\|\left(\sum_{k\in\mathbb{Z}} \sigma_k^q \left|\varphi_{2^{-k}} * u\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

with the usual modification made if $q = \infty$ and, when $p = \infty$,

$$\|u\|_{\dot{F}^{\sigma}_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n}\sup_{l\in\mathbb{Z}}\left\{\fint_{B(x,2^{-l})}\sum_{k\ge l}\sigma_k^q\left|\varphi_{2^{-k}} * u(y)\right|^q dy\right\}^{1/q}$$

with the usual modification made if $q = \infty$.

(ii) *The* homogeneous Besov space $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ with generalized smoothness *is defined as the set of all $u \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that*

$$\|u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} := \|\{\sigma_k\,\varphi_{2^{-k}} * u\}_{k\in\mathbb{Z}}\|_{l^q(\mathbb{R}^n,L^p)} := \left[\sum_{k\in\mathbb{Z}}\sigma_k^q\left\|\varphi_{2^{-k}} * u\right\|^q_{L^p(\mathbb{R}^n)}\right]^{1/q} < \infty$$

with the usual modification made if $q = \infty$.

(iii) *The* inhomogeneous Triebel–Lizorkin space $F^{\sigma}_{p,q}(\mathbb{R}^n)$ with generalized smoothness *is defined as the set of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|u\|_{F^{\sigma}_{p,q}(\mathbb{R}^n)}$ is finite, where $\|u\|_{F^{\sigma}_{p,q}(\mathbb{R}^n)}$ is defined as $\|u\|_{\dot{F}^{\sigma}_{p,q}(\mathbb{R}^n)}$ with $\{\sigma_k\,\varphi_{2^{-k}} * u\}_{k\in\mathbb{Z}}$ and φ_1 replaced, respectively, by $\{\sigma_k\,\varphi_{2^{-k}} * u\}_{k\in\mathbb{Z}_+}$ and Φ.*

(iv) *The* inhomogeneous Besov space $B^{\sigma}_{p,q}(\mathbb{R}^n)$ with generalized smoothness *is defined as the set of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|u\|_{B^{\sigma}_{p,q}(\mathbb{R}^n)}$ is finite, where $\|u\|_{B^{\sigma}_{p,q}(\mathbb{R}^n)}$ is defined as $\|u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}$ with $\{\sigma_k\,\varphi_{2^{-k}} * u\}_{k\in\mathbb{Z}}$ and φ_1 replaced, respectively, by $\{\sigma_k\,\varphi_{2^{-k}} * u\}_{k\in\mathbb{Z}_+}$ and Φ.*

We then have the following relation between homogeneous and inhomogeneous spaces.

Proposition 1. *Let $p \in [1,\infty]$, $q \in (0,\infty]$, and $\sigma := \{\sigma_j\}_{j\in\mathbb{Z}_+}$ be admissible sequences with $\alpha^+_\sigma \in (0,1)$. Then, for $A \in \{B, F\}$, $A^{\sigma}_{p,q}(\mathbb{R}^n) = [L^p(\mathbb{R}^n) \cap \dot{A}^{\widetilde{\sigma}}_{p,q}(\mathbb{R}^n)]$, where $\widetilde{\sigma} := \{\widetilde{\sigma}_j\}_{j\in\mathbb{Z}}$ is any given admissible sequence satisfying that, for any $j \in \mathbb{Z}_+$ and $\alpha^-_{\widetilde{\sigma}} \in (0,1)$, $\widetilde{\sigma}_j = \sigma_j$.*

Proof. By similarity, we only consider the Triebel–Lizorkin case.

First, we show $F^{\sigma}_{p,q}(\mathbb{R}^n) \subset [L^p(\mathbb{R}^n) \cap \dot{F}^{\widetilde{\sigma}}_{p,q}(\mathbb{R}^n)]$. From $p \in [1,\infty]$, $\alpha^+_\sigma < 1$, ([14] [Corollary 3.18]), or ([52] [Theorem 4.1]), we deduce that $B^{\sigma}_{p,\max\{p,q\}}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, which, together with the trivial embedding $F^{\sigma}_{p,q}(\mathbb{R}^n) \subset B^{\sigma}_{p,\max\{p,q\}}(\mathbb{R}^n)$, implies that $F^{\sigma}_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and, for any $u \in F^{\sigma}_{p,q}(\mathbb{R}^n)$, $\|u\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{F^{\sigma}_{p,q}(\mathbb{R}^n)}$. Moreover, if $p \in [1,\infty)$, applying (3) when $p/q \le 1$, the Minkowski inequality when $p/q > 1$, or the Minkowski integral inequality, we conclude that, for any $u \in F^{\sigma}_{p,q}(\mathbb{R}^n)$,

$$\left\|\left(\sum_{k\le 0}\widetilde{\sigma}_k^q\left|\varphi_{2^{-k}} * u\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \le \left[\sum_{k\le 0}\widetilde{\sigma}_k^{\min\{p,q\}}\|\varphi_{2^{-k}} * u\|^{\min\{p,q\}}_{L^p(\mathbb{R}^n)}\right]^{1/\min\{p,q\}}$$

$$\lesssim \left[\sum_{k\le 0}\widetilde{\sigma}_k^{\min\{p,q\}}\right]^{1/\min\{p,q\}}\|u\|_{L^p(\mathbb{R}^n)}.$$

By $\alpha_{\widetilde{\sigma}}^- \in (0,1)$, we know that there exists a $\delta_1 \in (0,\infty)$ small enough such that $\alpha_{\widetilde{\sigma}}^- + \delta_1 < 1$. Then we have, for any $k \le 0$ and $r \in (0,\infty]$,

$$\sum_{k\le 0}\widetilde{\sigma}_k^r \lesssim (\alpha_{\widetilde{\sigma}}^- + \delta_1)^{kr},$$

where the implicit positive constant only depends on $\widetilde{\sigma}$ and δ_1. Therefore, we obtain

$$\left\|\left(\sum_{k\le 0}\widetilde{\sigma}_k^q\left|\varphi_{2^{-k}} * u\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \lesssim \left[\sum_{k\le 0}(\alpha_{\widetilde{\sigma}}^- + \delta_1)^{k\min\{p,q\}}\right]^{1/\min\{p,q\}}\|u\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{L^p(\mathbb{R}^n)},$$

which implies that $\|u\|_{\dot{F}^{\widetilde{\sigma}}_{p,q}(\mathbb{R}^n)} \lesssim \|u\|_{L^p(\mathbb{R}^n)} + \|u\|_{F^{\widetilde{\sigma}}_{p,q}(\mathbb{R}^n)}$. Similar estimates also holds true for the case $p = \infty$. Altogether, we obtain the embedding $F^{\sigma}_{p,q}(\mathbb{R}^n) \subset [L^p(\mathbb{R}^n) \cap \dot{F}^{\widetilde{\sigma}}_{p,q}(\mathbb{R}^n)]$.

Conversely, let $u \in [L^p(\mathbb{R}^n) \cap \dot{F}^{\widetilde{\sigma}}_{p,q}(\mathbb{R}^n)]$. By the Minkowski integral inequality, we know that, for any given $p \in [1,\infty]$, $\|\Phi * u\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{L^p(\mathbb{R}^n)}$. This, combined with the obvious fact that $\|\{\sigma_k|\varphi_{2^{-k}} * u|\}_{k\ge 1}\|_{L^p(\mathbb{R}^n,l^q)} \le \|u\|_{\dot{F}^{\widetilde{\sigma}}_{p,q}(\mathbb{R}^n)}$, implies the embedding $[L^p(\mathbb{R}^n) \cap \dot{F}^{\widetilde{\sigma}}_{p,q}(\mathbb{R}^n)] \subset F^{\sigma}_{p,q}(\mathbb{R}^n)$. This finishes the proof of Proposition 1. □

As an application of Proposition 1 and ([28], Theorem 3.10), we immediately obtain the following conclusion; we omit the details.

Corollary 1. *Let $p \in [1,\infty]$, and $\sigma := \{\sigma_j\}_{j\in\mathbb{Z}_+}$ be an admissible sequence with $\alpha_\sigma^+ \in (0,1)$ and $\beta_\sigma^+ \in (0,2)$. Then, $F^{\sigma}_{p,q}(\mathbb{R}^n) = M^{\phi_{\widetilde{\sigma}}}_{p,q}(\mathbb{R}^n)$ for any $q \in (n/[n-\log_2\alpha_\sigma^+],\infty]$ and $B^{\sigma}_{p,q}(\mathbb{R}^n) = N^{\phi_{\widetilde{\sigma}}}_{p,q}(\mathbb{R}^n)$ for any $q \in (0,\infty]$, where $\widetilde{\sigma} := \{\widetilde{\sigma}_j\}_{j\in\mathbb{Z}}$ is any given admissible sequence satisfying $\widetilde{\sigma}_j = \sigma_j$ for any $j \in \mathbb{Z}_+$, $\alpha_{\widetilde{\sigma}}^- \in (0,1)$, and $\beta_{\widetilde{\sigma}}^- \in (0,2)$.*

3. Lebesgue Points of ϕ-Hajłasz-Type Functions

Let u be a function on the metric measure space $(\mathcal{X}, d, \mu)$. A point $x \in \mathcal{X}$ is called a *Lebesgue point* of u if it satisfies

$$\lim_{r\to 0^+} ⨍_{B(x,r)}\left|u(y) - u(x)\right| d\mu(y) = 0.$$

For such an x,

$$u(x) = \lim_{r\to 0^+} ⨍_{B(x,r)} u(y)\, d\mu(y).$$

Here and thereafter, $t \to 0^+$ means $t \in (0,\infty)$ and $t \to 0$. The classical Lebesgue differentiation theorem states that almost every point is a Lebegsue point of a locally integrable function on $\mathbb{R}^n$. If the function has higher regularity, one could expect a smaller exceptional set. In 2002, Kinnunen and Latvala [38] studied the Lebesgue point of functions of Hajłasz–Sobolev spaces on doubling metric measure spaces, which has led to a lot of related works; see, for instance [39–44].

In this section, we study the Lebesgue point of ϕ-Hajłasz–Besov and ϕ-Hajłasz–Triebel–Lizorkin functions on a given doubling metric measure space $(\mathcal{X}, d, \mu)$. To this end, one key tool is the maximal operators. Let $R \in (0,\infty]$. The *restricted maximal operator* $\mathcal{M}_R$ is defined by setting, for any $u \in L^0(\mathcal{X})$ and $x \in \mathcal{X}$,

$$\mathcal{M}_R u(x) := \sup_{B_r\ni x,\, r\in(0,R)} ⨍_{B_r} |u|\, d\mu, \tag{14}$$

where the supremum is taken over all balls B_r in $\mathcal{X}$ containing x with the radius $r \in (0, R)$. Obviously, $\mathcal{M} := \mathcal{M}_\infty$ is just the classical Hardy–Littlewood maximal operator, which is known to be bounded on $L^p(\mathcal{X})$ for any given $p \in (1, \infty]$ when $\mathcal{X}$ is a doubling measure space; see, for instance ([53], Theorem 14.13). We also need the discrete Hardy–Littlewood-type maximal operator defined via discrete convolutions (see, for instance [38,41,54]). To recall this, we first need the notion of the *partition of unity*.

Definition 8. *Let $r \in (0, \infty)$, $\mathcal{J} \subset \mathbb{N}$ be an index set, and balls $\{B_j\}_{j\in\mathcal{J}}$ be a covering of $\mathcal{X}$ with the radius r such that $\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j} \lesssim 1$, where the implicit positive constant is some positive absolute constant. A sequence $\{\varphi_j\}_{j\in\mathcal{J}}$ of functions is called a* partition of unity *with respect to the above ball covering $\{B_j\}_{j\in\mathcal{J}}$ if, for any $j \in \mathcal{J}$, φ_j is a Lipschitz function with the Lipschitz constant cr^{-1}, $\varphi_j \geq C > 0$ on B_j, $\operatorname{supp} \varphi_j \subset \overline{2B_j}$, $0 \leq \varphi_j \leq 1$, and $\sum_{j\in\mathcal{J}} \varphi_j \equiv 1$, where c and C are two positive constants depending only on the doubling constant.*

The existence of the partition of unity in Definition 8 with respect to any given ball covering of $\mathcal{X}$ can be seen, for instance, in ([38], p. 690).

Definition 9. (i) *Let $u \in L^0(\mathcal{X})$. The* discrete convolution of u at the scale $r \in (0, \infty)$ *is defined by setting*

$$u_r := \sum_{j\in\mathcal{J}} u_{B_j} \varphi_j,$$

where $\{B_j\}_{j\in\mathcal{J}}$ is a ball covering of $\mathcal{X}$ with the radius r and $\{\varphi_j\}_{j\in\mathcal{J}}$ a partition of unity with respect to $\{B_j\}_{j\in\mathcal{J}}$ as in Definition 8.

(ii) *The* discrete maximal operator $\mathcal{M}^*$ *is defined by setting, for any $u \in L^0(\mathcal{X})$,*

$$\mathcal{M}^* u := \sup_{k\in\mathbb{Z}} |u|_{2^{-k}},$$

where $|u|_{2^{-k}}$ is the discrete convolution of $|u|$ at the scale 2^{-k}.

(iii) *Let $R \in (0, \infty]$. The* restricted discrete maximal operator $\mathcal{M}_R^*$ *is defined by setting, for any $u \in L^0(\mathcal{X})$,*

$$\mathcal{M}_R^* u := \sup_{\{k\in\mathbb{Z}:\, 2^{-k}<R\}} |u|_{2^{-k}},$$

where $|u|_{2^{-k}}$ is the discrete convolution of $|u|$ at the scale 2^{-k}.

Obviously, $\mathcal{M}_\infty^* = \mathcal{M}^*$. Now, we present two Poincaré-type inequalities with respect to ϕ as below. The first one is easy to prove using the definition of Hajłasz gradients, and the other is provided in ([28], Lemma 3.7).

Lemma 3. *Let $\phi \in \mathcal{A}$. Then, there exists a positive constant $C = C_{(\phi, C_\mu)}$ such that, for any $x \in \mathcal{X}$, $k \in \mathbb{Z}$, $u \in L^0(B(x, 2^{-k}))$, and $g \in \mathcal{D}^\phi(u)$,*

$$\inf_{c\in\mathbb{R}} \fint_{B\,(x,2^{-k})} |u(y) - c|\, d\mu(y) \leq C\, \phi(2^{-k}) \fint_{B\,(x,2^{-k})} g(y)\, d\mu(y),$$

where C_μ is as in (1).

Proof. Let $x \in \mathcal{X}$, $k \in \mathbb{Z}$, $u \in L^0(B(x, 2^{-k}))$ and $g \in \mathcal{D}^\phi(u)$. Then,

$$\begin{aligned}
\inf_{c\in\mathbb{R}} \fint_{B\,(x,2^{-k})} |u(y) - c|\, d\mu(y) &\leq \fint_{B\,(x,2^{-k})} |u(y) - u_{B(x,2^{-k})}|\, d\mu(y) \\
&\leq \fint_{B\,(x,2^{-k})} \fint_{B\,(x,2^{-k})} |u(y) - u(z)|\, d\mu(z)\, d\mu(y) \\
&\leq \fint_{B\,(x,2^{-k})} \fint_{B\,(x,2^{-k})} \phi(2^{-k+1})[g(y) + g(z)]\, d\mu(z)\, d\mu(y)
\end{aligned}$$

$$\lesssim \phi(2^{-k}) \fint_{B(x,2^{-k})} g(y)\, d\mu(y).$$

This finishes the proof of Lemma 3. □

Lemma 4. *Let $\phi \in \mathcal{A}$ with $\alpha_\phi \in (0,1)$. Then, for any ε, $\varepsilon' \in (0, -\log_2 \alpha_\phi)$ with $\varepsilon < \varepsilon'$ and $p \in (0, D/\varepsilon)$, there exists a positive constant $C = C_{(\phi,p,\varepsilon',C_\mu)}$ such that, for any $x \in \mathcal{X}$, $k \in \mathbb{Z}$, $u \in L^0(B(x, 2^{-k+1}))$ and $\vec{g} := \{g_j\}_{j\in\mathbb{Z}} \in \mathbb{D}^\phi(u)$,*

$$\begin{aligned} &\inf_{c\in\mathbb{R}} \left[\fint_{B(x,2^{-k})} |u(y) - c|^{\frac{Dp}{D-\varepsilon p}}\, d\mu(y) \right]^{\frac{D-\varepsilon p}{np}} \\ &\quad \le C\, 2^{-k\varepsilon'} \sum_{j\ge k-2} 2^{j\varepsilon'} \phi(2^{-j}) \left\{ \fint_{B(x,2^{-k+1})} [g_j(y)]^p\, d\mu(y) \right\}^{1/p}, \end{aligned} \tag{15}$$

where D and C_μ are as in (1).

Remark 2. *Let D and C_μ be as in (1).*

(i) *Let ϕ, ε, and p be as in Lemma 4. By taking, for any $k \in \mathbb{Z}$, $x \in \mathcal{X}$, $u \in L^0(B(x, 2^{-k+1}))$, and $g \in \mathcal{D}^\phi(u)$, $\varepsilon' := (\varepsilon - \log_2 \alpha_\phi)/2$ and $\vec{g} := \{g_j := g\}_{j\in\mathbb{Z}}$ in (15), we obtain*

$$\inf_{c\in\mathbb{R}} \left[\fint_{B(x,2^{-k})} |u(y) - c|^{\frac{Dp}{D-\varepsilon p}}\, d\mu(y) \right]^{\frac{D-\varepsilon p}{Dp}} \lesssim \phi(2^{-k}) \left\{ \fint_{B(x,2^{-k+1})} [g(y)]^p\, d\mu(y) \right\}^{1/p}, \tag{16}$$

where the implicit positive constant depends only on ϕ, p, ε, and C_μ.

(ii) *Notice that, if $Dp/(D - \varepsilon p) = 1$, then $p = D/(D + \varepsilon)$. In this case, (15) and (16) become, respectively,*

$$\begin{aligned} &\inf_{c\in\mathbb{R}} \fint_{B(x,2^{-k})} |u(y) - c|\, d\mu(y) \\ &\quad \le C_{(\phi,p,\varepsilon',C_\mu)}\, 2^{-k\varepsilon'} \sum_{j\ge k-2} 2^{j\varepsilon'} \phi(2^{-j}) \left\{ \fint_{B(x,2^{-k+1})} [g_j(y)]^{\frac{D}{D+\varepsilon}}\, d\mu(y) \right\}^{\frac{D+\varepsilon}{D}} \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\inf_{c\in\mathbb{R}} \fint_{B(x,2^{-k})} |u(y) - c|\, d\mu(y) \\ &\quad \le C_{(\phi,p,\varepsilon,C_\mu)}\, \phi(2^{-k}) \left\{ \fint_{B(x,2^{-k+1})} [g(y)]^{\frac{D}{D+\varepsilon}}\, d\mu(y) \right\}^{\frac{D+\varepsilon}{D}}. \end{aligned} \tag{18}$$

Applying these Poincaré-type inequalities, we obtain the following estimates.

Lemma 5. *Let $\phi \in \mathcal{A}$, D, and C_μ be as in (1) and $\mathcal{M}$ be the Hardy–Littlewood maximal operator.*

(i) *Then, there exists a positive constant $C = C_{(\phi,C_\mu)}$ such that, for any $u \in L^1_{\mathrm{loc}}(\mathcal{X})$, $g \in \mathcal{D}^\phi(u)$, $i \in \mathbb{Z}$, $y \in \mathcal{X}$ with $u_{B(y,2^{-i})} < \infty$, and almost every $x \in B(y, 2^{-i+1})$,*

$$\left| u(x) - u_{B(y,2^{-i})} \right| \le C\, \phi(2^{-i}) \mathcal{M}(g)(x).$$

(ii) *Let $\alpha_\phi \in (0,1)$. Then, for any $\lambda \in (D/[D - \log_2 \alpha_\phi], \infty)$, there exists a positive constant $C = C_{(\phi,\lambda,C_\mu)}$ such that, for any $u \in L^1_{\mathrm{loc}}(\mathcal{X})$, $g \in \mathcal{D}^\phi(u)$, $i \in \mathbb{Z}$, $y \in \mathcal{X}$ with $u_{B(y,2^{-i})} < \infty$, and almost every $x \in B(y, 2^{-i+1})$,*

$$\left| u(x) - u_{B(y,2^{-i})} \right| \le C\, \phi(2^{-i}) \left[\mathcal{M}\left(g^\lambda\right)(x) \right]^{1/\lambda}. \tag{19}$$

(iii) *Let $\alpha_\phi \in (0,1)$. Then, for any $\lambda \in (D/[D-\log_2 \alpha_\phi], \infty)$, there exist an $\epsilon \in (0, -\log_2 \alpha_\phi)$ depending on λ, and a positive constant $C = C_{(\phi,\lambda,C_\mu)}$ such that, for any $u \in L^1_{\mathrm{loc}}(\mathcal{X})$, $\vec{g} := \{g_l\}_{l\in\mathbb{Z}} \in \mathbb{D}^\phi(u)$, $i \in \mathbb{Z}$, $y \in \mathcal{X}$ with $u_{B(y,2^{-i})} < \infty$, and almost every $x \in B(y, 2^{-i+1})$,*

$$\left|u(x) - u_{B(y,2^{-i})}\right| \le C \sum_{l\ge i-4} 2^{(l-i)\epsilon}\phi(2^{-l})\left[\mathcal{M}\left(g_l^\lambda\right)(x)\right]^{1/\lambda}. \tag{20}$$

Proof. Let u, g, i, y, and x be as in the present lemma. By the definition of Hajłasz gradients, the doubling property of μ, the geometrical observation that, for any $x \in B(y, 2^{-i+1})$, $B(y, 2^{-i+1}) \subset B(x, 2^{-i+2})$ and, for almost every $x \in \mathcal{X}$, $g(x) \le \mathcal{M}(g)(x)$, we have, for almost every $x \in B(y, 2^{-i+1})$,

$$\begin{aligned}\left|u(x) - u_{B(y,2^{-i})}\right| &\le \fint_{B(y,2^{-i})} |u(x) - u(z)|\, d\mu(z) \\ &\lesssim \phi(2^{-i}) \fint_{B(y,2^{-i})} [g(x) + g(z)]\, d\mu(z) \\ &\lesssim \phi(2^{-i})\left[g(x) + \fint_{B(x,2^{-i+2})} g(z)\, d\mu(z)\right] \\ &\lesssim \phi(2^{-i})\mathcal{M}(g)(x),\end{aligned}$$

which proves (i) of the present lemma.

To complete the proof of the present lemma, we observe that, for any $i \in \mathbb{Z}$, $y \in \mathcal{X}$ and $x \in B(y, 2^{-i+1})$, $B(y, 2^{-i}) \subset B(x, 2^{-i+2})$. Thus, by the Lebesgue differentiation theorem and the doubling property of μ, we find that, for almost every $x \in B(y, 2^{-i+1})$,

$$\begin{aligned}\left|u(x) - u_{B(y,2^{-i})}\right| &\le \left|u(x) - u_{B(x,2^{-i+2})}\right| + \left|u_{B(x,2^{-i+2})} - u_{B(y,2^{-i})}\right| \\ &\lesssim \sum_{k\ge i-2} \fint_{B(x,2^{-k})} \left|u(z) - u_{B(x,2^{-k})}\right| d\mu(z) \\ &\quad + \fint_{B(x,2^{-i+2})} \left|u(z) - u_{B(x,2^{-i+2})}\right| d\mu(z) \\ &\lesssim \sum_{k\ge i-2} \fint_{B(x,2^{-k})} \left|u(z) - u_{B(x,2^{-k})}\right| d\mu(z) \\ &\lesssim \sum_{k\ge i-2} \inf_{c\in\mathbb{R}} \fint_{B(x,2^{-k})} |u(z) - c|\, d\mu(z).\end{aligned} \tag{21}$$

If $\lambda \in (D/[D-\log_2 \alpha_\phi], 1)$, choose $\omega \in (0, -\log_2 \alpha_\phi)$ such that $\lambda = D/(D+\omega)$. By $\alpha_\phi \in (0,1)$, (21), and the definition of $\mathcal{M}$, we conclude that (19) and (20) follow from (18) and (17) with $\varepsilon = \omega$ therein, respectively.

If $\lambda \in [1, \infty)$, then, for any $\epsilon \in (0, -\log_2 \alpha_\phi)$, by the Hölder inequality, we also obtain the same estimate as the case $\lambda \in (D/[D-\log_2 \alpha_\phi], 1)$. This finishes the proof of Lemma 5. □

Remark 3. (i) *Let $\phi \in \mathcal{A}$ with $\alpha_\phi \in (0,1)$. Recall that, for any $p \in (D/(D-\log_2 \alpha_\phi), \infty]$, $q \in (0,\infty]$, and $u \in [\dot{M}^\phi_{p,q}(\mathcal{X}) \cup \dot{N}^\phi_{p,q}(\mathcal{X})]$, the integral of u on any ball in $\mathcal{X}$ is finite (see [28], Remark 3.8), where D is as in (1).*

(ii) *Let $\phi \in \mathcal{A}$. For any $u \in \dot{\mathcal{F}}$, the integral of $|u|^p$ on any ball $B := B(x, 2^{-k})$ in $\mathcal{X}$ with $k \in \mathbb{Z}$ is also finite, where*

$$\begin{aligned}\dot{\mathcal{F}} \in &\left\{\dot{M}^{\phi,p}(\mathcal{X}) : \ p \in [1,\infty)\right\} \cup \left\{\dot{M}^{\phi,p}(\mathcal{X}) : \ p \in (0,1),\ \alpha_\phi \in (0,1)\right\} \\ &\cup \left\{\dot{M}^\phi_{p,q}(\mathcal{X}),\ \dot{N}^\phi_{p,q}(\mathcal{X}) : \ p,\ q \in (0,\infty],\ \alpha_\phi \in (0,1)\right\}.\end{aligned}$$

To see this, by similarity, we only prove the case $\dot{\mathcal{F}} = \dot{M}^{\phi}_{p,q}(\mathcal{X})$ with $p,\ q \in (0,\infty]$ and $\alpha_\phi \in (0,1)$. Indeed, by (15), the Hölder inequality, Lemma 1(i), and the definition of $\mathcal{A}$, we find that

$$\begin{aligned}\inf_{c\in\mathbb{R}}\left[\fint_B |u(y)-c|^p\,d\mu(y)\right]^{1/p} &\lesssim 2^{-k\varepsilon'}\sum_{j\ge k-2} 2^{j\varepsilon'}\phi(2^{-j})\left\{\fint_{2B}[g_j(y)]^p\,d\mu(y)\right\}^{1/p}\\ &\lesssim 2^{-k\varepsilon'}\sum_{j\ge k-2} 2^{j\varepsilon'}\phi(2^{-j})[\mu(2B)]^{-1/p}\left\|\{g_j\}_{j\in\mathbb{Z}}\right\|_{L^p(2B,l^q)}\\ &\lesssim \phi(2^{-k})[\mu(2B)]^{-1/p}\left\|\{g_j\}_{j\in\mathbb{Z}}\right\|_{L^p(\mathcal{X},l^q)} < \infty,\end{aligned}$$

where $\varepsilon' \in (0, -\log_2\alpha_\phi)$ and $\{g_j\}_{j\in\mathbb{Z}} \in \mathbb{D}^\phi(u)\cap L^p(\mathcal{X},l^q)$. Let $c_0\in\mathbb{R}$ be such that

$$\fint_B |u(y)-c_0|^p\,d\mu(y) < \infty.$$

Then,

$$\int_B |u(y)|^p\,d\mu(y) \lesssim \mu(B)\fint_B |u(y)-c_0|^p\,d\mu(y) + \mu(B)c_0^p < \infty.$$

Thus, the above claim holds true.

Due to Remark 3(i), the classical Lebesgue differentiation theorem implies that almost every point is a Lebesgue point of u. As u has certain regularity, one would expect a smaller exceptional set than that of usual locally integrable functions. Inspired by [41,45], we introduce capacities related, respectively, to $M^{\phi}_{p,q}(\mathcal{X})$ and $N^{\phi}_{p,q}(\mathcal{X})$ to measure such exceptional sets.

Below, for simplicity, we use $\mathcal{F}$ to denote either $M^{\phi}_{p,q}(\mathcal{X})$ or $N^{\phi}_{p,q}(\mathcal{X})$, or $\dot{\mathcal{F}}$ to denote either $\dot{M}^{\phi}_{p,q}(\mathcal{X})$ or $\dot{N}^{\phi}_{p,q}(\mathcal{X})$.

Definition 10. *Let E be a subset of $\mathcal{X}$. Recall that a set U is called a* neighborhood *of E if it is open and $E\subset U$. Let $\mathcal{F}\in\{M^{\phi}_{p,q}(\mathcal{X}), N^{\phi}_{p,q}(\mathcal{X})\}$ with $\phi\in\mathcal{A}$ and $p,\ q\in(0,\infty]$, and*

$$\mathcal{G}_{\mathcal{F}}(E) := \{u\in\mathcal{F} :\ u\ge 1 \text{ on a neighborhood of } E\}.$$

The $\mathcal{F}$-capacity $\operatorname{Cap}_{\mathcal{F}}(E)$ of E is defined by setting

$$\operatorname{Cap}_{\mathcal{F}}(E) := \inf\left\{\|u\|^p_{\mathcal{F}} :\ u\in\mathcal{G}_{\mathcal{F}}(E)\right\}.$$

Remark 4. *Let $E,\ E_1,\ E_2\subset\mathcal{X}$ and $\mathcal{F}\in\{M^{\phi}_{p,q}(\mathcal{X}), N^{\phi}_{p,q}(\mathcal{X})\}$ with $\phi\in\mathcal{A}$ and $p,\ q\in(0,\infty]$.*

(i) *Let $\mathcal{G}'_{\mathcal{F}}(E) := \{u\in\mathcal{G}_{\mathcal{F}}(E) :\ 0\le u\le 1\}$. By Lemma 2(i), $\|\max\{\min\{u,1\},0\}\|_{\mathcal{F}}\le\|u\|_{\mathcal{F}}$, and an argument similar to that used in ([55], Remark 3.2), we have*

$$\operatorname{Cap}_{\mathcal{F}}(E) = \inf\left\{\|u\|^p_{\mathcal{F}} :\ u\in\mathcal{G}'_{\mathcal{F}}(E)\right\}.$$

(ii) *If $\operatorname{Cap}_{\mathcal{F}}(E)=0$ with $p\in(0,\infty)$, then $\mu(E)=0$. Indeed, for any $\epsilon\in(0,\infty)$, there always exists a neighborhood U_ϵ of E such that $\|\mathbf{1}_{U_\epsilon}\|_{\mathcal{F}}<\epsilon$, which implies that*

$$[\mu(E)]^{1/p} = \|\mathbf{1}_E\|_{L^p(\mathcal{X})} \le \|\mathbf{1}_E\|_{\mathcal{F}} \le \epsilon.$$

Letting $\epsilon\to 0^+$, we obtain $\mu(E)=0$.

(iii) *If $E_1\subset E_2$, then $\mathcal{G}_{\mathcal{F}}(E_2)\subset\mathcal{G}_{\mathcal{F}}(E_1)$, which means that $\operatorname{Cap}_{\mathcal{F}}(E_1)\le\operatorname{Cap}_{\mathcal{F}}(E_2)$.*

The following lemma provides a basic property of the capacity which is a slight generalization of ([41], Lemma 6.4); we omit the details.

Lemma 6. *Let $\mathcal{F} \in \{M^{\phi}_{p,q}(\mathcal{X}), N^{\phi}_{p,q}(\mathcal{X})\}$ with $\phi \in \mathcal{A}$ and $p \in (0,\infty)$ and $q \in (0,\infty]$. Let $\theta := \min\{1, q/p\}$. Then, there exists a positive constant $C = C_{(p,q)} \in [1,\infty)$ such that, for any sequence $\{E_i\}_{i\in\mathbb{N}}$ of subsets of $\mathcal{X}$,*

$$\left[\operatorname{Cap}_{\mathcal{F}}\left(\bigcup_{i\in\mathbb{N}} E_i\right)\right]^{\theta} \le C\sum_{i\in\mathbb{N}}[\operatorname{Cap}_{\mathcal{F}}(E_i)]^{\theta}.$$

Via $\mathcal{F}$-capacities, we introduce the $\mathcal{F}$-quasi-continuity as follows.

Definition 11. *Let $\mathcal{F} \in \{M^{\phi}_{p,q}(\mathcal{X}), N^{\phi}_{p,q}(\mathcal{X})\}$ with $\phi \in \mathcal{A}$ and $p,\ q \in (0,\infty]$. A function u is said to be* $\mathcal{F}$-quasi-continuous *if, for any $\varepsilon \in (0,\infty)$, there exists a set U_ε such that $\operatorname{Cap}_{\mathcal{F}}(U_\varepsilon) < \varepsilon$ and the restriction $u|_{\mathcal{X}\setminus U_\varepsilon}$ of u on $\mathcal{X}\setminus U_\varepsilon$ is continuous.*

The following theorem shows the convergence of discrete convolution approximations in $\mathcal{F}$, which generalizes ([41], Theorem 5.1).

Theorem 1. *Let $\phi \in \mathcal{A}_0$, $p \in (D/(D-\log_2 \alpha_\phi),\infty)$, $\mathcal{F} = M^{\phi}_{p,q}(\mathcal{X})$ [resp., $\dot{\mathcal{F}} = \dot{M}^{\phi}_{p,q}(\mathcal{X})$] with $q \in (D/(D-\log_2 \alpha_\phi),\infty)$, or $\mathcal{F} = N^{\phi}_{p,q}(\mathcal{X})$ [resp., $\dot{\mathcal{F}} = \dot{N}^{\phi}_{p,q}(\mathcal{X})$] with $q \in (0,\infty)$, and $u \in \dot{\mathcal{F}}$. Then, $\|u - u_{2^{-i}}\|_{\mathcal{F}} \to 0$ as $i \to \infty$, where $\{u_{2^{-i}}\}_{i\in\mathbb{Z}_+}$ are the discrete convolutions as in Definition 9(i).*

To prove Theorem 1, we need the following lemma, which generalizes ([41], Lemma 3.1) (see also [47], Lemma 3.10).

Lemma 7. *Let $E \subset \mathcal{X}$ be a measurable set, $L \in (0,\infty)$, φ be a bounded L-Lipschitz function supported in E, $u \in L^0(\mathcal{X})$, and $\phi \in \mathcal{A}_\infty$.*

(i) *If $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^{\phi}(u)$, then, for any $i \in \mathbb{Z}$, the sequence $\{h_k\}_{k\in\mathbb{Z}}$, defined by setting*

$$h_k := \begin{cases} \left\{2^{-k}\left[\phi(2^{-k})\right]^{-1} L\,|u| + \|\varphi\|_{L^\infty(\mathcal{X})}\, g_k\right\}\mathbf{1}_E, & k > i,\\ \|\varphi\|_{L^\infty(\mathcal{X})}\left[\phi(2^{-k})\right]^{-1} |u|\,\mathbf{1}_E, & k \le i, \end{cases} \tag{22}$$

is an element of $\mathbb{D}^{\phi}(u\varphi)$ modulo a positive constant that is independent of i and L.

(ii) *If $g \in \mathcal{D}^{\phi}(u)$, then*

$$h := \left\{\|\varphi\|_{L^\infty(\mathcal{X})}\, g + \left[\|\varphi\|_{L^\infty(\mathcal{X})} + 1\right]\left[\phi(L^{-1})\right]^{-1}|u|\right\}\mathbf{1}_E$$

is an element of $\mathcal{D}^{\phi}(u\varphi)$ modulo a positive constant that is independent of L.

Proof. We first prove (i). Let φ be a bounded L-Lipschitz function supported in E, $u \in L^0(\mathcal{X})$, and $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^{\phi}(u)$. For any $k \in \mathbb{Z}$ and $x,\ y \in \mathcal{X}$ with $d(x,y) \in [2^{-k-1}, 2^{-k})$, we have

$$d(x,y)/\phi(d(x,y)) \lesssim 2^{-k}/\phi(2^{-k}) \quad \text{and} \quad [\phi(d(x,y))]^{-1} \lesssim [\phi(2^{-k})]^{-1}.$$

Then, from the Lipschitz continuity of φ and the definition of $\mathbb{D}^{\phi}(u)$, it follows that, for any $k \in \mathbb{Z}$ and almost every $x,\ y \in E$ with $d(x,y) \in [2^{-k-1}, 2^{-k})$,

$$\begin{aligned} \left|u(x)\varphi(x) - u(y)\varphi(y)\right| &\le |u(x)|\left|\varphi(x) - \varphi(y)\right| + \|\varphi\|_{L^\infty}\left|u(x) - u(y)\right| \\ &\le \phi(d(x,y))\left\{\frac{Ld(x,y)|u(x)|}{\phi(d(x,y))} + \|\varphi\|_{L^\infty(\mathcal{X})}[g_k(x) + g_k(y)]\right\} \\ &\lesssim \phi(d(x,y))\left\{\frac{L2^{-k}|u(x)|}{\phi(2^{-k})} + \|\varphi\|_{L^\infty(\mathcal{X})}[g_k(x) + g_k(y)]\right\} \end{aligned}$$

and

$$\begin{aligned}\left|u(x)\varphi(x)-u(y)\varphi(y)\right| &\lesssim |u(x)|\|\varphi\|_{L^\infty(\mathcal{X})}+\|\varphi\|_{L^\infty}(|u(x)|+|u(y)|)\\ &\lesssim \phi(d(x,y))\frac{\|\varphi\|_{L^\infty(\mathcal{X})}(|u(x)|+|u(y)|)}{\phi(d(x,y))}\\ &\lesssim \phi(d(x,y))\frac{\|\varphi\|_{L^\infty(\mathcal{X})}(|u(x)|+|u(y)|)}{\phi(2^{-k})}.\end{aligned}$$

For any $k\in\mathbb{Z}$ and almost every $x\in E$ and $y\in\mathcal{X}\setminus E$ with $d(x,y)\in[2^{-k-1},2^{-k})$, we have

$$\begin{aligned}\left|u(x)\varphi(x)-u(y)\varphi(y)\right| &\le |u(x)|\left|\varphi(x)-\varphi(y)\right|\\ &\le \phi(d(x,y))\frac{Ld(x,y)|u(x)|}{\phi(d(x,y))}\lesssim \phi(d(x,y))\frac{L2^{-k}|u(x)|}{\phi(2^{-k})}\end{aligned}$$

and

$$\left|u(x)\varphi(x)-u(y)\varphi(y)\right|\le \|\varphi\|_{L^\infty(\mathcal{X})}|u(x)|\lesssim \phi(d(x,y))\frac{\|\varphi\|_{L^\infty(\mathcal{X})}|u(x)|}{\phi(2^{-k})}.$$

Similarly, for any $k\in\mathbb{Z}$ and almost every $x\in E$ and $y\in\mathcal{X}\setminus E$ with $d(x,y)\in[2^{-k-1},2^{-k})$, we have

$$\left|u(x)\varphi(x)-u(y)\varphi(y)\right|\lesssim \phi(d(x,y))\frac{L2^{-k}|u(y)|}{\phi(2^{-k})}$$

and

$$\left|u(x)\varphi(x)-u(y)\varphi(y)\right|\lesssim \phi(d(x,y))\frac{\|\varphi\|_{L^\infty(\mathcal{X})}|u(y)|}{\phi(2^{-k})}.$$

From these estimates, we deduce that $\{h_k\}_{k\in\mathbb{Z}}$ as in (22) is a positive constant multiple of an element in $\mathcal{D}^\phi(u\varphi)$, with the positive constant independent of i and L. This proves (i).

The item (ii) is easy to show using the result in (i) and choosing $h:=\sup_{k\in\mathbb{Z}}h_k$ and $i\in\mathbb{Z}$ such that $L\in[2^i,2^{i+1})$. This finishes the proof of Lemma 7. □

We now state some corollaries of Lemma 7 as follows.

Corollary 2. *Let $E\subset\mathcal{X}$ be a measurable set, $L\in[1/2,\infty)$, φ be a bounded L-Lipschitz function supported in E and $p\in(0,\infty)$. Let $\mathcal{F}\in\{M^\phi_{p,q}(\mathcal{X}),N^\phi_{p,q}(\mathcal{X})\}$ with $q\in(0,\infty)$ and $\phi\in\mathcal{A}_q$, or $\mathcal{F}\in\{M^\phi_{p,\infty}(\mathcal{X})=M^{\phi,p}(\mathcal{X}),N^\phi_{p,\infty}(\mathcal{X})\}$ with $\phi\in\mathcal{A}_\infty$. Then, for any $u\in\mathcal{F}$, $u\varphi\in\mathcal{F}$ with $\|u\varphi\|_{\mathcal{F}}\lesssim\|u\|_{\mathcal{F}}$, where the implicit positive constant is independent of u.*

Proof. By similarity, we only consider $\mathcal{F}=M^\phi_{p,q}(\mathcal{X})$ with $p,\ q\in(0,\infty)$ and $\phi\in\mathcal{A}_q$. Let $i\in\mathbb{Z}_+$ be such that $2^{i-1}\le L<2^i$, $u\in L^0(\mathcal{X})$, $\{g_k\}_{k\in\mathbb{Z}}\in\mathbb{D}^\phi(u)$ satisfy $\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)}\lesssim\|u\|_{\dot{M}^\phi_{p,q}(\mathcal{X})}$, and $\{h_k\}_{k\in\mathbb{Z}}$ be as in (22). By the definition of $\mathcal{A}_q$, we have

$$\sum_{k\le i}\frac{1}{[\phi(2^{-k})]^q}\lesssim X_L^q\quad\text{and}\quad\sum_{k>i}\frac{2^{-kq}}{[\phi(2^{-k})]^q}\lesssim Y_L^q,$$

where X_L and Y_L are two positive constants independent of ϕ. From this, we deduce that

$$\|\{h_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)}\lesssim\left\{\sum_{k>i}\left(2^{-k}\left[\phi(2^{-k})\right]^{-1}\right)^q\right\}^{1/q}L\|u\mathbf{1}_E\|_{L^p(\mathcal{X})}$$

$$
\begin{aligned}
&+ \|\varphi\|_{L^\infty(\mathcal{X})}\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(E,l^q)} \\
&+ \left\{\sum_{k\le i}\left(\left[\phi(2^{-k})\right]^{-1}\right)^q\right\}^{1/q}\|\varphi\|_{L^\infty(\mathcal{X})}\|u\mathbf{1}_E\|_{L^p(\mathcal{X})} \qquad (23)\\
\lesssim\ & \|\varphi\|_{L^\infty(\mathcal{X})}\,\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(E,l^q)} \\
&+ \left[X_L\,\|\varphi\|_{L^\infty(\mathcal{X})} + Y_L\,L\right]\|u\|_{L^p(E)},
\end{aligned}
$$

which, combined with Lemma 7 and $\|u\varphi\|_{L^p(\mathcal{X})} \le \|u\|_{L^p(\mathcal{X})}\|\varphi\|_{L^\infty(\mathcal{X})}$, implies that

$$
\begin{aligned}
\|u\varphi\|_{M^\phi_{p,q}(\mathcal{X})} &\lesssim \|u\varphi\|_{L^p(\mathcal{X})} + \|\{h_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \\
&\lesssim \left[(X_L+1)\|\varphi\|_{L^\infty(\mathcal{X})} + Y_L L\right]\|u\|_{M^\phi_{p,q}(\mathcal{X})},
\end{aligned}
$$

where the implicit positive constants are independent of L, φ, and u. This finishes the proof of Corollary 2. □

Corollary 3. *With the same assumptions as in Corollary 2, if the set E is bounded, then, for any $u \in \dot{\mathcal{F}}$, $u\varphi \in \mathcal{F}$.*

Proof. Again, by similarity, we only consider $\mathcal{F} = M^\phi_{p,q}(\mathcal{X})$ with $p,\ q \in (0,\infty)$ and $\phi \in \mathcal{A}_q$. Let $i \in \mathbb{Z}_+$ be such that $2^{i-1} \le L < 2^i$, $u \in L^0(\mathcal{X})$, and $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(u)$ be such that $\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \lesssim \|u\|_{\dot{M}^\phi_{p,q}(\mathcal{X})}$. Since E is bounded, we can find a ball B containing E. Then, by Remark 3(ii), we conclude that $\|u\|_{L^p(E)} \le \|u\|_{L^p(B)} < \infty$. Let $\{h_k\}_{k\in\mathbb{Z}}$ be as in (22). Then, from (23), we deduce that $\|\{h_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} < \infty$, which, combined with Lemma 7, implies that $\|u\varphi\|_{\dot{M}^\phi_{p,q}(\mathcal{X})} < \infty$. Notice that $\|u\varphi\|_{L^p(\mathcal{X})} = \|u\|_{L^p(E)}\|\varphi\|_{L^\infty(\mathcal{X})} < \infty$. We then obtain $\|u\varphi\|_{M^\phi_{p,q}(\mathcal{X})} < \infty$, which completes the proof of Corollary 3. □

Corollary 4. *Let $E \subset \mathcal{X}$ be a measurable set with $\mu(E) \in (0,\infty)$; $L \in (0,\infty)$; φ be a bounded L-Lipschitz function supported in E; and $\mathcal{F} \in \{M^\phi_{p,q}(\mathcal{X}), N^\phi_{p,q}(\mathcal{X})\}$ with $p,\ q \in (0,\infty)$, $\alpha_\phi \in (0,1)$, and $\beta_\phi \in (0,2)$ or $\mathcal{F} \in \{M^\phi_{p,\infty}(\mathcal{X}) = M^{\phi,p}(\mathcal{X}), N^\phi_{p,\infty}(\mathcal{X})\}$ with $p \in (0,\infty)$, $\phi \in \mathcal{A}_\infty$, and $u \in L^0(\mathcal{X})$. Then,*

$$
\|\varphi\|_{\mathcal{F}} \lesssim \left[1 + \|\varphi\|_{L^\infty(\mathcal{X})}\right]\left\{1 + \left[\phi(L^{-1})\right]^{-1}\right\}[\mu(E)]^{1/p} \qquad (24)
$$

with the implicit positive constant independent of L, φ, and E.

Proof. We first consider $\mathcal{F} = M^\phi_{p,q}(\mathcal{X})$ with $p,\ q \in (0,\infty)$, $\alpha_\phi \in (0,1)$, and $\beta_\phi \in (0,2)$. Let $\{h_k\}_{k\in\mathbb{Z}}$ be as in (22). From Lemma 7(i) and choosing $u \equiv 1$, $g_k \equiv 0$ for any $k \in \mathbb{Z}$, and $i \in \mathbb{Z}$ such that $2^i \le L < 2^{i+1}$ in (22), we deduce that

$$
\begin{aligned}
\|\varphi\|_{\dot{M}^\phi_{p,q}(\mathcal{X})} &\lesssim \|\{h_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \\
&\lesssim \left[\left\{\sum_{k>i}\left(2^{-k}\left[\phi(2^{-k})\right]^{-1}\right)^q\right\}^{1/q} L \right.\\
&\quad \left. + \left\{\sum_{k\le i}\left(\left[\phi(2^{-k})\right]^{-1}\right)^q\right\}^{1/q}\|\varphi\|_{L^\infty(\mathcal{X})}\right]\|\mathbf{1}_E\|_{L^p(\mathcal{X})} \\
&\lesssim \left\{\left[\phi(L^{-1})\right]^{-1} + \left[\phi(L^{-1})\right]^{-1}\|\varphi\|_{L^\infty(\mathcal{X})}\right\}[\mu(E)]^{1/p},
\end{aligned}
$$

where, in the last inequality, we used (9) and (8). This, combined with the fact that

$$
\|\varphi\|_{L^p(\mathcal{X})} \le \|\varphi\|_{L^\infty(\mathcal{X})}[\mu(E)]^{1/p},
$$

implies (24) with $\mathcal{F} = M^\phi_{p,q}(\mathcal{X})$.

By choosing $u \equiv 1$ and $g \equiv 0$ in Lemma 7(ii), the case

$$\mathcal{F} \in \left\{M^{\phi}_{p,\infty}(\mathcal{X}) = M^{\phi,p}(\mathcal{X}), N^{\phi}_{p,\infty}(\mathcal{X})\right\}$$

with $p \in (0, \infty)$ and $\phi \in \mathcal{A}_\infty$ can be similarly proved. This finishes the proof of Corollary 4. □

Now, we prove Theorem 1.

Proof of Theorem 1. By similarity, we only consider the case $\mathcal{F} = M^{\phi}_{p,q}(\mathcal{X})$. Let p, q, and ϕ be as in the present theorem; C_μ be as in (1); $i \in \mathbb{Z}_+$; $u \in \dot{M}^{\phi}_{p,q}(\mathcal{X})$; and $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^{\phi}(u) \cap L^p(\mathcal{X}, l^q)$. Let $\{B_j\}_{j\in\mathcal{J}}$ be any given ball covering of $\mathcal{X}$ with the radius 2^{-i} such that $\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j} \lesssim 1$ and $\{\varphi_j\}_{j\in\mathcal{J}}$, consisting of a sequence of $c2^i$-Lipschitz functions, be a partition of unity with respect to $\{B_j\}_{j\in\mathcal{J}}$ as in Definition 8, where c is a positive constant depending only on C_μ. For any $j \in \mathcal{J}$, let u_{B_j} be as in (2). By ([28], Remark 3.8), we have, for any $j \in \mathcal{J}$, $|u_{B_j}| < \infty$. Let $u_{2^{-i}}$ be as in Definition 9(i). Thus, by the properties of $\{\varphi_j\}_{j\in\mathcal{J}}$, we obtain

$$u - u_{2^{-i}} = \sum_{j\in\mathcal{J}} \left(u - u_{B_j}\right)\varphi_j. \tag{25}$$

Noticing that φ_j is a $c2^i$-Lipschitz function and $\|\varphi_j\|_{L^\infty(\mathcal{X})} \le 1$, from Lemma 7 with u and L replaced, respectively, by $u - u_{B_j}$ and $c2^i$, we deduce that, for any $j \in \mathcal{J}$, $\vec{h}_j := \{h_{k,j}\}_{k\in\mathbb{Z}}$, defined by setting, for any $k \in \mathbb{Z}$,

$$h_{k,j} := \begin{cases} \left\{2^{i-k}\left[\phi(2^{-k})\right]^{-1}|u - u_{B_j}| + g_k\right\}\mathbf{1}_{2B_j}, & k > i, \\ \left[\phi(2^{-k})\right]^{-1}\left|u - u_{B_j}\right|\mathbf{1}_{2B_j}, & k \le i, \end{cases}$$

is a positive constant multiple of an element of $\mathbb{D}^{\phi}([u - u_{B_j}]\varphi_j)$. By this, (25), and $\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j} \lesssim 1$, we conclude that, for almost every x, $y \in \mathcal{X}$ with $d(x, y) \in [2^{-k-1}, 2^{-k})$,

$$\begin{aligned} &\left|(u - u_{2^{-i}})(x) - (u - u_{2^{-i}})(y)\right| \\ &\quad = \sum_{j\in\mathcal{J}}\left(u(x) - u_{B_j}\right)\varphi_j(x) - \sum_{j\in\mathcal{J}}\left(u(y) - u_{B_j}\right)\varphi_j(y) \\ &\quad \le \sum_{j\in\mathcal{J},\, 2B_j\cap\{x,y\}\neq\emptyset} \left|\left(u(x) - u_{B_j}\right)\varphi_j(x) - \left(u(y) - u_{B_j}\right)\varphi_j(y)\right| \\ &\quad \lesssim \phi(d(x, y)) \sum_{j\in\mathcal{J},\, 2B_j\cap\{x,y\}\neq\emptyset} [h_{k,j}(x) + h_{k,j}(y)]. \end{aligned} \tag{26}$$

For any given $\epsilon \in (0, -\log_2 \alpha_\phi)$ and $\lambda \in (n/[n - \log_2 \alpha_\phi], \infty)$, by Lemma 5(iii), we obtain, for any $j \in \mathcal{J}$ and almost every $x \in 2B_j$,

$$\left|u(x) - u_{B_j}\right| \lesssim \sum_{l\ge i-4} 2^{(l-i)\epsilon}\phi(2^{-l})\left[\mathcal{M}\left(g_l^\lambda\right)(x)\right]^{1/\lambda}.$$

Then,

$$\begin{aligned} h_{k,j} &\le \begin{cases} \left\{2^{i-k}\left[\phi(2^{-k})\right]^{-1} \sum_{l\ge i-4} 2^{(l-i)\epsilon}\phi(2^{-l})\left[\mathcal{M}\left(g_l^\lambda\right)\right]^{1/\lambda} + g_k\right\}\mathbf{1}_{2B_j}, & k > i, \\ \left[\phi(2^{-k})\right]^{-1} \sum_{l\ge i-4} 2^{(l-i)\epsilon}\phi(2^{-l})\left[\mathcal{M}\left(g_l^\lambda\right)\right]^{1/\lambda}\mathbf{1}_{2B_j}, & k \le i \end{cases} \\ &=: \widetilde{h_{k,j}}. \end{aligned} \tag{27}$$

Define the sequence $\{h_k\}_{k\in\mathbb{Z}}$ by setting, for any $k \in \mathbb{Z}$,

$$h_k := \begin{cases} 2^{i-k}\, 2^{-i\epsilon}\Big[\phi(2^{-k})\Big]^{-1} \sum_{l\ge i-4} 2^{l\epsilon}\phi(2^{-l}) \Big[\mathcal{M}\Big(g_l^{\lambda}\Big)\Big]^{1/\lambda} + g_k, & k > i, \\ 2^{-i\epsilon}\Big[\phi(2^{-k})\Big]^{-1} \sum_{l\ge i-4} 2^{l\epsilon}\phi(2^{-l}) \Big[\mathcal{M}\Big(g_l^{\lambda}\Big)\Big]^{1/\lambda}, & k \le i. \end{cases} \tag{28}$$

Then, by (26), (27), and $\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j} \lesssim 1$, we conclude that, for almost every $x, y \in \mathcal{X}$,

$$\begin{aligned} \Big|(u - u_{2^{-i}})(x) - (u - u_{2^{-i}})(y)\Big| &\lesssim \phi(d(x,y)) \sum_{j\in\mathcal{J},\, 2B_j\cap\{x,y\}\neq\emptyset} \Big[\widetilde{h_{k,j}}(x) + \widetilde{h_{k,j}}(y)\Big] \\ &\lesssim \phi(d(x,y))[h_k(x) + h_k(y)], \end{aligned}$$

which implies that $\{h_k\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^{\phi}(u - u_{2^{-i}})$.

Let $\lambda \in (n/[n - \log_2 \alpha_\phi], \min\{p,q\})$. Using the Hölder inequality, the fact that $\alpha_\phi < 2^{-\epsilon}$, and Lemma 1, we have

$$\sum_{l\ge i-4} 2^{l\epsilon}\phi(2^{-l}) \Big[\mathcal{M}\Big(g_l^{\lambda}\Big)\Big]^{1/\lambda} \lesssim \Big[2^{i\epsilon}\phi(2^{-i})\Big]^{(q-1)/q} \left\{\sum_{l\ge i-4} 2^{l\epsilon}\phi(2^{-l}) \Big[\mathcal{M}\Big(g_l^{\lambda}\Big)\Big]^{q/\lambda}\right\}^{1/q} \tag{29}$$

with the implicit positive constant independent of i. Notice that, by (10) and $\beta_\phi^- < 2$,

$$\begin{aligned} \left\{\sum_{k>i}\Big\{2^{i-k}\, 2^{-i\epsilon}\Big[\phi(2^{-k})\Big]^{-1}\Big\}^q\right\}^{1/q} &= 2^{-i(\epsilon-1)} \left\{\sum_{k>i}\left[\frac{2^{-k}}{\phi(2^{-k})}\right]^q\right\}^{1/q} \\ &\lesssim \frac{2^{-i\epsilon}}{\phi(2^{-i})} \end{aligned} \tag{30}$$

and, by (9) and $\alpha_\phi < 1$,

$$\left\{\sum_{k\le i}\Big\{2^{-i\epsilon}\Big[\phi(2^{-k})\Big]^{-1}\Big\}^q\right\}^{1/q} \lesssim \frac{2^{-i\epsilon}}{\phi(2^{-i})}. \tag{31}$$

Thus, by (29)–(31), Lemma 1, and the Fefferman–Stein vector-valued maximal inequality in $L^{p/\lambda}(\mathcal{X}, l^{q/\lambda})$ (see ([56], Theorem 1.2) or ([57], Theorem 1.3)), we obtain

$$\begin{aligned} \|\{h_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} &\lesssim \left\|\left\{\sum_{l\ge i-4} 2^{(l-i)\epsilon}\frac{\phi(2^{-l})}{\phi(2^{-i})}\Big[\mathcal{M}(g_l^{\lambda})\Big]^{q/\lambda}\right\}^{1/q}\right\|_{L^p(\mathcal{X})} + \left\|\left(\sum_{k>i} g_k^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} \\ &\lesssim \left\|\left\{\sum_{l\ge i-4}\Big[\mathcal{M}(g_l^{\lambda})\Big]^{q/\lambda}\right\}^{1/q}\right\|_{L^p(\mathcal{X})} + \left\|\left(\sum_{k>i} g_k^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} \\ &\lesssim \left\|\left(\sum_{l\ge i-4} g_l^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} + \left\|\left(\sum_{k>i} g_k^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} \\ &\lesssim \left\|\left(\sum_{k\ge i-4} g_k^q\right)^{1/q}\right\|_{L^p(\mathcal{X})}, \end{aligned} \tag{32}$$

which, combined with $\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} < \infty$, implies that

$$\big\|u - u_{2^{-i}}\big\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})} \lesssim \left\|\left(\sum_{k\ge i-4} g_k^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} \to 0 \quad \text{as } i \to \infty.$$

On the other hand, from (25), Lemmas 5(iii), and 1(ii) with $\epsilon \in (0, -\log_2 \alpha_\phi)$, the properties of $\{\varphi_j\}_{j\in\mathcal{J}}$, the Fefferman–Stein vector-valued maximal inequality, and $\phi(0) = 0$, it follows that

$$\begin{aligned}
\left\|u - u_{2^{-i}}\right\|_{L^p(\mathcal{X})} &= \left\|\sum_{j\in\mathcal{J}}\left(u - u_{B_j}\right)\varphi_j\right\|_{L^p(\mathcal{X})} \\
&\lesssim \left\|\sum_{j\in\mathcal{J}}\left\{\sum_{l\geq i-4} 2^{(l-i)\epsilon}\phi(2^{-l})\left[\mathcal{M}\left(g_l^\lambda\right)\right]^{1/\lambda}\right\}\varphi_j\right\|_{L^p(\mathcal{X})} \\
&\lesssim \phi(2^{-i})\left\|\sum_{l\geq i-4} 2^{(l-i)\epsilon}\frac{\phi(2^{-l})}{\phi(2^{-i})}\left[\mathcal{M}(g_l^\lambda)\right]^{1/\lambda}\right\|_{L^p(\mathcal{X})} \\
&\lesssim \phi(2^{-i})\left\|\sum_{l\geq i-4}\left[\mathcal{M}(g_l^\lambda)\right]^{1/\lambda}\right\|_{L^p(\mathcal{X})} \\
&\lesssim \phi(2^{-i})\left\|\left(\sum_{l\geq i-4} g_l^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} \to 0 \quad \text{as } i \to \infty.
\end{aligned} \tag{33}$$

This finishes the proof of Theorem 1. □

Recall that, when $q = \infty$, $M^\phi_{p,\infty}(\mathcal{X}) = M^{\phi,p}(\mathcal{X})$ (see Remark 1(i)). Even in the classical case $\phi(t) := t$ for any $t \in [0, \infty)$, Theorem 1 is not true for $q = \infty$; we refer the reader to ([41], Example 3.5) with $m_u^\gamma(B_j)$ therein replaced by u_{B_j} for any $j \in \mathbb{N}$ for a counterexample. For any given Hajłasz–Sobolev function, to find a convergent sequence consisting of continuous functions to this given Hajłasz–Sobolev function in Hajłasz–Sobolev spaces, instead of Theorem 1, we turn to find a dense subspace of $M^\phi_{p,\infty}(\mathcal{X})$, which consists of some generalized Lipschitz continuous functions.

Definition 12. *Let $\phi \in \mathcal{A}$. A function u on $\mathcal{X}$ is said to be in the ϕ-*Lipschitz class $\mathrm{Lip}_\phi(\mathcal{X})$ *if there exists a positive constant C such that, for any $x, y \in \mathcal{X}$,*

$$|u(x) - u(y)| \leq C\,\phi(d(x, y)).$$

Observe that $\mathrm{Lip}_\phi(\mathcal{X})$ is just the classical Hölder space of order $s \in (0, 1]$ when $\phi(t) := t^s$ for any $t \in [0, \infty)$.

Recall that a function $\phi : [0, \infty) \to [0, \infty)$ is called a *modulus of continuity* if it is increasing, the function $\widetilde{\phi}$, defined by setting, for any $t \in [0, \infty)$, $\widetilde{\phi}(t) := \phi(t)/t$, is decreasing, $\phi(0) = 0$, and, for any $t \in (0, \infty)$, $\phi(t) > 0$; see [58]. Obviously, the collection of all moduli of continuity is contained in $\mathcal{A}_\infty$. It is well known that, if ϕ is a modulus of continuity, then, for any $x, y \in [0, \infty)$,

$$\phi(x + y) \leq \phi(x) + \phi(y).$$

Borrowing some ideas similar to that used in the proof of ([48], Theorem 5.19) (see also ([59], Proposition 4.5)), we can prove the following conclusion.

Theorem 2. *Let ϕ be a modulus of continuity, and $p \in (0, \infty)$. Then* $\mathrm{Lip}_\phi(\mathcal{X}) \cap M^{\phi,p}(\mathcal{X})$ *is a dense subspace of $M^{\phi,p}(\mathcal{X})$.*

Proof. Let $p \in (0, \infty)$, $u \in M^{\phi,p}(\mathcal{X})$, $g \in \mathcal{D}^\phi(u) \cap L^p(\mathcal{X})$, and E be the exceptional zero-measure set such that (13) holds true. For any $\lambda \in (0, \infty)$, let

$$E_\lambda := \{x \in \mathcal{X} \setminus E : \ g(x) \leq \lambda, \ |u(x)| \leq \lambda\}. \tag{34}$$

Then, the facts that $u \in L^p(\mathcal{X})$ and $g \in L^p(\mathcal{X})$ imply that, for any $\lambda \in (0, \infty)$,

$$\mu(\mathcal{X} \setminus E_\lambda) < \infty. \tag{35}$$

Moreover, by the definitions of $\mathcal{D}^\phi(u)$ and E_λ, we know that, for any $x,\ y \in E_\lambda$,

$$|u(x) - u(y)| \le \phi(d(x,y))[g(x) + g(y)] \le 2\lambda\phi(d(x,y)).$$

Thus, $u|_{E_\lambda}$ is ϕ-Lipschitz continuous on E_λ. By ([60], Theorem 2) with the function ω therein replaced by $2\lambda\phi$, we find that u_λ, defined by setting, for any $x \in \mathcal{X}$,

$$u_\lambda(x) := \sup\{u(y) - 2\lambda\phi(d(x,y)) :\ y \in E_\lambda\},$$

is a *ϕ-Lipschitz continuous extension* of $u|_{E_\lambda}$ from E_λ to $\mathcal{X}$ and, furthermore, for any $x_1,\ x_2 \in \mathcal{X}$,

$$\left|u_\lambda(x_1) - u_\lambda(x_2)\right| \le 2\lambda\phi(d(x_1, x_2)). \tag{36}$$

Define $v_\lambda := \operatorname{sgn}(u_\lambda) \min\{|u_\lambda|, \lambda\}$. By $u_\lambda|_{E_\lambda} = u|_{E_\lambda}$, (34), and the definition of v_λ, we find that

$$v_\lambda|_{E_\lambda} = u_\lambda|_{E_\lambda} = u|_{E_\lambda}. \tag{37}$$

By the definition of v_λ and (36), we find that, for any $x,\ y \in \mathcal{X}$,

$$\left|v_\lambda(x) - v_\lambda(y)\right| \le |u_\lambda(x) - u_\lambda(y)| \le 2\lambda\phi(d(x,y)), \tag{38}$$

which means that v_λ is still ϕ-Lipschitz continuous on $\mathcal{X}$.

We now show $v_\lambda \in M^{\phi,p}(\mathcal{X})$. If $x,\ y \in E_\lambda$, then, by (37) and the definition of $\mathcal{D}^\phi(u)$, we have

$$\begin{aligned}|v_\lambda(x) - v_\lambda(y)| &= |u(x) - u(y)| \\ &\le \phi(d(x,y))[g(x) + g(y)].\end{aligned} \tag{39}$$

Otherwise, if at least one of x and y lies in $\mathcal{X} \setminus E_\lambda$, then, by (38), we find that

$$\left|v_\lambda(x) - v_\lambda(y)\right| \le 2\lambda\phi(d(x,y)),$$

which, combined with (39) and the definition of $\mathcal{D}^\phi(v_\lambda)$, implies that

$$g_\lambda := g\,\mathbf{1}_{E_\lambda} + 2\lambda\,\mathbf{1}_{\mathcal{X}\setminus E_\lambda} \in \mathcal{D}^\phi(v_\lambda).$$

By the definitions of v_λ and g_λ, (37), $|v_\lambda| \le \lambda$, and (35), we conclude that

$$\begin{aligned}\|v_\lambda\|_{L^p(\mathcal{X})} &\lesssim \|v_\lambda \mathbf{1}_{E_\lambda}\|_{L^p(\mathcal{X})} + \|v_\lambda \mathbf{1}_{\mathcal{X}\setminus E_\lambda}\|_{L^p(\mathcal{X})} \\ &\lesssim \|u\|_{L^p(\mathcal{X})} + \lambda[\mu(\mathcal{X} \setminus E_\lambda)]^{1/p} < \infty\end{aligned}$$

and

$$\|g_\lambda\|_{L^p(\mathcal{X})} \lesssim \|g\|_{L^p(\mathcal{X})} + 2\lambda[\mu(\mathcal{X} \setminus E_\lambda)]^{1/p} < \infty,$$

which, combined with the definition of $\|\cdot\|_{M^{\phi,p}(\mathcal{X})}$, implies that $v_\lambda \in M^{\phi,p}(\mathcal{X})$.

Now, we consider $v_\lambda - u$. Let $x,\ y \in \mathcal{X} \setminus E$. If $x,\ y \in E_\lambda$, then, by (37), it is obvious that

$$\left|(v_\lambda - u)(x) - (v_\lambda - u)(y)\right| = 0.$$

If $x,\ y \in \mathcal{X} \setminus (E_\lambda \cup E)$, then, by (38) and the definition of $\mathcal{D}^\phi(u)$, we obtain

$$\begin{aligned}\left|(v_\lambda - u)(x) - (v_\lambda - u)(y)\right| &\le \left|v_\lambda(x) - v_\lambda(y)\right| + \left|u(x) - u(y)\right| \\ &\le \phi(d(x,y))[2\lambda + g(x) + g(y)].\end{aligned}$$

If $x \in E_\lambda$ and $y \in \mathcal{X} \setminus (E_\lambda \cup E)$, then, by (38) and the definitions of $\mathcal{D}^\phi(u)$ and E_λ, we conclude that

$$\begin{aligned}\left|(v_\lambda - u)(x) - (v_\lambda - u)(y)\right| &\le \left|v_\lambda(x) - v_\lambda(y)\right| + \left|u(x) - u(y)\right| \\ &\le \phi(d(x,y))[2\lambda + g(x) + g(y)] \\ &\le \phi(d(x,y))[3\lambda + g(y)]\end{aligned}$$

and, similarly, if $x \in \mathcal{X} \setminus (E_\lambda \cup E)$ and $y \in E_\lambda$, by (38) and the definitions of $\mathcal{D}^\phi(u)$ and E_λ again, we find that

$$\left|(v_\lambda - u)(x) - (v_\lambda - u)(y)\right| \le \phi(d(x,y))[3\lambda + g(x)].$$

Altogether, from the definition of $\mathcal{D}^\phi(v_\lambda - u)$ and $\mu(E) = 0$, we deduce that

$$\widetilde{g}_\lambda := (3\lambda + g)\, \mathbf{1}_{\mathcal{X}\setminus E_\lambda} \in \mathcal{D}^\phi(v_\lambda - u).$$

Moreover, by $|v_\lambda| \le \lambda$ and the definitions of $\widetilde{g}_\lambda$ and E_λ, we have

$$\begin{aligned}\left\|(v_\lambda - u)\mathbf{1}_{\mathcal{X}\setminus E_\lambda}\right\|_{L^p(\mathcal{X})} &\lesssim \left\|(g+u)\mathbf{1}_{\mathcal{X}\setminus E_\lambda}\right\|_{L^p(\mathcal{X})} \\ &\lesssim \|g\|_{L^p(\mathcal{X})} + \|u\|_{L^p(\mathcal{X})} < \infty\end{aligned}$$

and

$$\begin{aligned}\left\|\widetilde{g}_\lambda \mathbf{1}_{\mathcal{X}\setminus E_\lambda}\right\|_{L^p(\mathcal{X})} &\lesssim \left\|(3\lambda + g)\mathbf{1}_{\mathcal{X}\setminus E_\lambda}\right\|_{L^p(\mathcal{X})} \\ &\lesssim \|u\|_{L^p(\mathcal{X})} + \|g\|_{L^p(\mathcal{X})} < \infty.\end{aligned}$$

Then, using this, (37), the dominated convergence theorem with respect to μ, and $\mu(\mathcal{X} \setminus E_\lambda) \to 0$ as $\lambda \to \infty$, we conclude that

$$\lim_{\lambda\to\infty} \|v_\lambda - u\|_{L^p(\mathcal{X})} = \lim_{\lambda\to\infty} \left\|(v_\lambda - u)\mathbf{1}_{\mathcal{X}\setminus E_\lambda}\right\|_{L^p(\mathcal{X})} = 0$$

and

$$\lim_{\lambda\to\infty} \|\widetilde{g}_\lambda\|_{L^p(\mathcal{X})} = \lim_{\lambda\to\infty} \left\|\widetilde{g}_\lambda \mathbf{1}_{\mathcal{X}\setminus E_\lambda}\right\|_{L^p(\mathcal{X})} = 0,$$

which imply $\lim_{\lambda\to\infty} \|v_\lambda - u\|_{M^{\phi,p}(\mathcal{X})} = 0$. This finishes the proof of Theorem 2. □

Now, we state the main result of this section, which generalizes ([41], Theorem 8.1) from fractional Hajłasz-type spaces to those with generalized smoothness.

Theorem 3. *Let $\phi \in \mathcal{A}$ and $\mathcal{F}$ be one of the following cases:*

(i) $\mathcal{F} = M^\phi_{p,\infty}(\mathcal{X}) = M^{\phi,p}(\mathcal{X})$ *with ϕ being a modulus of continuity and $p \in (1, \infty)$;*
(ii) $\mathcal{F} = M^\phi_{p,\infty}(\mathcal{X}) = M^{\phi,p}(\mathcal{X})$ *with ϕ being a modulus of continuity, $\alpha_\phi \in (0, 1)$, and $p \in (D/(D - \log_2 \alpha_\phi), 1]$;*
(iii) $\mathcal{F} = M^\phi_{p,q}(\mathcal{X})$ *with $\alpha_\phi \in (0, 1)$, $\beta_\phi \in (0, 2)$, and $p,\ q \in (D/(D - \log_2 \alpha_\phi), \infty)$;*
(iv) $\mathcal{F} = N^\phi_{p,q}(\mathcal{X})$ *with $\alpha_\phi \in (0, 1)$, $\beta_\phi \in (0, 2)$, $p \in (D/(D - \log_2 \alpha_\phi), \infty)$, and $q \in (0, \infty)$,*

where D is as in (1). If $u \in \dot{\mathcal{F}}$, then there exist a set E with $\mathrm{Cap}_{\mathcal{F}}(E) = 0$ and an $\mathcal{F}$-quasi-continuous function u^* on $\mathcal{X}$ such that, for any $x \in \mathcal{X} \setminus E$,

$$u^*(x) = \lim_{r\to 0^+} u_{B(x,r)}. \tag{40}$$

To prove Theorem 3, we need a weak-type estimate of the $\mathcal{F}$-capacity. To this end, we need several technical lemmas. The first one is on the Hajłasz gradient of $\mathcal{M}^* u$ for any u in which the integral on any ball is finite. Recall that, for any $u \in L^1_{\mathrm{loc}}(\mathcal{X})$, either $\mathcal{M}^* u \equiv \infty$ or

$\mathcal{M}^*u < \infty$ almost everywhere (see ([54], (3.1) and Lemma 4.8) or ([61], Remark 2.2)), where $\mathcal{M}^*$ is as in Definition 9(ii).

Lemma 8. (i) *Let $\phi \in \mathcal{A}_\infty$. Then, for any $u \in L^1_{\mathrm{loc}}(\mathcal{X})$ satisfying that its integral on any ball of $\mathcal{X}$ is finite and $\mathcal{M}^*u \not\equiv \infty$ and for any $g \in \mathcal{D}^\phi(u)$, $\mathcal{M}(g)$ is an element of $\mathcal{D}^\phi(\mathcal{M}^*u)$ modulo a positive constant independent of u and g, where $\mathcal{M}$ is the classical Hardy–Littlewood maximal operator and $\mathcal{M}^*$ as in Definition 9(ii).*

(ii) *Let $\phi \in \mathcal{A}_\infty$ with $\alpha_\phi \in (0,1)$. Then, for any $\lambda \in (D/[D-\log_2 \alpha_\phi], \infty)$, any $u \in L^1_{\mathrm{loc}}(\mathcal{X})$ satisfying that its integral on any ball of $\mathcal{X}$ is finite and $\mathcal{M}^*u \not\equiv \infty$, and for any $g \in \mathcal{D}^\phi(u)$, $[\mathcal{M}(g^\lambda)]^{1/\lambda}$ is an element of $\mathcal{D}^\phi(\mathcal{M}^*u)$ modulo a positive constant independent of u and g.*

Proof. Due to similarity, we only prove (ii). For any given $r \in (0,\infty)$, let $\{B_j\}_{j\in\mathcal{J}}$ be any given sequence of balls as in the definition of $\mathcal{M}^*$ with the radius r, and $\{\varphi_j\}_{j\in\mathcal{J}}$ be a partition of unity with respect to $\{B_j\}_{j\in\mathcal{J}}$ as in Definition 8, where $\mathcal{J}$ is an index set. Let u and g be as in the present lemma. From the definition of $\mathcal{M}^*$ and the observation that $\mathcal{D}^\phi(u) \subset \mathcal{D}^\phi(|u|)$, without loss of generality, we may assume that $u \geq 0$.

Let u_r be as in Definition 9(i). By $\sum_{j\in\mathcal{J}} \varphi_j \equiv 1$, we have

$$u_r = u + \sum_{j\in\mathcal{J}} \left(u_{B_j} - u\right)\varphi_j. \tag{41}$$

Therefore, for any $j \in \mathcal{J}$, using Lemma 7(ii) with u, E, and L^{-1} therein replaced, respectively, by $u - u_{B_j}$, $2B_j$, and r, and the properties of φ_j, we find that, for any $j \in \mathcal{J}$,

$$\widetilde{g^{(j)}} := \left\{g + [\phi(r)]^{-1}\left|u - u_{B_j}\right|\right\}\mathbf{1}_{2B_j}$$

is a positive constant multiple of an element in $\mathcal{D}^\phi([u - u_{B_j}]\varphi_j)$, where the positive constant is independent of r, u, and g. Let $\lambda \in (D/[D-\log_2 \alpha_\phi], \infty)$. Notice that, for any $j \in \mathcal{J}$, by Lemma 5(ii) with $B(y, 2^{-i})$ and 2^{-i} therein replaced, respectively, by B_j and r, we have, for any $x \in 2B_j$,

$$\left|u(x) - u_{B_j}\right| \lesssim \phi(r)\left[\mathcal{M}\left(g^\lambda\right)(x)\right]^{1/\lambda}$$

with the implicit positive constant independent of u, g, x, j, and r. From this; the proven conclusion that, for any $j \in \mathcal{J}$, $\widetilde{g^{(j)}}$ is a positive constant multiple of an element in $\mathcal{D}^\phi([u - u_{B_j}]\varphi_j)$; the definition of $\widetilde{g^{(j)}}$, $\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j} \lesssim 1$; and $g \leq [\mathcal{M}(g^\lambda)]^{1/\lambda}$, we deduce that, for almost every $x, y \in \mathcal{X}$,

$$\begin{aligned}
&\left|\sum_{j\in\mathcal{J}}\left[u_{B_j} - u(x)\right]\varphi_j(x) - \sum_{j\in\mathcal{J}}\left[u_{B_j} - u(y)\right]\varphi_j(y)\right| \\
&\quad\lesssim \phi(d(x,y)) \sum_{j\in\mathcal{J}}\left[\widetilde{g^{(j)}}(x) + \widetilde{g^{(j)}}(y)\right] \\
&\quad\lesssim \phi(d(x,y)) \sum_{j\in\mathcal{J}}\left[\left\{g(x) + \left[\mathcal{M}\left(g^\lambda\right)(x)\right]^{1/\lambda}\right\}\mathbf{1}_{2B_j}(x) + \left\{g(y) + \left[\mathcal{M}\left(g^\lambda\right)(y)\right]^{1/\lambda}\right\}\mathbf{1}_{2B_j}(y)\right] \\
&\quad\lesssim \phi(d(x,y))\left\{g(x) + \left[\mathcal{M}\left(g^\lambda\right)(x)\right]^{1/\lambda} + g(y) + \left[\mathcal{M}\left(g^\lambda\right)(y)\right]^{1/\lambda}\right\} \\
&\quad\lesssim \phi(d(x,y))\left\{\left[\mathcal{M}\left(g^\lambda\right)(x)\right]^{1/\lambda} + \left[\mathcal{M}\left(g^\lambda\right)(y)\right]^{1/\lambda}\right\},
\end{aligned}$$

which implies that $[\mathcal{M}(g^\lambda)]^{1/\lambda}$ is a positive constant multiple of an element of $\mathcal{D}^\phi(\sum_{j\in\mathcal{J}}[u_{B_j} - u]\varphi_j)$. By this, (41), the definition of $\mathcal{D}^\phi(u_r)$, $g \in \mathcal{D}^\phi(u)$, and $g \leq [\mathcal{M}(g^\lambda)]^{1/\lambda}$, we further conclude that $[\mathcal{M}(g^\lambda)]^{1/\lambda}$ is a positive constant multiple of an element in $\mathcal{D}^\phi(u_r)$ with the positive constant independent of u, g, and r. Moreover, if $\mathcal{M}^*u \not\equiv \infty$, then by the definition of $\mathcal{M}^*$ and Lemma 2(ii), we conclude that $[\mathcal{M}(g^\lambda)]^{1/\lambda}$ is an element of $\mathcal{D}^\phi(\mathcal{M}^*u)$ modulo a positive constant independent of u and g. This finishes the proof of (ii) and hence of Lemma 8. □

Borrowing some ideas from the proof of ([41], Lemma 7.1), we can prove the following lemma on the Hajłasz gradient sequence of $\mathcal{M}^*u$ for any $u \in L^1_{\mathrm{loc}}(\mathcal{X})$ with its integral on any ball being finite.

Lemma 9. *Let $\phi \in \mathcal{A}_0$ with $\beta_\phi^+ \in (0,2)$, $\epsilon \in (0, -\log_2 \alpha_\phi)$, and*

$$\delta \in \left(0, \min\left\{1 - \log_2 \beta_\phi, -\log_2 \alpha_\phi - \epsilon\right\}\right).$$

Then, for any $\lambda \in (D/[D+\epsilon], \infty)$, any $u \in L^1_{\mathrm{loc}}(\mathcal{X})$ such that its integral on any ball in $\mathcal{X}$ is finite and $\mathcal{M}^*u \not\equiv \infty$, and any $\vec{g} := \{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(u)$, the sequence $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ of functions, defined by setting, for any $k \in \mathbb{Z}$,

$$\widetilde{g_k} := \sum_{l\in\mathbb{Z}} 2^{-|l-k|\delta} \left[\mathcal{M}\left(g_l^\lambda\right)\right]^{1/\lambda}, \tag{42}$$

is a positive constant multiple of an element in $\mathbb{D}^\phi(\mathcal{M}^*u)$, where the positive constant is independent of u and $\vec{g}$, D as in (1), and $\mathcal{M}^*$ as in Definition 9(ii).

Proof. Let all of the symbols be as in the present lemma. By the definition of $\mathcal{M}^*u$ and the observation that $\mathbb{D}^\phi(u) \subset \mathbb{D}^\phi(|u|)$, without loss of generality, we may assume that $u \ge 0$. Moreover, by Lemma 2 and the definition of $\mathcal{M}^*$, to prove the present lemma, it suffices to show that, for any $i \in \mathbb{Z}$, $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^\phi(u_{2^{-i}})$ with the positive constant independent of i, where $u_{2^{-i}}$ is as in Definition 9(i).

To this end, we first recall that, in the proof of Theorem 1, we have shown that, for any $i \in \mathbb{Z}$, $\{h_k\}_{k\in\mathbb{Z}}$, defined as in (28), is a positive constant multiple of an element in $\mathbb{D}^\phi(u - u_{2^{-i}})$. From this, $\vec{g} \in \mathbb{D}^\phi(u)$, the definitions of $\mathbb{D}^\phi(u)$ and $\mathbb{D}^\phi(u - u_{2^{-i}})$, and, for any $x,\ y \in \mathcal{X}$,

$$\begin{aligned}&\left|u_{2^{-i}}(x) - u_{2^{-i}}(y)\right|\\ &\quad\le |u(x) - u(y)| + \left|(u - u_{2^{-i}})(x) - (u - u_{2^{-i}})(y)\right|,\end{aligned}$$

it follows that, for any $i \in \mathbb{Z}$, $\{g_k + h_k\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^\phi(u_{2^{-i}})$, where the positive constant is independent of i, u, and $\vec{g}$. Thus, to prove that $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^\phi(u_{2^{-i}})$ for any $i \in \mathbb{Z}$, it suffices to show that

$$g_k + h_k \lesssim \widetilde{g_k}, \quad \forall\, k \in \mathbb{Z}. \tag{43}$$

Indeed, by the definition of $\widetilde{g_k}$ and the fact that, for almost every $x \in \mathcal{X}$, $g_k(x) \le [\mathcal{M}(g_k^\lambda)(x)]^{1/\lambda}$, we have $g_k \le \widetilde{g_k}$ for any $k \in \mathbb{Z}$ almost everywhere. Then, to show (43), it suffices to prove that, for any $k \in \mathbb{Z}$, $h_k \lesssim \widetilde{g_k}$ almost everywhere. Let ϵ and δ be as in the present lemma. By $1 - \delta > \log_2 \beta_\phi$, $\epsilon + \delta < -\log_2 \alpha_\phi$, and Lemma 1(ii) with δ and ϵ therein replaced, respectively, by $1 - \delta$ and $\epsilon + \delta$, we find that, for any $k,\ l \in \mathbb{Z}$ with $l \le k$,

$$2^{l-k}\phi(2^{-l})\left[\phi(2^{-k})\right]^{-1} \lesssim 2^{(l-k)\delta} \tag{44}$$

and, for any $k,\ l \in \mathbb{Z}$ with $l \ge k - 4$,

$$2^{(l-k)\epsilon}\phi(2^{-l})\left[\phi(2^{-k})\right]^{-1} \lesssim 2^{-(l-k)\delta}. \tag{45}$$

Let $i \in \mathbb{Z}$. Observe that, for any $l \ge i - 4$, $2^{(i-l)(1-\epsilon)} \lesssim 1$ and, for any $k \le i$, $2^{(k-i)\epsilon} \lesssim 1$. By this, (44) and (45), we obtain, for any $x \in \mathcal{X}$ and $k > i$,

$$\frac{2^{i-k}\, 2^{-i\epsilon}}{\phi(2^{-k})} \sum_{i-4\le l\le k} 2^{l\epsilon}\phi(2^{-l}) \left[\mathcal{M}\left([g_l(x)]^\lambda\right)\right]^{1/\lambda}$$

$$\begin{aligned}
&\lesssim \sum_{i-4\le l\le k} 2^{(i-l)(1-\epsilon)}2^{(l-k)\delta}\Big[\mathcal{M}\Big([g_l(x)]^{\lambda}\Big)\Big]^{1/\lambda}\\
&\lesssim \sum_{i-4\le l\le k} 2^{(l-k)\delta}\Big[\mathcal{M}\Big([g_l(x)]^{\lambda}\Big)\Big]^{1/\lambda}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2^{i-k}\,2^{-i\epsilon}}{\phi(2^{-k})}\sum_{l>k}2^{l\epsilon}\phi(2^{-l})\,\Big[\mathcal{M}\Big([g_l(x)]^{\lambda}\Big)\Big]^{1/\lambda}\\
&\quad\lesssim \sum_{l>k} 2^{(i-k)(1-\epsilon)}2^{-(l-k)\delta}\Big[\mathcal{M}\Big([g_l(x)]^{\lambda}\Big)\Big]^{1/\lambda}\\
&\quad\lesssim \sum_{l>k} 2^{-(l-k)\delta}\Big[\mathcal{M}\Big([g_l(x)]^{\lambda}\Big)\Big]^{1/\lambda}
\end{aligned}$$

and, for any $x\in\mathcal{X}$ and $k\le i$,

$$\begin{aligned}
&\frac{2^{-i\epsilon}}{\phi(2^{-k})}\sum_{l\ge i-4}2^{l\epsilon}\phi(2^{-l})\,\Big[\mathcal{M}\Big([g_l(x)]^{\lambda}\Big)\Big]^{1/\lambda}\\
&\quad\lesssim \sum_{l\ge i-4} 2^{(k-i)\epsilon}2^{-(l-k)\delta}\,\Big[\mathcal{M}\Big([g_l(x)]^{\lambda}\Big)\Big]^{1/\lambda}\\
&\quad\lesssim \sum_{l\ge i-4} 2^{-(l-k)\delta}\,\Big[\mathcal{M}\Big([g_l(x)]^{\lambda}\Big)\Big]^{1/\lambda},
\end{aligned}$$

which, combined with the proved conclusion that, for any $k\in\mathbb{Z}$, $g_k\le\widetilde{g_k}$ almost everywhere, implies that, for any $k\in\mathbb{Z}$, $h_k\lesssim\widetilde{g_k}$ almost everywhere. Thus, for any $k\in\mathbb{Z}$, $g_k+h_k\lesssim\widetilde{g_k}$ almost everywhere. Furthermore, noticing that $\{g_k+h_k\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^{\phi}(u_{2^{-i}})$, from the definition of $\mathbb{D}^{\phi}(u_{2^{-i}})$, we deduce that $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ is also a positive constant multiple of an element in $\mathbb{D}^{\phi}(u_{2^{-i}})$, where the positive constant is independent of u, $\vec{g}$, and i. Thus, by Lemma 2 and the definition of $\mathcal{M}^*$, we conclude that $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^{\phi}(\mathcal{M}^*u)$, which completes the proof of Lemma 9. □

The next two lemmas are used to show the boundedness of the discrete maximal operator $\mathcal{M}^*$ on ϕ-Hajłasz-type spaces, which is a generalization of ([61], Theorem 4.7) and ([41], Lemma 8.3), respectively.

Lemma 10. *With the assumptions same as in Theorem 3, there exists a positive constant C, independent of u, such that, for any $u\in\dot{\mathcal{F}}$ with $\mathcal{M}^*u\not\equiv\infty$,*

$$\left\|\mathcal{M}^*u\right\|_{\dot{\mathcal{F}}}\le C\|u\|_{\dot{\mathcal{F}}}, \tag{46}$$

where $\mathcal{M}^*$ is as in Definition 9(ii).

Proof. If $\mathcal{F}$ belongs to the case (i) of Theorem 3, then (46) follows from Lemma 8(i) and the boundedness of the Hardy–Littlewood maximal operator on $L^p(\mathcal{X})$.

If $\mathcal{F}$ belongs to the case (ii) of Theorem 3, then (46) follows from Lemma 8(ii) and the boundedness of the classical Hardy–Littlewood maximal operator $\mathcal{M}$ on $L^{p/\lambda}(\mathcal{X})$ with $\lambda\in(D/[D-\log_2\alpha_\phi],p)$.

Now, let $\mathcal{F}$ belong to the case (iii) of Theorem 3. Let $u\in\dot{M}^{\phi}_{p,q}(\mathcal{X})$, $\{g_k\}_{k\in\mathbb{Z}}\in\mathbb{D}^{\phi}(u)$ with $\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)}\lesssim\|u\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})}$, and $r:=\min\{p,q\}$. Let D be as in (1), $\lambda\in(D/[D-\log_2\alpha_\phi],r)$, and $\epsilon'\in(0,-\log_2\alpha_\phi)$ be such that $\lambda=D/(D+\epsilon')$. We also choose $\epsilon:=(\epsilon'-\log_2\alpha_\phi)/2$. From $\alpha_\phi<1$, it follows that $0<\epsilon'<\epsilon<-\log_2\alpha_\phi$ and hence $\lambda\in(D/[D+\epsilon],r)$.

Let $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ be as in (42) with $\delta \in (0, \min\{1-\log_2\beta_\phi, -\log_2\alpha_\phi - \epsilon\})$. Then, by the definition of $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ and the Fefferman–Stein vector-valued maximal inequality on $L^{\frac{p}{\lambda}}(\mathcal{X}, l^{\frac{q}{\lambda}})$ (see ([56], Theorem 1.2) or ([57], Theorem 1.3)), we have

$$\left\|\{\widetilde{g_k}\}_{k\in\mathbb{Z}}\right\|_{L^p(\mathcal{X},l^q)} \le \left\|\left\{\left[\mathcal{M}\left(g_k^\lambda\right)\right]^{1/\lambda}\right\}_{k\in\mathbb{Z}}\right\|_{L^p(\mathcal{X},l^q)} \lesssim \left\|\{g_k\}_{k\in\mathbb{Z}}\right\|_{L^p(\mathcal{X},l^q)}.$$

Thus, using this, the definition of $\|\cdot\|_{\dot{M}^\phi_{p,q}(\mathcal{X})}$, Lemma 9, and $\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \lesssim \|u\|_{\dot{M}^\phi_{p,q}(\mathcal{X})}$, we obtain

$$\left\|\mathcal{M}^* u\right\|_{\dot{M}^\phi_{p,q}(\mathcal{X})} \lesssim \left\|\{\widetilde{g_k}\}_{k\in\mathbb{Z}}\right\|_{L^p(\mathcal{X},l^q)} \lesssim \left\|\{g_k\}_{k\in\mathbb{Z}}\right\|_{L^p(\mathcal{X},l^q)} \lesssim \|u\|_{\dot{M}^\phi_{p,q}(\mathcal{X})}.$$

This finishes the proof of Lemma 10. □

Lemma 11. *Let $x_0 \in \mathcal{X}$, $r \in (0,\infty)$, $B_0 := B(x_0, r)$, $\phi \in \mathcal{A}$, and*

$$\mathcal{F} \in \left\{M^\phi_{p,q}(\mathcal{X}),\ N^\phi_{p,q}(\mathcal{X}) :\ p,\ q \in (0,\infty]\right\}.$$

If $\{y \in \mathcal{X} :\ d(x_0, y) = \tau r\}$ for some $\tau \in (2,\infty)$ is not empty, then there exists a positive constant C, depending only on τ, ϕ, and C_μ, such that, for any $u \in \dot{\mathcal{F}}$ supported in B_0,

$$\|u\|_{\mathcal{F}} \le C[1+\phi(r)]\|u\|_{\dot{\mathcal{F}}},$$

where C_μ is as in (1).

Proof. By similarity, we only prove the case $\mathcal{F} = M^\phi_{p,q}(\mathcal{X})$. Let $B_0 := B(x_0, r)$; τ, C_μ, and u be as in the present lemma; E be the exceptional zero-measure set such that (13) holds true; and $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(u)$ with $\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \lesssim \|u\|_{\dot{M}^\phi_{p,q}(\mathcal{X})}$. Notice that, for any $x \in B_0$ and $y \in 2\tau B_0 \setminus 2B_0$, we have $d(x,y) \in (r, [1+2\tau]r)$. From this, the fact that $u|_{2\tau B_0\setminus 2B_0} \equiv 0$, and the definitions of both $\mathcal{A}$ and $\mathbb{D}^\phi(u)$, we deduce that, for any $x \in B_0 \setminus E$,

$$\begin{aligned}\left|u(x)\right| &= \inf_{y\in 2\tau B_0\setminus(2B_0\cup E)}\left|u(x)-u(y)\right| \\ &\lesssim \phi([1+2\tau]r)\left[g(x) + \inf_{y\in 2\tau B_0\setminus(2B_0\cup E)} g(y)\right],\end{aligned} \tag{47}$$

where $g := \sup_{\{k:\, r\le 2^{-k}\le(1+2\tau)r\}} g_k$ and $g \ge 0$.

Let $z \in \mathcal{X}$ be such that $d(x_0, z) = \tau r$. Then, by a geometrical observation, we have

$$B_0 \subset B(z, [1+\tau]r) \quad \text{and} \quad B(z, [\tau-2]r/2) \subset (2\tau B_0 \setminus 2B_0),$$

which, together with the doubling property of μ, implies that

$$\mu(B_0) \le \mu(B(z,[1+\tau]r)) \lesssim \mu(B(z,[\tau-2]r/2)) \lesssim \mu(2\tau B_0 \setminus 2B_0), \tag{48}$$

where the implicit positive constants depend only on τ and C_μ. Thus, from $u_{\mathcal{X}\setminus B_0} \equiv 0$, $\mu(E) = 0$, (47), (48), and the definitions of g and $\mathcal{A}$, we deduce that

$$\begin{aligned}\|u\|_{L^p(\mathcal{X})} &= \|u\|_{L^p(B_0)} \\ &\lesssim \phi([1+2\tau]r)\left\{\|g\|_{L^p(B_0)} + [\mu(B_0)]^{1/p}\inf_{y\in 2\tau B_0\setminus(2B_0\cup E)} g(y)\right\} \\ &\lesssim \phi([1+2\tau]r)\left\{\|g\|_{L^p(B_0)} + [\mu(2\tau B_0\setminus 2B_0)]^{1/p}\inf_{y\in 2\tau B_0\setminus(2B_0\cup E)} g(y)\right\} \\ &\lesssim \phi([1+2\tau]r)\|g\|_{L^p(2\tau B_0)} \lesssim \phi(r)\|\{g_k\}\|_{L^p(\mathcal{X},l^q)}\end{aligned}$$

with the usual modification made when $p = \infty$, which, together with the assumption of $\{g_k\}_{k\in\mathbb{Z}}$, implies that

$$\begin{aligned}\|u\|_{M^{\phi}_{p,q}(\mathcal{X})} &\le \|u\|_{L^p(\mathcal{X})} + \|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \\ &\lesssim \phi(r)\|\{g_k\}\|_{L^p(\mathcal{X},l^q)} + \|u\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})} \\ &\lesssim [1+\phi(r)]\|u\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})}.\end{aligned}$$

This finishes the proof of Lemma 11. □

Based on the above lemmas, we can obtain the following localized weak-type capacitary estimate for the restricted maximal operator $\mathcal{M}_R$, where $R \in (0,\infty]$. Recall that there exists a positive constant c, depending only on C_μ, such that, for any $u \in L^0(\mathcal{X})$,

$$c^{-1}\mathcal{M}_{R/c}u \le \mathcal{M}^*_R u \le c\mathcal{M}_{cR}u \tag{49}$$

(see, for instance, [41], [(8.1)]), where $\mathcal{M}_R$ is as in (14), $\mathcal{M}^*_R$ as in Definition 9(iii), and C_μ as in (1).

Lemma 12. *With the same assumptions as in Theorem 3, let $x_0 \in \mathcal{X}$, $R \in (0,\infty)$, and $B := B(x_0,R)$. If $\tau B \setminus 10B$ for some $\tau \in (10,\infty)$ is not empty, then there exist positive constants $c = c_{(C_\mu)}$ and $C = C_{(\mathcal{F},R,\tau,C_\mu)}$ such that, for any $u \in \mathcal{F}$ and $\kappa \in (0,\infty)$,*

$$\mathrm{Cap}_{\mathcal{F}}\left(\left\{x \in B:\ \mathcal{M}_{R/c}u(x) > \kappa\right\}\right) \le C\kappa^{-p}\|u\|^p_{\mathcal{F}},$$

where $\mathcal{M}_R$ is as in (14) and C_μ as in (1).

Proof. Let all of the symbols be as in the present lemma, $\mathcal{M}^*_R$ as in Definition 9(iii), and $u \in \mathcal{F}$. Let φ be a Lipschitz function supported in $4B$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $3B$. By the definition of $\mathcal{M}^*_R$ and the assumption of φ, we have $\mathcal{M}^*_R u = \mathcal{M}^*_R(u\varphi)$ on B and $\mathcal{M}^*_R(u\varphi) \equiv 0$ on $\mathcal{X} \setminus 5B$. Then, from (49), we deduce that

$$\begin{aligned}\left\{x \in B:\ \mathcal{M}_{R/c}u(x) > \kappa\right\} &\subset \left\{x \in B:\ c\mathcal{M}^*_R u(x) > \kappa\right\} \\ &\subset \left\{x \in \mathcal{X}:\ c\mathcal{M}^*_R(u\varphi)(x) > \kappa\right\} \\ &= \left\{x \in \mathcal{X}:\ c\kappa^{-1}\mathcal{M}^*_R(u\varphi)(x) > 1\right\} =: Q,\end{aligned} \tag{50}$$

where $c = c_{(C_\mu)}$ is just the positive constant as in (49).

By the lower semi-continuity of $\mathcal{M}^*_R(u\varphi)$ (see [54], p. 376), we conclude that, for any $x \in Q$, there exists a $\delta_x \in (0,1)$ such that, for any $y \in B(x,\delta_x)$, $c\kappa^{-1}\mathcal{M}^*_R(u\varphi)(y) > 1$. Thus, $Q' := \bigcup_{x\in Q} B(x,\delta_x)$ is a neighborhood of Q and $c\kappa^{-1}\mathcal{M}^*_R(u\varphi) > 1$ on Q'. By this; (50); Remark 4(iii); Definition 10; Lemma 11 with u and B_0 therein replaced, respectively, by $\mathcal{M}^*_R(u\varphi)$ and $5B$; Lemma 10; and Corollary 2, we obtain

$$\begin{aligned}\mathrm{Cap}_{\mathcal{F}}\left(\left\{x \in B:\ \mathcal{M}_{R/c}u(x) > \kappa\right\}\right) &\le \mathrm{Cap}_{\mathcal{F}}(Q) \\ &\le \left\|c\kappa^{-1}\mathcal{M}^*_R(u\varphi)\right\|^p_{\mathcal{F}} \lesssim \kappa^{-p}\left\|\mathcal{M}^*_R(u\varphi)\right\|^p_{\dot{\mathcal{F}}} \\ &\lesssim \kappa^{-p}\left\|\mathcal{M}^*(u\varphi)\right\|^p_{\dot{\mathcal{F}}} \lesssim \kappa^{-p}\|u\varphi\|^p_{\dot{\mathcal{F}}} \\ &\lesssim \kappa^{-p}\|u\|^p_{\mathcal{F}},\end{aligned}$$

where the implicit positive constants depend on $\mathcal{F}$, R, τ, and C_μ. This finishes the proof of Lemma 12. □

Now, we show Theorem 3.

Proof of Theorem 3. Again, by similarity, we only consider the case $\mathcal{F} = M^{\phi}_{p,q}(\mathcal{X})$. Without loss of generality, we may assume that $\mathcal{X}$ contains at least two points. By this, we easily know that there exist balls $\{B(x_l, r_l)\}_{l\in\mathcal{I}}$ with $\mathcal{I} \subset \mathbb{N}$ being an index set such that $\mathcal{X} \subset \bigcup_{l\in\mathcal{I}} B(x_l, r_l)$ and, for any $l \in \mathcal{I}$, $5B(x_l, r_l) \setminus 4B(x_l, r_l)$ is not empty.

Let $\mathcal{F}$ be any given function space as in (i), (ii), or (iii) of the present theorem, and $u \in \mathcal{F}$. Then, from Theorem 2 when $\mathcal{F}$ is as in either (i) or (ii), or from Theorem 1 when $\mathcal{F}$ is as in case (iii), we deduce that there exists a sequence $\{v_i\}_{i\in\mathbb{N}}$ of continuous functions such that, for any $i \in \mathbb{N}$,

$$\|u - v_i\|^p_{\mathcal{F}} < 2^{-i(1+p)}. \tag{51}$$

Let $\{B(x_l, r_l)\}_{l\in\mathcal{I}}$ be a ball covering of $\mathcal{X}$ as above and $c = c(C_\mu)$ the positive constant as in Lemma 12. For any $l \in \mathcal{I}$, any $i, j \in \mathbb{N}$, and any $u \in \mathcal{F}$, let

$$A_{l,i} := \left\{x \in B(x_l, r_l) : \ \mathcal{M}_{r_l/c}(u - v_i)(x) > 2^{-i}\right\}$$

and

$$B_{l,j} := \bigcup_{i\geq j} A_{l,i}.$$

Then, by Lemma 12 and (51), we have

$$\mathrm{Cap}_{\mathcal{F}}(A_{l,i}) \lesssim 2^{ip}\|u - v_i\|^p_{\mathcal{F}} \lesssim 2^{-i}$$

and, furthermore, by Lemma 6, we obtain

$$\mathrm{Cap}_{\mathcal{F}}(B_{l,j}) \lesssim \left\{\sum_{i\geq j}[\mathrm{Cap}_{\mathcal{F}}(A_{l,i})]^\theta\right\}^{1/\theta} \lesssim 2^{-j},$$

where $\theta := \min\{1, q/p\}$. Thus, the set $F_l := \bigcap_{j\in\mathbb{N}} B_{l,j}$ is of zero $\mathcal{F}$-capacity.

Let $l \in \mathcal{I}$. For any $i \in \mathbb{N}$, using the continuity of v_i and the Lebesgue differentiation theorem, we conclude that, for any $x \in \mathcal{X}$,

$$\lim_{r\to 0^+} \fint_{B(x,r)} |v_i(y) - v_i(x)|\, d\mu(y) = 0. \tag{52}$$

Since u is locally integrable (see Remark 3(i)), then, for any $i \in \mathbb{N}$, from (52) and the definition of $A_{l,i}$, we deduce that, for any $x \in B(x_l, r_l) \setminus A_{l,i}$,

$$\begin{aligned}
\limsup_{r\to 0^+}\left|v_i(x) - u_{B(x,r)}\right| &\leq \limsup_{r\to 0^+} \fint_{B(x,r)} |v_i(x) - u(y)|\, d\mu(y) \\
&\leq \limsup_{r\to 0^+} \fint_{B(x,r)} |v_i(y) - u(y)|\, d\mu(y) \\
&\leq \mathcal{M}_{r_l/c}(u - v_i)(x) \leq 2^{-i}.
\end{aligned} \tag{53}$$

Therefore, by (53), we find that, for any $j \in \mathbb{N}$, $i_1, i_2 \in \mathbb{N}$ with $i_1, i_2 \geq j$ and $x \in B(x_l, r_l) \setminus B_{l,j} = \bigcap_{i\geq j}[B(x_l, r_l) \setminus A_{l,i}]$,

$$\begin{aligned}
\left|v_{i_1}(x) - v_{i_2}(x)\right| &\leq \limsup_{r\to 0^+}\left|v_{i_1}(x) - u_{B(x,r)}\right| + \limsup_{r\to 0^+}\left|v_{i_2}(x) - u_{B(x,r)}\right| \\
&\leq 2^{-i_1} + 2^{-i_2},
\end{aligned}$$

which means that, for any given $j \in \mathbb{N}$, $\{v_i|_{B(x_l,r_l)\setminus B_{l,j}}\}_{i\in\mathbb{N}}$ is a Cauchy sequence uniformly in $B(x_l, r_l) \setminus B_{l,j}$. Thus, for any $j \in \mathbb{N}$, $\{v_i|_{B(x_l,r_l)\setminus B_{l,j}}\}_{i\in\mathbb{N}}$ converge to some continuous function

$v_{l,j}$ uniformly in $B(x_l, r_l) \setminus B_{l,j}$ as $i \to \infty$. Due to the observation that $B(x_l, r_l) \setminus B_{l,j}$ increases on j and the uniqueness of the limit, we conclude that, for any $j_1,\ j_2 \in \mathbb{N}$ with $j_1 \le j_2$,

$$v_{l,j_2}|_{B(x_l,r_l)\setminus B_{l,j_1}} = v_{l,j_1}.$$

Therefore, the function v_l^*, defined by setting, for any $x \in B(x_l, r_l) \setminus F_l$,

$$v_l^*(x) := \lim_{j\to\infty} v_{l,j}(x),$$

exists and, for any given $j \in \mathbb{N}$, $v_l^*|_{B(x_l,r_l)\setminus B_{l,j}} = v_{l,j}$. Since $v_{l,j}$ is continuous in $B(x_l, r_l) \setminus B_{l,j}$, we deduce that, for any given $j \in \mathbb{N}$, v_l^* is continuous in $B(x_l, r_l) \setminus B_{l,j}$. By the definitions of v_l^* and $v_{l,j}$, and (53) with $i \to \infty$, we conclude that, for any $x \in B(x_l, r_l) \setminus F_l = \bigcup_{j\in\mathbb{N}}[B(x_l, r_l) \setminus B_{l,j}] = \bigcup_{j\in\mathbb{N}} \bigcap_{i\ge j}[B(x_l, r_l) \setminus A_{l,i}]$,

$$v_l^*(x) = \lim_{j\to\infty} v_{l,j}(x) = \lim_{j\to\infty} \lim_{i\to\infty} v_i|_{B(x_l,r_l)\setminus B_{l,j}}(x) = \lim_{r\to0^+} u_{B(x,r)}.$$

Altogether, we find a function v_l^* and a set F_l with $\mathrm{Cap}_{\mathcal{F}}(F_l) = 0$ such that

$$v_l^*(\cdot) = \lim_{r\to0^+} u_{B(\cdot,r)}$$

in $B(x_l, r_l) \setminus F_l$ and, for any $\epsilon \in (0, \infty)$, there exist a $j \in \mathbb{N}$ and a set $B_{l,j}$ with $\mathrm{Cap}_{\mathcal{F}}(B_{l,j}) < \epsilon$ such that v_l^* is continuous in $B(x_l, r_l) \setminus B_{l,j}$.

Next, let $u \in \dot{\mathcal{F}}$. For any given $\widetilde{x} \in \mathcal{X}$ and $k \in \mathbb{N}$, let φ_k be a Lipschitz function such that $\varphi_k \mathbf{1}_{B(\widetilde{x},2k)} = 1$ and $\varphi_k \mathbf{1}_{\mathcal{X}\setminus B(\widetilde{x},3k)} = 0$. By the boundedness of the support of φ_k and Corollary 3, we find that $u\varphi_k \in \mathcal{F}$. Thus, from the conclusion proved in the above paragraph, we deduce that, for any $k \in \mathbb{N}$, there exist a set $E_{l,k}$ with $\mathrm{Cap}_{\mathcal{F}}(E_{l,k}) = 0$ and a function $u_{l,k}$ defined on $B(x_l, r_l) \setminus E_{l,k}$ such that, for any $x \in B(x_l, r_l) \setminus E_{l,k}$,

$$u_{l,k}(x) = \lim_{r\to0^+} (u\varphi_k)_{B(x,r)}$$

and, for any $\epsilon \in (0, \infty)$, there exists a set $U_{l,k}$ with $\mathrm{Cap}_{\mathcal{F}}(U_{l,k}) < 2^{-k-l}\epsilon$ such that $u_{l,k}$ is continuous in $B(x_l, r_l) \setminus U_{l,k}$.

Define $E_l := \bigcup_{k\in\mathbb{N}} E_{l,k}$ and $U_l := \bigcup_{k\in\mathbb{N}} U_{l,k}$. Then, by Lemma 6, we have $\mathrm{Cap}_{\mathcal{F}}(E_l) = 0$ and, for the above given $\epsilon \in (0, \infty)$, $\mathrm{Cap}_{\mathcal{F}}(U_l) \le 2^{-l}\epsilon$ and, moreover, $\mathrm{Cap}_{\mathcal{F}}(E_l \cup U_l) \le 2^{-l}\epsilon$. For any $x \in B(x_l, r_l) \setminus E_l = \bigcap_{k\in\mathbb{N}} B(x_l, r_l) \setminus E_{l,k}$ and any $k_x \in \mathbb{N}$ big enough such that $x \in B(\widetilde{x}, k_x)$, since, for any $r \in (0, k_x]$, we have $B(x, r) \subset B(\widetilde{x}, 2k_x)$, then, from the fact that $\varphi_{k_x} \mathbf{1}_{B(\widetilde{x},2k_x)} = 1$, we deduce that

$$\begin{aligned}\lim_{r\to0^+} (u\varphi_{k_x})_{B(x,r)} &= \lim_{r\to0^+,\, r\in(0,k_x]} \fint_{B(x,r)} u\varphi_{k_x}\, d\mu \\ &= \lim_{r\to0^+,\, r\in(0,k_x]} \fint_{B(x,r)} u\, d\mu \\ &= \lim_{r\to0^+} u_{B(x,r)}.\end{aligned} \tag{54}$$

Define u_l by setting, for any $x \in B(x_l, r_l) \setminus E_l$, $u_l(x) := \lim_{r\to0^+} u_{B(x,r)}$. Then, by (54) and the definition of $u_{l,k}$, we conclude that, for any $k \in \mathbb{N}$, $u_l = u_{l,k}$ in $[B(x_l, r_l) \cap B(\widetilde{x}, k)] \setminus E_l$. From this, the fact that $u_{l,k}$ is continuous in $B(x_l, r_l) \setminus U_{l,k}$, and the definition of U_l, we deduce that, for any $k \in \mathbb{N}$, u_l is continuous in $[B(x_l, r_l) \cap B(\widetilde{x}, k)] \setminus (E_l \cup U_l)$. Therefore, u_l is continuous in $B(x_l, r_l) \setminus (E_l \cup U_l)$.

Finally, we turn to the whole space $\mathcal{X}$ using the covering $\mathcal{X} \subset \bigcup_{l\in\mathcal{I}} B(x_l, r_l)$. Let $u \in \dot{\mathcal{F}}$. On the one hand, we have shown that, for any $l \in \mathcal{I}$, there exists a set E_l with $\mathrm{Cap}_{\mathcal{F}}(E_l) = 0$ such that $u_l(\cdot) := \lim_{r\to0^+} u_{B(\cdot,r)}$ exists on $B(x_l, r_l) \setminus E_l$. Define $E := \bigcup_{l\in\mathcal{I}} E_l$ and, for any $x \in \mathcal{X} \setminus E$, $\widetilde{u}(x) := \lim_{r\to0^+} u_{B(x,r)}$. Then, for any $l \in \mathcal{I}$, $\widetilde{u} = u_l$ in $B(x_l, r_l) \setminus E$.

On the other hand, by the above proof, we conclude that, for any given $\epsilon \in (0, \infty)$ and any $l \in \mathcal{I}$, there exists a set $\widetilde{U_l}$ with $\mathrm{Cap}_{\mathcal{F}}(\widetilde{U_l}) \le 2^{-l}\epsilon$ such that u_l is continuous in $B(x_l, r_l) \setminus \widetilde{U_l}$. Define $U := \bigcup_{l\in\mathcal{I}} \widetilde{U_l}$. Then, for any $l \in \mathcal{I}$, u_l is continuous in $B(x_l, r_l) \setminus U$. From this and the fact that, for any $l \in \mathcal{I}$, $\widetilde{u} = u_l$ in $B(x_l, r_l) \setminus E$, we deduce that $\widetilde{u}$ is continuous in $B(x_l, r_l) \setminus (E \cup U)$ for any $l \in \mathcal{I}$ and hence in $\mathcal{X} \setminus (E \cup U)$.

By Lemma 6, we have $\mathrm{Cap}_{\mathcal{F}}(E) = 0$ and $\mathrm{Cap}_{\mathcal{F}}(U) \le \epsilon$ and, furthermore,

$$\mathrm{Cap}_{\mathcal{F}}(E \cup U) \le \epsilon.$$

Let u^* be any function defined in $\mathcal{X}$ such that $u^* = \widetilde{u}$ in $\mathcal{X} \setminus E$. Then, u^* is continuous in $\mathcal{X} \setminus (E \cup U)$. Thus, u^* is one of the desired $\mathcal{F}$-quasi-continuous functions on $\mathcal{X}$, which completes the proof of Theorem 3. □

Remark 5. *With the same assumptions as in Theorem 3, by (40), the local integrability of u (see [28], Remark 3.8), Remark 4(ii), and the Lebesgue differentiation theorem, we have the following two obvious observations:*

(i) *$u^* = u$ almost everywhere;*
(ii) *every point outside E is a Lebesgue point of u^*.*

In this sense, u^* is called an $\mathcal{F}$-quasi-continuous representative *of u. Furthermore, from the conclusion in (ii) of the present remark and ([45], Lemma 17), we deduce that, for any given $\mathcal{F}$-quasi-continuous function u in $\mathcal{F}$, there exists a set of zero $\mathcal{F}$-capacity such that all the outside points are Lebesgue points of u. Observe that, by Remark 4(ii), any set of zero $\mathcal{F}$-capacity is of zero measure. This implies that, for any $\mathcal{F}$-quasi-continuous function, compared with only locally integrable functions, there exist more Lebesgue points.*

4. Generalized Lebesgue Points of ϕ-Hajłasz-Type Functions

If a function fails to be locally integrable, which may happen, for instance, when the index p of the ϕ-Hajłasz-type space is close to zero, the γ-median serves as a reasonable substitute of the integral average (see, for instance [41,45,46]). That is because the γ-median is defined, instead of integrals, only by the distribution sets of functions and their measures, which removes the necessity for the local integrability of functions. Due to the similarity between the behavior of the γ-median and that of the integral average, the Lebesgue point can naturally be generalized to the γ-median case; see (56). In this section, we still use the capacity to measure the set of such generalized Lebesgue points of ϕ-Hajłasz-type functions. We first recall the notion of the γ-median and some of its basic properties; see ([41], Section 2.4) (see also ([46], Section 1) for a different definition).

Definition 13. *Let $u \in L^0(\mathcal{X})$ and $\gamma \in (0, 1/2]$. The γ*-median *$m_u^\gamma(E)$ of u over a set $E \subset \mathcal{X}$ of finite measure is defined by setting*

$$m_u^\gamma(E) := \inf\{\lambda \in \mathbb{R} : \ \mu(\{x \in E : u(x) > \lambda\}) < \gamma\mu(E)\}.$$

Observe that, if $E \subset \mathcal{X}$, $\mu(E) \in (0, \infty)$ and $u \in L^0(E)$, then $m_u^\gamma(E)$ is finite.

Lemma 13. *Let $E,\ E_1,\ E_2 \subset \mathcal{X}$ be sets of finite measure, $\gamma,\ \gamma_1,\ \gamma_2 \in (0, 1/2]$, and $u,\ v \in L^0(\mathcal{X})$. The following statements hold true:*

(i) *If $\gamma_1 \le \gamma_2$, then $m_u^{\gamma_1}(E) \ge m_u^{\gamma_2}(E)$.*
(ii) *If $u \le v$ almost everywhere, then $m_u^\gamma(E) \le m_v^\gamma(E)$.*
(iii) *If $E_1 \subset E_2$ and, for some positive constant c, $\mu(E_2) \le c\mu(E_1)$, then*

$$m_u^\gamma(E_1) \le m_u^{\gamma/c}(E_2).$$

(iv) *For any $c \in \mathbb{R}$, $m_u^\gamma(E) + c = m_{u+c}^\gamma(E)$.*
(v) *For any $c \in (0, \infty)$, $m_{cu}^\gamma(E) = cm_u^\gamma(E)$.*

(vi) $\left|m_u^\gamma(E)\right| \le m_{|u|}^\gamma(E)$.

(vii) $m_{u+v}^\gamma(E) \le m_u^{\gamma/2}(E) + m_v^{\gamma/2}(E)$.

(viii) *For any* $t \in (0,\infty)$,

$$m_{|u|}^\gamma(E) \le \left(\gamma^{-1} \fint_E |u|^t \, d\mu\right)^{1/t}. \tag{55}$$

The following lemma (see, for instance, ([46], Theorem 2.1)) implies that the γ-median over small balls can behave similar to the classical integral average of locally integrable functions at Lebesgue points and becomes a reasonable substitute of the classical Lebesgue differentiation theorem when the function fails to be locally integrable.

Lemma 14. *Let* $u \in L^0(\mathcal{X})$. *Then, there exists a set* $E \subset \mathcal{X}$ *with* $\mu(E) = 0$ *such that, for any* $\gamma \in (0, 1/2]$ *and* $x \in \mathcal{X} \setminus E$,

$$\lim_{r\to 0^+} m_u^\gamma(B(x,r)) = u(x). \tag{56}$$

In particular, (56) *holds true at every continuous point* x *of* u.

Let $u \in L^0(\mathcal{X})$. Recall that a point $x \in \mathcal{X}$ is called a *generalized Lebesgue point* of u if (56) holds true for x and any $\gamma \in (0, 1/2]$; see, for instance [41,44,45]. If u is locally integrable, as was pointed by ([46], p. 231), any Lebesgue point of u is a generalized Lebesgue point of u. This means that the generalized Lebesgue point is a more extensive notion than the Lebesgue point.

Next, we recall the variants of both $\mathcal{M}$ and $\mathcal{M}^*$ in the γ-median version (see, for instance [41,45]), where $\mathcal{M} = \mathcal{M}_\infty$ is as in (14), and $\mathcal{M}^*$ as in Definition 9(ii).

Definition 14. *Let* $\gamma \in (0, 1/2]$ *and* $u \in L^0(\mathcal{X})$. *The* γ-median maximal function $\mathcal{M}^\gamma(u)$ *of* u *is defined by setting, for any* $x \in \mathcal{X}$,

$$\mathcal{M}^\gamma(u)(x) := \sup_{r\in(0,\infty)} m_{|u|}^\gamma(B(x,r)).$$

Definition 15. *Let* $\gamma \in (0, 1/2]$ *and* $u \in L^0(\mathcal{X})$.

(i) *The* discrete γ-median convolution u_r^γ *of* u *at scale* $r \in (0,\infty)$ *is defined by setting, for any* $x \in \mathcal{X}$,

$$u_r^\gamma(x) := \sum_{j\in\mathcal{J}} m_u^\gamma(B_j)\, \varphi_j(x),$$

where $\mathcal{J}$ *is an index set,* $\{B_j\}_{j\in\mathcal{J}}$ *is a ball covering of* $\mathcal{X}$ *with the radius* r *such that* $\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j} \lesssim 1$, *and* $\{\varphi_j\}_{j\in\mathcal{J}}$ *is a partition of unity with respect to* $\{B_j\}_{j\in\mathcal{J}}$ *as in Definition 8.*

(ii) *The* discrete γ-median maximal function $\mathcal{M}^{\gamma,*}u$ *of* u *is defined by setting, for any* $x \in \mathcal{X}$,

$$\mathcal{M}^{\gamma,*}u(x) := \sup_{k\in\mathbb{Z}} |u|_{2^{-k}}^\gamma(x),$$

where $|u|_{2^{-k}}^\gamma$ *is as in (i) with* u *and* r *replaced, respectively, by* $|u|$ *and* 2^{-k}.

Remark 6. *Let* $\mathcal{M}^\gamma$ *and* $\mathcal{M}^{\gamma,*}$ *be as in Definitions 14 and 15. Recall that there exists a positive constant* c *such that, for any* $u \in L^0(\mathcal{X})$,

$$\mathcal{M}^\gamma u \le c\mathcal{M}^{\gamma/c,*}u \le c^2\mathcal{M}^{\gamma/c^2}u; \tag{57}$$

see ([41], (2.10)). Additionally, recall that either $\mathcal{M}^\gamma u \equiv \infty$ *or* $\mathcal{M}^\gamma u < \infty$ *almost everywhere in* $\mathcal{X}$ *and either* $\mathcal{M}^{\gamma,*}u \equiv \infty$ *or* $\mathcal{M}^{\gamma,*}u < \infty$ *almost everywhere in* $\mathcal{X}$*; see ([41], (2.10)) and ([41], Remark 2.11).*

The following two lemmas are the variants of Poincaré-type inequalities, respectively, in Lemma 3, (18), and (17), where the second lemma is a generalization of ([41], Lemma 3.2).

Lemma 15. *Let $\gamma \in (0, 1/2]$, $\phi \in \mathcal{A}$, and C_μ be as in (1).*

(i) *Then, there exists a positive constant $C = C_{(\phi,C_\mu)}$ such that, for any $k \in \mathbb{Z}$, $u \in L^0(\mathcal{X})$, $g \in \mathcal{D}^\phi(u)$, and $x \in \mathcal{X}$,*

$$\inf_{c\in\mathbb{R}} m^\gamma_{|u-c|}(B(x, 2^{-k})) \leq C\gamma^{-1}\phi(2^{-k}) ⨍_{B(x,2^{-k})} g(y)\,d\mu(y).$$

(ii) *If $\alpha_\phi \in (0, 1)$, then, for any given $\lambda \in (0, \infty)$, there exists a positive constant $C = C_{(\gamma,\lambda,\phi,C_\mu)}$ such that, for any $k \in \mathbb{Z}$, $u \in L^0(\mathcal{X})$, $g \in \mathcal{D}^\phi(u)$, and $x \in \mathcal{X}$,*

$$\inf_{c\in\mathbb{R}} m^\gamma_{|u-c|}(B(x, 2^{-k})) \leq C\phi(2^{-k})\left\{⨍_{B(x,2^{-k+1})} [g(y)]^\lambda\,d\mu(y)\right\}^{1/\lambda}.$$

Proof. We first prove (i). For any $k \in \mathbb{Z}$, $u \in L^0(\mathcal{X})$, $g \in \mathcal{D}^\phi(u)$, $x \in \mathcal{X}$, and $c \in \mathbb{R}$, from (55) with $t = 1$, and E and u therein replaced, respectively, by $B(x, 2^{-k})$ and $u - c$, we deduce that

$$m^\gamma_{|u-c|}(B(x, 2^{-k})) \leq \gamma^{-1} ⨍_{B(x,2^{-k})} |u(y) - c|\,d\mu(y). \tag{58}$$

Taking the infimum of $c \in \mathbb{R}$ in (58), and using Lemma 3, we obtain (i) of the present lemma.

Now we prove (ii). By $\alpha_\phi < 1$, we choose $\varepsilon := -(\log_2 \alpha_\phi)/2 > 0$. For any $k \in \mathbb{Z}$, $\lambda \in (0, D/\varepsilon)$, $u \in L^0(\mathcal{X})$, $g \in \mathcal{D}^\phi(u)$, $x \in \mathcal{X}$, and $c \in \mathbb{R}$, applying (55) with $t = (D\lambda)/(D - \varepsilon\lambda) \in (0, \infty)$, and E and u therein replaced, respectively, by $B(x, 2^{-k})$ and $u - c$, we conclude that

$$\begin{aligned} &m^\gamma_{|u-c|}(B(x, 2^{-k})) \\ &\quad\leq \left\{\gamma^{-1} ⨍_{B(x,2^{-k})} [u(y) - c]^{(D\lambda)/(D-\varepsilon\lambda)}\,d\mu(y)\right\}^{(D-\varepsilon\lambda)/(D\lambda)}. \end{aligned} \tag{59}$$

Taking the infimum of $c \in \mathbb{R}$ in (59) and using (16) with $p = \lambda$, we obtain the conclusion of (ii) when $\lambda \in (0, D/\varepsilon)$. From this and the Hölder inequality, we deduce that the conclusion of (ii) also holds true when $\lambda \in [D/\varepsilon, \infty)$, which completes the proof of Lemma 15. □

Lemma 16. *Let $\gamma \in (0, 1/2]$, $\phi \in \mathcal{A}$ with $\alpha_\phi \in (0, 1)$, and C_μ be as in (1). Then, for any given $\lambda \in (0, \infty)$ and $\epsilon \in (0, -\log_2 \alpha_\phi)$, there exists a positive constant $C = C_{(\gamma,\phi,\epsilon,\lambda,C_\mu)}$ such that, for any $k \in \mathbb{Z}$, $u \in L^0(\mathcal{X})$, $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(u)$, and $x \in \mathcal{X}$,*

$$\begin{aligned} &\inf_{c\in\mathbb{R}} m^\gamma_{|u-c|}(B(x, 2^{-k})) \\ &\quad\leq C2^{-k\epsilon} \sum_{l\geq k-2} 2^{l\epsilon}\phi(2^{-l}) \left\{⨍_{B(x,2^{-k+1})} [g_l(y)]^\lambda\,d\mu(y)\right\}^{1/\lambda}. \end{aligned} \tag{60}$$

Proof. Let $\lambda \in (0, \infty)$ and $\nu \in (0, \epsilon)$, where ϵ is given as in Lemma 16. When $\lambda \in (0, D/\nu)$, (60) follows from (55) with $t = (D\lambda)/(D - \nu\lambda) \in (0, \infty)$, E and u therein replaced, respectively, by $B(x, 2^{-k})$ and $u - c$ for arbitrary $c \in \mathbb{R}$, and from Lemma 4 with p and ε' therein replaced, respectively, by λ and ϵ. This, combined with the Hölder inequality, further implies (60) when $\lambda \in [D/\nu, \infty)$. This finishes the proof of Lemma 16. □

The following lemma is a variant of Lemma 5 in the γ-median version.

Lemma 17. *Let $\gamma \in (0, 1/2]$, $\phi \in \mathcal{A}$, C_μ be as in (1), and $\mathcal{M}$ the classical Hardy–Littlewood maximal operator.*

(i) *Then, there exists a positive constant $C = C_{(\phi, C_\mu)}$ such that, for any $k \in \mathbb{Z}$, $u \in L^0(\mathcal{X})$, $g \in \mathcal{D}^\phi(u)$, $y \in \mathcal{X}$, and almost every $x \in B(y, 2^{-k+1})$,*

$$\left|u(x) - m_u^\gamma(B(y, 2^{-k}))\right| \le C\gamma^{-1}\phi(2^{-k})\, \mathcal{M}(g)(x).$$

(ii) *Let $\alpha_\phi \in (0, 1)$. Then, for any given $\lambda \in (0, \infty)$, there exists a positive constant $C = C_{(\gamma, \phi, \lambda, C_\mu)}$ such that, for any $k \in \mathbb{Z}$, $u \in L^0(\mathcal{X})$, $g \in \mathcal{D}^\phi(u)$, $y \in \mathcal{X}$, and any generalized Lebesgue point $x \in B(y, 2^{-k+1})$,*

$$\left|u(x) - m_u^\gamma(B(y, 2^{-k}))\right| \le C\phi(2^{-k}) \left[\mathcal{M}\left(g^\lambda\right)(x)\right]^{1/\lambda}.$$

(iii) *Let $\alpha_\phi \in (0, 1)$. Then, for any given $\lambda \in (0, \infty)$ and $\epsilon \in (0, -\log_2 \alpha_\phi)$, there exists a positive constant $C = C_{(\gamma, \phi, \lambda, \epsilon, C_\mu)}$ such that, for any $k \in \mathbb{Z}$, $u \in L^0(\mathcal{X})$, $\{g_l\}_{l \in \mathbb{Z}} \in \mathbb{D}^\phi(u)$, $y \in \mathcal{X}$, and any generalized Lebesgue point $x \in B(y, 2^{-k+1})$,*

$$\left|u(x) - m_u^\gamma(B(y, 2^{-k}))\right| \le C2^{-k\epsilon} \sum_{l \ge k-4} 2^{l\epsilon}\phi(2^{-l}) \left[\mathcal{M}\left(g_l^\lambda\right)(x)\right]^{1/\lambda}.$$

Proof. Let all of the symbols be as in the present lemma. We first prove (i). For any $k \in \mathbb{Z}$, $y \in \mathcal{X}$ and almost every $x \in B(y, 2^{-k+1})$, by (iv) and (vi) of Lemma 13; (55) with $t = 1$; and E and u therein replaced, respectively, by $B(y, 2^{-k})$ and $u - u(x)$; the geometric observation that, for any $x \in B(y, 2^{-k+1})$, $B(y, 2^{-k}) \subset B(x, 2^{-k+2})$; the doubling property of μ; the definitions of $\mathcal{D}^\phi(u)$ and $\mathcal{A}$; and $g \le \mathcal{M}(g)$ almost everywhere, we have, for almost every $x \in B(y, 2^{-k+1}) \setminus E$,

$$\begin{aligned}
&\left|u(x) - m_u^\gamma(B(y, 2^{-k}))\right| \\
&\quad = \left|m_{u-u(x)}^\gamma(B(y, 2^{-k}))\right| \le m_{|u-u(x)|}^\gamma(B(y, 2^{-k})) \\
&\quad \le \gamma^{-1} \fint_{B(y, 2^{-k})} |u(z) - u(x)|\, d\mu(z) \lesssim \gamma^{-1} \fint_{B(x, 2^{-k+2})} |u(z) - u(x)|\, d\mu(z) \\
&\quad \lesssim \gamma^{-1}\phi(2^{-k})\left[g(x) + \fint_{B(x, 2^{-k+2})} g(z)\, d\mu(z)\right] \lesssim \gamma^{-1}\phi(2^{-k})\mathcal{M}(g)(x),
\end{aligned}$$

which completes the proof of (i).

Now, we prove (ii) and (iii). Let λ and ϵ be as in (ii) and (iii) of the present lemma. Similar to ([41] (3.3)), by (ii), (iv), and (vi) of Lemma 13, we have, for any γ, $\gamma' \in (0, 1/2]$ and any ball B,

$$\begin{aligned}
m_{|u - m_u^\gamma(B)|}^{\gamma'}(B) &\le \inf_{c \in \mathbb{R}} m_{|u-c| + |c - m_u^\gamma(B)|}^{\gamma'}(B) \\
&= \inf_{c \in \mathbb{R}}\left[m_{|u-c|}^{\gamma'}(B) + |c - m_u^\gamma(B)|\right] \\
&\le \inf_{c \in \mathbb{R}}\left[m_{|u-c|}^{\gamma'}(B) + m_{|u-c|}^{\gamma}(B)\right].
\end{aligned} \tag{61}$$

Moreover, by the geometrical observation that, for any $k \in \mathbb{Z}$, $y \in \mathcal{X}$, and $x \in B(y, 2^{-k+1})$; $B(x, 2^{-k}) \subset B(y, 2^{-k+2})$; and the doubling property of μ, we obtain

$$\mu\left(B(x, 2^{-k+2})\right) \le C_\mu^2\mu\left(B(x, 2^{-k})\right) \le C_\mu^2\mu\left(B(y, 2^{-k+2})\right) \le C_\mu^4\mu\left(B(y, 2^{-k})\right).$$

Therefore, from this, the definition of generalized Lebesgue points; the doubling property of μ; (i), (iii), (iv), and (vi) of Lemma 13; $C_\mu \in [1,\infty)$; and (61) with $\gamma' = \gamma/C_\mu^4$ and B replaced by $B(x,2^{-j})$, we deduce that, for any generalized Lebesgue point $x \in B(y,2^{-k+1})$,

$$\begin{aligned}
&\left|u(x) - m_u^\gamma(B(y,2^{-k}))\right| \\
&\quad \le \left|u(x) - m_u^\gamma(B(x,2^{-k+2}))\right| + \left|m_u^\gamma(B(x,2^{-k+2})) - m_u^\gamma(B(y,2^{-k}))\right| \\
&\quad \le \sum_{j\ge k-2} \left|m_u^\gamma(B(x,2^{-j-1})) - m_u^\gamma(B(x,2^{-j}))\right| + \left|m_u^\gamma(B(y,2^{-k})) - m_u^\gamma(B(x,2^{-k+2}))\right| \\
&\quad \le \sum_{j\ge k-2} m^\gamma_{|u-m_u^\gamma(B(x,2^{-j}))|}(B(x,2^{-j-1})) + m^\gamma_{|u-m_u^\gamma(B(x,2^{-k+2}))|}(B(y,2^{-k})) \qquad (62)\\
&\quad \le \sum_{j\ge k-2} m^{\gamma/C_\mu}_{|u-m_u^\gamma(B(x,2^{-j}))|}(B(x,2^{-j})) + m^{\gamma/C_\mu^4}_{|u-m_u^\gamma(B(x,2^{-k+2}))|}(B(x,2^{-k+2})) \\
&\quad \lesssim \sum_{j\ge k-2} m^{\gamma/C_\mu^4}_{|u-m_u^\gamma(B(x,2^{-j}))|}(B(x,2^{-j})) \\
&\quad \lesssim \sum_{j\ge k-2} \inf_{c\in\mathbb{R}}\left[m^{\gamma/C_\mu^4}_{|u-c|}(B(x,2^{-j})) + m^{\gamma}_{|u-c|}(B(x,2^{-j}))\right].
\end{aligned}$$

On the one hand, (62), combined with Lemma 15(ii) with k therein replaced by j, (9) with k and k_0 therein replaced, respectively, by $-j$ and $-k+2$ and the definitions of $\mathcal{M}$ and $\mathcal{A}$, implies (ii) of the present lemma. On the other hand, (62), combined with Lemma 16, the definition of $\mathcal{M}$, and $\sum_{j\ge k-2} 2^{-j\epsilon} \lesssim 2^{-k\epsilon}$, implies (iii) of the present lemma, which completes the proof of Lemma 17. □

We now establish the convergence of approximations by discrete γ-median convolutions as below, which is a generalization of ([41], Theorem 1.1) from fractional Hajłasz-type spaces to those with generalized smoothness.

Theorem 4. *Let $\gamma \in (0,1/2]$, $\mathcal{F} \in \{M^\phi_{p,q}(\mathcal{X}), N^\phi_{p,q}(\mathcal{X})\}$ with $\phi \in \mathcal{A}_0$ and p, $q \in (0,\infty)$, and $u \in \dot{\mathcal{F}}$. Then, $\|u - u^\gamma_{2^{-i}}\|_{\mathcal{F}} \to 0$ as $i \to \infty$, where $\{u^\gamma_{2^{-i}}\}_{i\ge 0}$ are the discrete γ-median convolutions as in Definition 15(i).*

Proof. By similarity, we only consider the case $\mathcal{F} = M^\phi_{p,q}(\mathcal{X})$. Let $\gamma \in (0,1/2]$, $i \in \mathbb{Z}_+$, $u^\gamma_{2^{-i}}$ be as in Definition 15(i), $u \in \dot{M}^\phi_{p,q}(\mathcal{X})$, and $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^\phi(u) \cap L^p(\mathcal{X}, l^q)$.

Let $\lambda \in (0, \min(p,q))$, $\epsilon \in (0, -\log_2 \alpha_\phi)$, $\{B_j\}_{j\in\mathcal{J}}$ be any given ball covering of $\mathcal{X}$ with the radius 2^{-i} such that $\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j} \lesssim 1$, and $\{\varphi_j\}_{j\in\mathcal{J}}$, consisting of a sequence of $c2^i$-Lipschitz functions, a partition of unity with respect to $\{B_j\}_{j\in\mathcal{J}}$ as in Definition 8. For any $j \in \mathcal{J}$, let $m^\gamma_u(B_j)$ be as in Definition 13. Then, by the properties of $\{\varphi_j\}_{j\in\mathcal{J}}$, we have

$$u - u^\gamma_{2^{-i}} = \sum_{j\in\mathcal{J}} \left(u - m^\gamma_u(B_j)\right)\varphi_j. \qquad (63)$$

Using Lemma 7 with u and L^{-1} therein replaced, respectively, by $u - m^\gamma_u(B_j)$ and $c2^i$, we conclude that, for any $j \in \mathcal{J}$, $\{h^*_{k,j}\}_{k\in\mathbb{Z}}$, defined by setting, for any $k \in \mathbb{Z}$,

$$h^*_{k,j} := \begin{cases} \left\{2^{i-k}\left[\phi(2^{-k})\right]^{-1}\left|u - m^\gamma_u(B_j)\right| + g_k\right\}\mathbf{1}_{2B_j}, & k > i, \\ \left[\phi(2^{-k})\right]^{-1}\left|u - m^\gamma_u(B_j)\right|\mathbf{1}_{2B_j}, & k \le i, \end{cases}$$

is a positive constant multiple of an element of $\mathbb{D}^\phi([u - m^\gamma_u(B_j)]\varphi_j)$. From this, (63), an argument similar to that used in the estimation of (26) with $u_{2^{-i}}$, u_{B_j}, and $h_{k,j}$ therein replaced, respectively, by $u^\gamma_{2^{-i}}$, $m^\gamma_u(B_j)$, and $h^*_{k,j}$, Lemma 17(iii), (27), and $\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j} \lesssim 1$, we deduce that $\{h_k\}_{k\in\mathbb{Z}}$, defined as in (28) with the above λ and ϵ, is also a positive constant multiple of an element in $\mathbb{D}^{\phi}(u-u^{\gamma}_{2^{-i}})$. By this, (32),

$$\left\|\left(\sum_{k\geq i-4} g_k^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} \leq \|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} < \infty$$

with $i\in\mathbb{Z}_+$, and the dominated convergence theorem with respect to μ, we obtain

$$\left\|u-u^{\gamma}_{2^{-i}}\right\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})} \lesssim \|h_k\|_{L^p(\mathcal{X},l^q)} \lesssim \left\|\left(\sum_{k\geq i-4} g_k^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} \to 0$$

as $i\to\infty$. Then, using (63), Lemma 17(iii) instead of Lemma 5(iii), the properties of $\{\varphi_j\}_{j\in\mathcal{J}}$, Lemma 1(ii) with $\epsilon\in(0,-\log_2\alpha_\phi)$, the Fefferman–Stein vector-valued maximal inequality on $L^{p/\lambda}(\mathcal{X},l^{q/\lambda})$ (see ([56], Theorem 1.2) or ([57], Theorem 1.3)), $\phi(0)=0$, and an argument similar to that used in the estimation of (33), we conclude that

$$\left\|u-u^{\gamma}_{2^{-i}}\right\|_{L^p(\mathcal{X})} \lesssim \phi(2^{-i})\left\|\left(\sum_{l\geq i-4} g_l^q\right)^{1/q}\right\|_{L^p(\mathcal{X})} \to 0$$

as $i\to\infty$. This finishes the proof of Theorem 4. □

Now, we state the following variant of Theorem 3 for γ-medians.

Theorem 5. *Let $\gamma\in(0,1/2]$, $\phi\in\mathcal{A}$, and $\mathcal{F}$ be one of the following cases:*

(i) $\mathcal{F}=M^{\phi}_{p,\infty}(\mathcal{X})=M^{\phi,p}(\mathcal{X})$ *with ϕ being a modulus of continuity and $p\in(1,\infty)$;*

(ii) $\mathcal{F}=M^{\phi}_{p,\infty}(\mathcal{X})=M^{\phi,p}(\mathcal{X})$ *with ϕ being a modulus of continuity, $\alpha_\phi\in(0,1)$, and $p\in(0,1]$;*

(iii) $\mathcal{F}\in\{M^{\phi}_{p,q}(\mathcal{X}),N^{\phi}_{p,q}(\mathcal{X})\}$ *with $\alpha_\phi\in(0,1)$, $\beta_\phi\in(0,2)$, and $p,\ q\in(0,\infty)$.*

Then, for any $u\in\dot{\mathcal{F}}$, there exists a set E with $\mathrm{Cap}_{\mathcal{F}}(E)=0$ satisfying that, for any $\gamma\in(0,1/2]$, there exists an $\mathcal{F}$-quasi-continuous function u^* on $\mathcal{X}$ such that, for any $x\in\mathcal{X}\setminus E$,

$$u^*(x)=\lim_{r\to 0} m^{\gamma}_u(B(x,r)). \tag{64}$$

To show Theorem 5, similar to the proof of Theorem 3, we need a weak-type capacitary estimate with respect to $\mathcal{M}^{\gamma}$. To this end, we first prove an auxiliary lemma as below, which is about the boundedness of $\mathcal{M}^{\gamma,*}$ in ϕ-Hajłasz-type spaces and generalizes ([41], Theorem 7.6). Here and thereafter, $\mathcal{M}^{\gamma}$ and $\mathcal{M}^{\gamma,*}$ are as in Definitions 14 and 15(ii), respectively.

Lemma 18. *With the same assumptions as in Theorem 5, there exists a positive constant $C=C_{(\mathcal{F},\gamma,C_\mu)}$ such that, for any $u\in\mathcal{F}$,*

$$\left\|\mathcal{M}^{\gamma,*}u\right\|_{\mathcal{F}}\leq C\|u\|_{\mathcal{F}}, \tag{65}$$

where $\mathcal{M}^{\gamma,*}$ is as in Definition 15.

Proof. Let all of the symbols be as in the present lemma. Without loss of generality, by the definition of $\mathcal{M}^{\gamma,*}$, $\mathcal{D}^{\phi}(u)\subset\mathcal{D}^{\phi}(|u|)$, and $\mathbb{D}^{\phi}(u)\subset\mathbb{D}^{\phi}(|u|)$, we may assume that $u\geq 0$.

Let $i\in\mathbb{Z}$, $\{B_j\}_{j\in\mathcal{J}}$ be any given ball covering of $\mathcal{X}$ with the radius 2^{-i} such that

$$\sum_{j\in\mathcal{J}} \mathbf{1}_{2B_j}\lesssim 1,$$

$\{\varphi_j\}_{j\in\mathcal{J}}$ be a partition of unity with respect to $\{B_j\}_{j\in\mathcal{J}}$ as in Definition 8, $u^{\gamma}_{2^{-i}}$ be as in Definition 15, and $\mathcal{M}^{\gamma}$ be as in Definition 14. Then, by (57) and ([41], (2.7)), we have, for any given $p \in (0,\infty)$,

$$\left\|\mathcal{M}^{\gamma,*}u\right\|_{L^p(\mathcal{X})} \lesssim \left\|\mathcal{M}^{\gamma/c}u\right\|_{L^p(\mathcal{X})} \lesssim \|u\|_{L^p(\mathcal{X})} < \infty, \tag{66}$$

where c is the same positive constant as in (57). From this and Remark 6, we deduce that $\mathcal{M}^{\gamma,*}u < \infty$ almost everywhere.

Let $\mathcal{F} = M^{\phi,p}(\mathcal{X})$ and $g \in \mathcal{D}^{\phi}(u)$. Using (i) and (ii) of Lemma 17 instead of (i) and (ii) of Lemma 5, and $\mathcal{M}^{\gamma,*}u < \infty$ almost everywhere, from an argument similar to that used in the proof of Lemma 8 with $\{u_{B_j}\}_{j\in\mathcal{J}}$ and $u_{2^{-i}}$ therein replaced, respectively, by $\{m^{\gamma}_u(B_j)\}_{j\in\mathcal{J}}$ and $u^{\gamma}_{2^{-i}}$, we deduce that $\mathcal{M}(g)$ is a positive constant multiple of an element in $\mathcal{D}^{\phi}(\mathcal{M}^{\gamma,*}u)$ and, if $\alpha_\phi \in (0,1)$, then for any $\lambda \in (0,\infty)$, $[\mathcal{M}(g^{\lambda})]^{1/\lambda}$ is a positive constant multiple of an element in $\mathcal{D}^{\phi}(\mathcal{M}^{\gamma,*}u)$, where both of the positive constants are independent of u and g. Below, we let $\lambda \in (0,\min(p,q))$. Thus, by the boundedness of $\mathcal{M}$ on $L^p(\mathcal{X})$ when $p \in (1,\infty)$, and on $L^{p/\lambda}(\mathcal{X})$ with $\lambda \in (0,p)$ when $p \in (0,1]$, we obtain, when $p \in (1,\infty)$,

$$\left\|\mathcal{M}^{\gamma,*}u\right\|_{\dot{M}^{\phi,p}(\mathcal{X})} \lesssim \left\|\mathcal{M}(g)\right\|_{L^p(\mathcal{X})} \lesssim \|g\|_{L^p(\mathcal{X})}$$

and, when $\alpha_\phi \in (0,1)$ and $p \in (0,1]$,

$$\left\|\mathcal{M}^{\gamma,*}u\right\|_{\dot{M}^{\phi,p}(\mathcal{X})} \lesssim \left\|[\mathcal{M}(g^{\lambda})]^{1/\lambda}\right\|_{L^p(\mathcal{X})} \lesssim \|g\|_{L^p(\mathcal{X})}.$$

This, combined with (66), proves (65) when $\mathcal{F}$ belongs to either (i) or (ii) of the assumptions of Theorem 5.

Next, we prove (65) when $\mathcal{F}$ belongs to the case (iii) of Theorem 5. By similarity, we only consider the case $\mathcal{F} = M^{\phi}_{p,q}(\mathcal{X})$ with $\alpha_\phi \in (0,1)$, $\beta_\phi \in (0,2)$, and $p,\ q \in (0,\infty)$. To prove (65), by (66), it suffices to show

$$\left\|\mathcal{M}^{\gamma,*}u\right\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})} \lesssim \|u\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})}.$$

Let $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^{\phi}(u)$ be such that $\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \lesssim \|u\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})}$, and $\epsilon \in (0,-\log_2 \alpha_\phi)$. Recall that we have proved in the proof of Theorem 4 that $\{h_k\}_{k\in\mathbb{Z}}$, defined as in (28) with the above λ and ϵ, is a positive constant multiple of an element in $\mathbb{D}^{\phi}(u - u^{\gamma}_{2^{-i}})$. Thus, by $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^{\phi}(u)$, we conclude that $\{g_k + h_k\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^{\phi}(u^{\gamma}_{2^{-i}})$.

Let $\delta \in (0,\min\{1-\log_2\beta_\phi, -\log_2\alpha_\phi - \epsilon\})$ and $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ be as in (42) with the above λ and δ. Similar to the proof of Lemma 9, we know that, for any $k \in \mathbb{Z}$, $g_k + h_k \lesssim \widetilde{g_k}$ almost everywhere. By this and the proved conclusion that $\{g_k + h_k\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^{\phi}(u^{\gamma}_{2^{-i}})$, we conclude that $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ is also a positive constant multiple of an element in $\mathbb{D}^{\phi}(u^{\gamma}_{2^{-i}})$ with the positive constant independent of i. Furthermore, using the fact that $\mathcal{M}^{\gamma,*}u < \infty$ almost everywhere and Lemma 2(ii), we find that $\{\widetilde{g_k}\}_{k\in\mathbb{Z}}$ is a positive constant multiple of an element in $\mathbb{D}^{\phi}(\mathcal{M}^{\gamma,*}u)$. From this, the Fefferman–Stein vector-valued maximal inequality on $L^{p/\lambda}(\mathcal{X}, l^{q/\lambda})$ (see ([56], Theorem 1.2) or ([57], Theorem 1.3)), and the choice of $\{g_k\}_{k\in\mathbb{Z}}$, we deduce that

$$\begin{aligned}\left\|\mathcal{M}^{\gamma,*}u\right\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})} &\lesssim \|\{\widetilde{g_k}\}_{k\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \\ &\lesssim \left\|\left\{\sum_{l\in\mathbb{Z}}\left[\mathcal{M}(g_l^{\lambda})\right]^{q/\lambda}\right\}^{\lambda/q}\right\|^{1/\lambda}_{L^{p/\lambda}(\mathcal{X})} \\ &\lesssim \|\{g_l\}_{l\in\mathbb{Z}}\|_{L^p(\mathcal{X},l^q)} \lesssim \|u\|_{\dot{M}^{\phi}_{p,q}(\mathcal{X})}.\end{aligned}$$

Thus, by (66), we conclude that (65) holds true for $\mathcal{F} = M^{\phi}_{p,q}(\mathcal{X})$ with $\alpha_\phi \in (0,1)$, $\beta_\phi \in (0,2)$, and $p,\ q \in (0,\infty)$. This finishes the proof of Lemma 18. □

The following weak-type capacitary estimate plays a crucial role in the proof of Theorem 5. Since it is just a generalization of ([41], Theorem 7.7), and a straight corollary of both Lemma 18 and the lower semi-continuity of $\mathcal{M}^{\gamma,*}u$ for any $u \in L^0(\mathcal{X})$, we omit its proof.

Lemma 19. *With the assumptions same as in Theorem 5, there exists a positive constant C, depending only on $\mathcal{F}$, γ, and C_μ, such that, for any $u \in \mathcal{F}$ and $\kappa \in (0,\infty)$,*

$$\mathrm{Cap}_{\mathcal{F}}(\{x \in \mathcal{X} :\ \mathcal{M}^{\gamma}u(x) > \kappa\}) \le C\kappa^{-p}\|u\|^p_{\mathcal{F}},$$

where $\mathcal{M}^{\gamma}$ is as in Definition 14 and C_μ as in (1).

Now, we turn to prove Theorem 5. Since the proof of Theorem 5 is quite similar to that of Theorem 3, we only sketch the main steps.

Proof of Theorem 5. Let $\mathcal{F}$ be any given function space as in (i), (ii), or (iii) of the present theorem, and $p \in (0,\infty)$. We first let $u \in \mathcal{F}$. By Theorems 2 and 4, we find that, in any case as above, there always exists a sequence $\{u_i\}_{i\in\mathbb{N}}$ of continuous functions such that, for any $i \in \mathbb{N}$,

$$\|u - u_i\|^p_{\mathcal{F}} < 2^{-i(1+p)}.$$

For any $k,\ i \in \mathbb{N}$, define

$$A_{k,i} := \left\{x \in \mathcal{X} :\ \mathcal{M}^{1/(2k)}(u - u_i)(x) > 2^{-i}\right\}$$

and

$$E := \bigcup_{k\ge2} E_k := \bigcup_{k\ge2}\bigcap_{j\in\mathbb{N}} B_{k,j} := \bigcup_{k\ge2}\bigcap_{j\in\mathbb{N}}\bigcup_{i\ge j} A_{k,i}.$$

Then, by Lemma 19, we have, for any given $k \in \mathbb{N}$, $\mathrm{Cap}_{\mathcal{F}}(A_{k,i}) \lesssim 2^{-i}$ and, by Lemma 6, for any $j \in \mathbb{N}$, $\mathrm{Cap}_{\mathcal{F}}(B_{k,j}) \lesssim 2^{-j}$, which implies that, for any given $k \in \mathbb{N}$, $\mathrm{Cap}_{\mathcal{F}}(E_k) = 0$ and hence $\mathrm{Cap}_{\mathcal{F}}(E) = 0$.

For any given $k \in \mathbb{N} \setminus \{1\}$ and any $i \in \mathbb{N}$, by the continuity of u_i and (55) with $t = 1$, we find that, for any $x \in \mathcal{X}$,

$$\begin{aligned}\limsup_{r\to0^+} m^{1/k}_{|u_i - u_i(x)|}(B(x,r)) &\le k \lim_{r\to0^+} \fint_{B(x,r)} \left|u(y) - u(x)\right| dy \\ &= 0.\end{aligned}$$

From this, (i), (iv), (vi), and (vii) of Lemma 13 and the definitions of $\mathcal{M}^{1/(2k)}$ and $A_{k,i}$, we deduce that, for any given $k \in \mathbb{N} \setminus \{1\}$, any $\gamma \in [1/k, 1/2]$, $i \in \mathbb{N}$, and $x \in \mathcal{X} \setminus A_{k,i}$,

$$\begin{aligned}\limsup_{r\to0^+}\left|u_i(x) - m^{\gamma}_u(B(x,r))\right| &\le \limsup_{r\to0^+} m^{\gamma}_{|u-u_i(x)|}(B(x,r)) \\ &\le \limsup_{r\to0^+}\left[m^{1/(2k)}_{|u-u_i|}(B(x,r)) + m^{1/(2k)}_{|u_i-u_i(x)|}(B(x,r))\right] \\ &\le \mathcal{M}^{1/(2k)}(u - u_i)(x) \le 2^{-i}.\end{aligned} \tag{67}$$

By an argument similar to that used in the proof of Theorem 3, with (53) replaced by (67), we conclude that, for any given $k \in \mathbb{N} \setminus \{1\}$, there exists a function v_k on $\mathcal{X} \setminus E_k$ such that, for any $\gamma \in [1/k, 1/2]$ and $x \in \mathcal{X} \setminus E_k$,

$$v_k(x) = \lim_{r\to0^+} m^{\gamma}_u(B(x,r))$$

and, moreover, for any $j \in \mathbb{N}$, v_k is continuous on $\mathcal{X} \setminus B_{k,j}$.

For any given $\gamma \in (0, 1/2]$, define v_γ^* by setting, for any $x \in \mathcal{X} \setminus E$,

$$v_\gamma^*(x) := \lim_{r \to 0^+} m_u^\gamma(B(x, r)).$$

Then, for any $k \in \mathbb{N}$ with $k \geq 2$, $v_\gamma^* = v_k$ in $\mathcal{X} \setminus E$ and hence v_γ^* is continuous in $\mathcal{X} \setminus (E \cup B_{k,j})$ for any $j \in \mathbb{N}$. Notice that, by Lemma 6, for any $j \in \mathbb{N}$,

$$\mathrm{Cap}_{\mathcal{F}}(E \cup B_{k,j}) \lesssim 2^{-j}.$$

By choosing j big enough, we conclude that any function u^* satisfying $u^* = v_\gamma^*$ in $\mathcal{X} \setminus E$ is $\mathcal{F}$-quasi-continuous in $\mathcal{X}$ and hence the desired function in the present theorem.

Similar to the proof of Theorem 3, by Corollary 3, the proved conclusion for the case $u \in \mathcal{F}$, and Lemma 6, via choosing a sequence of Lipschitz continuous functions supported in balls, we obtain the desired conclusion of the present theorem when $u \in \dot{\mathcal{F}}$. This finishes the proof of Theorem 5. □

Remark 7. *With the same assumptions as in Theorem 5, by Lemma 14, (64), and Remark 4(ii), we have the following two observations:*

(i) *$u^* = u$ almost everywhere;*

(ii) *every point outside E is a generalized Lebesgue point of u.*

From (ii) and ([45], Lemma 17), we further deduce that, if $u \in \mathcal{F}$ is $\mathcal{F}$-quasi-continuous, then there exists a set E with $\mathrm{Cap}_{\mathcal{F}}(E) = 0$ such that every point outside E is a generalized Lebesgue point of u. This means that $\mathcal{F}$-quasi-continuous functions may have more Lebesgue points, compared with the functions that are only locally integrable.

In the following, we consider another technical tool, the generalized Hausdorff measure, which can also be applied to measure the exceptional set of (generalized) Lebesgue points. To see this, we study the comparison between the capacity and the above generalized Hausdorff measure. We refer the reader to [55,62,63] for more studies on the comparison between the capacity and the generalized Hausdorff measure, and to [64] for a study on measuring the exceptional set of Lebesgue points via the generalized Hausdorff measure straightly.

Let $h \in \mathcal{A}$, $\theta \in (0, 1]$, and $R \in (0, \infty]$. The *Netrusov–Hausdorff cocontent* $\mathcal{H}_R^{h,\theta}$, related to h, θ, and R, is defined by setting, for any $E \subset \mathcal{X}$,

$$\mathcal{H}_R^{h,\theta}(E) := \inf\left\{ \left[\sum_{i \in \mathcal{I}} \left\{ \frac{\mu(B(x_i, r_i))}{h(r_i)} \right\}^\theta \right]^{1/\theta} : E \subset \bigcup_{i \in \mathcal{I}} B(x_i, r_i),\ r_i \leq R \right\}, \tag{68}$$

where the infimum is taken over all coverings $\{B(x_i, r_i)\}_{i \in \mathcal{I}}$ of E, and $\mathcal{I} \subset \mathbb{N}$ an index set. Then, the *generalized Hausdorff measure* $\mathcal{H}^{h,\theta}(E)$, related to h and θ, is defined by setting, for any $E \subset \mathcal{X}$,

$$\mathcal{H}^{h,\theta}(E) := \limsup_{R \to 0^+} \mathcal{H}_R^{h,\theta}(E). \tag{69}$$

Recall that the Netrusov–Hausdorff content on $\mathbb{R}^n$ defined via the powers of the radius was first considered by Netrusov [65] and generalized to metric spaces via an increasing function h by Nuutinen ([55], Definition 5.1).

Observe that some lower bound and upper bound estimates for the $N_{p,q}^s$-capacity and the $M_{p,q}^s$-capacity with $p,\ q \in (0, \infty)$, in terms of the related Netrusov–Hausdorff contents, have been established, respectively, in ([55], Theorems 5.4 and 5.5) and ([63], Theorems 3.6 and 3.7) where $N_{p,q}^s$ and $M_{p,q}^s$ denote the classical fractional Hajłasz–Besov and Hajłasz–Triebel–Lizorkin spaces, respectively. By some arguments similar to those used in the proofs of ([55], Theorems 5.4 and 5.5) and ([63], Theorems 3.6 and 3.7), we have the following conclusions (Theorems 6 and 7) on the generalized spaces $M^{\phi}_{p,q}(\mathcal{X})$ and $N^{\phi}_{p,q}(\mathcal{X})$; we omit the details of their proofs.

Theorem 6. *Let $\phi \in \mathcal{A}_0$, $p \in (0,\infty)$, $q \in (0,\infty]$, $\theta := \min\{1, q/p\}$, $\mathcal{F} \in \{M^{\phi}_{p,q}(\mathcal{X}), N^{\phi}_{p,q}(\mathcal{X})\}$, and C_μ be as in (1). Then, there exists a positive constant $C = C_{(\mathcal{F})}$ such that, for any $E \subset \mathcal{X}$ and $R \in (0,\infty)$,*

$$\mathrm{Cap}_{\mathcal{F}}(E) \le C\mathcal{H}^{h,\theta}_R(E), \tag{70}$$

where $\mathcal{H}^{h,\theta}_R$ is as in (68).

Remark 8. *Let $\phi(r) := r^s$ with $s \in (0,1)$ for any $r \in [0,\infty)$. In this case, (70) with $\mathcal{F} = N^{\phi}_{p,q}(\mathcal{X})$ becomes*

$$\mathrm{Cap}_{N^{\phi}_{p,q}(\mathcal{X})}(E) \lesssim \mathcal{H}^{h,\theta}_R(E)$$

with the implicit positive constant independent of E, which is just ([55] Theorem 5.4); moreover, taking $\mathcal{F} = M^{\phi}_{p,q}(\mathcal{X})$ and letting $R \to 0^+$ in (70), we obtain

$$\mathrm{Cap}_{M^{\phi}_{p,q}(\mathcal{X})}(E) \lesssim \mathcal{H}^{h,\theta}(E)$$

with the implicit positive constant independent of E, which is just ([63] Theorem 3.6), where $\mathcal{H}^{h,\theta}$ is as in (69).

Theorem 7. *Let $\phi \in \mathcal{A}_0$, $p,\ q \in (0,\infty)$, $\mathcal{F} \in \{M^{\phi}_{p,q}(\mathcal{X}), N^{\phi}_{p,q}(\mathcal{X})\}$, ω be any given function of admissible growth such that, for any $L \in \mathbb{Z}_+$,*

$$\sum_{k\ge L} \frac{1}{\omega(2^{-k})} < \infty;$$

and C_μ as in (1). Let $x_0 \in \mathcal{X}$, $R \in (0,1)$, and $B_0 := B(x_0, R)$. If there exist two positive constants $\kappa_1 \in (2,\infty)$ and $\kappa_2 \in (\kappa_1,\infty)$ such that $\kappa_2 B_0 \setminus \kappa_1 B_0 \neq \emptyset$, then there exist two positive constants τ and $C = C_{(\kappa_1,\kappa_2,R,\omega,\mathcal{F},C_\mu)}$ such that, for any compact set $E \subset B_0$,

$$\mathcal{H}^{h_\omega,1}_{\tau R}(E) \le C\,\mathrm{Cap}_{\mathcal{F}}(E),$$

where, for any $r \in (0,R]$, $h_\omega(r) := [\phi(r)\omega(r)]^p$, and $\mathcal{H}^{h_\epsilon,1}_{\tau R}(E)$ is as in (68) with $\theta = 1$.

Remark 9. *Let $\phi(r) := r^s$ with $s \in (0,1)$ for any $r \in [0,\infty)$. When $\mathcal{F} = N^{\phi}_{p,q}(\mathcal{X})$, if h_ω, as in Theorem 7, satisfies that, for any $N \in \mathbb{Z}$,*

$$\int_0^{2^N} [h_\omega(t)]^{-1/p} t^{s-1}\,dt < \infty$$

(which is just the assumption in ([55], Theorem 5.5)), then, for any $L \in \mathbb{Z}_+$,

$$\begin{aligned}\sum_{k\ge L} \frac{1}{\omega(2^{-k})} &\sim \sum_{k\ge L} \int_{2^{-k-1}}^{2^{-k}} [\omega(t)]^{-1} t^{-1}\,dt \\ &\sim \int_0^{2^{-L}} [\omega(t)]^{-1} t^{-1}\,dt \\ &\sim \int_0^{2^{-L}} [h_\omega(t)]^{-1/p} t^{s-1}\,dt < \infty.\end{aligned}$$

Thus, Theorem 7 implies ([55], Theorem 5.5) with $\kappa_1 = 4$ and $\kappa_2 = 8$.

When $\mathcal{F} = N^{\phi}_{p,q}(\mathcal{X})$, for any given $\varepsilon \in (0,\infty)$, let $\omega(r) := [\log(1/r)]^{-1-\varepsilon/p}$ for any $r \in [0,\infty)$. Obviously, we have

$$\sum_{k\geq L}[\log(2^k)]^{-1-\varepsilon/p} < \infty.$$

Moreover, if $\mathrm{Cap}_{\mathcal{F}}(E) = 0$, then, by (71), we obtain $\mathcal{H}^{h_\omega,1}_{\infty}(E) = 0$, which implies ([63], Theorem 3.7) with $\kappa_1 = 4$ and $\kappa_2 = 8$, where $\mathcal{H}^{h_\omega,1}_{\infty}(E)$ is as in (68) with $R = \infty$.

Finally, we concentrate on the space $M^{\phi,p}(\mathcal{X})$ with $\alpha_\phi \in (0,1)$ and $p \in (D/(-\log_2 \alpha_\phi),\infty)$, where D is as in (1). We point out that, similarly to ([55], Theorems 5.4 and 5.5) and ([63], Theorems 3.6 and 3.7), the proofs of Theorems 6 and 7 rely on some equivalent characterizations of the related capacities $\mathrm{Cap}_{N^{\phi}_{p,q}(\mathcal{X})}$ and $\mathrm{Cap}_{M^{\phi}_{p,q}(\mathcal{X})}$, in which the counterpart for the capacity $\mathrm{Cap}_{M^{\phi,p}(\mathcal{X})}$ is unknown. Instead, we use Lemma 14 and the doubling property of the measure to obtain the following result.

Theorem 8. *Let $\mathcal{F} = M^{\phi}_{p,\infty}(\mathcal{X}) = M^{\phi,p}(\mathcal{X})$ with $\phi \in \mathcal{A}_\infty$, $\alpha_\phi \in (0,1)$, $p \in (D/(-\log_2 \alpha_\phi),\infty)$, and D and C_μ be as in (1). Let B_0 be a ball with the radius $\widetilde{R_0} \in (0,\infty)$. If there exist an $R_0 \in (0,\widetilde{R_0}]$ and a $\tau \in (2,\infty)$ such that, for any ball $B \subset 2B_0$ with the radius no more than R_0, $\tau B \setminus 2B \neq \emptyset$, then, for any compact set $E \subset B_0$,*

$$\mathrm{Cap}_{\mathcal{F}}(E) = 0 \iff \mathcal{H}^{h,1}(E) = 0, \tag{71}$$

where, for any $r \in (0,R_0]$, $h(r) := [\phi(r)]^p$, and $\mathcal{H}^{h,1}(E)$ is as in (69) with $\theta = 1$.

Proof. Let all the symbols be as in the present theorem and $L \in \mathbb{Z}$ such that $R_0 \in (2^{L-1},2^L]$.

We first prove $\mathcal{H}^{h,1}(E) = 0 \Longrightarrow \mathrm{Cap}_{\mathcal{F}}(E) = 0$. To this end, let $R \in (0,\min\{1,R_0\}]$ and $\{B(x_i,r_i) : r_i \leq R\}_{i\in\mathcal{I}}$ be a ball covering of E, where $\mathcal{I}$ is an index set. For any $i \in \mathcal{I}$, we let φ_i be an r_i^{-1}-Lipschitz function supported in $2B(x_i,r_i)$ such that $0 \leq \varphi_i \leq 1$ and $\varphi_i|_{B(x_i,r_i)} \equiv 1$. The existence of such $\{\varphi_i\}_{i\in\mathcal{I}}$ can be found in the proof of ([63], Theorem 3.6). For any $i \in \mathcal{I}$, by Definition 10; the continuity of φ_i; Corollary 4 with L^{-1} and E therein replaced, respectively, by r_i and $2B(x_i,r_i)$; the doubling property of μ; the definition of $\mathcal{A}$; and $r_i \leq 1$, we have

$$\begin{aligned}
\mathrm{Cap}_{\mathcal{F}}(B(x_i,r_i)) &\leq \|2\varphi\|^p_{\mathcal{F}} \\
&\lesssim \left\{1 + [\phi(r_i)]^{-1}\right\}^p \mu(2B(x_i,r_i)) \\
&\lesssim [\phi(r_i)]^{-p}\mu(B(x_i,r_i))
\end{aligned}$$

with the implicit positive constants independent of x_i and r_i. From this, Remark 4(iii), and Lemma 6 with $\theta = 1$ and E_i replaced by $B(x_i,r_i)$, we deduce that

$$\mathrm{Cap}_{\mathcal{F}}(E) \leq \mathrm{Cap}_{\mathcal{F}}\left(\bigcup_{i\in\mathcal{I}} B(x_i,r_i)\right) \lesssim \sum_{i\in\mathcal{I}} \frac{\mu(B(x_i,r_i))}{[\phi(r_i)]^p} \sim \sum_{i\in\mathcal{I}} \frac{\mu(B(x_i,r_i))}{h(r_i)},$$

which, combined with (68) with $\theta = 1$, implies that $\mathrm{Cap}_{\mathcal{F}}(E) \lesssim \mathcal{H}^{h,1}_R(E)$ with the implicit positive constant independent of R and E. Letting $R \to 0^+$, we obtain $\mathrm{Cap}_{\mathcal{F}}(E) \lesssim \mathcal{H}^{h,1}(E)$, which implies that, if $\mathcal{H}^{h,1}(E) = 0$, then $\mathrm{Cap}_{\mathcal{F}}(E) = 0$.

Conversely, if $\mathrm{Cap}_{\mathcal{F}}(E) = 0$, then by the definition of $\mathrm{Cap}_{\mathcal{F}}(E)$, we find that, for any given $\varepsilon \in (0,\infty)$, there exists a function v such that $v \geq 1$ in a neighborhood of E and

$$\|v\|^p_{M^{\phi,p}(\mathcal{X})} < \mathrm{Cap}_{\mathcal{F}}(E) + \varepsilon = \varepsilon. \tag{72}$$

For any given generalized Lebesgue point $x \in E$ and any given $k \in \mathbb{Z}$ with $k \geq -L+1$, take $B := B(x,2^{-k})$. Then $B \subset 2B_0$, which together with the assumption of the present theorem, means that $\tau B \setminus 2B \neq \emptyset$. Let φ be a Lipschitz function such that $\varphi|_B \equiv 1$ and

$\varphi|_{\mathcal{X}\setminus 2B}\equiv 0$. Define $u:=v\varphi$. Then, by Lemma 7(ii) with E and u therein replaced, respectively, by $2B$ and v, we conclude that there exists a $g\in\mathcal{D}^{\phi}(u)$, supported in $2B$, such that

$$\|g\|_{L^p(\mathcal{X})}\lesssim\|v\|_{M^{\phi,p}(2B)}, \tag{73}$$

where the implicit positive constant depends only on ϕ, p, and K.

Since $\tau B\setminus 2B\neq\emptyset$, it follows that there always exists a point $z\in\tau B\setminus 2B$. Observe that, for any $y\in 2B$, we have $d(y,z)<(\tau+2)2^{-k}$, $u(z)=0$, and $g(z)=0$. Then, by the definition of $\mathcal{D}^{\phi}(u)$ and $\phi\in\mathcal{A}$, we conclude that, for almost every $y\in 2B$,

$$\left|u(y)\right|=\inf_{z\in\tau B\setminus 2B}\left|u(y)-u(z)\right|\le\inf_{z\in\tau B\setminus 2B}\phi(d(y,z))[g(y)+g(z)]\lesssim\phi(2^{-k})g(y),$$

which combined with (ii), (v), and (vi) of Lemma 13 and the doubling property of μ, implies that

$$\left|m_u^{\gamma}(B)\right|\le m_{|u|}^{\gamma}(B)\lesssim\phi(2^{-k})\,m_g^{\gamma}(B).$$

From this; the definition of the generalized Lebesgue point; the doubling property of μ; (iii), (iv), and (vi) of Lemma 13; (61); Lemma 15(ii) with $\lambda=p$; and (55) with $t=p$ and $E=B$, we deduce that, for the above given x,

$$\begin{aligned}
\left|u(x)\right|&\le\left|u(x)-m_u^{\gamma}(B)\right|+\left|m_u^{\gamma}(B)\right|\\
&\le\sum_{j\ge k-2}\left|m_u^{\gamma}(B(x,2^{-j-1}))-m_u^{\gamma}(B(x,2^{-j}))\right|+\left|m_u^{\gamma}(B)\right|\\
&\le\sum_{j\ge k-2}m^{\gamma/C_\mu}_{|u-m_u^{\gamma}(B(x,2^{-j}))|}(B(x,2^{-j}))+\left|m_u^{\gamma}(B)\right|\\
&\lesssim\sum_{j\ge k-2}\inf_{c\in\mathbb{R}}\left[m^{\gamma/C_\mu}_{|u-c|}(B(x,2^{-j}))+m^{\gamma}_{|u-c|}(B(x,2^{-j}))\right]+\phi(2^{-K})\,m_g^{\gamma}(B)\\
&\lesssim\sum_{j\ge k-2}\phi(2^{-j})\left\{\fint_{B(x,2^{-j+1})}[g(y)]^p\,d\mu(y)\right\}^{1/p}\\
&\quad+\phi(2^{-K})\left\{\fint_{B}[g(y)]^p\,d\mu(y)\right\}^{1/p}\\
&\lesssim\sum_{j\ge k-2}\phi(2^{-j})\left\{\fint_{B(x,2^{-j+1})}[g(y)]^p\,d\mu(y)\right\}^{1/p}\\
&\sim\sum_{j\ge k-2}\phi(2^{-j})\left[\mu(B(x,2^{-j+1}))\right]^{-1/p}\|g\|_{L^p(B(x,2^{-j+1}))}.
\end{aligned}$$

Using this and $u|_{E\cap B}\ge 1$, we conclude that, for this x,

$$1\lesssim\sum_{j\ge k-2}\phi(2^{-j})\left[\mu(B(x,2^{-j+1}))\right]^{-1/p}\|g\|_{L^p(B(x,2^{-j+1}))}, \tag{74}$$

where the implicit positive constant is independent of x and k. Moreover, by the doubling property of μ, we find that, for any $j\ge k-2$,

$$\left[\mu\left(B(x,2^{-j+1})\right)\right]^{-1}\lesssim 2^{(j-k)D}\left[\mu\left(B(x,2^{-k+2})\right)\right]^{-1},$$

where the implicit positive constant depends only on C_μ. From this, (74), the fact that g is supported in $2B$, the doubling property of μ, $p\in(D/(-\log_2\alpha_\phi),\infty)$, and Lemma 1(i) with $\varepsilon=D/p$, it follows that, for x and k as above,

$$1\lesssim\sum_{j\ge k-2}2^{(j-k)D/p}\phi(2^{-j})\left[\mu(B(x,2^{-k+2}))\right]^{-1/p}\|g\|_{L^p(B(x,2^{-k+1}))}$$

$$\lesssim \phi(2^{-k})\Big[\mu(B(x,2^{-k}))\Big]^{-1/p}\|g\|_{L^p(B(x,2^{-k+1}))}, \tag{75}$$

where the implicit positive constant is independent of x and k. By (73), (75), and the definition of h, we conclude that, for any given $k \in \mathbb{Z}$ with $k \geq -L+1$ and any generalized Lebesgue point $x \in E$,

$$\begin{aligned} \frac{\mu(B(x,2^{-k}))}{h(2^{-k})} &\lesssim \frac{[\phi(2^{-k})]^p}{h(2^{-k})}\|g\|^p_{L^p(B(x,2^{-k+1}))} \\ &\lesssim \frac{[\phi(2^{-k})]^p}{h(2^{-k})}\|v\|^p_{M^{\phi,p}(B(x,2^{-k+1}))} \\ &\sim \|v\|^p_{M^{\phi,p}(B(x,2^{-k+1}))}, \end{aligned} \tag{76}$$

where the implicit positive constants depend only on k, γ, ϕ, p, and C_μ.

Recall that, for any ball B' with the radius $r \in (0,\infty)$, $\mu(B') \in (0,\infty)$. Then, by Lemma 14, we have that, for any $k \in \mathbb{Z}$ with $k \geq -L+1$ and $x' \in E$, there always exists a generalized Lebesgue point y in $B(x',2^{-k})$. Thus, $B(x',2^{-k}) \subset B(y,2^{-k+1})$ and $B(y,2^{-k}) \subset B(x',2^{-k+1})$. Using this, (76) with x therein replaced by y, the definition of h, $\phi \in \mathcal{A}$, and the doubling property of μ, we further conclude that, for any given $k \in \mathbb{Z}$ with $k \geq -L+1$ and any $x' \in E$,

$$\begin{aligned} \frac{\mu(B(x',2^{-k}))}{h(2^{-k})} &\leq \frac{\mu(B(y,2^{-k+1}))}{h(2^{-k})} \\ &\lesssim \frac{\mu(B(y,2^{-k-1}))}{h(2^{-k-1})} \lesssim \|v\|^p_{M^{\phi,p}(B(y,2^{-k}))} \lesssim \|v\|^p_{M^{\phi,p}(B(x',2^{-k+1}))}. \end{aligned} \tag{77}$$

For any given $R \in (0,2^{L-1}]$, let $k_0 \in \mathbb{Z}$ be such that $2^{-k_0} \leq R < 2^{-k_0+1}$. Obviously, $\{B(x,2^{-k_0}) : x \in E\}$ is a covering, consisting of balls with uniformly bounded diameter, of E. Thus, by a covering lemma for doubling metric spaces (see, for instance, ([66], Theorem 3.1.3) and ([67], Lemma 2.9)), we obtain a countable subfamily $\{B(x_i,2^{-k_0}) : x_i \in E,\ i \in \mathcal{I}\}$ of disjoint balls with the radius no more than R such that

$$E \subset \bigcup_{i\in\mathcal{I}} 5B(x_i,2^{-k_0}),$$

where $\mathcal{I}$ is an index set. From this, (68) with $\theta = 1$ and R replaced by $5R$, the doubling property of μ, $\phi \in \mathcal{A}$, (77) with $k = k_0+1$, the property of $\{B(x_i,2^{-k_0}) : x_i \in E,\ i \in \mathcal{I}\}$, and (72), we deduce that

$$\begin{aligned} \mathcal{H}^{h,1}_{5R}(E) &\leq \sum_{i\in\mathcal{I}} \frac{\mu(5B(x_i,2^{-k_0}))}{h(5\cdot 2^{-k_0})} \\ &\lesssim \sum_{i\in\mathcal{I}} \frac{\mu(B(x_i,2^{-k_0-1}))}{h(2^{-k_0-1})} \lesssim \sum_{i\in\mathcal{I}} \|v\|^p_{M^{\phi,p}(B(x_i,2^{-k_0}))} \lesssim \|v\|^p_{M^{\phi,p}(\mathcal{X})} \lesssim \varepsilon, \end{aligned}$$

where the implicit positive constants depend only on ϕ, p, C_μ, and R. Letting $\varepsilon \to 0^+$, we then conclude that, for any $R \in (0,2^{L-1}]$, $\mathcal{H}^{h,1}_{5R}(E) = 0$, which further implies that

$$\mathcal{H}^{h,1}(E) = \limsup_{R\to 0^+} \mathcal{H}^{h,1}_{5R}(E) = 0.$$

This finishes the proof of Theorem 8. □

Remark 10. *Let $\mathcal{F}$ and h be as in Theorem 8, and D and C_μ be as in (1). We point out that, by the proof of Theorem 8, the implication*

$$\mathrm{Cap}_{\mathcal{F}}(E) = 0 \Longrightarrow \mathcal{H}^{h,1}(E) = 0$$

holds true for any set $E \subset \mathcal{X}$.

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