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Optimal Effort on Self-Insurance-Cum-Protection: A New Analysis Using Yaari's Dual Theory

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Abstract: People take different measures to control risks. The measures that can simultaneously reduce loss probability and loss size are called self-insurance-cum-protection. This paper studies self-insurance-cum-protection using Yaari's dual theory. We analyze the comparative statics of increased risk aversion. Two different sufficient conditions are found in the two-state model, from which an increase in the level of risk aversion will lead to an increase in the level of self-insurance-cum-protection. The first condition is a new result under Yaari's dual theory and its implication is that the more risk-averse individual is willing to exert greater effort on self-insurance-cum-protection if the probability of loss can be reduced to very small by a less risk-averse individual with optimal effort. The second condition depends on the forms of the self-insurance-cum-protection cost and the loss. This condition is the same as that obtained under expected utility in existing literature. Our study therefore assures the robustness of this result. We also study comparative statics in the continuous model and find out that the results are analogous to that in the two-state model. In addition, we consider how the availability of market insurance affects the self-insurance-cum-protection level. When the probability of loss is small, the self-insurance-cum-protection and market insurance are substitutes. This means when market insurance is available, people tend to exert less effort on self-insurance-cum-protection.

Keywords: self-insurance-cum-protection; dual theory; risk aversion; market insurance



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1. Introduction

In daily life, people purchase insurance to manage different risks. For most of the risks, people can also exercise effort on their own to reduce the loss size (self-insurance) or to reduce the loss probability (self-protection). This idea was first formally studied by Ehrlich and Gary [1]. Self-insurance or self-protection are especially important for new risks when market insurance is not yet available, e.g., at the early stage of COVID-19 pandemic. People take different measures to protect themselves from the pandemic. These measures may reduce the loss size and the loss probability at the same time. For example, vaccination not just only reduce the probability of contracting disease, but also reduce severity of disease and hospitalization if infected. This recent situation motivates us to have a closer look on the idea of self-insurance-cum-protection.

Self-insurance and self-protection activities have been popular topics in the literature. One aspect that is widely examined is the comparative statics of increased risk aversion on these activities. Dionne and Eeckhoudt [2], McGuire et al. [3], and Lohse et al. [4] showed that increasing risk aversion results in a higher level of self-insurance but its effect on self-protection is ambiguous. Lee [5] suggested studying self-insurance-cum-protection activities that serve the purposes of both self-insurance and self-protection. Except vaccination, there are other examples of self-insurance-cum-protection. A good fire fighting system reduces the probability of a fire and the damage brought by a fire if it occurs. An effective community warning system on extreme weather can alert the

people to take appropriate preventive measures and reduce the severity of the accidents occurred. It was shown in Lee [5] that increasing risk aversion leads to a higher level of self-insurance-cum-protection if certain conditions are satisfied. Jullien et al. [6] combined self-insurance and self-protection in one model and considered the effect of risk aversion by using the single-crossing condition.

The expected utility framework laid the foundation for the results discussed above. Ever since the development of the expected utility theory as an axiomatic theory of choice under risk from von Neumann and Morgenstern [7], expected utility theory has become a prominent tool in analyzing individuals' choices and economic behavior. However, the validity of the independence axiom in the expected utility hypothesis has been challenged by different empirical studies. The well-known Allais paradox given in Allais [8] showed the violation of independence axiom systematically observed in behavior. Therefore, alternative utility theories have been developed and a comprehensive review of the development can be found in Starmer [9]. In this paper, we propose studying self-insurance-cum-protection using the dual theory of risk introduced by Yaari [10].

Unlike in the expected utility framework, where the attitude towards risk is incorporated through the utility on wealth, the dual theory of risk allows the attitude towards risk to be considered independently of wealth through a distortion in probabilities. Moreover, the dual theory of risk modifies the independence axiom that was challenged in the expected utility theory. There has not been much work on self-insurance or self-protection using the dual theory. However, the few existing literature still provides us insight on individual's insurance decision making and why it is valuable to conduct further research work in this area. Konrad [11] interpreted the condition for a more risk averse individual to exercise more effort on self-protection. Doherty and Eeckhoudt [12] determined the optimal market insurance, i.e., the optimal coinsurance rate and deductible amount, respectively, using the dual theory. Courbage [13] filled the gap between these two papers by studying the interaction between self-protection and market insurance, as well as the interaction between self-insurance and market insurance under dual theory. The existing literature studied self-insurance and self-protection separately as two lines. Here, we model self-insurance-cum-protection using dual theory, which has not been considered explicitly before.

The research questions in this paper are as follows. First, we study the effect of increased risk aversion on self-insurance-cum-protection from comparative statics. The objective is to determine conditions from which increasing risk aversion leads to greater effort on self-insurance-cum-protection. Second, we study the interaction of self-insurance-cum-protection with market insurance. The objective is to find out whether self-insurance-cum-protection are substitutes or complements to each other. In the following, we first provide a background of Yaari's dual theory. Next, we introduce a two-state model to include self-insurance-cum-protection. The result will be compared to that obtained under expected utility framework and its implications will also be discussed. Subsequently, a continuous model will be investigated. It will be shown that the results in the two-state model can be generalized to the continuous model. In order to approach a real-life situation, we also allow the existence of market insurance in later section. The substitutability between self-insurance-cum-protection and market insurance under certain condition will be shown and explained.

2. Yaari's Dual Theory

Consider an individual who faces a random wealth W with cumulative distribution function K . In the well-known expected utility hypothesis, expected utility is given by:

$$EU = \int U(w) dK(w).$$

The continuously increasing utility function U is a transformation of wealth, with $U' > 0$ and $U'' < 0$ for a risk averse individual. If the wealth equals w_1 with probability p and equals w_2 with probability $1 - p$, where $w_1 < w_2$, the expected utility reduces to

$$EU = pU(w_1) + (1 - p)U(w_2).$$

Unlike in the expected utility framework, where the attitude towards risk is incorporated through the utility on wealth, Yaari’s dual theory allows the attitude towards risk to be considered independently of wealth through a distortion to probabilities. The individual can subjectively weight the probabilities by using a distortion function g . Yaari’s utility function is given by

$$Y_g = \int w d(g \circ K)(w),$$

where utility is assumed to be a linear function. A concave function $g : [0, 1] \rightarrow [0, 1]$ represents a risk averse individual, where $g(0) = 0, g(1) = 1, g' > 0$ and $g'' < 0$. With a concave g , the probability of having a low wealth is enlarged while the probability of having a high wealth is lowered. Yaari’s formulation also modifies the independence axiom in the expected utility theory. For details of the characteristics of Yaari’s framework, please refer to Yaari [10].

3. A Two-State Model

In the binary case where the wealth only takes on two values, Yaari’s utility function reduces to

$$Y_g = g(p)w_1 + (1 - g(p))w_2. \tag{1}$$

To start the analysis, we consider a basic model with two states - low wealth state and high wealth state. An individual faces random wealth with the following bivariate distribution:

$$W(e) = \begin{cases} w - l(e), & \text{with probability } p(e), \\ w, & \text{with probability } 1 - p(e), \end{cases}$$

In the low wealth state, initial wealth w is subject to a loss $l(e) > 0$ with probability $0 \leq p(e) \leq 1$. In the high wealth state, there is no loss and the individual ends up with w . Let e denote the effort spent on self-insurance-cum-protection. Increasing effort reduces the probability of loss in a diminishing manner, i.e., $p'(e) < 0$ and $p''(e) > 0$. In case a loss happens, the effort also reduces the loss size with diminishing effect, i.e., $l'(e) < 0$ and $l''(e) > 0$.

The cost of the effort spent is represented by an increasing convex function $c(e)$ (i.e., $c'(e) > 0$ and $c''(e) > 0$). The individual is assumed to be risk averse and to choose e to maximize the following Yaari’s utility:

$$Y_g(e) = g(p(e))(w - c(e) - l(e)) + (1 - g(p(e)))(w - c(e)).$$

To determine the optimal level of effort, we evaluate

$$\begin{aligned} \frac{\partial Y_g(e)}{\partial e} &= g'(p(e))p'(e)(w - c(e) - l(e)) - g(p(e))(c'(e) + l'(e)) \\ &\quad - g'(p(e))p'(e)(w - c(e)) - \{1 - g(p(e))\}c'(e) \\ &= -g'(p(e))p'(e)l(e) - g(p(e))l'(e) - c'(e) \end{aligned} \tag{2}$$

The first-order condition is therefore

$$-g'(p(e))p'(e)l(e) - g(p(e))l'(e) = c'(e). \tag{3}$$

The left hand side of (3) shows the marginal benefit brought by the effort spent. The probability that a loss would occur is lowered while the size of the loss is also reduced in

the low wealth state. The marginal benefit is equal to the marginal cost of the effort spent, which is the right hand side of (3). Note that the following second-order condition is not always true for a risk averse individual:

$$\begin{aligned}
 & - [g''(p(e))\{p'(e)\}^2 + g'(p(e))p''(e)]l(e) \\
 & - 2g'(p(e))p'(e)l'(e) - g(p(e))l''(e) - c''(e) < 0.
 \end{aligned}$$

We assume that $g''(p(e))\{p'(e)\}^2 + g'(p(e))p''(e) > 0$, which is sufficient for the second-order condition to hold.

Let us now compare the above result to that of a more risk averse individual. For Yaari's utility, an individual with probability distortion function $h = v(g)$ is (weakly) more risk averse than the one with probability distortion function g , if v is a concave function fulfilling $v' > 0$ and $v'' < 0$ (see Pratt [14]). For h to be a valid probability distribution, $v : [0, 1] \rightarrow [0, 1]$ with $v(0) = 0$ and $v(1) = 1$.

Theorem 1. *Let e_g be the interior solution of (3). Increasing risk aversion results in larger effort spent on self-insurance-cum-protection if $e_g > e^*$, where e^* satisfies $v'(g(p(e^*))) = 1$.*

Proof. Consider the concave transformation $h = v(g)$ which indicates an increase in risk aversion. From (2), it can be seen that the optimal level of effort for the more risk averse individual is

$$\frac{\partial Y_h(e)}{\partial e} = -v'(g(p(e)))g'(p(e))p'(e)l(e) - v(g(p(e)))l'(e) - c'(e) \tag{4}$$

If we evaluate (4) at e_g and use the the first-order condition (3), we obtain

$$\begin{aligned}
 \left. \frac{\partial Y_h(e)}{\partial e} \right|_{e=e_g} &= - \left(v'(g(p(e_g))) - 1 \right) g'(p(e_g))p'(e_g)l(e_g) \\
 &\quad - \left(v(g(p(e_g))) - g(p(e_g)) \right) l'(e_g).
 \end{aligned} \tag{5}$$

Since v is concave, $v(g(p(e_g))) \geq g(p(e_g))$. Therefore, $v'(g(p(e_g))) > 1$ is a sufficient condition for (5) to be positive. To see this, note that $\lim_{p(e) \rightarrow 0} v'(g(p(e))) > 1$ and $\lim_{p(e) \rightarrow 1} v'(g(p(e))) < 1$.

For a continuous v , there exists a unique e^* such that $v'(g(p(e^*))) = 1$. Together with the fact that $p'(e) < 0$, this implies that $e_g > e^*$ for $v'(g(p(e_g))) > 1$ to hold. The sufficient condition means that the more risk averse individual would put more effort in self-insurance-cum-protection than the less risk averse individual if the less risk averse individual puts in a sufficiently large effort. □

There are additional conditions resulting in a positive value of Equation (5). Lee [5] studied the condition that relates to the loss size and the cost of the effort spent. We shall show in the following proposition that the condition below, which is also given in Lee [5] under the expected utility framework, also holds here under Yaari's Dual Theory.

Theorem 2. *Increasing risk aversion results in larger effort spent on self-insurance-cum-protection if $c'(e_g) + l'(e_g) \leq 0$.*

Proof. Note that the first-order condition (3) can be rewritten as

$$\frac{\partial Y_g(e)}{\partial e} = -g'(p(e))p'(e)l(e) - g(p(e))(c'(e) + l'(e)) - \{1 - g(p(e))\}c'(e) = 0. \tag{6}$$

Recall that $h = v(g)$ represents an increase in risk aversion. To study its effect on optimal effort, we consider

$$\begin{aligned} \left. \frac{\partial Y_h(e)}{\partial e} \right|_{e=e_g} &= -v'(g(p(e_g)))g'(p(e_g))p'(e_g)l(e_g) \\ &\quad - v[g(p(e_g))](c'(e_g) + l'(e_g)) - \{1 - v[g(p(e_g))]\}c'(e_g) \end{aligned}$$

Using (6), the above becomes

$$\begin{aligned} \left. \frac{\partial Y_h(e)}{\partial e} \right|_{e=e_g} &= v'(g(p(e_g))) \left\{ g(p(e_g))(c'(e_g) + l'(e_g)) + \{1 - g(p(e_g))\}c'(e_g) \right\} \\ &\quad - v[g(p(e_g))](c'(e_g) + l'(e_g)) - \{1 - v[g(p(e_g))]\}c'(e_g) \\ &= \left\{ c'(e_g) + l'(e_g) \right\} \left\{ v'(g(p(e_g)))g(p(e_g)) - v[g(p(e_g))] \right\} \\ &\quad + c'(e_g) \left\{ v'(g(p(e_g)))[1 - g(p(e_g))] - \{1 - v[g(p(e_g))]\} \right\} \end{aligned} \tag{7}$$

Since v is concave, $v(0) = 0$ and $v(1) = 1$, we can apply the first-order Taylor expansion to arrive at the following bounds

$$\begin{aligned} 0 &< v(g(p(e))) - v'(g(p(e)))g(p(e)), \\ 1 &< v(g(p(e))) + v'(g(p(e)))[1 - g(p(e))]. \end{aligned}$$

It follows from the bounds and $c'(e) > 0$ that $c'(e_g) + l'(e_g) \leq 0$ is a sufficient condition for (7) to be positive. The result shows that for more risk-averse individuals, the marginal reduction in loss size is an important factor that they will consider. Although the effort is spent on self-insurance-cum-protection, in deciding they interestingly neglect its effect on the loss probability. □

4. Results and Discussion

Theorems 1 and 2 show two sufficient conditions such that increasing risk aversion results in larger effort spent on self-insurance-cum-protection. The reason to exploit the results under dual theory is two-fold. First, dual theory is opposite to expected utility theory such that the distortion lies in the probabilities rather than in the wealth levels. By examining dual theory, we test the robustness of the behavior shown under expected utility. As proved in Theorem 2, the result obtained under expected utility theory remains true in the setting of dual theory. The second reason to study self-insurance-cum-protection under dual theory is that a new sufficient condition, i.e., Theorem 1, can be obtained. This result has not been proved before under expected utility. In the following, we shall point out the implications of this new condition.

Since v is a continuous increasing concave function, it follows from the condition $v[g(p(e^*))] = 1$ that $v'[g(p(e_g))] > 1$. This happens when $p(e_g) \rightarrow 0$. Thus, this condition implies that for a loss that can be reduced to a very small probability by the optimal effort spent by the less risk-averse individual, the more risk-averse individual is willing to spend an even greater effort on self-insurance-cum-protection. The more risk-averse individual has a stronger desire to reduce the probability and the size of loss. However, for a high probability of loss, the effect of increasing risk-aversion on the effort spent is ambiguous. Further assumptions are needed before obtaining conclusions.

5. A Continuous Model

Suppose now that the loss is continuous. Let e continue to denote the effort spent on self-insurance-cum-protection and $c(e)$ denote the cost of the effort. Assume that there is a state random variable $\Theta \in [0, \bar{\theta}]$, with density function $f(\theta; e)$ and distribution function $F(\theta; e)$ given e .

The loss size $l(\theta; e)$ is a function of the state happened and the effort spent, where a higher θ implies a greater loss, i.e., $\frac{\partial l(\theta; e)}{\partial \theta} > 0$. The loss size has lower bound $l(0; e) = 0$ and $l(\bar{\theta}; e) = \bar{l}$ for all e , where $\bar{l} < \infty$ is a constant. Given θ , the loss size relates to e through $\frac{\partial l(\theta; e)}{\partial e} < 0$. Effort e has also an effect on the loss probability distribution by lowering the probability of large losses. By first-order stochastic dominance, this implies that $\frac{\partial F(\theta; e)}{\partial e} > 0$ for all θ .

In this continuous model, loss size and probability distribution are determined by the state θ . Therefore, it is more straightforward to apply the distribution of the state directly in Yaari’s utility than considering the wealth status:

$$Y_\psi(e) = \int_0^{\bar{\theta}} \left(w - c(e) - l(\theta; e) \right) \psi'(F(\theta; e)) f(\theta; e) d\theta, \tag{8}$$

where $\psi(0) = 0$, $\psi(1) = 1$ and ψ is a convex function for a risk averse individual ($\psi' > 0$ and $\psi'' > 0$).

As in the two-state model, we will investigate how increasing risk aversion affects the optimal effort. Note that ψ is a distortion function on the state distribution instead of the wealth distribution. A more risk averse individual should therefore be represented by the distortion function $\phi = \zeta(\psi)$, where $\zeta : [0, 1] \rightarrow [0, 1]$ is a convex function with $\zeta' > 0$ and $\zeta'' > 0$.

Theorem 3. *Let e_ψ be the interior solution that maximizes (8). Increasing risk aversion results in larger effort spent on self-insurance-cum-protection if $e_\psi > e^{**}(\theta)$ is true for all $\theta \in [0, \bar{\theta}]$, where $e^{**}(\theta)$ satisfies $\zeta'(\psi(F(\theta; e^{**}(\theta)))) = 1$ for a given θ .*

Proof. The utility (8) can also be written as

$$Y_\psi(e) = w - c(e) - \int_0^{\bar{\theta}} l(\theta; e) \psi'(F(\theta; e)) f(\theta; e) d\theta.$$

To evaluate the optimal effort, we find

$$\begin{aligned} \frac{\partial Y_\psi(e)}{\partial e} &= -c'(e) \\ &\quad - \int_0^{\bar{\theta}} \left\{ l(\theta; e) \left(\psi''(F(\theta; e)) F_e(\theta; e) f(\theta; e) + \psi'(F(\theta; e)) f_e(\theta; e) \right) \right. \\ &\quad \left. + l_e(\theta; e) \psi'(F(\theta; e)) f(\theta; e) \right\} d\theta = 0, \end{aligned} \tag{9}$$

where partial derivatives is denoted by a subscript. By using integration by parts, note that

$$\begin{aligned} \int_0^{\bar{\theta}} l(\theta; e) \psi'(F(\theta; e)) f_e(\theta; e) d\theta &= \left[l(\theta; e) \psi'(F(\theta; e)) F_e(\theta; e) \right]_0^{\bar{\theta}} \\ &\quad - \int_0^{\bar{\theta}} \left(l(\theta; e) \psi''(F(\theta; e)) f(\theta; e) \right. \\ &\quad \left. + l_e(\theta; e) \psi'(F(\theta; e)) \right) F_e(\theta; e) d\theta. \end{aligned} \tag{10}$$

By definition, $F(0; e) = 0$ and $F(\bar{\theta}; e) = 1$ for all e . Therefore, $F_e(0; e) = 0$ and $F_e(\bar{\theta}; e) = 0$, from which (10) can be simplified to become

$$\begin{aligned} & \int_0^{\bar{\theta}} l(\theta; e)\psi'(F(\theta; e))f_e(\theta; e) d\theta \\ &= - \int_0^{\bar{\theta}} \left(l(\theta; e)\psi''(F(\theta; e))f(\theta; e) + l_\theta(\theta; e)\psi'(F(\theta; e)) \right) F_e(\theta; e) d\theta \end{aligned} \tag{11}$$

We substitute (11) into (9) and get the following first-order condition

$$\frac{\partial Y_\psi(e)}{\partial e} = -c'(e) + \int_0^{\bar{\theta}} \left(l_\theta(\theta; e)F_e(\theta; e) - l_e(\theta; e)f(\theta; e) \right) \psi'(F(\theta; e)) d\theta = 0 \tag{12}$$

Let $\phi = \zeta(\psi)$ denote the distortion function for a more risk averse individual. To see the effect of risk aversion, we evaluate

$$\begin{aligned} \left. \frac{\partial Y_\phi(e)}{\partial e} \right|_{e=e_\psi} &= -c'(e_\psi) \\ &+ \int_0^{\bar{\theta}} \left(l_\theta(\theta; e)F_e(\theta; e) - l_e(\theta; e)f(\theta; e) \right) \zeta'(\psi(F(\theta; e)))\psi'(F(\theta; e)) d\theta \Big|_{e=e_\psi} . \end{aligned} \tag{13}$$

By using the first-order condition (12), we obtain

$$\begin{aligned} \left. \frac{\partial Y_\phi(e)}{\partial e} \right|_{e=e_\psi} &= \int_0^{\bar{\theta}} l_\theta(\theta; e)F_e(\theta; e)\psi'(F(\theta; e)) \left\{ \zeta'(\psi(F(\theta; e))) - 1 \right\} d\theta \Big|_{e=e_\psi} \\ &- \int_0^{\bar{\theta}} l_e(\theta; e)f(\theta; e)\psi'(F(\theta; e)) \left\{ \zeta'(\psi(F(\theta; e))) - 1 \right\} d\theta \Big|_{e=e_\psi} \end{aligned} \tag{14}$$

It can be seen that $\zeta'(\psi(F(\theta; e_\psi))) > 1$ for all $\theta \in [0, \bar{\theta}]$ is a sufficient condition for (14) to be positive. Let us now give an interpretation to this condition. Since ζ is a convex function, we have $\lim_{F \rightarrow 0} \zeta'(\psi(F(\theta; e))) < 1$ and $\lim_{F \rightarrow 1} \zeta'(\psi(F(\theta; e))) > 1$. Assuming that ζ is a continuous function, there exists a unique $e^{**}(\theta)$ such that $\zeta'(\psi(F(\theta; e^{**}(\theta)))) = 1$ for a given θ . As mentioned above, $\partial F(\theta; e) / \partial e > 0$ by first-order stochastic dominance. Therefore, we conclude that $e_\psi > e^{**}(\theta)$ for $\zeta'(\psi(F(\theta; e_\psi))) > 1$ to hold for all θ . As an analogy of Theorem 1, this means that the more risk averse individual exerts a greater effort in self-insurance-cum-protection than the less risk averse individual if the less risk averse individual exercises a sufficiently large effort. □

As in the two-state model, the condition that we can conclude on the effect of increasing risk aversion is not unique. In the following proposition, we shall demonstrate that the condition given in Lee [5], i.e., a continuous analogy to Theorem 2, also holds here under Yaari’s Dual Theory.

Theorem 4. Recall that the interior solution that maximizes (8) is denoted by e_ψ . Define $\sigma(\theta; e) = l_\theta(\theta; e) \frac{F_e(\theta; e)}{f(\theta; e)} - l_e(\theta; e) - c'(e)$, which represents the marginal net benefit of exercising effort. Increasing risk aversion results in larger effort spent on self-insurance-cum-protection if $l_{e_\psi}(\bar{\theta}; e_\psi) + c'(e_\psi) \leq 0$ and there exists a unique $\theta^* \in [0, \bar{\theta}]$ such that $\sigma(\theta^*; e_\psi) = 0$.

Proof. From (8), we obtain the first-order condition

$$\begin{aligned} \frac{\partial Y_\psi(e)}{\partial e} &= \int_0^{\bar{\theta}} \left\{ \left(w - c(e) - l(\theta; e) \right) \left(\psi''(F(\theta; e)) F_e(\theta; e) f(\theta; e) + \psi'(F(\theta; e)) f_e(\theta; e) \right) \right. \\ &\quad \left. + \left(-c'(e) - l_e(\theta; e) \right) \psi'(F(\theta; e)) f(\theta; e) \right\} d\theta = 0. \end{aligned} \tag{15}$$

Using integration by parts and the same argument in obtaining (11), Equation (15) becomes

$$\begin{aligned} &\int_0^{\bar{\theta}} \left(w - c(e) - l(\theta; e) \right) \psi'(F(\theta; e)) f_e(\theta; e) d\theta \\ &= - \int_0^{\bar{\theta}} \left\{ \left(w - c(e) - l(\theta; e) \right) \psi''(F(\theta; e)) f(\theta; e) - l_\theta(\theta; e) \psi'(F(\theta; e)) \right\} F_e(\theta; e) d\theta. \end{aligned} \tag{16}$$

From (16) and in analogy to the argument below (10), one can see that (15) reduces to

$$\begin{aligned} \frac{\partial Y_\psi(e)}{\partial e} &= \int_0^{\bar{\theta}} \left(l_\theta(\theta; e) \frac{F_e(\theta; e)}{f(\theta; e)} - l_e(\theta; e) - c'(e) \right) \psi'(F(\theta; e)) f(\theta; e) d\theta \\ &= \int_0^{\bar{\theta}} \sigma(\theta; e) \psi'(F(\theta; e)) f(\theta; e) d\theta = 0. \end{aligned} \tag{17}$$

Now, consider a more risk averse individual with distortion function $\phi = \zeta(\psi)$. We will examine

$$\left. \frac{\partial Y_\phi(e)}{\partial e} \right|_{e=e_\psi} = \int_0^{\bar{\theta}} \sigma(\theta; e) \zeta'(\psi(F(\theta; e))) \psi'(F(\theta; e)) f(\theta; e) d\theta \Big|_{e=e_\psi}. \tag{18}$$

Note that $\sigma(0; e_\psi) = -c'(e_\psi) < 0$. If we assume $\sigma(\bar{\theta}; e_\psi) = -l_{e_\psi}(\bar{\theta}; e_\psi) - c'(e_\psi) \geq 0$ and that there exists a unique $\theta^* \in [0, \bar{\theta}]$ such that $\sigma(\theta^*; e_\psi) = 0$, then we get $\sigma(\theta; e_\psi) < 0$ for $\theta < \theta^*$ and $\sigma(\theta; e_\psi) > 0$ for $\theta > \theta^*$. Using (17), let us rewrite (18) to become

$$\left. \frac{\partial Y_\phi(e)}{\partial e} \right|_{e=e_\psi} = \int_0^{\bar{\theta}} \sigma(\theta; e) \left\{ \zeta'(\psi(F(\theta; e))) - \zeta'(\psi(F(\theta^*; e))) \right\} \psi'(F(\theta; e)) f(\theta; e) d\theta \Big|_{e=e_\psi}. \tag{19}$$

For $\theta < \theta^*$, we have $F(\theta; e_\psi) < F(\theta^*; e_\psi)$ and $\psi(F(\theta; e_\psi)) < \psi(F(\theta^*; e_\psi))$ which lead to $\zeta'(\psi(F(\theta; e_\psi))) < \zeta'(\psi(F(\theta^*; e_\psi)))$. As for $\theta > \theta^*$, we have $F(\theta; e_\psi) > F(\theta^*; e_\psi)$ and $\psi(F(\theta; e_\psi)) > \psi(F(\theta^*; e_\psi))$ which lead to $\zeta'(\psi(F(\theta; e_\psi))) > \zeta'(\psi(F(\theta^*; e_\psi)))$. Therefore, for any $\theta \in [0, \bar{\theta}]$,

$$\sigma(\theta; e_\psi) \left\{ \zeta'(\psi(F(\theta; e_\psi))) - \zeta'(\psi(F(\theta^*; e_\psi))) \right\} > 0.$$

Then we can conclude that (19) is positive, which implies that a more risk averse individual exerts a larger effort on self-insurance-cum-protection. \square

As shown, the results in Theorems 2 and 4 are similar. Interested readers may refer to Lee [5] for a detailed discussion on the economic meaning of these results. In our analysis here using Yaari’s dual theory, a new set of sufficient conditions, Theorem 1 for two-state model and Theorem 3 for continuous model, is proposed. The implication of Theorem 3 is similar to that of Theorem 1 that is discussed in Section 3 above. From Theorem 3, it means that increasing risk aversion would result in greater effort in self-insurance-cum-protection if the probability of bad states is low enough in first stochastic order.

In Jullien et al. [6], the continuous model is considered under expected utility theory. The authors assume that wealth follows a continuous distribution and the single-crossing property holds. Here, a continuous distribution is defined over states, with the size of loss depending on the state.

6. Optimal Effort under Market Insurance

In addition to self-insurance-cum-protection activities, people may need market insurance to reduce their loss exposure. In this section, the individual can decide to have x of the loss insured, with $0 \leq x \leq 1$. We will analyze the behavior of the individual under the two-state model described in Section 3 and study under what condition will market insurance substitute or complement self-insurance-cum-protection. Assuming a market with perfect information, the insurance premium takes the following form:

$$\pi = (1 + a)xp(e)l(e),$$

where a is the loading charged by the insurer. The individual chooses x to maximize the following Yaari’s utility:

$$Y_g(e, x) = g(p(e))(w - c(e) - \pi - (1 - x)l(e)) + [1 - g(p(e))](w - c(e) - \pi). \quad (20)$$

To determine the optimal level of market insurance, we take the partial derivative of (20) with respect to x :

$$\begin{aligned} \frac{\partial Y_g(e, x)}{\partial x} &= g(p(e))(-(1 + a)p(e)l(e) + l(e)) - [1 - g(p(e))](1 + a)p(e)l(e) \\ &= l(e)\{g(p(e)) - (1 + a)p(e)\}, \end{aligned} \quad (21)$$

which is independent of the insurance coverage x . Together with the fact that (20) is linear in x , the optimal insurance coverage will only be $x = 0$ (no coverage) or $x = 1$ (full coverage). If the loading a is large such that the individual considers the unit cost of insurance $(1 + a)p(e)$ to be greater than the subjective probability of loss $g(p(e))$, then (21) is negative and the optimal coverage is $x = 0$. On the other hand, if the individual considers the unit cost of insurance to be lower than the subjective probability of loss, then (21) is positive and the individual will buy full coverage. This result points out the factors that affect the decision of an individual on buying insurance or not, which are the loading level and the risk aversion level of the individual. As mentioned in Doherty and Eeckhoudt [12], the expected utility theory normally provides a partial insurance coverage solution while Yaari’s dual theory fills this gap by explaining the often-observed “full or no coverage” decision in real-life consumer behavior.

Now, let us take a closer look at the case that the individual buys full coverage of insurance, i.e., when $(1 + a)p(e) < g(p(e))$ is satisfied. To determine the optimal self-insurance-cum-protection level, we evaluate:

$$\frac{\partial Y_g(e, 1)}{\partial e} = -c'(e) - (1 + a)(p'(e)l(e) + p(e)l'(e)) \quad (22)$$

Recall that e_g is the optimal effort obtained from (3). To study how the presence of market insurance would affect the self-insurance-cum-protection level, we evaluate (22) at e_g by applying the first-order condition (3):

$$\left. \frac{\partial Y_g(e, 1)}{\partial e} \right|_{e=e_g} = (g'(p(e)) - (1 + a))p'(e)l(e) + (g(p(e)) - (1 + a)p(e))l'(e) \Big|_{e=e_g} \quad (23)$$

Note that we are considering the case when $(1 + a)p(e) < g(p(e))$. Therefore, the sufficient condition for (23) to be negative is $g'(p(e)) > 1 + a$. If (23) is negative, it implies that the individual will exercise less effort on self-insurance-cum-protection when there market insurance available is available. This means that when the slope of g exceeds a certain level, which occurs when the probability of loss is small, the self-insurance-cum-protection and market insurance are substitutes.

7. Examples

In this section, we will discuss the implications of this study with mathematical examples.

Example 1. Consider the model in Section 3. Assume that the probability of loss and the loss size are $p(e) = \exp\{-0.5e\}$ and $l(e) = \exp\{-0.3e\}$ respectively given effort level e . Following Eeckhoudt and Laeven [15], we consider the well-known Prelec [16] probability weighting function:

$$g(p(e)) = 1 - \exp\{-[-\log(1 - p(e))]^\alpha\}, \quad 0 < \alpha < 1,$$

which is a concave function with $g(0) = 0$ and $g(1) = 1$. Also, let $c(e) = \exp\{0.8e\}$ be the cost of the effort spent. For $\alpha = 0.5$, one can solve Equation (3) to obtain the optimal level of self-insurance-cum-protection effort. The optimal level of effort is $e_g = 0.06968$.

To compare the result with a more risk-averse individual, let

$$h(p(e)) = 1 - \exp\{-(-\log(1 - p(e)))^\beta\}, \quad 0 < \beta < 1, \tag{24}$$

such that

$$v' = \frac{h'}{g'} = \frac{\beta}{\alpha} ((-\log(1 - p(e))^{\beta-\alpha}) \exp\{-(-\log(1 - p(e)))^\beta\} + (-\log(1 - p(e)))^\alpha).$$

According to Theorem 1, increasing risk aversion results in larger effort if $e_g > e^*$. Let $\beta = 0.65$, then $e^* = 0.052$ which satisfies $v'(g(p(e^*))) = 1$. This means that increasing risk aversion should result in larger effort. We can confirm this result by solving Equation (3) again with probability weighting function (24), keeping all other assumptions the same. The optimal effort in this case is 0.08169, which is larger than $e_g = 0.06968$.

On the other hand, if the sufficient condition in Theorem 1 is not satisfied, the effect of increasing risk aversion may not be positive. For example, let $\beta = 0.9$ in (24), and again keeping all other assumptions the same. In this case, $e^* = 0.081$ which is greater than e_g . If one solves for the optimal effort from Equation (3) with probability weighting function (24) and $\beta = 0.9$, the optimal effort in this case is 0.06529. This means that increasing risk aversion does not imply a higher level of self-insurance-cum-protection effort.

Example 2. To compare the results under dual theory and expected utility theory, let us consider the expected utility

$$EU = p(e)U(w - c(e) - l(e)) + [1 - p(e)]U(w - c(e)) \tag{25}$$

Let the probability of loss and the loss size be $p(e) = \exp\{-0.5e\}$ and $l(e) = \exp\{-0.3e\}$ respectively given effort level e . The first-order condition from (25) is given by

$$0 = p'(e)[U(w - c(e) - l(e)) - U(w - c(e))] - p(e)U'(w - c(e) - l(e))[c'(e) + l'(e)] - (1 - p(e))c'(e)U'(w - c(e)) \tag{26}$$

Assume the exponential utility function $U(w) = -\exp\{-\gamma w\}$ and let $\gamma = 0.5$. Also, let $c(e) = \exp\{0.8e\}$ be the cost of the effort spent. Different from the first-order condition obtained in dual theory, the first-order condition obtained in expected utility framework depends on the value of the initial wealth w . Now, let $w = 100$. The optimal effort obtained from (26) is 0.097477. The sufficient condition in Proposition 1 of Lee [5] is satisfied, which means that a more risk-averse individual would exercise a larger effort. Under the dual theory, this condition can also be obtained and has been stated in Theorem 2. Using the basic assumption in Example 1, the optimal effort is 0.06968. The sufficient condition in Theorem 2 is satisfied. The same condition carries over from the expected utility framework to the dual theory framework. In addition, we provide a new sufficient

condition as demonstrated in Example 1 and this condition was not shown in expected utility theory before.

Dionne and Eeckhoudt [2] showed that increasing risk aversion leads to an increasing level of self-insurance, while the effect on the level of self-protection is ambiguous. This is consistent with our results under dual theory such that the self-protection component renders the effect of increasing risk aversion ambiguous.

8. Conclusions

Self-insurance and self-protection are usually studied separately in classical models. Yet, there have been but few contributions under the expected utility framework that combines self-insurance and self-protection jointly, a case that frequently occurs in real life such as vaccination. Here, we provide a new analysis on self-insurance-cum-protection under the framework of Yaari's dual theory.

Yaari's dual theory is opposite to expected utility theory in that the former considers risk through distortion in probabilities while the latter considers risk through the utility on wealth. Dual theory also modifies the independence axiom that was challenged in expected utility theory. In this paper, we characterized the optimal effort on self-insurance-cum-protection in both a two-state model and a continuous model. Predictions concerning the effect of increased risk aversion are derived from comparative statics.

In the two-state model, two independent sufficient conditions are derived for a more risk-averse individual to spend greater effort. The first condition is newly proved in dual theory. A more risk-averse individual exerts more effort on self-insurance-cum-protection than that exerted by a less risk-averse individual whose effort exceeds a certain level. There are other sufficient conditions for risk-averse individuals to spend greater effort, which however depend on the shapes of the cost function and the loss function. This condition was obtained under expected utility framework and is now shown to carry over to dual theory. This finding assures the robustness of the comparative statics results obtained from expected utility. In the continuous model, the loss is assumed to depend on the state which has a continuous distribution. Under this setting, we find that the sufficient conditions for a more risk-averse individual to exert more effort are analogous to those obtained in the two-state model. One limitation of the present analysis is that it focuses exclusively on an increase in risk aversion. Future work should be devoted to an extension of Yaari's dual theory to changes in higher-order risk preferences such as prudence and temperance (see e.g., Eeckhoudt and Schlesinger [17]).

In addition to self-insurance-cum-protection activities, we introduced market insurance to the model. It is found out that the optimal level of market insurance purchased is either no coverage or full coverage. We further studied the case when an individual chooses full coverage of market insurance. Given certain assumptions concerning the probability distortion, market insurance and self-insurance-cum-protection are substitutes. That means people who purchase market insurance tend to exert less effort on self-insurance-cum-protection. This is a newly obtained result about market insurance and self-insurance-cum-protection in the dual theory framework.

9. List of Mathematical Notations

$g(\cdot)$	Probability distortion function
$p(e)$	Loss probability given effort e
$l(e)$	Loss size given effort e in two-state model
$c(e)$	Cost of the effort e
$Y_g(e)$	Yaari's utility given probability distortion function g and effort e in two-state model
e_g	Optimal level of self-insurance-cum-protection for an individual with probability distortion function g
$h(\cdot)$	Probability distortion function for a more risk-averse individual
$v(\cdot)$	Concave function fulfilling $v' > 0$, $v'' < 0$, and $h(\cdot) = v(g(\cdot))$
$Y_h(e)$	Yaari's utility given probability distortion function h and effort e in two-state model

$f(\theta, e)$	Density function of the state random variable Θ given effort e
$F(\theta, e)$	Distribution function of the state random variable Θ given effort e
$l(\theta, e)$	Loss size given state θ and effort e
$\psi(\cdot)$	Probability distortion function on the state distribution
$Y_\psi(e)$	Yaari's utility given probability distortion function ψ and effort e in continuous model
e_ψ	Optimal level of self-insurance-cum-protection for an individual with probability distortion function ψ
$\phi(\cdot)$	Probability distortion function on the state distribution for a more risk averse individual
$\xi(\cdot)$	Convex function with $\xi' > 0$ and $\xi'' > 0$
$Y_\phi(e)$	Yaari's utility given probability distortion function ϕ and effort e in continuous model
π	Market insurance premium
a	Market insurance loading
x	Proportion of loss insured by market insurance
$Y_g(e, x)$	Yaari's utility given probability distortion function g , effort e and proportion of loss insured x

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