

Article

Computing the Number of Failures for Fuzzy Weibull Hazard Function

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Abstract: The number of failures plays an important factor in the study of maintenance strategy of a manufacturing system. In the real situation, this number is often affected by some uncertainties. Many of the uncertainties fall into the possibilistic uncertainty, which are different from the probabilistic uncertainty. This uncertainty is commonly modeled by applying the fuzzy theoretical framework. This paper aims to compute the number of failures for a system which has Weibull failure distribution with a fuzzy shape parameter. In this case two different approaches are used to calculate the number. In the first approach, the fuzziness membership of the shape parameter propagates to the number of failures so that they have exactly the same values of the membership. While in the second approach, the membership is computed through the α -cut or α -level of the shape parameter approach in the computation of the formula for the number of failures. Without loss of generality, we use the Triangular Fuzzy Number (*TFN*) for the Weibull shape parameter. We show that both methods have succeeded in computing the number of failures for the system under investigation. Both methods show that when we consider the function of the number of failures as a function of time then the uncertainty (the fuzziness) of the resulting number of failures becomes larger and larger as the time increases. By using the first method, the resulting number of failures has a *TFN* form. Meanwhile, the resulting number of failures from the second method does not necessarily have a *TFN* form, but a *TFN-like* form. Some comparisons between these two methods are presented using the Generalized Mean Value Defuzzification (*GMVD*) method. The results show that for certain weighting factor of the *GMVD*, the cores of these fuzzy numbers of failures are identical.

Keywords: Weibull hazard function; number of failures; *TFN*; α -cut; defuzzification



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1. Introduction

Uncertainty is present in almost all decision problems, including in the field of reliability and maintenance. This is due to unknown future events and imprecision as well as human subjectivity in a decision process [1]. There are some important factors that significantly affect the decision-making in any field. In the field of reliability and maintenance, the number of failures plays important roles in the study of maintenance strategy of a manufacturing system. In the real situation, this number is often affected by some uncertainties. Many of the uncertainties fall into the possibilistic uncertainty, which is different from the probabilistic uncertainty. In many cases, at least one of the parameters or variables of the decision function has fuzzy value, instead of crisp value. This uncertainty is commonly modeled by applying fuzzy theoretical framework, e.g., the variable and parameter have fuzzy values and the calculation is done using extension principle approach [2].

As an important factor, the number of failures is essential to obtain, and subsequently is used as a base for further decision processes in reliability and maintenance analysis. As an example, this “number” is used in the calculation to design optimal maintenance strategies which are directed to minimize the number of failures while also minimizing the costs of operation [3–5]. For this reason, the knowledge on how to compute or predict the number of failures becomes vital. Considering the occurrence of uncertainty and

imprecision—together with complexity of the system under investigation, failure data are often difficult to obtain. In this case, the theory of fuzzy sets has been widely used to provide a framework to deal with these uncertainty and imprecision [6]. Among the important questions needed to be addressed related to the number of failures of a system having a possibilistic uncertainty is, first, how to compute this number for a given possibility distribution with fuzzy parameters. Nowadays, some calculator for fuzzy numbers are readily available [2]. Second, it is also important to know how the degree of uncertainty of the parameters propagates to the resulting failure numbers. This is commonly known as the propagation of fuzziness, which is defined as “the way in which the amount of imprecision in the model’s inputs affects the changes in the model’s output” [7], (p. 163). Technically the propagation of uncertainty happens through mathematical operations involved in the model and in the computation. Knowing the method to calculate the number and its degree of uncertainty, will significantly improve the quality of the decision being sought (see also [8,9] for similar cases in other area). In general, fuzziness propagation in complex engineering systems may constitute a significant challenge [10].

The aims of the paper are two-fold, namely, to calculate the number of failures for a system which has Weibull failure distribution with a fuzzy shape parameter and to understand how the fuzziness of this shape parameter propagates to the resulting number of failures. These two objectives constitute the importance and contributions of the work presented in this paper. In addition, in this paper we look for the number of failures and two different approaches are used to calculate this number. In the first approach, the fuzziness membership of the shape parameter propagates to the number of failures so that they have exactly the same values of the membership. While in the second approach, the membership is computed through the α -cut or α -level of the shape parameter.

Literature Review

As it is explained earlier, the motivation of the paper is due the importance of finding the number of failures in the field of maintenance strategy. Some examples of such importance can be seen in [11–16] from various perspectives. It is often found that most of the problems in maintenance engineering are finding optimal strategies that minimize the cost of operation to manipulate the system as well as minimize the number of failures of the system (e.g., [17]). In many cases, the number of failures is represented in its distribution function. Several type of distribution functions are commonly used to model the failures of an industrial equipment, among others is the Weibull distribution function together with its hazard function [18,19]. This distribution function could appear either in two-parameters model or in three-parameters model [20,21]. Hence, the Weibull distribution function plays vital roles in areas of research related to maintenance strategy in which understanding a system, predicting the outcome of a system, and prescribing an optimal intervention to obtain the best performance of a system are being sought. In fact, the spectrum of the area applications of the Weibull distribution is quite broad from engineering, social sciences, to biological and health problems.

Apart from the abundant usage of the Weibull distribution in many areas of research, especially in maintenance strategy, most of the analysis only consider the crisp form of data, i.e., ignoring the presence of possibilistic uncertainty which might be often found in many real phenomena. For example, in maintenance engineering, most maintenance models in literature mainly consider certain or crisp condition, e.g., [22,23]. However, as mentioned earlier, these kinds of models do not seem to fit in the real condition. Readers may find a brief review of the importance of the possibilistic uncertainty in [24]. In reliability and maintenance problems, uncertainty may affect the models, the nonhomogeneous Poisson process (NHPP), and the Weibull generalized renewal process parameters [25,26], and the probability distribution parameters [8], and it is important to know how this uncertainty propagates through the models which likely affect the insight and prediction from the models [27].

There are several approaches to model the possibilistic uncertainty, one of them is by applying the fuzzy number theory, as suggested by Zadeh [28,29] and his followers. The popularity of fuzzy number theory in reliability and maintenance literatures is now getting higher [30] resulting in new modeling approach in many aspects, like maintenance risk-based inspection interval optimization with fuzzy failure interaction for two-component repairable system [31] and many others. The authors in [32] are among the few authors who consider fuzzy Weibull distribution in their works. They consider the Weibull distribution function as a fuzzy function and use it for analyzing the behavior of an industrial system stochastically by utilizing vague, imprecise, and uncertain data, which in turn result in the reliability indices (such as hazard function, maintainability, etc.) of time of time varying failure rate instead of the constant failure rate for the system.

In general, the author in [33] (pp. 152–157) shows several methods on how to implement a fuzzy function in addressing problems with possibilistic uncertainty. The methods can be classified into three different ways depending to which aspect of the crisp function the fuzzy concept was applied, namely (i) crisp function with fuzzy constraint, (ii) crisp function which propagates the fuzziness of independent variable to dependent variable, and (iii) function that is itself fuzzy. However, in this paper we will only look for the number of failures by using the first and the second approaches above. The fuzziness of the shape parameter is assumed to propagate to the number of failures with the same form of fuzzy number membership in the first approach, as found in [34,35]. While in the second approach, the concept of α -cut or α -level of the fuzziness of the shape parameter is used in the computation to calculate the number of failures, as found in [36]. An example of the methodology on how to compare fuzzy numbers, such as those resulting from different approaches of fuzzy function concepts above can be seen in [37].

In this paper we re-visit the model in [34,35] by giving some more detail analysis and results discussed in those papers. The authors in [34] discussed the Weibull hazard function by assuming a fuzzy shape parameter, which conceptually can be used to compute the number of failures without actually showing the resulting number of failures (either in crisp number form or fuzzy number form). They show how to compute the fuzzy number of failures of Weibull hazard function in [35] by assuming a fuzzy shape parameter in the Weibull hazard function via the second approach in [33], (p. 154), i.e., by considering the Weibull function as a crisp function which propagates the fuzziness of independent variable to dependent variable. In this paper we use different approaches by considering the fuzziness of the shape parameter in the computation of the number of failures directly, through the concept of α -cut or α -level [33] (p. 130) and [38], (pp. 7–16). Further we discuss the generalized mean value defuzzification (GMVD) and use it to compare the resulting fuzzy number of failures from different approaches of computation. The proposed defuzzification method (GMVD) is able to find a crisp number which is close to the core of the triangular fuzzy number (TFN).

We organize the presentation of the paper as follows. Section 2 presents briefly some basic methods that are utilized in the preceding sections, namely, the Weibull distribution function, fuzzy number and its membership function, α -cut of a fuzzy number, defuzzification process with Generalized Mean Value Defuzzification (GMVD), and the number of failures for Weibull hazard function with fuzzy parameter. Section 3 gives the main results together with numerical examples to show the visual illustration of the main results. This includes the comparisons from two different methods, i.e., the results from the method considering propagation of the fuzziness of independent variable to dependent variable and the results from the α -cut method. Section 4 presents the discussions of the results and it is finally followed by concluding remarks and further direction of research in Section 5.

2. Materials and Methods

The object being investigated in this paper is the Weibull distribution function as a mathematical model describing the deterioration of life cycle of an industrial system or an equipment. This deterioration or failure data are commonly modeled by the Weibull

distribution function such as found in [39]. The reason of popularity of the Weibull function is its flexibility, so that it can be regarded as the generalization of exponential and Rayleigh distribution functions, which are also commonly used in reliability and maintenance studies [40]. The Weibull distribution is a continuous probability distribution function having the form:

$$f(t) = \begin{cases} \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{(\beta-1)} \exp\left[-\left(\frac{t}{\theta}\right)^\beta\right], & x > 0, \theta > 0, \beta > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

where θ is the scale parameter and β is the shape parameter. The first mathematician who described it in detail is Waloddi Weibull in 1951. The Weibull distribution has a flexibility to model various lifetime data by changing the value of the shape parameter, e.g., if $\beta = 1$ the Weibull distribution is reduced to an exponential distribution and if $\beta = 2$ the Weibull distribution is identical to Rayleigh distribution [40]. Throughout the paper we will assume the scale parameter $\theta = 1$ for some reasons. For example, this choice is sufficient in our context of maintenance modeling if we assume that the average of first failure of the equipment/system under investigation happens within one unit of time—say one month or one year, because of its warranty and good quality control. The authors in [41] give an example with $\theta = 1$ in their simulation.

Using this Weibull distribution function we can calculate some reliability indices, such as hazard function, number of failures, mean time between failures, preventive maintenance time, and replacement time. The standard methods on the calculation of these indices, both for standard and complex systems, can be found among others in [42,43]. Details theory and applications of the Weibull distribution function can be found in [44,45]. In the Section 2.1 we present some concepts of fuzzy theoretical framework which are used in the subsequent method and analysis, namely, fuzzy number and its membership function, α -cut of a fuzzy number, defuzzification process with Generalized Mean Value Defuzzification.

2.1. Fuzzy Number and Its Membership Function

As an introduction to the section that follows we define several concepts of fuzzy number theory that will be used later on. A fuzzy number can be regarded as an extension of a real number in the sense that it has a membership function other than binary to represent uncertainty. Binary membership gives a crisp value for the membership, i.e., either a member or not a member. Fuzzy number gives a wide spectrum of membership from zero (definitely not a member of a set S) to one (definitely a member of a set S). Technically, a fuzzy number \tilde{A} refers to a connected set of possible values, where each possible value of \tilde{A} , say a , has its own membership value in the interval $[0,1]$. This value that measures the degree of possibility for a to be a member of \tilde{A} is called the membership function, usually written as $\mu : a \in A \rightarrow x \in [0, 1]$. This fuzzy number is commonly written with the symbol $\tilde{A} = (A, \mu(A))$ or alternatively $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$ representing the underlying connected set A with the membership function $\mu(A)$. In this regards, the fuzzy number is viewed as a pair of mathematical objects comprising of a set together with its grade or membership function. The fuzzy number is purportedly designed to represent the possibilistic uncertainty and to quantify the unclear and inaccuracies of the abundance of information.

The membership of a fuzzy number can be determined by several functional approaches, which can be classified into the linear and the non-linear functional forms. Among the most popular functional form of fuzzy number are the Triangular Fuzzy Number (TFN) which is often written as $(a;b;c)$ and the Trapezoidal Fuzzy Number (TrFN) which is often written as $(a;b;c;d)$. The functional forms or the membership functions of these fuzzy numbers are given in Equations (2) and (3), which are graphically shown in Figures 1 and 2. Note that for the TrFN in Equation (3), the membership function within the intervals $[a,b]$ and $[c,d]$ are given by increasing and decreasing linear curves respectively. This concept is

generalized by the LR-Flat Fuzzy Number which is then used as a new method for solving fuzzy transportation problems [2,46,47].

- The membership function of a triangular fuzzy number (TFN):

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{x-c}{b-c}, & b \leq x \leq c \\ 0, & x \geq c. \end{cases} \tag{2}$$

- The membership function of a trapezoidal fuzzy number (TrFN):

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{x-d}{c-d}, & c \leq x \leq d \\ 0, & x \geq d. \end{cases} \tag{3}$$

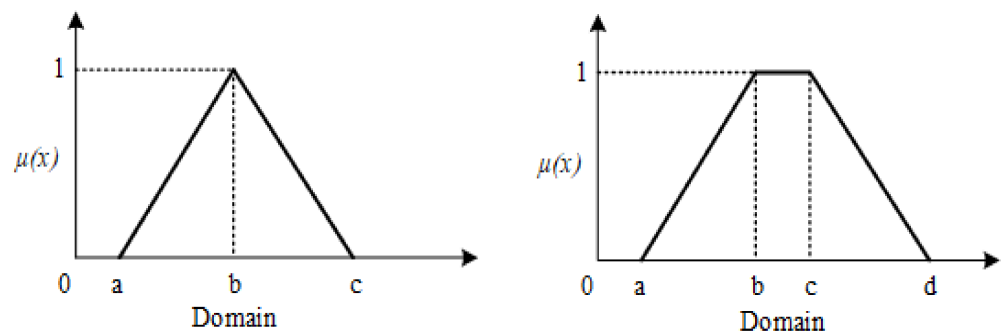


Figure 1. Graphical representation of a triangular fuzzy number $(a;b;c)$ —left figure, and a trapezoidal fuzzy number $(a;b;c;d)$ —right figure.

In Equation (2), $a, b,$ and c are real numbers satisfying $a < b < c$ which constitute the TFN core and support components. In this case b is called the core of the fuzzy number and the sets $[a,b)$ and $(b,c]$ are called the support of the fuzzy number. Similarly, for TrFN, in Equation (3) the core of the fuzzy number is given by $[b,c]$ and the support is given by the set $[a,b)$ and $(c,d]$. Other forms of fuzzy numbers are piecewise quadratic fuzzy number [48], pentagonal fuzzy number [49], Bell shaped fuzzy number [50], parabolic trapezoidal fuzzy number [51], new bell shaped fuzzy number [52], and many others. A good reference on how some new methods and techniques are developed to advance fuzzy numbers concepts for modern analytics can be found in [46]. However, for simplicity, to emphasize the methodological aspect all examples in this paper assume the triangular fuzzy numbers (TFN). In the next section we briefly describe the α -cut of a triangular fuzzy numbers (α -cut, α -level cut, α -level set or sometimes simply is called a cut).

2.2. The α -Cut of a Fuzzy Number

Each fuzzy number is associated with its α -cut. This α -cut sometimes is also called the α -level set. It is technically defined as the set of objects in the associated fuzzy set which have the membership with the values which are at least α . This actually can be seen as a crisp set representation of a fuzzy number. Following this definition, it can be shown that the α -cut of the triangular fuzzy number (1) is given by:

$$\tilde{A}_\alpha = [a_1^\alpha, a_2^\alpha] = [(b - a)\alpha + a, (b - c)\alpha + c] \tag{4}$$

for all $\alpha \in [0, 1]$.

2.3. Generalized Mean Value Defuzzification

For some reasons, the information regarding the best representation of a crisp number for a fuzzy number is needed. In this case, defuzzification of the fuzzy number is done. It is a mathematical calculation which converts the fuzzy number into a single crisp value with respect to a fuzzy set. Some defuzzification formulas are available in literature, such as basic defuzzification distributions, center of area, center of gravity, fuzzy mean, last of maxima, weighted fuzzy mean, etc., [53–55]. In this paper we will use the generalized mean value defuzzification method (GMVD) which is defined as

$$N(\tilde{A}) = \frac{a + nb + c}{n + 2}, \quad (5)$$

where $\tilde{A} = (a; b; c)$ is a TFN and n can be regarded as the weight of the core of the fuzzy number. The larger the weight of the core, the closer the resulting crisp number from the GMVD to the core of the fuzzy number. The properties of this GMVD will be discussed later on and used in the comparison of the resulting number of failures.

3. Results

3.1. Number of Failures for Weibull Hazard Function with Fuzzy Parameter

As explained in the previous section we consider the one-parameter Weibull distribution function, since this choice is sufficient in our context of maintenance modeling if we assume that the average of first failure of the equipment/system under investigation happens within one unit of time—say one month or one year—because of its warranty and good quality control. By considering this assumption ($\theta = 1$) and fuzzy parameter $\tilde{\beta}$ the number of failure is computed using the first method, in which the calculation of the fuzzy number is done point-wise (will be defined later), and we only need the crisp function for the computation. From Equation (1) we have the following one-parameter Weibull cumulative distribution, g , and its hazard function, h :

$$g(t) = 1 - e^{-t^\beta}, \quad (6)$$

and

$$h(t) = \beta t^{\beta-1}, \quad (7)$$

so that the number of failures is given by

$$N(t) = t^\beta. \quad (8)$$

The parameter $\tilde{\beta}$ is the fuzzy number of the shape parameter of the Weibull function. We will treat the fuzziness of the shape parameter in two different approaches: (i) Crisp function which propagates the fuzziness of independent variable to dependent variable and, in which the computation is done point-wise; (ii) crisp function with fuzzy constraint through the level-set computation.

The First Method (Point-wise Method): Let $\tilde{\beta}$ be a TFN which is identified by three crisp numbers a , b , and c , i.e., $\tilde{\beta} = (a; b; c)$ satisfying Equation (2). We compute the number of failures point-wise, i.e., by substituting these crisp numbers one at a time to obtain the crisp output, say a' , b' , and c' . By assuming the same fuzzy measure propagates to the output, we will have $\mu(a') = \mu(a)$, $\mu(b') = \mu(b)$, and $\mu(c') = \mu(c)$, which give a TFN fuzzy output $(a'; b'; c')$ for the function $g(t)$, $h(t)$, and $N(t)$ [34].

The Second Method (α -Cut Method): In the second approach, the fuzzy number $\tilde{\beta}$ is identified as an α -cut satisfying Equation (4). As it is explained in [34], the fuzzy number of the shape parameter is approximated by a sequence of interval associated with the number α in $[0,1]$. This sequence consists of crisp numbers in the interval indicating the support of the fuzzy number for every α in $[0, 1)$. If α is one then the supports converge to/become the core of the fuzzy number. The calculation to determine the number of failures is done at the end points of the interval. In this case, the stack of the end points of the intervals

need not to be a *TFN*, which in many cases forms a *TFN-like* form (see numerical examples for the details).

To facilitate comparison between the results from the two methods, we use the *GMVD* defined in Equation (5). This *GMVD* has the properties as described in Theorem 1.

Theorem 1. Let a *TFN* is given by $(a;b;c)$, then the generalized mean value defuzzification (*GMVD*) defined by Equation (5) has the following properties:

1. For a symmetrical case, i.e., $b - a = c - b = \Delta$ then $N(\tilde{A}) = b$
2. For an asymmetrical case, i.e., $b - a = \Delta_a \neq c - b = \Delta_c$ then
 - a. $N(\tilde{A}) > b$ if $\Delta_a < \Delta_c$
 - b. $N(\tilde{A}) < b$ if $\Delta_a > \Delta_c$
3. If $n \rightarrow \infty$ then $N(\tilde{A}) = b$ regardless the value of p and q .

Proof of Theorem 1:

1. Case 1: symmetrical *TFN*, i.e., $b - a = c - b = \Delta$ then

$$\begin{aligned}
 N(\tilde{A}) &= \frac{a+nb+c}{n+2} = \frac{a+nb+(a+2\Delta)}{n+2} = \frac{2a+nb+2\Delta}{n+2} \\
 &= \frac{2(a+\Delta)+nb}{n+2} = \frac{2b+nb}{n+2} = \frac{(2+n)b}{n+2} = b.
 \end{aligned}$$

Hence, $N(\tilde{A}) = b$.

2. Case 2: non-symmetrical *TFN*, i.e., $b - a = \Delta_a \neq c - b = \Delta_c$ then

- a. if $\Delta_a < \Delta_c$ then

$$N(\tilde{A}) = \frac{a+nb+c}{n+2} = \frac{a+nb+(a+\Delta_a+\Delta_c)}{n+2} > \frac{a+nb+(a+2\Delta_a)}{n+2} = \frac{2a+nb+2\Delta_a}{n+2}$$

Hence, $N(\tilde{A}) > b$.

- b. if $\Delta_a > \Delta_c$ then

$$N(\tilde{A}) = \frac{a+nb+c}{n+2} = \frac{a+nb+(a+\Delta_a+\Delta_c)}{n+2} < \frac{a+nb+(a+2\Delta_a)}{n+2} = \frac{2a+nb+2\Delta_a}{n+2}$$

Hence, $N(\tilde{A}) < b$.

3. If $n \rightarrow \infty$ then $\lim_{n \rightarrow \infty} N(\tilde{A}) = \lim_{n \rightarrow \infty} \frac{a+nb+c}{n+2} = b$. \square

As shown by the theorem, the *GMVD* above has a special characteristic, i.e., it is able to find a crisp number which is close to the core of the triangular fuzzy number (*TFN*). As examples, first consider the symmetrical *TFN* in Figure 2(left), i.e., $\tilde{\beta} = (p = 1.25; q = 1.55; s = 1.85)$. It has *GMVD* = 1.55 for $n = 1$ and *GMVD* = 1.55 for $n = 1000$. Since it is symmetrical, the values of *GMVD* are the same as the core of the *TFN* for all n . However, for the non-symmetrical *TFN*, such as skewed left *TFN* $\tilde{\beta} = (p = 2.50; q = 2.75; s = 2.80)$ in Figure 2(right), it has *GMVD* = 2.6833 for $n = 1$ and *GMVD* = 2.74980 for $n = 1000$. In this case, the larger is n the closer it is to the core of the *TFN*, i.e., 2.75. We will use this method of defuzzification for comparing the fuzzy output from two different methods in this paper.

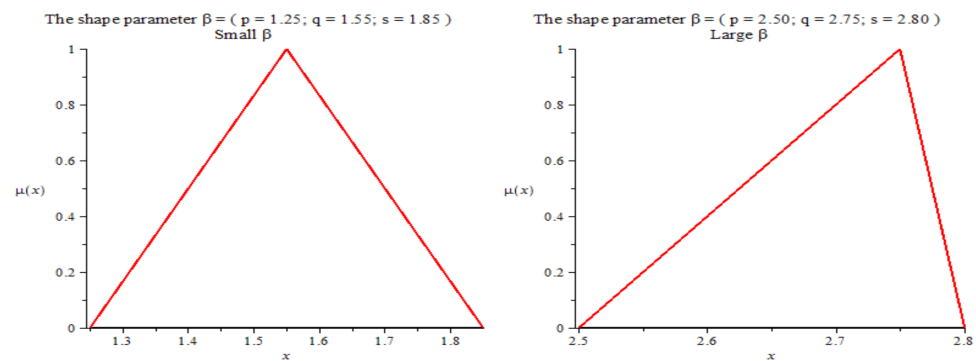


Figure 2. On the **left** figure is shown the relatively small shape parameter $\tilde{\beta} = (p = 1.25; q = 1.55; s = 1.85)$ and on the **right** figure is shown the relatively large shape parameter $\tilde{\beta} = (p = 2.50; q = 2.75; s := 2.80)$. The vertical axis is the fuzzy membership μ of the shape parameter’s TFN. The first shape parameter is a symmetrical TFN and the second shape parameter is a nonsymmetrical TFN. These TFNs are used to calculate their respective number of failures in the subsequent figures.

Next we look at the fuzzy number of failures generated by the Weibull distribution via the α -cut method. Let us recall the α -cut of the triangular fuzzy number $\tilde{A} = (a; b; c)$ is given by $\tilde{A}_\alpha = [a_1^\alpha, a_2^\alpha] = [(b - a)\alpha + a, (b - c)\alpha + c]$ then the shape parameter, the Weibull cumulative distribution, the Weibull hazard function, and the number of failures are, respectively, given in the form of α -cut as follows. The shape parameter will have the form

$$\beta_\alpha = [x_1 + x_3^\alpha, x_2 - x_3^\alpha], \tag{9}$$

for some $x_1, x_2, x_3 \in \mathbb{R}$. By considering the α -cut in Equation (9) and substituting it into Equations (6) and (7) using the fuzzy arithmetic give rise to the cumulative distribution

$$g(t)_\alpha = [1 - \exp(-t^{y_1+y_3^\alpha}), 1 - \exp(-t^{y_2-y_3^\alpha})], \tag{10}$$

for some $y_1, y_2, y_3 \in \mathbb{R}$ and the hazard function

$$h(t)_\alpha = [(z_1 + z_3^\alpha)t^{z_4+z_6^\alpha}, (z_2 - z_3^\alpha)t^{z_5-z_6^\alpha}], \tag{11}$$

for some $z_1, z_2, z_3, z_4, z_5, z_6 \in \mathbb{R}$. So that by integrating both sides of Equation (11) we end up with the number of failures, which is given by

$$N(t)_\alpha = [t^{u_1+u_3^\alpha}, t^{u_2-u_3^\alpha}] \tag{12}$$

for some $u_1, u_2, u_3 \in \mathbb{R}$.

The following theorem shows that as time goes, the GMVD of the number of failures increases and the support of the number of failures becomes wider. This means that the degree of uncertainty becomes larger.

Theorem 2. For $\Delta t > 0$ let $N(t)_\alpha$ and $N(t + \Delta t)_\alpha$ be the fuzzy number of failures at time t and $t + \Delta t$, respectively, then:

1. $N(t)_\alpha = (t^{p_\alpha}, t^{s_\alpha})$ and $N(t + \Delta t)_\alpha = ((t + \Delta t)^{p_\alpha}, (t + \Delta t)^{s_\alpha})$,
2. $GMVD(N(t + \Delta t)_\alpha) \geq GMVD(N(t)_\alpha)$ for all $t \in \mathbb{R}^+$,
3. $((t + \Delta t)^{p_\alpha} - (t + \Delta t)^{s_\alpha}) - (t^{p_\alpha} - t^{s_\alpha}) \geq 0$ for all $t \in \mathbb{Z}^+$.

Proof of Theorem 2:

1. It is clear.
2. It can be proved by using Theorem 1.

- Note that for every $\alpha \in [0, 1]$, the interval in Equation (12) has the form $(t^{p_\alpha}, t^{s_\alpha})$ for some $p_\alpha, s_\alpha \in R$. Without loss of generality, we will drop the index α , so that to prove the theorem we need $((t + \Delta t)^p - (t + \Delta t)^s) - (t^p - t^s) \geq 0$.

Consider the following binomial rule,

$$(x + \Delta x)^n = \sum_{k=0}^n \frac{n!}{(n-k)!k!} \Delta x^{n-k} x^k. \text{ Then we have}$$

$$\begin{aligned} (x + \Delta x)^n &= \sum_{k=0}^{n-1} \frac{(n-1)!}{((n-1)-k)!k!} \Delta x^{(n-1)-k} x^k + \frac{n!}{(n-k)!k!} \Delta x^{n-k} x^k \\ &= \sum_{k=0}^{n-1} \frac{(n-1)!}{((n-1)-k)!k!} \Delta x^{(n-1)-k} x^k + x^n. \end{aligned}$$

Using this rule then for $p, s \in Z^+$ we have

$$\begin{aligned} (t + \Delta t)^p &= \sum_{k=0}^{p-1} \frac{(p-1)!}{((p-1)-k)!k!} \Delta t^{(p-1)-k} t^k + t^p \\ (t + \Delta t)^s &= \sum_{k=0}^{s-1} \frac{(s-1)!}{((s-1)-k)!k!} \Delta t^{(s-1)-k} t^k + t^s \end{aligned}$$

A little algebraic manipulation gives

$$\begin{aligned} ((t + \Delta t)^p - (t + \Delta t)^s) - (t^p - t^s) &= \sum_{k=0}^{p-1} \frac{(p-1)!}{((p-1)-k)!k!} \Delta t^{(p-1)-k} t^k - \sum_{k=0}^{s-1} \frac{(s-1)!}{((s-1)-k)!k!} \Delta t^{(s-1)-k} t^k, \\ &= \sum_{k=s}^{p-1} \frac{(p-1)!}{((p-1)-k)!k!} \Delta t^{(p-1)-k} t^k \geq 0, \end{aligned}$$

Which shows that $((t + \Delta t)^p - (t + \Delta t)^s) - (t^p - t^s) \geq 0$ for all $t \in Z^+$. Note that the theorem can be extended to any case of $p, s \in R^+$. One can prove this using the Newton’s generalized Binomial theorem [56,57] in the form of infinite series rather than an infinite sum such as in the above case of $t \in Z^+$. □

3.2. Numerical Examples

To obtain better insight regarding the results presented in the previous section we illustrate the concept above by using two different values for the shape parameters, the relatively small value $\tilde{\beta} = (p = 1.25; q = 1.55; s = 1.85)$ and the relatively large value $\tilde{\beta} = (p = 2.50; q = 2.75; s := 2.80)$. Here p, q , and s are the TFN components which constitute the TFN defined just the same as a, b , and c in Equation (2). The graphs of these TFNs are shown in Figure 2. For the first method, the number of failures for the shape parameters in Figure 2 at $t = 10$ is presented in Figure 3 while Figure 4 (top figures) shows the number of failures for t in $[0,100]$ with 10 steps size. Figure 4 (bottom figures) shows the nonlinearity of the failure numbers as a function of t . Similarly, for the second method, the number of failures for the shape parameters in Figure 2 at $t = 10$ is presented in Figure 5 while Figure 6 shows the number of failures for t in $[0,100]$ with all steps of time. For the finer step size, i.e., 100 steps size, the graph of the number of failures from the second method is presented in Figure 7. Clearly the number of failures in Figure 3 are in triangular forms since the first method assumes that the fuzziness of the shape parameter propagates to the number of failures with the same form of fuzzy number membership, while the number of failures in Figure 5 does not have a triangular form since the fuzziness uncertainty is considered and affects the functional calculation of the number of failures through the α -cut arithmetic. Figure 8 gives the comparisons between these two relatively different shapes. Further, if we plot the numbers of failures over time (see bottom figures in Figure 4), then the curves are non-linear and seem to be “exponentially” increase as expected in the theory. The bottom graphs in Figure 4 actually show the numbers of failures

over time for the end points and core of the shape parameter TFNs. To be exact these figures show the graphs of Weibull’s numbers of failures bands, which analytically is given by Equation (8) and comparable to Equations (10) and (14) for the α -cut, hence it has a power curve. This is consistent with the curve for Weibull’s number of failures with crisp parameters [58]. This is also true for the second method (the α -cut approach), but we do not show the graphs here.

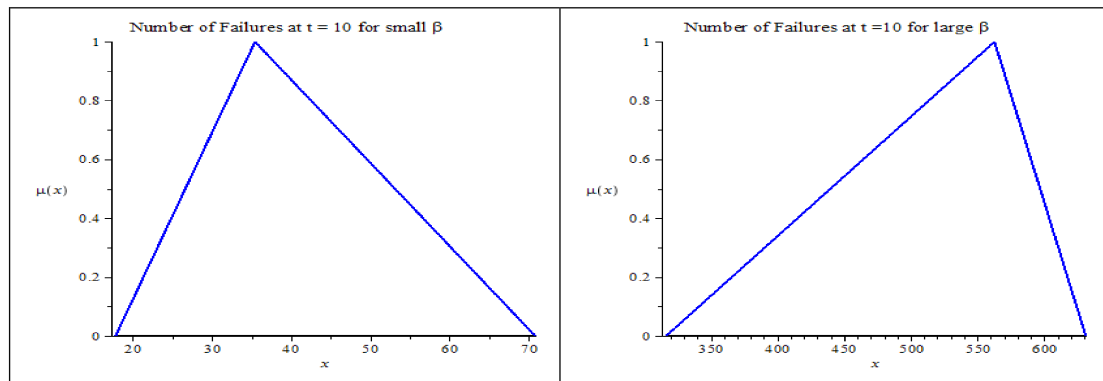


Figure 3. The **left** figure is the number of failures for the shape parameter $\tilde{\beta} = (p = 1.25; q = 1.55; s = 1.85)$ at $t = 10$ —see left figure in Figure 1. The **right** figure is the number of failures for the shape parameter $\tilde{\beta} = (p = 2.50; q = 2.75; s = 2.80)$ at $t = 10$ —see right figure in Figure 2. Note that the vertical axis indicates the fuzzy membership μ .

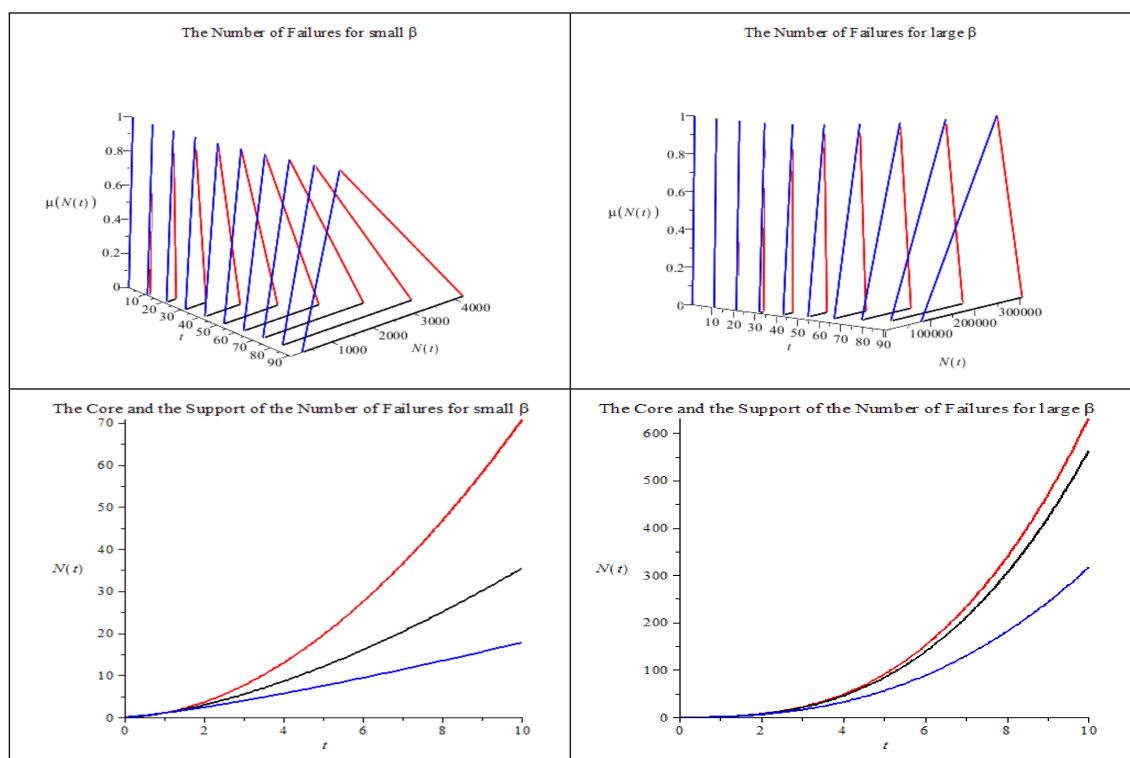


Figure 4. The description is as in Figure 3 above but with $t = 0$ to $t = 100$ and step size of t is 10. The left axis is time, the right axis is the number of failures, and the vertical axis is the fuzzy membership degree of the number of failures (above). The figures in the bottom show the core (black), the lower bound (blue), and the upper bound (red) for the resulting number of failures with small shape parameter (**left**) and large shape parameter (**right**).

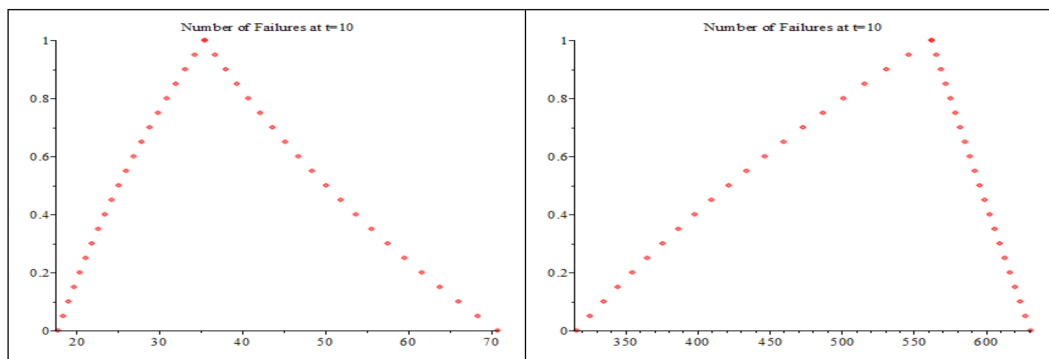


Figure 5. The left figure is the number of failures for the shape parameter $\beta = (p = 1.25; q = 1.55; s = 1.85)$ at $t = 10$. The right figure is the number of failures for the shape parameter $\beta = (p = 2.50; q = 2.75; s = 2.80)$ at $t = 10$. Both figures are generated by the second method with 20 levels of α , i.e., $\alpha_0 = 0$ as the base to $\alpha_{21} = 1$ as the peak.

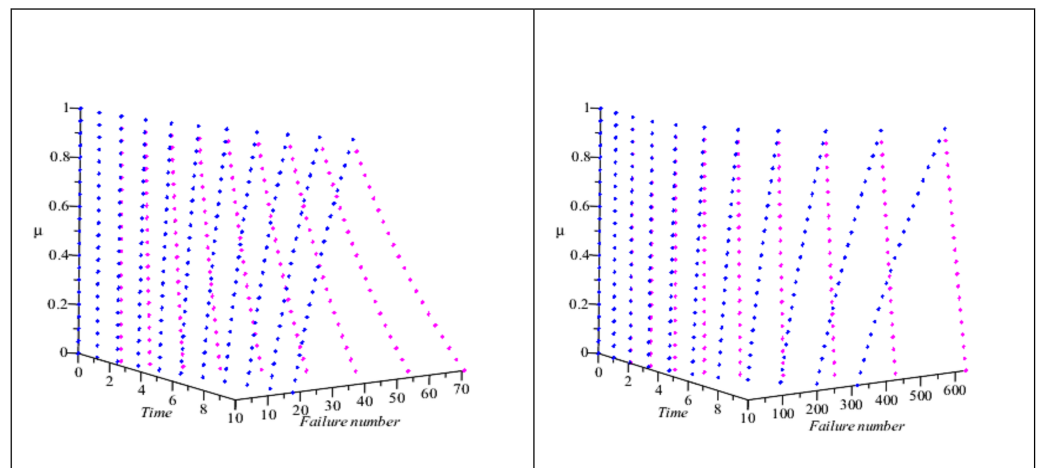


Figure 6. The description is as in Figure 5 above but with complete steps from $t = 0$ to $t = 10$. The left axis is time, the right axis is the number of failures, and the vertical axis is the fuzzy membership degree of the number of failures.

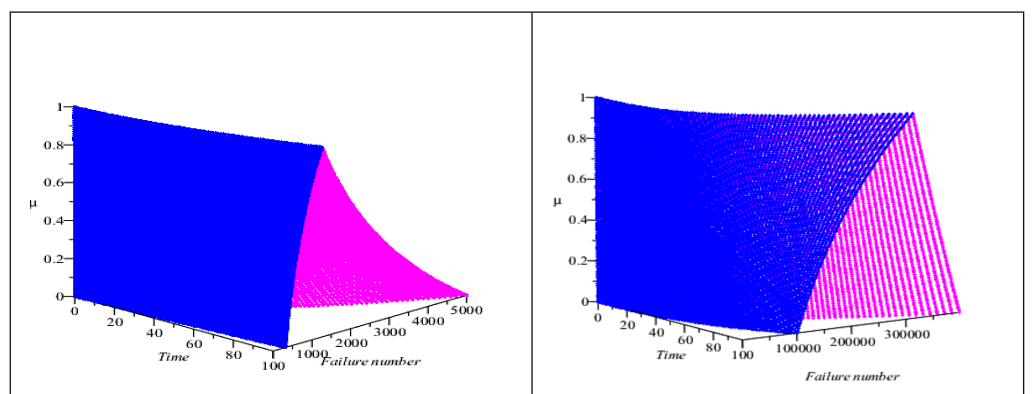


Figure 7. The plots of the number of failures for the shape parameter $\beta = (p = 1.25; q = 1.55; s = 1.85)$ and $\beta = (p = 0.9; q = 1.0; s = 1.5)$ from the second method against time from $t = 0$ to $t = 100$ as in Figure 6 but with a finer step size of t (other parameters are the same as in Figures 5 and 6).

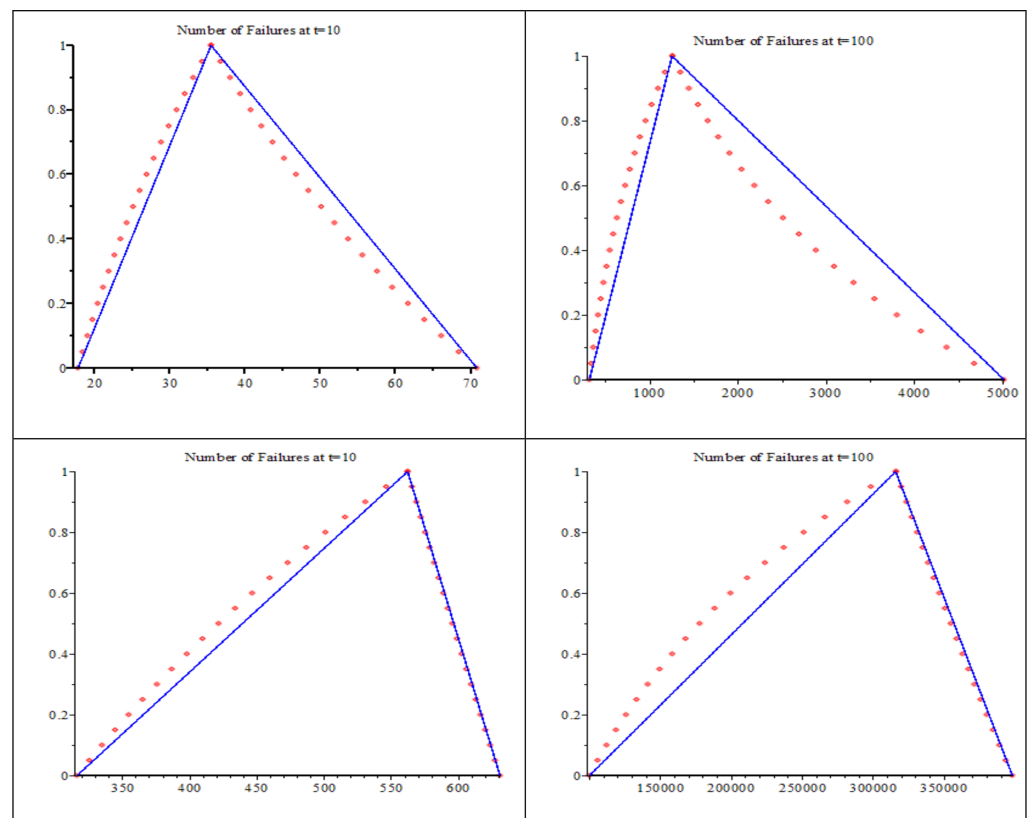


Figure 8. The top and bottom figures are plots of the number of failures for $\tilde{\beta} = (p = 1.25; q = 1.55; s = 1.85)$ and $\tilde{\beta} = (p = 2.50; q = 2.75; s = 2.80)$, respectively, with the left hand side is for $t = 10$ and the right hand side is for $t = 100$.

The figures show that for both values of fuzzy shape parameters $\tilde{\beta}$, the relatively small value $\tilde{\beta} = (p = 1.25; q = 1.55; s = 1.85)$ and the relatively large value $\tilde{\beta} = (p = 2.50; q = 2.75; s = 2.80)$, the length of the fuzziness of the resulting number of failures get bigger as the time t increases. This means the increase of the possibilistic uncertainty of the number of failures. This phenomenon also appears in the α -cut method as is shown in the next section.

3.3. Results from the α -Cut Method

The following results are plotted from the calculation of the number of failures using the α -cut method. Recall the α -cut of the triangular fuzzy number $\tilde{A} = (a; b; c)$ is given by $\tilde{A}_\alpha = [a_1^\alpha, a_2^\alpha] = [(b - a)\alpha + a, (b - c)\alpha + c]$ hence for the fuzzy shape parameter $\tilde{\beta} = (p = 1.25; q = 1.55; s = 1.85)$ we obtain its α -cut is

$$\beta_\alpha = [1.25 + 0.30\alpha, 1.85 - 0.30\alpha], \tag{13}$$

as the fuzzy number of the shape parameter. By considering the α -cut in Equation (7) and substituting it into Equations (5) and (6) using the fuzzy arithmetic give rise to the cumulative distribution

$$g(t)_\alpha = [1 - \exp(-t^{1.25+0.30\alpha}), 1 - \exp(-t^{1.85-0.30\alpha})], \tag{14}$$

and the hazard function

$$h(t)_\alpha = [(1.25 + 0.30\alpha)t^{0.25+0.30\alpha}, (1.85 - 0.30\alpha)t^{0.85-0.30\alpha}], \tag{15}$$

So that by integrating both sides of Equation (9) we end up with the number of failures, which is given by

$$N(t)_\alpha = [t^{1.25+0.30\alpha}, t^{1.85-0.30\alpha}]. \tag{16}$$

When we use the α -cut method, we will have a triangular-like fuzzy number which is comparable (not necessarily the same) to the triangular fuzzy number $(p; q; r)$ defined by:

$$p = \min N(t)_{\alpha=0} = t^{5/4}, \tag{17}$$

$$q = N(t)_{\alpha=1} = t^{31/20}, \tag{18}$$

$$r = \min N(t)_{\alpha=0} = t^{37/20}, \tag{19}$$

We enumerate the fuzzy number of failures in Table 1 based on the calculation of these formulas for $t = 0$ to $t = 10$.

Table 1. Number of failures comparisons for $\beta = (p = 1.25; q = 1.55; s = 1.85)$. Note that for the α -cut method we use $\alpha = 0$ to obtain the support (a, c) and $\alpha = 1$ to find the core b of the resulting fuzzy number so that we have an analogous TFN $(a; b; c)$.

Time t	Crisp Method	Fuzzy Propagation Method		Fuzzy α -Cut Method	
		TFN (p;q;s)	Defuzzification (p + 4q + s)/6	TFN-like (p;q;s)	Defuzzification (p + 4q + s)/6
0	0	0	0	$p = 0$ $q = 0$ $s = 0$	0
1	1	1	1	$p = 1$ $q = 1$ $s = 1$	1
2	2.949350275	$p = 2.378414230$ $q = 2.928171392$ $s = 3.605001850$	2.949350275	$p = 2.378414230$ $q = 2.928171392$ $s = 3.605001850$	2.949350275
3	5.589852442	$p = 3.948222039$ $q = 5.489565165$ $s = 7.632631956$	5.589852442	$p = 3.948222039$ $q = 5.489565165$ $s = 7.632631956$	5.589852442
4	8.824940564	$p = 5.656854248$ $q = 8.574187700$ $s = 12.99603834$	8.824940564	$p = 5.656854248$ $q = 8.574187700$ $s = 12.99603834$	8.824940564
5	12.59725950	$p = 7.476743905$ $q = 12.11723434$ $s = 19.63787576$	12.59725950	$p = 7.476743905$ $q = 12.11723434$ $s = 19.63787576$	12.59725950
6	16.86728508	$p = 9.390507480$ $q = 16.07438767$ $s = 27.51565232$	16.86728508	$p = 9.390507480$ $q = 16.07438767$ $s = 27.51565232$	16.86728508
7	21.60548840	$p = 11.38603593$ $q = 20.41277093$ $s = 36.59581083$	21.60548840	$p = 11.38603593$ $q = 20.41277093$ $s = 36.59581083$	21.60548840
8	26.78864158	$p = 13.45434265$ $q = 25.10669114$ $s = 46.85074227$	26.78864158	$p = 13.45434265$ $q = 25.10669114$ $s = 46.85074227$	26.78864158
9	32.39780510	$p = 15.58845727$ $q = 30.13532570$ $s = 58.25707056$	32.39780510	$p = 15.58845727$ $q = 30.13532570$ $s = 58.25707056$	32.39780510
10	38.41712138	$p = 17.78279410$ $q = 35.48133892$ $s = 70.79457844$	38.41712138	$p = 17.78279410$ $q = 35.48133892$ $s = 70.79457844$	38.41712138

Table 1 also gives the counterpart of the fuzzy number of failure calculated by the first method. Note that in Table 1, *TFN* is a triangle fuzzy number, while *FN* is not necessarily a triangle fuzzy number. However, they both have the same core and the same support but the shapes are different (see Figure 8).

Further, to compare the resulting fuzzy number of failures among the methods, we defuzzified them using the generalized mean value defuzzification (GMVD) which is defined by (4) with $n = 4$. The comparison shows that the defuzzified numbers both from the first method and the second method are exactly the same to the results from the crisp method. Table 2 shows that if n is getting larger, then, the defuzzified number gets closer to the core of the fuzzy number, e.g., for $t = 10$, with $n = 1,000,000$ the defuzzified number is 35.4813565346595 which approaches the core of its fuzzy number, i.e., $q = 35.48133892$. This agrees with Theorem 1. We plot the resulting number of failures for $t = 10$ in Figure 5 and for $t = 0$ to $t = 10$ in Figure 6. The same procedure is done for the relatively large value of the shape parameter $\tilde{\beta} = (p = 2.50; q = 2.75; s = 2.80)$ but the details are not presented here. The plots are presented in the righthand side of Figures 5–7.

Table 2. The illustration of Theorem 1 of the GMVD for non-symmetrical fuzzy number in Figure 5 (left). The fuzzy number is (17.782794100; 35.481338920; 70.794578440).

n	GMVD	n	GMVD
0	44,28868627	6	37,68317576
1	41,35290382	7	37,43852722
2	39,88501260	8	37,24280839
3	39,00427786	9	37,08267480
4	38,41712137	10	36,94923015
5	37,99772388	10,000,000	35,48135653

The time-series plots of the Cumulative Distribution Function, the Hazard Function, and the Number of Failures are presented in Figure 9. The shape parameter on the upper-left of Figure 9 is $\beta = (p = 1.25; q = 1.55; s = 1.85)$ and on the upper-right of Figure 9 is $\beta = (p = 2.50; q = 2.75; s = 2.80)$. The figure shows the plots for a short period of time, up to $t = 1.5$.

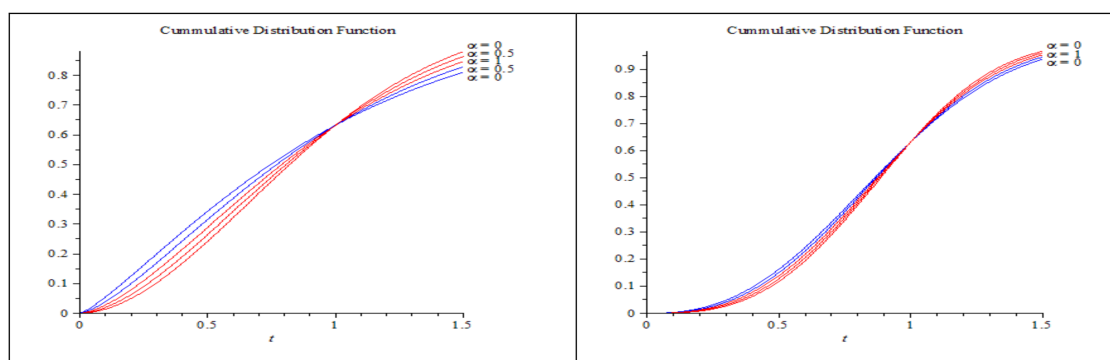


Figure 9. Cont.

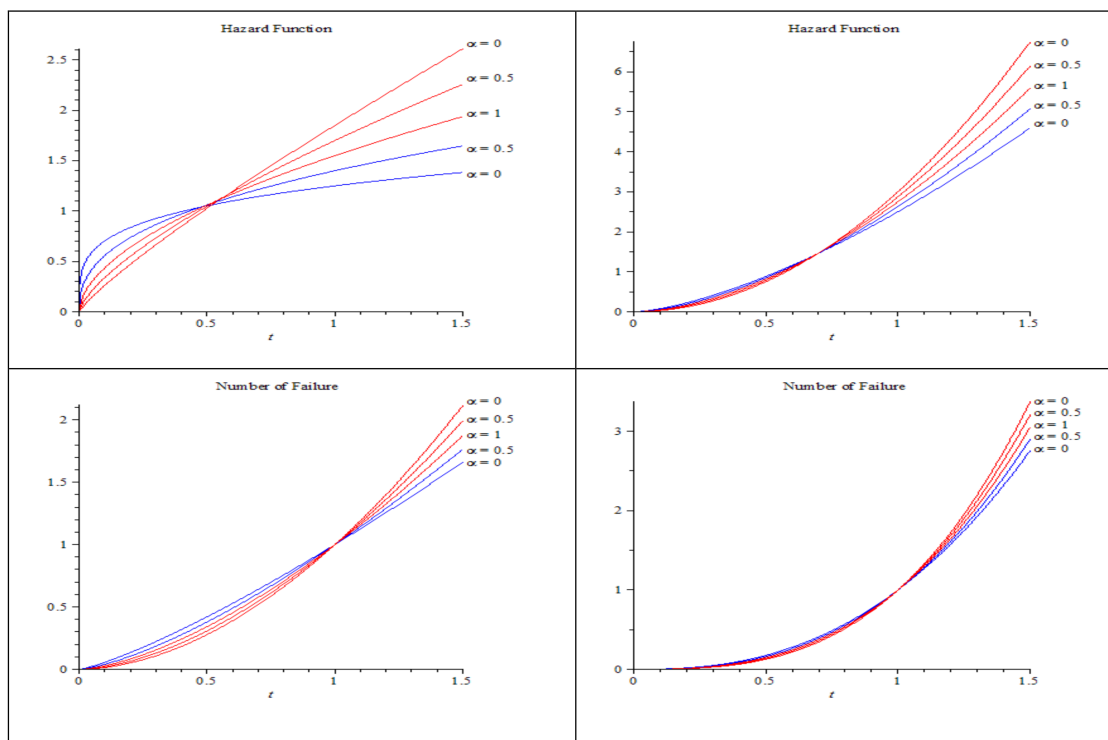


Figure 9. The time-series plots of the Cumulative Distribution Function, the Hazard Function, and the Number of Failures. The shape parameter on **left** figure is $\beta = (p = 1.25; q = 1.55; s = 1.85)$ and on the **right** figure is $\beta = (p = 2.50; q = 2.75; s = 2.80)$.

4. Discussions

The analytical results in Theorems 1 to 3 are illustrated by numerical examples to gain visual understanding regarding the analytical finding above by using $\tilde{\beta} = (p = 1.25; q = 1.55; s = 1.85)$ and $\tilde{\beta} = (p = 2.50; q = 2.75; s := 2.80)$ reflecting a relatively small and a relatively large shape parameter, respectively. Here $p, q,$ and s are the TFN components which constitute the TFN defined just the same as $a, b,$ and c in Equation (1). See Figure 2 for the graphs of these TFNs and Figure 3 for the resulting number of failures at $t = 10$ from the first method and Figure 5 (for the second method). While Figure 4 (top figures) shows the number of failures for t in $[0,100]$ for the first method and Figure 6 for the second method with 10 steps size, (for the finer step size, i.e., 100 steps size see Figure 7). Clearly the number of failures in Figure 3 are in triangular forms due to the assumption in the first method in which the fuzziness of the shape parameter propagates with the same form of fuzzy number membership to the number of failures, while the number of failures in Figure 5 does not have a triangular form since the fuzziness uncertainty is considered and affecting the functional calculation of the number of failures through the α -cut arithmetic. Figure 8 gives the comparisons between these two relatively different shapes. The time-series plots of the cumulative distribution function, the hazard function, and the number of failures are presented in Figure 9. All curves are familiar in shape as it conform to their crisp parameter of Weibull distribution, but here they form twisted-cumulative band, -hazard band, and -number of failures band instead of single curve, respectively.

Furthermore, if we plot the numbers of failures over time (see bottom figures in Figure 4), then the curves are non-linear and seem to “exponentially” increase as expected in the theory. The bottom graphs in Figure 4 actually show the numbers of failures over time for the end points and core of the shape parameter TFNs. To be exact these figures show the graphs of Weibull’s numbers of failures bands, which analytically is given by Equations (12) and (16), hence it has a power curve shape which conforms to the known curve for Weibull’s number of failures with crisp parameters [58]. This is also true for the second method (the α -cut approach), but we do not show the graphs here.

When considering a Weibull distribution with fuzzy shape parameter to calculate the fuzzy number of failures, usually in such imprecise situations, extension principle approach is often used as one choice of calculation though it could lead to a complex form. Here we have proposed a simple method (the first method) to calculate the number of failures, by assuming that the fuzziness of the shape parameter propagates to the number of failures with the same form of fuzzy number membership, and also proposed an alternative method (the second method) which is the calculation done using the α -cut method. This method could be extended to the Weibull distribution with more parameters to enlarge the applicability to other area [59].

5. Conclusions

In this paper we have discussed the Weibull hazard function by assuming a fuzzy shape parameter to calculate the fuzzy number of failures. Here we have proposed a simple method (the first method) to calculate the number of failures, by assuming that the fuzziness of the shape parameter propagates to the number of failures with the same form of fuzzy number membership, and also proposed an alternative method (the second method) which is the calculation done using the α -cut method.

We have shown that both methods have succeeded in computing the number of failures for the system under investigation. Both methods show that when we consider the function of the number of failures as a function of time then the uncertainty (the fuzziness) of the resulting number of failures becomes larger and larger as the time increases. This indicates the propagation of uncertainty in the shape parameter into the resulting number of failures, in which for large values of t , a small value of uncertainty in the shape parameter will produce a large support to the fuzzy number of failures. In practical implication, one should be aware of these properties when using the resulting number of failures as a base for the further process of decision-making.

In this paper we have used a *TFN* for the shape parameter and by using the first method, the resulting number of failures has a *TFN* form. Meanwhile, the resulting number of failures from the second method does not necessarily have a *TFN* form, but a *TFN*-like form. Some comparisons between these two methods are presented using the Generalized Mean Value Defuzzification (*GMVD*) method. The results show that for certain weighting factor of the *GMVD*, the cores of these fuzzy numbers of failures are identical. We did the comparison between the two methods after we use the *GMVD* which produces crisp number of failures. This can be regarded as a shortcoming of the study since once we defuzzify the resulting number of failures we lose the information of the uncertainty. Further study can be done by considering the comparison with a method that preserves the uncertainty.

The *TFN* form and value of the shape parameter used in the Weibull distribution function was taken for granted. For the practical applications this would be not easy. The true form of the fuzzy number should be correctly decided from the available real data and the value should be estimated from the same data. These issues are among the limitations of the methods presented here and could also lead to future direction of research. Other concern is that here we only consider one parameter which has fuzzy value. In reality all of the Weibull parameters could also have imprecise measure or uncertainty. This also will lead to important future venue of research (currently four-parameter Weibull distribution has already around in crisp value application ref). Here we only consider one-parameter Weibull distribution by assuming the scale parameter is assumed to be one. This is sufficient in our context of maintenance modeling if we assume that the average of first failure of the equipment/system under investigation happens within one unit of time. However, in general case this may not be true, so we need to extend the analysis into Weibull distribution having arbitrary values of the scale parameters. Further studies can also be done for different approaches with different forms of fuzzy numbers, different uses of defuzzification methods, and explore the applications of the theory in different related field, such as the number of failures in biological processes (e.g., failure in protecting

healthy status (susceptibility) for people who are infected by COVID-19 disease), which currently we are working on.

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