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Complete CSC Hypersurfaces Satisfying an Okumura-Type Inequality in Ricci Symmetric Manifolds

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Abstract: We investigate the spacelike hypersurface with constant scalar curvature (SCS) immersed in a Ricci symmetric manifold obeying standard curvature constraints. By supposing these hypersurfaces satisfy a suitable Okumura-type inequality recently introduced by Meléndez, which is a weaker hypothesis than to assume that they have two distinct principal curvatures, we obtain a series of umbilicity and pinching results. In particular, when the Ricci symmetric manifold is an Einstein manifold, then we further obtain some rigidity classifications of such hypersufaces.

Keywords: Ricci symmetric manifolds; Einstein manifolds; Okumura-type inequality; constant scalar curvature; spacelike hypersurface

MSC: 53C20; 53C50

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1. Introduction

Let L_1^{n+1} be an $(n+1)$-dimensional Lorentz manifold, i.e., an indefinite Riemannian manifold of index 1. A hypersurface M^n of L_1^{n+1} is said to be spacelike if the induced metric on M^n from that of L_1^{n+1} is positive definite.

The problem of characterizing spacelike hypersurfaces immersed in a Lorentz space form is an important and fruitful topic in the theory of isometric immersions, which is originated from the seminal paper by Calabi in [1] and Cheng-Yau in [2]. As a generalization of their studies, it motivated a great deal of work of several authors to research the problem of hypersurface with constant mean curvature (CMC), such as [3–5], or constant scalar curvature (CSC), such as [6–11]. Meanwhile, some rigidity classifications and pinching results were obtained, by using Omori-Yau's maximum principle for the Laplace operator in [12], or the generalized Omori-Yau's maximum principle for the self-adjoint differential operator introduced by Cheng-Yau in [13], respectively.

The above problems have been studied in the more general spaces, such as in locally symmetric Lorentz manifolds (whose curvature tensors are parallel), which is supposed to obey some appropriate curvature constraints. We recall that, for constants c_1 and c_2, Choi et al. [14,15] introduced the class of $(n+1)$-dimensional Lorentz spaces L_1^{n+1} with the following two additional conditions (here, $\overline{R}(u,v)$ denotes the sectional curvature of L_1^{n+1}):

$$\overline{R}(u,v) = \frac{c_1}{n} \tag{1}$$

for any unit spacelike vector u and timelike vector v and

$$\overline{R}(u,v) \geq c_2 \tag{2}$$

for any unit spacelike vectors u and v.

It should be noted that the locally symmetric Lorentz manifolds satisfying (1) and (2) are generalization of the Lorentz space forms and some non-trivial examples are given in [14–16]. In this setting, many authors work in this type of ambient manifolds and a series of similar results for totally umbilical and pinching results are obtained (see [17–19]), but they could not give the rigidity classification results due to the fact that there are no nice symmetry properties for the ambient manifold.

Motivated by the works described above, our aims, in this paper, are to establish the umbilicity and pinching results by considering hypersurfaces immersed in a Lorentz Ricci symmetric manifold satisfying (1) and (2). Here, we call it *Lorentz Ricci symmetric manifold* if it is a Lorentz space whose Ricci tensors are parallel. Moreover, when the Ricci symmetric manifold is an Einstein manifold, we further give such hypersurfaces some rigidity classifications. In the following, we give a large class of examples of Lorentz Ricci symmetric manifolds satisfying (1) and (2) which are not locally symmetric or space forms. In this sense, it is worth characterizing the spacelike hypersurfaces in such class of ambient manifolds.

2. Models

Example 1. *Let $(\mathbb{R}^k_1, g_0)$ be a Lorentz–Minkowski space and (N^{n+1-k}, g_N) be a Riemannian manifold. We consider the semi-Riemannian direct product manifold*

$$\mathbb{R}^k_1 \times N^{n+1-k}$$

with the metric $\overline{g} = g_0 + g_N$. Then, we claim that this direct product manifold is a Ricci symmetric manifold satisfying (1) and (2) if and only if N^{n+1-k} is a Ricci symmetric manifold with sectional curvature bounded from below. Moreover, $\mathbb{R}^k_1 \times N^{n+1-k}$ is not a locally symmetric manifold if and only if N^{n+1-k} is not locally symmetric.

Proof. In fact, we know $\mathbb{R}^k_1 \times N^{n+1-k}$ is a Ricci symmetric manifold if and only if N^{n+1-k} is a Ricci symmetric manifold.

For any unit vector fields u, v on $\mathbb{R}^k_1$, as in [20], we also denote by u, v the vector fields $(u, 0)$, $(v, 0)$ on $\mathbb{R}^k_1 \times N^{n+1-k}$. Likewise, for any unit vector fields η, ζ on N^{n+1-k}, we also denote vector fields $(0, \eta)$ and $(0, \zeta)$ on $\mathbb{R}^k_1 \times N^{n+1-k}$ by η, ζ. Obviously, u, v are either spacelike or timelike and η, ζ must be spacelike. Then, the sectional curvatures of $\mathbb{R}^k_1 \times N^{n+1-k}$ are given by

$$\overline{R}(u, v) = \overline{R}(u, \eta) = 0, \quad \overline{R}(\eta, \zeta) = R^N(\eta, \zeta), \tag{3}$$

where $R^N(\eta, \zeta)$ is the sectional curvature of N^{n+1-k}; u, v and η, ζ are linear independent respectively. Therefore, from (3), we conclude that (1) always holds and (2) holds if and only if $R^N(\eta, \zeta) \geq c_2$ and $0 \geq c_2$, that is to say, the sectional curvature of N^{n+1-k} is bounded from below.

On the other hand, by (Remark 0.26, [21]), $\mathbb{R}^k_1 \times N^{n+1-k}$ is locally symmetric if and only if both $\mathbb{R}^k_1$ and N^{n+1-k} are locally symmetric manifolds, which confirms our claim. □

Example 2. *Let $(\mathbb{F}^k_1(\frac{c_1}{n}), g_F)$ $(k \neq 1)$ be a Lorentz space form with the constant sectional curvature $\frac{c_1}{n}$ and (N^{n+1-k}, g_N) be a Riemannian manifold. We consider the semi-Riemannian warped product manifold*

$$\mathbb{F}^k_1(\frac{c_1}{n}) \times_f N^{n+1-k}$$

with the metric $\overline{g} = g_F + f^2 g_N$, where $f > 0$ is a smooth function defined on $\mathbb{F}^k_1(\frac{c_1}{n})$. Then, the warped product manifold is an Einstein manifold (with the constant c_1) satisfying (1) and (2) if and only if:

(i) *The Hessian H^f of the function f satisfies*

$$\frac{H^f}{f} = \frac{c_1}{n} g_F, \tag{4}$$

(ii) N^{n+1-k} *is an Einstein manifold with its Ricci tensor satisfying*

$$Ric_N = (n-k)\Big(\frac{c_1}{n}f^2 + g_F(\nabla f, \nabla f)\Big)g_N, \tag{5}$$

and the sectional curvature satisfying

$$R^N(\eta,\zeta) \geq c_2 f^2 + g_F(\nabla f, \nabla f) \tag{6}$$

for any linear independent vector fields η, ζ;

(iii)

$$\frac{c_1}{n} \geq c_2. \tag{7}$$

Proof. Firstly, we give two basic facts. Following the notations in Example 1, the sectional curvatures of $\mathbb{F}_1^k(\frac{c_1}{n}) \times_f N^{n+1-k}$, $k \neq 1$, are given by (see [22], Proposition 4.2)

$$\overline{R}(u,v) = \frac{c_1}{n}, \quad \overline{R}(u,\eta) = \frac{H^f(u,u)}{f g_F(u,u)}, \quad \overline{R}(\eta,\zeta) = \frac{R^N(\eta,\zeta) - g_F(\nabla f, \nabla f)}{f^2}, \tag{8}$$

where u, v on $\mathbb{F}_1^k(\frac{c_1}{n})$ and η, ζ on N^{n+1-k} are linear independent, respectively.

Moreover, based on ([23], Corollary 3), $\mathbb{F}_1^k(\frac{c_1}{n}) \times_f N^{n+1-k}$ is an Einstein manifold with Einstein constant c_1 if and only if:

(a) The Ricci tensor of $\mathbb{F}_1^k(\frac{c_1}{n})$ satisfies

$$Ric_F = c_1 g_F - \frac{n+1-k}{f}H^f,$$

(b) N^{n+1-k} is an Einstein manifold with

$$Ric_N = \mu g_N,$$

where μ is a constant given by

$$\mu = -f\Delta f + (n-k)g_F(\nabla f, \nabla f) + c_1 f^2. \tag{9}$$

Now, we assert that (a) together with (b) are equivalent to (4) and (5). Since $Ric_F = \frac{(k-1)c_1}{n} g_F$, we know that (a) is equivalent to (4). Owing to (a), we get $\Delta f := \mathrm{tr}(H^f) = \frac{kc_1}{n} f$, and hence (b) is equivalent to (5). So, we can say that $\mathbb{F}_1^k(\frac{c_1}{n}) \times_f N^{n+1-k}$ is an Einstein manifold with the constant c_1 if and only if (4) and (5) hold.

Then, we prove its sufficiency and necessity. Due to the obviousness of the sufficiency ($\Leftarrow$), i.e., if (i), (ii) and (iii) hold, $\mathbb{F}_1^k(\frac{c_1}{n}) \times_f N^{n+1-k}$ is an Einstein manifold with the constant c_1 and satisfies (1) and (2), we next prove the necessity ($\Rightarrow$).

The two basic facts above show that, if $\mathbb{F}_1^k(\frac{c_1}{n}) \times_f N^{n+1-k}$ is an Einstein manifold with the constant c_1, then (4) and (5) hold and, using (4), we further know (8) reduces to

$$\overline{R}(u,v) = \overline{R}(u,\eta) = \frac{c_1}{n}, \tag{10}$$

which means (1) is automatic.

On the other hand, since $\mathbb{F}_1^k(\frac{c_1}{n}) \times_f N^{n+1-k}$ satisfies (2), we obtain, from (8) and (10), that

$$\frac{R^N(\eta,\zeta) - g_F(\nabla f, \nabla f)}{f^2} \geq c_2 \text{ and } \frac{c_1}{n} \geq c_2,$$

that is, (6) and (7).

To sum up, $\mathbb{F}_1^k(\frac{c_1}{n}) \times_f N^{n+1-k}$ is an Einstein manifold with the constant c_1 and satisfies (1) and (2) if and only if (i), (ii) and (iii) hold. □

Let us suppose that $k = 1$; then, Example 2 becomes the so-called generalized Robertson-Walker spacetime. Then, we have the following Example 3.

Example 3. *We consider the generalized Robertson–Walker spacetime*

$$I \times_f N^n$$

endowed with metric $\overline{g} = -dt^2 + f^2(t)g_N$, *where* $I \subset \mathbb{R}$ *is an open interval and* $f : I \to \mathbb{R}$ *is a smooth function. Then, the generalized Robertson–Walker spacetime is an Einstein manifold satisfying (1) and (2) if and only if*

(i) N^n *is an Einstein manifold with the constant* ρ^N *and its sectional curvature*

$$R^N(\xi,\eta) \geqslant c_2 f^2 + f'^2, \ \ (c_2 \leqslant \frac{c_1}{n})$$

for any linear independent vectors ξ *and* η*;*

(ii) f *satisfies one of the following items:*

- $f = a_1 t + a_2$, $\rho^N = -(n-1)a_1^2$ *when* $c_1 = 0$;
- $f = a_1 \exp(\sqrt{\frac{c_1}{n}}t) + a_2 \exp(-\sqrt{\frac{c_1}{n}}t)$, $\rho^N = \frac{(n-1)c_1}{n} 4a_1 a_2$ *when* $c_1 > 0$;
- $f = a_1 \sin(\sqrt{-\frac{c_1}{n}}t) + a_2 \cos(\sqrt{-\frac{c_1}{n}}t)$, $\rho^N = \frac{(n-1)c_1}{n}(a_1^2 + a_2^2)$ *when* $c_1 < 0$

for any constants a_1 *and* a_2.

Proof. Firstly, by [24,25], $I \times_f N^n$ is an Einstein manifold with the constant ρ if and only if N^n has constant Ricci curvature ρ^N and f satisfies the differential equations

$$\frac{f''}{f} = \frac{\rho}{n} \text{ and } \frac{\rho(n-1)}{n} = \frac{\rho^N + (n-1)f'^2}{f^2}. \tag{11}$$

On the other hand, following the notations above, the sectional curvatures of the generalized Robertson–Walker spacetime are given by (see [22], Lemma 5.2)

$$\overline{R}(u,\eta) = \frac{f''}{f}, \quad \overline{R}(\xi,\eta) = \frac{R^N(\xi,\eta) - f'^2}{f^2} \tag{12}$$

for any timelike vectors u on I and any spacelike vectors ξ, η on N^n. So, the conditions (1) and (2) hold if and only if

$$\frac{f''}{f} = \frac{c_1}{n} \geqslant c_2 \text{ and } \frac{R^N(\xi,\eta) - f'^2}{f^2} \geqslant c_2. \tag{13}$$

Solving the first equation of (13), we obtain the expression of the function f and, substituting f into the second equation of (11), we obtain the value of the constant ρ^N; thus, together with (11) and (13), we finally confirm our proof. □

For more complicated examples, we can construct other warped product manifolds or twisted product manifolds.

3. Main Theorems

In this section, we only present our characterization results of spacelike hypersurfaces with constant scalar curvature in L_1^{n+1}, then presenting their proofs in Sections 5 and 6.

Before giving our main theorems, we need some basic facts and notations. Let us denote as $\overline{R}_{AB}$ the components of the Ricci tensor of L_1^{n+1} under a suitable local orthonormal frame $\{e_A\}_{A=1}^{n+1}$; using (1), the scalar curvature $\overline{R}$ of L_1^{n+1} is given by

$$\overline{R} = \sum_{A=1}^{n+1} \overline{R}_{AA} = \sum_{i,j=1}^{n} \overline{R}_{ijij} + 2\sum_{i=1}^{n} \overline{R}_{n+1in+1i} = \sum_{i,j=1}^{n} \overline{R}_{ijij} + 2c_1. \tag{14}$$

Since the scalar curvature of a Ricci symmetric manifold is constant, we know from (14) that $\sum_{i,j=1}^{n} \overline{R}_{ijij}$ is also a constant.

Let us consider the spacelike hypersurface M^n of L_1^{n+1}; we may choose e_{n+1} as the normal vector, then the second fundamental form $B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \otimes e_{n+1}$ with its square length $S = |B|^2 = \sum_{i,j} h_{ij}^2$ and the mean curvature $H = \frac{1}{n}\sum_i h_{ii}$. Thus, the Gauss equation of M^n is given by

$$R_{ijkl} = \overline{R}_{ijkl} - (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{15}$$

The components R_{ij} of the Ricci curvature tensor and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ij} = \sum_k \overline{R}_{ikjk} - nHh_{ij} + \sum_k h_{ik}h_{kj},$$

$$n(n-1)R = \sum_{i,k} \overline{R}_{ikik} - n^2H^2 + S. \tag{16}$$

If we assume the normalized scalar curvature R of M^n in L_1^{n+1} is a constant and define

$$P := R - \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij} + c,$$

then P is a constant and (16) becomes

$$n(n-1)P = n(n-1)c - n^2H^2 + S. \tag{17}$$

In particular, if L_1^{n+1} is a Lorentz space form with constant sectional curvature c, then $\sum_{i,j} \overline{R}_{ijij} = n(n-1)c$ and $P = R$; then, (17) is just the Gauss Equation (16).

Let Φ be a symmetric tensor on M^n defined by $\Phi_{ij} = h_{ij} - H\delta_{ij}$ with $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$. It follows, from (17), that

$$|\Phi|^2 = S - nH^2 = n(n-1)(H^2 + P - c). \tag{18}$$

A well-known fact is that $|\Phi|^2 = 0$ if and only if M^n is totally umbilical.

Now, with $c := 2c_2 - \frac{c_1}{n}$, we are in the position to state our main results.

Theorem 1. *Let $M^n (n \geq 3)$ be a complete spacelike hypersurface with constant normalized scalar curvature R in a Ricci symmetric manifold L_1^{n+1} satisfying (1) and (2). Let us suppose that H is bounded on M^n and $c > 0$.*

(i) If $\frac{(n-2)c}{n} \leq P \leq c$, then M^n is totally umbilical and M^n is totally geodesic if and only if $P = c$;

(ii) If $0 < P < \frac{(n-2)c}{n}$, then either $\sup|\Phi|^2 = 0$ and M^n is totally umbilical, or

$$\sup|\Phi|^2 \geq \tau(P, n, c) := \frac{(n-1)P^2}{(n-2)\left(\frac{n-2}{n}c - P\right)}.$$

The equality $\sup|\Phi|^2 = \tau(P,n,c)$ holds and this supremum attains at some point on M^n, if and only if M^n is isoparametric with two distinct constant principle curvatures, one of which is simple.

In particular, if L_1^{n+1} is a (geodesically) complete simply-connected Einstein manifold, then such a totally umbilical (or, totally geodesic) hypersurface in (i) is a sphere $\mathbb{S}^n(R)$ (or, $\mathbb{S}^n(c)$) and such an isoparametric hypersurface in (ii) is a hyperbolic cylinder $\mathbb{H}^1(a) \times \mathbb{S}^{n-1}(b) \to \mathbb{S}_1^{n+1}(c)$, with a, b defined by (57).

Theorem 2. *Let $M^n (n \geq 3)$ be a complete spacelike hypersurface with constant normalized scalar curvature R in a Ricci symmetric manifold L_1^{n+1} satisfying (1) and (2). Let us suppose that H is bounded on M^n, $c > 0$, and*

$$\left|\mathrm{tr}(\Phi^3)\right| \leq \frac{n-2k}{\sqrt{nk(n-k)}}|\Phi|^3 \tag{19}$$

for the integer $2 \leq k < \frac{n}{2}$. $D(n,k,c)$ is a positive constant defined by (32):

(i) If $D(n,k,c) < P \leq c$, then $\sup|\Phi|^2 = 0$ and M^n is totally umbilical;

(ii) If $0 < P \leq D(n,k,c)$, then either $\sup|\Phi|^2 = 0$ and M^n is totally umbilical, or

$$\alpha(P,n,k,c) \leq \sup|\Phi|^2 \leq \beta(P,n,k,c),$$

where $\alpha(P,n,k,c)$ and $\beta(P,n,k,c)$ are two constants defined by (35). The equality $\sup|\Phi|^2 = \alpha(P,n,k,c)$ holds and this supremum attains at some point on M^n, or the equality $|\Phi|^2 = \beta(P,n,k,c)$ holds, if and only if M^n is isoparametric and has exactly two distinct constant principal curvatures, with multiplicities k and $n-k$.

In particular, if L_1^{n+1} is a (geodesically) complete simply-connected Einstein manifold, then such a totally umbilical hypersurface in (i) is a sphere $\mathbb{S}^n(R)$ and such an isoparametric hypersurface in (ii) is a hyperbolic cylinder $\mathbb{H}^k(a) \times \mathbb{S}^{n-k}(b) \to \mathbb{S}_1^{n+1}(c)$, with a, b defined by (47), when $\sup|\Phi|^2 = \alpha(P,n,k,c)$, or a hyperbolic cylinder $\mathbb{H}^k(a) \times \mathbb{S}^{n-k}(b) \to \mathbb{S}_1^{n+1}(c)$, with a, b defined by (48), when $|\Phi|^2 = \beta(P,n,k,c)$.

Theorem 3. *Let $M^{2m} (m \geq 2)$ be a complete spacelike hypersurface with constant normalized scalar curvature R in a Ricci symmetric manifold L_1^{2m+1} satisfying (1) and (2). Let us suppose that H is bounded on M^{2m}, $0 < P \leq c$, $c > 0$ and $\mathrm{tr}(\Phi^3) = 0$; then, M^{2m} is totally umbilical and it is totally geodesic if and only if $P = c$. In particular, if L_1^{n+1} is a (geodesically) complete simply-connected Einstein manifold, then such totally umbilical hypersurface is a sphere $\mathbb{S}^{2m}(R)$ and such totally geodesic hypersurface is a sphere $\mathbb{S}^{2m}(c)$.*

Remark 1. *The Okumura-type inequality (19) in Theorem 2 was introduced by Meléndez in [26]; it is weaker than to assume the spacelike hypersurface has two distinct principal curvatures with multiplicities k and $n-k$.*

Remark 2. *Concerning the integer k in (19), it is originally assumed that $1 \leq k \leq \frac{n}{2}$. By the classical Okumura's lemma ([27], Lemma 2.1), the inequality (19) is automatically true when $k = 1$. So, Theorem 1 is just the case of (19) that holds for $k = 1$ because of $D(n,1,c) = \frac{(n-2)c}{n}$, while Theorem 3, corresponding to the case of (19), is true for $k = \frac{n}{2}$ because of the assumption $tr(\Phi^3) = 0$. Keeping these in mind, we only assume, in Theorem 2, that (19) holds for $2 \leq k < \frac{n}{2}$.*

Remark 3. *Theorems 1–3 greatly generalize the previous case that the ambient manifold is a space form, an Einstein manifold or a locally symmetric manifold. At the same time, they are also the generalization of the case in which the hypersurface has two distinct principal curvatures. See the literature [6,7,9–11,17–19] and references.*

4. Lemmas

Taking an appropriate orthonormal frame $\{e_i\}_{i=1}^n$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, then we cite directly from [19] the following Simons type formula:

$$\begin{aligned}\frac{1}{2}\Delta S =& |\nabla B|^2 + \sum_i \lambda_i (nH)_{ii} + \sum_{i,j} (\lambda_i - \lambda_j)^2 \overline{R}_{ijij} + S^2 - nH \sum_i \lambda_i^3 \\ &- \sum_i \overline{R}_{n+1in+1i}(nH\lambda_i - S) + \sum_{i,k} \big(\lambda_i \overline{R}_{n+1iik;k} + \lambda_i \overline{R}_{n+1kik;i}\big),\end{aligned} \tag{20}$$

where $\overline{R}_{n+1ijk;l}$ is the covariant derivative of $\overline{R}_{n+1ijk}$ on L_1^{n+1}.

Now, following Cheng-Yau [13], we recall the self-adjoint operator acting on any C^2-function f by $\Box f = \sum_{i,j}(nH\delta_{ij} - h_{ij}) f_{ij}$. Taking $f = nH$ on M^n, we have

$$\Box(nH) = \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{ii}.$$

Consequently, combining with (20), we obtain

$$\begin{aligned}\Box(nH) =& |\nabla B|^2 - n^2|\nabla H|^2 + \sum_{i,j} (\lambda_i - \lambda_j)^2 \overline{R}_{ijij} + S^2 - nH \sum_i \lambda_i^3 \\ &- \sum_i \overline{R}_{n+1in+1i}(nH\lambda_i - S) + \sum_{i,k} \big(\lambda_i \overline{R}_{n+1iik;k} + \lambda_i \overline{R}_{n+1kik;i}\big).\end{aligned} \tag{21}$$

By the same idea as [10] or [7], we directly have Lemma 1.

Lemma 1. *Let $M^n (n \geq 3)$ be a spacelike hypersurface with constant normalized scalar curvature in a Ricci symmetric manifold L_1^{n+1} which satisfies (1) and (2). Let us suppose that $P \leq c$; then*

$$|\nabla B|^2 \geq n^2 |\nabla H|^2. \tag{22}$$

Now, we give some key lemmas in order to prove our main results.

Lemma 2. *Let $M^n (n \geq 3)$ be a spacelike hypersurface with constant normalized scalar curvature in a Ricci symmetric manifold L_1^{n+1} satisfying (1) and (2). Let us assume that the inequality (19) holds for the integer $1 \leq k \leq \frac{n}{2}$; then, we have*

$$\Box(nH) \geq |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{n-1}|\Phi|^2 Q_{P,n,k,c}(|\Phi|),$$

where

$$Q_{P,n,k,c}(x) = (n-2)x^2 - \sqrt{\frac{n-1}{k(n-k)}}(n-2k)x\sqrt{x^2 + n(n-1)(c-P)} + n(n-1)P. \tag{23}$$

Proof. Using curvature conditions (1) and (2), we obtain

$$\sum_{i,j} (\lambda_i - \lambda_j)^2 \overline{R}_{ijij} \geq \sum_{i,j} (\lambda_i - \lambda_j)^2 c_2 = 2nc_2(S - nH^2), \tag{24}$$

$$- \sum_i \overline{R}_{n+1in+1i}(nH\lambda_i - S) = \sum_i (nH\lambda_i - S)\frac{c_1}{n} = -c_1(S - nH^2). \tag{25}$$

Since L_1^{n+1} is a Ricci symmetric manifold, then the components of the Ricci tensor satisfy $\overline{R}_{AB;C} \equiv 0$. Based on differential Bianchi identity, we have

$$\begin{aligned}\sum_{i,k} \lambda_i \overline{R}_{n+1iik;k} &= -\sum_{i,k} \lambda_i \left(\overline{R}_{ikik;n+1} + \overline{R}_{kn+1ik;i}\right) \\ &= -\sum_i \lambda_i \left(\overline{R}_{ii;n+1} - \overline{R}_{n+1i;i}\right) = 0\end{aligned} \tag{26}$$

and

$$\sum_{i,k} \lambda_i \overline{R}_{n+1kik;i} = \sum_i \lambda_i \overline{R}_{n+1i;i} = 0, \tag{27}$$

where $\overline{R}_{ijkl;m}$ are the covariant derivatives of $\overline{R}_{ijkl}$ on L_1^{n+1}.

On the other hand, by inequality (19), we have

$$\begin{aligned}S^2 - nH\sum_i \lambda_i^3 &= S^2 - nH\left(\mathrm{tr}(\Phi^3) + 3H|\Phi|^2 + nH^3\right) \\ &\geq |\Phi|^4 - nH^2|\Phi|^2 - n|H|\left|\mathrm{tr}(\Phi^3)\right| \\ &\geq |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2k)}{\sqrt{nk(n-k)}}|H||\Phi| - nH^2\right).\end{aligned} \tag{28}$$

Thus, combining (21), (24)–(28), we obtain

$$\Box(nH) \geq |\nabla B|^2 - n^2|\nabla H|^2 + |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2k)}{\sqrt{nk(n-k)}}|H||\Phi| + n(c - H^2)\right). \tag{29}$$

In addition, from (18), we have

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + c - P. \tag{30}$$

Substituting (30) into (29), Lemma 2 follows. □

Lemma 3. *For any integer k with $2 \leq k < \frac{n}{2}$ and the constant $D(n,k,c)$ defined by (32), the function $Q_{P,n,k,c}(x)$ of x has the following properties:*

(i) If $P > D(n,k,c)$, then $Q_{P,n,k,c}(x) > 0$ for any $x \geq 0$;

(ii) If $0 < P \leq D(n,k,c)$, then:

- *$Q_{P,n,k,c}(x) > 0$, for $x^2 < \alpha(P,n,k,c)$ or $x^2 > \beta(P,n,k,c)$;*
- *$Q_{P,n,k,c}(x) \leq 0$, for $\alpha(P,n,k,c) \leq x^2 \leq \beta(P,n,k,c)$.*

where the constants $\alpha(P,n,k,c)$ and $\beta(P,n,k,c)$ are defined by (35).

Proof. For any $x \geq 0$, let us observe, from (23), that $Q_{P,n,k,c}(x) = 0$ is equivalent to

$$\sqrt{\frac{n-1}{k(n-k)}}(n-2k)x\sqrt{x^2 + n(n-1)(c-P)} = (n-2)x^2 + n(n-1)P. \tag{31}$$

Note that $P > 0$ and $2 \leq k < \frac{n}{2}$, so (31) is equivalent to the following quadratic equation:

$$h(y) := Ay^2 + By + C = 0 \text{ with } y = x^2,$$

where

$$\begin{aligned}A &= \frac{n(k-1)(n-k-1)}{k(n-k)} > 0, \quad C = n(n-1)^2P^2 > 0, \\ B &= \frac{(n-1)^2(n-2k)^2}{k(n-k)}(P-c) + 2(n-1)(n-2)P.\end{aligned}$$

Likewise, we also have that $Q_{P,n,k,c}(x) > 0$ (resp., $Q_{P,n,k,c}(x) < 0$) if and only if $h(y) > 0$ (resp., $h(y) < 0$). Note that $A, C > 0$ and $y \geq 0$; then:

- If $B \geq 0$, or $B < 0$ and $B^2 - 4AC < 0$, i.e., $B > -2\sqrt{AC}$, then $Q_{P,n,k,c}(x)$ has no positive root and $Q_{P,n,k,c}(x) > 0$ for any $x \geq 0$;
- If $B < 0$ and $B^2 - 4AC = 0$, i.e., $B = -2\sqrt{AC}$, then $Q_{P,n,k,c}(x)$ has one positive root and $Q_{P,n,k,c}(x) \geq 0$ for any $x \geq 0$;
- If $B < 0$ and $B^2 - 4AC > 0$, i.e., $B < -2\sqrt{AC}$, then $Q_{P,n,k,c}(x)$ has two distinct positive roots; $Q_{P,n,k,c}(x) > 0$ when x lies outside the two roots and $Q_{P,n,k,c}(x) < 0$ when x lies between the two roots.

Now, we explicitly calculate the solution, denoted by x_0, of $Q_{P,n,k,c}(x) = 0$. By a direct calculation, $B \geq -2\sqrt{AC}$ $\big(\text{resp.}, B < -2\sqrt{AC}\big)$ if and only if

$$P \geq D(n,k,c) \ \ \big(\text{resp.}, P < D(n,k,c)\big),$$

where

$$D(n,k,c) = \frac{(n-1)(n-2k)^2 c}{n(n^2 - 2kn - n + 2k^2) + 2n\sqrt{k(k-1)(n-k)(n-k-1)}}. \tag{32}$$

It is not hard to verify that $D(n,k,c)$ is a strictly decreasing function of k for $1 \leq k \leq \frac{n}{2}$; hence,

$$0 = D(n, \frac{n}{2}, c) \leq D(n,k,c) \leq D(n,1,c) = \frac{n-2}{n}c,$$

i.e., $D(n,k,c)$ is a positive constant and $D(n,k,c) < c$ for $2 \leq k < \frac{n}{2}$.

For the case of $P < D(n,k,c)$, i.e., $B < -2\sqrt{AC}$, the two positive roots of $Q_{P,n,k,c}(x) = 0$ are given by

$$x_0^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{33}$$

It is straightforward to check that $B^2 - 4AC = (n-1)^2(n-2k)^2\Delta$, where Δ is

$$\Delta = (n-1)^2\left(\big(nP - (n-2k)c\big)^2 - \frac{4nk(k-1)Pc}{n-1}\right). \tag{34}$$

So, substituting A, B and C into (33), the squared of the two roots, denoted by $\alpha(P,n,k,c)$ and $\beta(P,n,k,c)$, of the solution $Q_{P,n,k,c}(x) = 0$ are given, respectively, by

$$\begin{aligned} \alpha(P,n,k,c) =& \frac{(n-1)\Big((n-1)(n-2k)^2(P-c) + 2k(n-k)(n-2)P - (n-2k)\sqrt{\Delta}\Big)}{2n(k-1)(n-k-1)}, \\ \beta(P,n,k,c) =& \frac{(n-1)\Big((n-1)(n-2k)^2(P-c) + 2k(n-k)(n-2)P + (n-2k)\sqrt{\Delta}\Big)}{2n(k-1)(n-k-1)}. \end{aligned} \tag{35}$$

For the cases of $P = D(n,k,c)$, i.e., $B = -2\sqrt{AC}$, then $\Delta = 0$ and $\alpha(P,n,k,c) = \beta(P,n,k,c)$, that is to say, $Q_{P,n,k,c}(x)$ has one positive root. To sum up, Lemma 3 follows. □

Lemma 4. *Let $M^n (n \geq 3)$ be a complete spacelike hypersurface with constant normalized scalar curvature in a Ricci symmetric manifold L_1^{n+1} satisfying (1) and (2). Let us suppose that $P \leq c$ and H is bounded on M^n; then, there exists a sequence of points $\{q_\ell\}_{\ell \in \mathbb{N}} \subset M^n$ such that*

$$\lim_{\ell \to \infty} nH(q_\ell) = \sup_M nH, \quad \lim_{\ell \to \infty} |\nabla nH(q_\ell)| = 0 \ \ \textit{and} \ \ \limsup_{\ell \to \infty} \square(nH)(q_\ell) \leq 0.$$

Proof. We observe that, if H vanishes identically on M^n, then the result is valid. So, let us suppose that H is not identically zero. This way, we can choose the orientation of M^n such that $\sup H > 0$.

Let us choose a local orthonormal frame field $\{e_i\}_{i=1}^n$ such that $h_{ij} = \lambda_i \delta_{ij}$. Since $P \leq c$, it follows, from (17), that $\lambda_i^2 \leq S \leq n^2 H^2$, which shows that

$$0 \leq n|H| - |\lambda_i|. \tag{36}$$

Taking together (36) with (2) and (15) leads to $R_{ijij} \geq c_2 - \lambda_i \lambda_j \geq c_2 - n^2 H^2$, i.e., the sectional curvatures of M^n are bounded from below because H is bounded. Thus, we may apply Omori's maximum principle [28] to the function nH and obtain a sequence of points $\{q_\ell\} \subset M^n$ such that

$$\lim_{\ell\to\infty} nH(q_\ell) = \sup nH, \quad \lim_{\ell\to\infty} \left|\nabla nH(q_\ell)\right| = 0 \text{ and } \limsup_{\ell\to\infty} nH_{ii}(q_\ell) \leq 0. \tag{37}$$

Since $\sup H > 0$, taking subsequences if necessary, we can arrive to a sequence $\{q_\ell\} \subset M^n$ which satisfies (37) and such that $H(q_\ell) \geq 0$. Then, from (36), we obtain

$$0 \leq nH(q_\ell) - |\lambda_i(q_\ell)| \leq nH(q_\ell) + |\lambda_i(q_\ell)| \leq 2nH(q_\ell). \tag{38}$$

Note that H is bounded; hence, $\{nH(q_\ell) - \lambda_i(q_\ell)\}$ is a non-negative and bounded sequence. Evaluating $\square(nH)$ at q_ℓ, taking the limit and using (37) and (38), we have

$$\limsup_{\ell\to\infty} \left(\square(nH)(q_\ell)\right) \leq \sum_i \limsup_{\ell\to\infty} \left((nH - \lambda_i)(q_\ell) nH_{ii}(q_\ell)\right) \leq 0.$$

This completes the proof of Lemma 4. □

5. Proof of Theorem 2

Proof of Theorem 2. Since the constant $P \leq c$ for any $2 \leq k < \frac{n}{2}$, by the inequality (19) and Lemmas 1 and 2, we have

$$\square(nH) \geq \frac{1}{n-1} |\Phi|^2 Q_{P,n,k,c}(|\Phi|). \tag{39}$$

Using Lemma 4, there exists a sequence of points $\{q_\ell\}$; evaluating (39) at this sequence, we obtain

$$0 \geq \limsup_{\ell\to\infty} \square(nH(q_\ell)) \geq \frac{1}{n-1} \sup |\Phi|^2 Q_{P,n,k,c}(\sup |\Phi|). \tag{40}$$

Now, by considering the range of the constant P, we prove Theorem 2 in two cases.

(i) Let us suppose that $D(n,k,c) < P \leq c$; then, from Lemma 3, $Q_{P,n,k,c}(\sup |\Phi|) > 0$. Hence, by (40), we obtain $\sup |\Phi|^2 = 0$ and M^n is totally umbilical.

In particular, if L_1^{n+1} is an Einstein manifold, then (30) indicates that H is also a constant; hence, (39) becomes

$$0 = \square(nH) \geq \frac{1}{n-1} |\Phi|^2 Q_{P,n,k,c}(|\Phi|) \geq 0. \tag{41}$$

Therefore, the inequalities in (41) hold for equalities, that is to say, all the inequalities that we have obtained are, in fact, equalities, as well as the curvature condition (2). As a result, (1) and (2) indicate that the Ricci curvature of L_1^{n+1} is

$$\overline{\text{Ric}}(e_j) = \sum_i \overline{R}(e_j, e_i) + \overline{R}(e_j, e_{n+1}) = (n-1)c_2 + \frac{c_1}{n},$$
$$\overline{\text{Ric}}(e_{n+1}) = \sum_i \overline{R}(e_{n+1}, e_i) = c_1.$$

Therefore, we have $c_2 = \frac{c_1}{n}$, because of L_1^{n+1} being an Einstein manifold, and, by the hypothesis of geodesic completeness and connectivity, the ambient space L_1^{n+1} must be the de Sitter space $\mathbb{S}_1^{n+1}(c)$. Thus, by (15) and (30), we know this totally umbilical hypersurface must be a sphere $\mathbb{S}^n(R)$.

(ii) When $0 < P \leq D(n,k,c)$, it follows, from Lemma 3 and (40), that either $\sup|\Phi|^2 = 0$ and M^n is totally umbilical, or $Q_{P,n,k,c}(\sup|\Phi|) \leq 0$ with

$$\alpha(P,n,k,c) \leq \sup|\Phi|^2 \leq \beta(P,n,k,c).$$

- If the equality $\sup|\Phi|^2 = \alpha(P,n,k,c)$ holds, then $|\Phi|^2 \leq \alpha(P,n,k,c)$. Using Lemma 3, we have $Q_{P,n,k,c}(|\Phi|) \geq 0$. Inserting this into (39) yields $\square(nH) \geq 0$ on M^n.

Moreover, since $P \leq D(n,k,c) < c$, then (17) gives $n^2H^2 > S$, which means $H \neq 0$, by choosing an appropriate orientation such that $H > 0$ on M^n, so we have $nH - \lambda_i > 0$; hence, the operator $\square$ is elliptic.

By means of (30), the assumption that $\sup|\Phi|^2$ attains at some points on M^n assures that $\sup H^2$ also attains at some points on M^n. Thus, based on the strong maximum principle, H is a constant. Moreover, (41) becomes trivially an equality, which means all the inequalities we have obtained become equalities; hence, (22) must be also an equality or, equivalently, $|\nabla B|^2 = n^2|\nabla H|^2 = 0$, that is, M^n is an isoparametric hypersurface. In addition, (41) assures that the equality in (28) holds, which implies, by (19) and J. Meléndez ([26], Lemma 2.2), that the hypersurface has exactly two distinct constant principal curvatures, with multiplicities k and $n-k$.

- If the equality $|\Phi|^2 = \beta(P,n,k,c)$ holds, then $Q_{P,n,k,c}(|\Phi|) = 0$ and (41) becomes trivially an equality, by a similar way as above; M^n is an isoparametric hypersurface of two distinct constant principal curvatures with multiplicities k and $n-k$.

In the following, we classify the isoparametric hypersurface mentioned above which satisfies $\sup|\Phi|^2 = \alpha(P,n,k,c)$ or $|\Phi|^2 = \beta(P,n,k,c)$ under the assumption of L_1^{n+1} being a geodesically complete simply-connected Einstein manifold. Since we have proved that (41) becomes trivially an equality in this setting, similar to (i), we know $L_1^{n+1} = \mathbb{S}_1^{n+1}(c)$. By a classical congruence theorem (in [29]), we conclude that M^n must be isometric to a standard product $\mathbb{H}^k(a) \times \mathbb{S}^{n-k}(b) \hookrightarrow \mathbb{S}_1^{n+1}(c)$, where $2 \leq k < \frac{n}{2}$ or $\frac{n}{2} < k \leq n-2$, $a < 0$, $b > 0$ and $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. Let us denote its principal curvatures by

$$\lambda_1 = \cdots = \lambda_k = \sqrt{c-a} \text{ and } \lambda_{k+1} = \cdots = \lambda_n = \sqrt{c-b}. \tag{42}$$

Let $\lambda = \sqrt{c-a}$ and $\mu = \sqrt{c-b}$; then, together with $c = \lambda\mu$, by (15),

$$H = \frac{(n-k)\mu^2 + kc}{n\mu}, \quad |\Phi|^2 = \frac{k(n-k)}{n}\left(\frac{c}{\mu} - \mu\right)^2 \tag{43}$$

and

$$P = c - \frac{2k(n-k)}{n(n-1)}c - \frac{k(k-1)c^2}{n(n-1)\mu^2} - \frac{(n-k)(n-k-1)}{n(n-1)}\mu^2. \tag{44}$$

Since $0 < \mu^2 < c$ because of (42); hence, solving the Equation (44), we obtain

$$\begin{aligned} &0 < P < D(n,k,c), \quad \text{when } 2 \leq k < \frac{n}{2}; \\ &-\infty < P < c, \qquad \text{when } \frac{n}{2} < k \leq n-2, \end{aligned}$$

where $D(n,k,c)$ is given by (32). Together with the range of P in Theorem 2, we finally obtain $2 \leq k < \frac{n}{2}$ and M^n is a product $\mathbb{H}^k(a) \times \mathbb{S}^{n-k}(b)$ with $2 \leq k < \frac{n}{2}$.

Now, we give the values of the constants a, b. For any integer $2 \leq k < \frac{n}{2}$, solving the Equation (44), we have

$$\mu^2 = \frac{n(n-1)(c-P) - 2k(n-k)c + \sqrt{\Delta}}{2(n-k)(n-k-1)}, \tag{45}$$

or,

$$\mu^2 = \frac{n(n-1)(c-P) - 2k(n-k)c - \sqrt{\Delta}}{2(n-k)(n-k-1)}, \tag{46}$$

where Δ is given by (34).

Substituting (45) and (46) into the second equation of (43) and comparing with (35), we obtain, respectively,

$$|\Phi|^2 = \alpha(P,n,k,c), \text{ or } |\Phi|^2 = \beta(P,n,k,c).$$

Thus, $\sup|\Phi|^2 = \alpha(P,n,k,c)$ or $|\Phi|^2 = \beta(P,n,k,c)$ holds if and only if (45) or (46) holds. Solving (45) and together with $c = \lambda\mu$ and $\mu^2 = c - b$, we obtain

$$\begin{aligned} b &= \frac{(n-1)\big((n-2k)c+nP\big) - \sqrt{\Delta}}{2(n-k)(n-k-1)}, \\ a &= \frac{(n-1)\big(nP-(n-2k)c\big) + \sqrt{\Delta}}{2k(k-1)}. \end{aligned} \tag{47}$$

Similarly, it follows, from (46), that

$$\begin{aligned} b &= \frac{(n-1)\big((n-2k)c+nP\big) + \sqrt{\Delta}}{2(n-k)(n-k-1)}, \\ a &= \frac{(n-1)\big(nP-(n-2k)c\big) - \sqrt{\Delta}}{2k(k-1)}. \end{aligned} \tag{48}$$

Therefore, we obtain that M^n is isometric to $\mathbb{H}^k(a) \times \mathbb{S}^{n-k}(b)$, $2 \le k < \frac{n}{2}$, when the equality $\sup|\Phi|^2 = \alpha(P,n,k,c)$ holds with a, b defined by (47), or the equality $|\Phi|^2 = \beta(P,n,k,c)$ holds with a, b defined by (48). We complete the proof of Theorem 2. □

6. Proofs of Theorems 1 and 3

Proof of Theorem 1. By the classical algebraic inequality due to M. Okumura in ([27], Lemma 2.1), (19) holds automatically for $k = 1$. Then, Lemmas 1 and 2 imply that

$$\square(nH) \ge \frac{1}{n-1}|\Phi|^2 Q_{P,n,1,c}(|\Phi|), \tag{49}$$

where

$$Q_{P,n,1,c}(x) = (n-2)x^2 - (n-2)x\sqrt{x^2 + n(n-1)(c-P)} + n(n-1)P.$$

It is easy to see that $Q_{P,n,1,c}(x)$ is decreasing for any $x \ge 0$, and $Q_{P,n,1,c}(0) = n(n-1)P > 0$.

(i) Let us suppose that $\frac{(n-2)c}{n} \le P \le c$; then, we claim that $Q_{P,n,1,c}(x) > 0$ for every $x \ge 0$. Indeed, if there exists a point x_0 such that $Q_{P,n,1,c}(x_0) = 0$, a straightforward computation gives

$$x_0^2 = \frac{(n-1)P^2}{(n-2)\big(\frac{n-2}{n}c - P\big)}, \tag{50}$$

which indicates that $P < \frac{(n-2)c}{n}$, a contradiction. By the continuity of the function $Q_{P,n,1,c}(x)$, the claim is true.

Using Lemma 4, there exists a sequence of points $\{q_\ell\} \subset M^n$; evaluating (49) at the sequence $\{q_\ell\}$, we obtain

$$0 \ge \limsup_{\ell\to\infty} \square(nH)(q_\ell) \ge \frac{1}{n-1}\sup|\Phi|^2 Q_{P,n,1,c}(\sup|\Phi|). \tag{51}$$

So, we immediately conclude that $\sup|\Phi|^2 = 0$ and M^n is totally umbilical. Moreover, we further obtain that H is constant because of (30); further, (49) must be

$$0 = \square(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_{P,n,1,c}(|\Phi|) \geq 0. \tag{52}$$

That is to say that (52) becomes trivially an equality. So, when L_1^{n+1} is a geodesically complete simply-connected Einstein manifold, with the same discussion as the proof of Theorem 2, L_1^{n+1} must be the de Sitter space $\mathbb{S}_1^{n+1}(c)$. By (15) and (30), we know such totally umbilical hypersurface must be the sphere $\mathbb{S}^n(R)$.

(ii) Let us suppose that $0 < P < \frac{(n-2)c}{n}$; then, $Q_{P,n,1,c}(x) = 0$ has one positive root given by (50). From (51), we obtain that either $\sup|\Phi|^2 = 0$ and M^n is totally umbilical, or

$$\sup|\Phi|^2 \leq \tau(P,n,c) := \frac{(n-1)P^2}{(n-2)\left(\frac{n-2}{n}c - P\right)}.$$

Let us consider the case in which the equality $\sup|\Phi|^2 = \tau(P,n,c)$ holds; then, $|\Phi|^2 \leq \tau(P,n,c)$ and $Q_{P,n,1,c}(|\Phi|) \geq 0$ on M^n. Since $\sup|\Phi|^2$ attains at some points on M^n, so does $\sup H$ because of (30). Besides, $P < c$ guarantees that $\square$ is elliptic and, by the strong maximum principle, H is a constant. Thus, (52) becomes trivially an equality and M^n is an isoparametric hypersurface. In addition, the inequality in (28) holds for equality. By Lemma 2.1 in [27], we conclude that M^n is an isoparametric hypersurface with two distinct constant principal curvatures, one of which is simple.

In particular, if L_1^{n+1} is an Einstein manifold with geodesic completeness and simplified connectivity, then we further give such isoparametric hypersurface a rigidity classification. Under the assumption $0 < P < \frac{(n-2)c}{n}$ and based on [29], M^n must be isometric to $\mathbb{H}^k(a) \times \mathbb{S}^{n-k}(b) \hookrightarrow \mathbb{S}_1^{n+1}(c)$, where $k \in \{1, n-1\}$, $a < 0$, $b > 0$ and $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$; its principal curvatures are given by

$$\lambda = \sqrt{c-a} \quad \text{and} \quad \mu = \sqrt{c-b} \tag{53}$$

with multiplicities k and $n-k$, respectively. So, together with $c = \lambda\mu$, H, $|\Phi|^2$ and P are given, respectively, by

$$H = \frac{(n-k)\mu^2 + kc}{n\mu}, \quad |\Phi|^2 = \frac{n-1}{n}\left(\frac{c}{\mu} - \mu\right)^2 \tag{54}$$

and

$$P = \frac{(n-2)c}{n} - \frac{1}{n(n-1)}\left(\frac{k(k-1)c^2}{\mu^2} + (n-k)(n-k-1)\mu^2\right). \tag{55}$$

Here, the last two equations hold because of the equality $k(n-k) = n-1$ with $k \in \{1, n-1\}$.

Since $0 < \mu^2 < c$, from (55), we have

$$P = \begin{cases} \frac{n-2}{n}\left(c - \mu^2\right) \in \left(0, \frac{n-2}{n}c\right), & \text{when } k = 1, \\ \frac{n-2}{n}\left(c - \frac{c^2}{\mu^2}\right) \in \left(-\infty, 0\right), & \text{when } k = n-1. \end{cases} \tag{56}$$

Hence, we conclude that $k = 1$ because of $0 < P < \frac{n-2}{n}c$ and M^n is a hyperbolic cylinder $\mathbb{H}^1(a) \times \mathbb{S}^{n-1}(b)$. It is easy to check $|\Phi|^2 = \tau(P,n,c)$ when substituting (56) into the second equation of (54).

Together with (53), (56) and $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, we finally obtain

$$b = \frac{nP}{n-2}, \quad a = \frac{ncP}{nP - (n-2)c}, \tag{57}$$

and complete the proof of Theorem 1. □

Proof of Theorem 3. Since $\mathrm{tr}(\Phi^3) = 0$, i.e., (19) holds for $k = \frac{n}{2}$, then Lemma 2, together with Lemma 1, implies that

$$\square(nH) \geq \frac{1}{n-1}|\Phi|^2\Big((n-2)|\Phi|^2 + n(n-1)P\Big).$$

Using Lemma 4 and following the proof of Theorem 2, M^n is a totally umbilical hypersurface and, by (18), it is totally geodesic if and only if $P = c$. In particular, if L_1^{n+1} is a geodesically complete simply-connected Einstein manifold, applying the same process as in the proof of Theorem 1 or Theorem 2, we obtain that such totally umbilical hypersurface must be a sphere $\mathbb{S}^n(R)$ and it is a totally geodesic sphere $\mathbb{S}^n(c)$ if and only if $P = c$. This completes the proof of Theorem 3. □

7. Conclusions

In this paper, we investigate the spacelike hypersurface immersed in Lorentz manifolds. One often solves this problem by using the Bochner technique combined with the maximum principle. Here, with some appropriate skills, we extend the ambient manifold to a more generalized Ricci symmetric manifold; then, we obtain some rigidity classifications when the ambient manifold is an Einstein manifold. These skills are also applicable to (spacelike) submanifolds in (pseudo) Riemannian manifolds, which means that many results of the isometric immersion theory of submanifolds can be generalized.

Meanwhile, we give several non-trivial examples in order to prove the existence of the Ricci symmetric manifolds satisfying the curvature conditions (1) and (2). The Okumura-type inequality (19) introduced in [26] also implies the case of hypersurfaces with two distinct principal curvatures. However, we were not able to point out whether this inequality has a certain geometric significance.

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References

1. Calabi, E. Examples of Bernstein problems for some nonlinear equations. *Proc. Sympos. Pure Math.* **1970**, *15*, 223–230.
2. Cheng, S.Y.; Yau, S.T. Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces. *Ann. Math.* **1976**, *104*, 407–419. [CrossRef]
3. Akutagawa, K. On spacelike hypersurfaces with constant mean curvature in the de Sitter space. *Math. Z.* **1987**, *196*, 13–19. [CrossRef]
4. Goddard, A.J. Some remarks on the existence of spacelike hypersurfaces of constant mean curvature. *Math. Proc. Camb. Philos. Soc.* **1977**, *82*, 489–495. [CrossRef]
5. Montiel, S. An integral inequality for compact spacelike hypersurfaces in the de Sitter space and applications to the case of constant mean curvature. *Indiana Univ. Math. J.* **1988**, *37*, 909–917. [CrossRef]
6. Brasil, A., Jr.; Colares, A.G.; Palmas, O. A gap theorem for complete constant scalar curvature hypersurfaces in the de Sitter space. *J. Geom. Phys.* **2001**, *37*, 237–250. [CrossRef]

7. Camargo, F.E.C.; Chaves, R.M.B.; Sousa, L.A.M., Jr. Rigidity theorems for complete spacelike hypersurfaces with constant scalar curvature in de Sitter space. *Differ. Geom. Its Appl.* **2008**, *26*, 592–599. [CrossRef]
8. Caminha, A. A rigidity theorem for complete CMC hypersurfaces in Lorentz manifolds. *Differ. Geom. Its Appl.* **2006**, *24*, 652–659. [CrossRef]
9. Cheng, Q.M.; Ishikawa, S. Spacelike hypersurfaces with constant scalar curvature. *Manuscripta Math.* **1998**, *95*, 499–505. [CrossRef]
10. Li, H. Global rigidity theorems of hypersurfaces. *Ark. Mat.* **1997**, *35*, 327–351. [CrossRef]
11. Zheng, Y. Spacelike hypersurfaces with constant scalar curvature in the de Sitter spaces. *Differ. Geom. Its Appl.* **1996**, *6*, 51–54.
12. Yau, S.T. Harmonic functions on complete Riemannian manifolds. *Commun. Pure Appl. Math.* **1975**, *28*, 201–228. [CrossRef]
13. Cheng, S.Y.; Yau, S.T. Hypersurfaces with constant scalar curvature. *Math. Ann.* **1977**, *225*, 195–204. [CrossRef]
14. Choi, S.M.; Kwon, J.H.; Suh, Y.J. Complete spacelike hypersurfaces in a Lorentz manifold. *Math. J. Toyama Univ.* **1999**, *22*, 53–76.
15. Suh, Y.J.; Choi, Y.S.; Yang, H.Y. On spacelike hypersurfaces with constant mean curvature in Lorentz manifold. *Houst. J. Math.* **2002**, *28*, 47–70.
16. Baek, J.O.; Cheng, Q.M.; Suh, Y.J. Complete space-like hypersurfaces in locally symmetric Lorentz spaces. *J. Geom. Phys.* **2004**, *49*, 231–247. [CrossRef]
17. Liu, J.; Sun, Z. On spacelike hypersurfaces with constant scalar curvature in locally symmetric Lorentz spaces. *J. Math. Anal. Appl.* **2010**, *364*, 195–230. [CrossRef]
18. Wang, Y.; Liu, X. Compact space-like hypersurfaces with constant scalar curvature in locally symmetric Lorentz spaces. *Arch. Math.* **2012**, *48*, 163–172. [CrossRef]
19. Zhang, S.; Wu, B. Rigidity theorems for complete spacelike hypersurfaces with constant scalar curvature in locally symmetric Lorentz spaces. *J. Geom. Phys.* **2010**, *60*, 333–340. [CrossRef]
20. Tanno, S. A class of Riemannian manifolds satisfying $R(X,Y)\cdot R = 0$. *Nagoya Math. J.* **1971**, *42*, 67–77. [CrossRef]
21. Kowalski, O. *Generalized Symmetric Spaces (Lecture Notes in Mathematics, No. 805)*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1980.
22. Chen, B.-Y. *Pseudo-Riemannian Geometry, δ-Invariants and Applications*; World Scientic: Hackensack, NJ, USA, 2011.
23. Kim, D.-S.; Kim, Y.H. Compact Einstein warped product spaces with nonpositive scalar curvature. *Proc. Am. Math. Soc.* **2003**, *131*, 2573–2576. [CrossRef]
24. El-Sayied, H.K.; Shenawy, S.; Syied, N. On symmetries of generalized Robertson-Walker spacetimes and applications. *J. Dyn. Syst. Geom. Theor.* **2017**, *15*, 51–69. [CrossRef]
25. Alías, L.J.; Romero, A.; Sánchez, M. Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems. *Tohôku Math. J.* **1997**, *49*, 337–345. [CrossRef]
26. Meléndez, J. Rigidity theorems for hypersurfaces with constant mean curvature. *Bull. Braz. Math. Soc.* **2014**, *45*, 385–404. [CrossRef]
27. Okumura, M. Hypersurfaces and a pinching problem on the second fundamental tensor. *Am. J. Math.* **1974**, *96*, 207–213. [CrossRef]
28. Omori, H. Isometric immersions of Riemannian manifolds. *J. Math. Soc. Jpn.* **1967**, *19*, 205–214. [CrossRef]
29. Abe, K.; Koike, N.; Yamaguchi, S. Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form. *Yokohama Math. J.* **1987**, *35*, 123–136.