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Exact Solutions and Conservation Laws of a Generalized (1 + 1) Dimensional System of Equations via Symbolic Computation

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Abstract: The aim of this paper is to compute the exact solutions and conservation of a generalized (1 + 1) dimensional system. This can be achieved by employing symbolic manipulation software such as Maple, Mathematica, or MATLAB. In theoretical physics and in many scientific applications, the mentioned system naturally arises. Time, space, and scaling transformation symmetries lead to novel similarity reductions and new exact solutions. The solutions obtained include solitary waves and cnoidal and snoidal waves. The familiarity of closed-form solutions of nonlinear ordinary and partial differential equations enables numerical solvers and supports stability analysis. Although many efforts have been dedicated to solving nonlinear evolution equations, there is no unified method. To the best of our knowledge, this is the first time that Lie point symmetry analysis in conjunction with an ansatz method has been applied on this underlying equation. It should also be noted that the methods applied in this paper give a unique solution set that differs from the newly reported solutions. In addition, we derive the conservation laws of the underlying system. It is also worth mentioning that this is the first time that the conservation laws for the equation under study are derived.

Keywords: auxiliary equations; associated solutions; Lie symmetry analysis; conservation laws

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1. Introduction

The illustrious equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

is an example of a nonlinear evolution equation (NLEE) [1,2]. The term u_t describes the time evolution of the wave [3,4] and, as such, (1) is considered an evolution equation. The nonlinear term uu_x accounts for the steepening of the wave [5,6], whereas the linear dispersive term u_{xxx} describes the spreading of the wave [7,8]. Moreover, this essential equation illustrates the subtleties of solitary waves. Primarily, it was developed to portray shallow water waves of long wavelength and small amplitude. Equation (1) is a significant equation in the theory of integrable systems since it has an infinite number of conservation laws, multiple-soliton solutions, and many other physical properties [1].

The modified KdV (mKdV) equation is similar

$$u_t + 6u^2u_x + u_{xxx} = 0 \tag{2}$$

to the KdV equation in the sense that both of them are completely integrable, and each has many conserved quantities. The mKdV equation arises in electric circuits and multi-component plasmas. The mKdV equation yields algebraic soliton solutions in the form of rational functions. The stability and instability circumstances of algebraic solitons of the mKdV equation have been investigated extensively.

Numerous variations of the above equations [9–12] have been introduced by many scientists, especially coupled systems of the above equations. Among these variations is the coupled KdV-mKdV system [12]:

$$u_t + u_{xxx} - 6uu_x + 3vv_{xxx} + 3v_x v_{xx} - 3u_x v^2 + 6uvv_x = 0, \tag{3a}$$
$$v_t + v_{xxx} - 3v^2 v_x - 3uv_x + 3vu_x = 0. \tag{3b}$$

The aim of this paper is to compute the exact solutions and conservation of a generalized (1 + 1) dimensional system:

$$u_t + u_{xxx} + auu_x + bvv_{xxx} + bv_x v_{xx} - bu_x v^2 + auvv_x = 0, \tag{4a}$$
$$v_t + v_{xxx} - bv^2 v_x - buv_x - bvu_x = 0, \tag{4b}$$

where a and b are arbitrary constants. The motivation for computing closed-form solutions of nonlinear ordinary and partial differential equations is that it enables numerical solvers and supports stability analysis. Although many efforts have been dedicated to solving nonlinear evolution equations, there is no unified method. To the best of our knowledge, this is the first time that Lie point symmetry analysis in conjunction with an ansatz method has been applied to this underlying equation. It should also be noted that the methods applied in this paper give a unique solution set that differs from the newly reported solutions. In addition, we derive the conservation laws of the underlying system. It is also worth mentioning that this is the first time that the conservation laws for the equation under study are derived.

2. Lie Point Symmetries of (4)

The infinitesimal generator [13–15]

$$\mathbf{\Theta} = \xi^1(x,t,u,v)\frac{\partial}{\partial x} + \xi^2(x,t,u,v)\frac{\partial}{\partial t} + \eta^1(x,t,u,v)\frac{\partial}{\partial u} + \eta^2(x,t,u,v)\frac{\partial}{\partial v}, \tag{5}$$

where $\xi^1(x,t,u,v), \xi^2(x,t,u,v), \eta^1(x,t,u,v), \eta^2(x,t,u,v)$ are coefficient functions, is a point symmetry of (4) if

$$\mathbf{\Theta}^{[3]}\left\{ u_t + u_{xxx} + auu_x + bvv_{xxx} + bv_x v_{xx} - bu_x v^2 + auvv_x, v_t + v_{xxx} - bv^2 v_x - buv_x - bvu_x \right\}\Big|_{(u_t+u_{xxx}+auu_x+bvv_{xxx}+bv_xv_{xx}-bu_xv^2+auvv_x=0),(v_t+v_{xxx}-bv^2v_x-buv_x-bvu_x=0)} = 0, \tag{6}$$

where $\mathbf{\Theta}^{[3]}$ is the third extension of (5). The infinitesimal invariance criterion (6) is considered the determining equation as it generates the defining or the determining equations of (4). The above algorithmic procedure leads to following theorem courtesy of symbolic manipulation.

Theorem 1. *The infinitesimal point symmetries of (4) form the three-dimensional Lie algebra spanned by the following linearly independent operators:*

$$\mathbf{\Theta}_1 = \frac{\partial}{\partial t},$$
$$\mathbf{\Theta}_2 = \frac{\partial}{\partial x},$$
$$\mathbf{\Theta}_3 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

Remark 1. *The operators $\mathbf{\Theta}_1, \mathbf{\Theta}_2$ denote time translation and space translation, respectively, and, finally, the vector $\mathbf{\Theta}_3$ represents a scaling transformation.*

3. Point Symmetry Reductions of (4)

In this section, we derive point symmetry reductions. In order to achieve this goal, one has to solve the associated system of Lagrange equations given by

$$\frac{dx}{ds} = \xi^1(x,t,u,v), \tag{7a}$$
$$\frac{dt}{ds} = \xi^1(x,t,u,v), \tag{7b}$$
$$\frac{du}{ds} = \eta^1(x,t,u,v), \tag{7c}$$
$$\frac{dv}{ds} = \eta^1(x,t,u,v). \tag{7d}$$

We consider the following cases.
Case 1. $k_1\mathbf{\Theta}_1 + k_2\mathbf{\Theta}_2 + k_3\mathbf{\Theta}$. Here, k_1, k_2, k_3 are non-zero arbitrary real constants.

$$\zeta = \frac{k_3x + k_2}{\sqrt[3]{3k_3t + k_1}k_3},\ u(x,t) = \frac{F(\zeta)}{(3k_3t + k_1)^{2/3}},\ v(x,t) = \frac{G(\zeta)}{\sqrt[3]{3k_3t + k_1}}, \tag{8}$$

where $F(\zeta), G(\zeta)$ satisfy the following system:

$$G^2bG_\zeta + FbG_\zeta + GbF_\zeta + \zeta G_\zeta k_3 + Gk_3 - G_{\zeta\zeta\zeta} = 0,$$
$$-FGaG_\zeta + G^2bF_\zeta - FaF_\zeta - GbG_{\zeta\zeta\zeta} - G_\zeta G_{\zeta\zeta}b + \zeta F_\zeta k_3 + 2Fk_3 - F_{\zeta\zeta\zeta} = 0.$$

Case 2. $k_1\mathbf{\Theta}_1 - k_2\mathbf{\Theta}_2$

$$\zeta = k_1x + k_2t + k_3,\ \ u(x,t) = F(\zeta),\ \ v(x,t) = G(\zeta) \tag{9}$$

where the functions $F(\zeta), G(\zeta)$ satisfy

$$GbG_{\zeta\zeta\zeta}{k_1}^3 + bG_\zeta {k_1}^3 G_{\zeta\zeta} + FGaG_\zeta k_1 - G^2bF_\zeta k_1 + FaF_\zeta k_1 + F_{\zeta\zeta\zeta}{k_1}^3 + F_\zeta k_2 = 0, \tag{10a}$$
$$-G^2bG_\zeta k_1 - FbG_\zeta k_1 - GbF_\zeta k_1 + G_{\zeta\zeta\zeta}{k_1}^3 + G_\zeta k_2 = 0. \tag{10b}$$

4. Exact Solutions Using an Anstaz Method

An ansatz method is used to solve the system (10) and, as a result, we obtain the exact solutions of our system (4).

Let us consider the solutions of (10) in the form

$$F(\zeta) = \sum_{i=0}^{M} A_i(\Psi(\zeta))^i, \tag{11}$$
$$G(\zeta) = \sum_{i=0}^{M} B_i(\Psi(\zeta))^i, \tag{12}$$

where $\Psi(\zeta)$ satisfies the associated auxiliary equation, M is a positive integer, and A_i, B_i $(i = 0, 1, \cdots, M)$ are parameters to be determined. The auxiliary equations and their associated solutions are given as follows:

$$\Psi'(\zeta) = \alpha\Psi(\zeta) + \beta\Psi^2(\zeta). \tag{13}$$

$$\Psi'(\zeta) = \alpha\Psi^2(\zeta) + \beta\Psi(\zeta) + \gamma \tag{14}$$

$$\Psi'(\zeta)^2 = \Psi(\zeta)^2(\alpha + \beta\Psi(\zeta)), \tag{15}$$

$$\Psi'(\zeta)^2 = \left(1 - \Psi^2(\zeta)\right)\left(1 - \omega + \omega\Psi^2(\zeta)\right) \tag{16}$$

$$\Psi'(\zeta)^2 = \left(1 - \Psi^2(\zeta)\right)\left(1 - \omega\Psi^2(\zeta)\right) \tag{17}$$

$$\Psi(\zeta) = \alpha\left\{\frac{\cosh[\alpha(\zeta + C)] + \sinh[\alpha(\zeta + C)]}{1 - \beta\cosh[\alpha(\zeta + C)] - \beta\sinh[\alpha(\zeta + C)]}\right\}. \tag{18}$$

$$\Psi(\zeta) = -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left[\frac{1}{2}\theta(\zeta + C)\right] \tag{19}$$

$$\Psi(\zeta) = -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left(\frac{1}{2}\theta\zeta\right) + \frac{\operatorname{sech}\left(\frac{\theta\zeta}{2}\right)}{C\cosh\left(\frac{\theta\zeta}{2}\right) - \frac{2\alpha}{\theta}\sinh\left(\frac{\theta\zeta}{2}\right)}, \tag{20}$$

$$\theta^2 = \beta^2 - 4\alpha\gamma \tag{21}$$

$$\Psi(\zeta) = -\frac{\alpha}{\beta}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\alpha}(\zeta + C)\right]. \tag{22}$$

$$\Psi(\zeta) = \operatorname{cn}(\zeta|\omega) \tag{23}$$

$$\Psi(\zeta) = \operatorname{sn}(\zeta|\omega), \tag{24}$$

respectively, and α, β, γ are non-zero arbitrary real constants.

4.1. Solutions of (4) Using (13)

The balancing procedure yields $M = 4, N = 2$ so the solutions of (10) are of the form

$$F(\zeta) = A_0 + A_1\Psi(\zeta) + A_2\Psi(\zeta)^2 + A_3\Psi(\zeta)^3 + A_4\Psi(\zeta)^4, \tag{25a}$$
$$G(\zeta) = B_0 + B_1\Psi(\zeta) + B_2\Psi(\zeta)^2 \tag{25b}$$

Substituting (25) into (10), making use of (13), and then equating all coefficients of the functions $\Psi(\zeta)^i$ to zero, we obtain an algebraic system of equations in terms of A_0, A_1, A_2, A_3, A_4 and B_0, B_1, B_2. Upon solving the system of algebraic equations with the aid of Mathematica, we obtain the following solutions:

$$\begin{aligned} u(x,t) = A_0 + A_1 a&\left\{\frac{\cosh[a(\zeta + C)] + \sinh[a(\zeta + C)]}{1 - b\cosh[a(\zeta + C)] - b\sinh[a(\zeta + C)]}\right\} \\ &+ A_2 a^2\left\{\frac{\cosh[a(\zeta + C)] + \sinh[a(\zeta + C)]}{1 - b\cosh[a(\zeta + C)] - b\sinh[a(\zeta + C)]}\right\}^2 \\ &+ A_3 a^3\left\{\frac{\cosh[a(\zeta + C)] + \sinh[a(\zeta + C)]}{1 - b\cosh[a(\zeta + C)] - b\sinh[a(\zeta + C)]}\right\}^3 \\ &+ A_4 a^4\left\{\frac{\cosh[a(\zeta + C)] + \sinh[a(\zeta + C)]}{1 - b\cosh[a(\zeta + C)] - b\sinh[aFC(\zeta + C)]}\right\}^4 \end{aligned} \tag{26a}$$

$$\begin{aligned} v(x,t) = B_0 + B_1 a&\left\{\frac{\cosh[a(\zeta + C)] + \sinh[a(\zeta + C)]}{1 - b\cosh[a(\zeta + C)] - b\sinh[a(\zeta + C)]}\right\} \\ &+ B_2 a^2\left\{\frac{\cosh[a(\zeta + C)] + \sinh[a(\zeta + C)]}{1 - b\cosh[a(\zeta + C)] - b\sinh[a(\zeta + C)]}\right\}^2, \end{aligned} \tag{26b}$$

$$A_0 = -\frac{13909320\,{k_1}^8\beta^4\alpha^2 + 2925\,{k_1}^6{B_2}^2\alpha^4 + 5826168\,{k_1}^5\beta^4 k_2 - 50310\,{k_1}^3{B_2}^2\alpha^2 k_2 - 17199\,{B_2}^2{k_2}^2}{42250\,{k_1}^4{B_2}^2\alpha^2 - 25350\,k_1{B_2}^2 k_2}$$

$$
\begin{aligned}
A_1 &= -\frac{972\,{k_1}^5\beta^4\alpha + 65\,{k_1}^3{B_2}^2\alpha^3 + 39\,k_2{B_2}^2\alpha}{1170\,{k_1}^3\beta^3},\\
A_2 &= -\frac{972\,{k_1}^5\beta^4 + 455\,{k_1}^3{B_2}^2\alpha^2 + 39\,k_2{B_2}^2}{1170\,{k_1}^3\beta^2},\\
A_3 &= -\frac{2\,{B_2}^2\alpha}{3\,\beta},\\
A_4 &= -\frac{{B_2}^2}{3},\\
B_0 &= -\frac{-5832\,{k_1}^5\beta^4 - 65\,{k_1}^3{B_2}^2\alpha^2 - 39\,k_2{B_2}^2}{780\,{k_1}^3B_2\beta^2},\\
B_1 &= \frac{B_2\alpha}{\beta},\\
a &= \frac{26}{3},\\
b &= \frac{13}{9},\\
\zeta &= k_1x + k_2t + k_3,\\
&510,183,360\,{k_1}^{10}\beta^8 - 4225\,{k_1}^6{B_2}^4\alpha^4 - 1667952\,{B_2}^2k_2{k_1}^5\beta^4 + 1521\,{B_2}^4{k_2}^2 = 0.
\end{aligned}
$$

The parameter B_2 can be computed from the above fourth-degree polynomial in B_2.

4.2. Solutions of (4) Using (14)

This subsection employs the methodology of the previous subsection and, consequently, one obtains the following desired solutions. It should be pointed out that this procedure is used in the following subsections. In this particular subsection, the solution is as follows:

$$
\begin{aligned}
A_0 = &-\frac{1}{3,380,000\,{k_1}^7\alpha^2\gamma^2{B_2}^2 - 1,690,000\,\beta^2{k_1}^7\alpha\,\gamma\,{B_2}^2 + 211,250\,\beta^4{k_1}^7{B_2}^2 - 76,050\,k_1{k_2}^2{B_2}^2}\Big\{3,947,097,600\,{k_1}^{11}\alpha^6\gamma^2\\
&+860,803,200\,\beta^2{k_1}^{11}\alpha^5\gamma + 69,546,600\,\beta^4{k_1}^{11}\alpha^4 + 1,872,000\,{k_1}^9\alpha^3\gamma^3{B_2}^2 - 702,000\,\beta^2{k_1}^9\alpha^2\gamma^2{B_2}^2 + 14,625\,\beta^6{k_1}^9{B_2}^2\\
&+566,870,400\,{k_1}^8\alpha^5\gamma\,k_2 + 70,858,800\,\beta^2{k_1}^8\alpha^4k_2 - 13,150,800\,{k_1}^6\alpha^2\gamma^2k_2{B_2}^2 - 2,691,000\,\beta^2{k_1}^6\alpha\,\gamma\,k_2{B_2}^2\\
&-242775\,\beta^4{k_1}^6k_2{B_2}^2 + 17,478,504\,{k_1}^5\alpha^4{k_2}^2 - 1,895,400\,{k_1}^3\alpha\,\gamma\,{k_2}^2{B_2}^2 - 236,925\,\beta^2{k_1}^3{k_2}^2{B_2}^2 - 51,597\,{k_2}^3{B_2}^2\Big\},\\
A_1 = &-\frac{972\,\beta\,{k_1}^5\alpha^4 + 520\,\beta\,{k_1}^3\gamma\,{B_2}^2\alpha + 65\,{k_1}^3{B_2}^2\beta^3 + 39\,k_2{B_2}^2\beta}{1170\,\alpha^3{k_1}^3},\\
A_2 = &-\frac{972\,{k_1}^5\alpha^4 + 520\,{B_2}^2{k_1}^3\gamma\,\alpha + 455\,{B_2}^2{k_1}^3\beta^2 + 39\,k_2{B_2}^2}{1170\,{k_1}^3\alpha^2},\\
A_3 = &-\frac{2\,{B_2}^2\beta}{3\,\alpha},\\
A_4 = &-\frac{{B_2}^2}{3},\\
B_0 = &-\frac{-5832\,{k_1}^5\alpha^4 - 520\,{B_2}^2{k_1}^3\gamma\,\alpha - 65\,{B_2}^2{k_1}^3\beta^2 - 39\,k_2{B_2}^2}{780\,\alpha^2B_2{k_1}^3},\\
B_1 = &\frac{B_2\beta}{\alpha},\\
a = &\frac{26}{3},\\
b = &\frac{13}{9},\\
\zeta = &k_1x + k_2t + k_3,\\
&-510,183,360\,\alpha^8{k_1}^{10} + 67,600\,\alpha^2\gamma^2{B_2}^4{k_1}^6 - 33,800\,\alpha\,\beta^2\gamma\,{B_2}^4{k_1}^6 + 4225\,\beta^4{B_2}^4{k_1}^6 + 1,667,952\,\alpha^4{B_2}^2{k_1}^5k_2\\
&-1521\,{B_2}^4{k_2}^2 = 0.
\end{aligned}
$$

Note that parameter B_2 can be computed from the fourth-degree polynomial in B_2.

$$u(x,t) = A_0 + A_1\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left[\frac{1}{2}\theta(\zeta+C)\right]\right\} + A_2\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left[\frac{1}{2}\theta(\zeta+C)\right]\right\}^2 + A_3\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left[\frac{1}{2}\theta(\zeta+C)\right]\right\}^3 + A_4\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left[\frac{1}{2}\theta(\zeta+C)\right]\right\}^4 \quad (27a)$$

$$v(x,t) = B_0 + B_1\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left[\frac{1}{2}\theta(\zeta+C)\right]\right\} + B_2\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left[\frac{1}{2}\theta(\zeta+C)\right]\right\}^2, \quad (27b)$$

$$u(x,t) = A_0 + A_1\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left(\frac{1}{2}\theta\zeta\right) + \frac{\operatorname{sech}\left(\frac{\theta\zeta}{2}\right)}{C\cosh\left(\frac{\theta\zeta}{2}\right) - \frac{2\alpha}{\theta}\sinh\left(\frac{\theta\zeta}{2}\right)}\right\} + A_2\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left(\frac{1}{2}\theta\zeta\right) + \frac{\operatorname{sech}\left(\frac{\theta\zeta}{2}\right)}{C\cosh\left(\frac{\theta\zeta}{2}\right) - \frac{2\alpha}{\theta}\sinh\left(\frac{\theta\zeta}{2}\right)}\right\}^2 + A_3\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left(\frac{1}{2}\theta\zeta\right) + \frac{\operatorname{sech}\left(\frac{\theta\zeta}{2}\right)}{C\cosh\left(\frac{\theta\zeta}{2}\right) - \frac{2\alpha}{\theta}\sinh\left(\frac{\theta\zeta}{2}\right)}\right\}^3 + A_4\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left(\frac{1}{2}\theta\zeta\right) + \frac{\operatorname{sech}\left(\frac{\theta\zeta}{2}\right)}{C\cosh\left(\frac{\theta\zeta}{2}\right) - \frac{2\alpha}{\theta}\sinh\left(\frac{\theta\zeta}{2}\right)}\right\}^4, \quad (28a)$$

$$v(x,t) = B_0 + B_1\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left(\frac{1}{2}\theta\zeta\right) + \frac{\operatorname{sech}\left(\frac{\theta\zeta}{2}\right)}{C\cosh\left(\frac{\theta\zeta}{2}\right) - \frac{2\alpha}{\theta}\sinh\left(\frac{\theta\zeta}{2}\right)}\right\} + B_2\left\{ -\frac{\beta}{2\alpha} - \frac{\theta}{2\alpha}\tanh\left(\frac{1}{2}\theta\zeta\right) + \frac{\operatorname{sech}\left(\frac{\theta\zeta}{2}\right)}{C\cosh\left(\frac{\theta\zeta}{2}\right) - \frac{2\alpha}{\theta}\sinh\left(\frac{\theta\zeta}{2}\right)}\right\}^2. \quad (28b)$$

4.3. Solutions of (4) Using (15)

This subsection aims to employ the methodology in Section 4.1. This leads to the following solutions of the system (4):

$$u(x,t) = A_0 + A_1\left\{ -\frac{\alpha}{\beta}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\alpha}(\zeta+C)\right]\right\} + A_2\left\{ -\frac{\alpha}{\beta}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\alpha}(\zeta+C)\right]\right\}^2 + A_3\left\{ -\frac{\alpha}{\beta}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\alpha}(\zeta+C)\right]\right\}^3 + A_4\left\{ -\frac{\alpha}{\beta}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\alpha}(\zeta+C)\right]\right\}^4, \quad (29a)$$

$$v(x,t) = B_0 + B_1\left\{ -\frac{\alpha}{\beta}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\alpha}(\zeta+C)\right]\right\} + B_2\left\{ -\frac{\alpha}{\beta}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\alpha}(\zeta+C)\right]\right\}^2, \quad (29b)$$

$$A_0 = -\frac{1,738,665\,k_1{}^8\beta^2\alpha + 5850\,k_1{}^6\alpha^2 B_1{}^2 + 728,271\,k_1{}^5\beta^2 k_2 - 100,620\,k_1{}^3 k_2\alpha\,B_1{}^2 - 34,398\,k_2{}^2 B_1{}^2}{84,500\,k_1{}^4\alpha\,B_1{}^2 - 50,700\,k_1 k_2 B_1{}^2},$$
$$A_1 = -\frac{243\,k_1{}^5\beta^2 + 260\,k_1{}^3\alpha\,B_1{}^2 + 156\,k_2 B_1{}^2}{1170\,k_1{}^3\beta},$$
$$A_2 = -\frac{B_1{}^2}{3},$$
$$A_3 = 0,$$
$$A_4 = 0,$$
$$B_0 = -\frac{-729\,k_1{}^5\beta^2 - 130\,k_1{}^3\alpha\,B_1{}^2 - 78\,k_2 B_1{}^2}{390\,\beta\,B_1 k_1{}^3},$$
$$B_2 = 0,$$
$$a = \frac{26}{3},$$
$$b = \frac{13}{9},$$
$$\zeta = k_1 x + k_2 t + k_3,$$
$$7,971,615\,\beta^4 k_1{}^{10} - 16,900\,\alpha^2 B_1{}^4 k_1{}^6 - 416,988\,\beta^2 B_1{}^2 k_1{}^5 k_2 + 6084\,B_1{}^4 k_2{}^2 = 0,$$

It must be mentioned that parameter B_1 can be computed symbolically from the above fourth-degree polynomial in B_1.

4.4. *Solutions of (4) Using (16)*

This subsection aims to substitute (25) into (10) by making use of (16) and then equating all coefficients of the functions $\Psi(\zeta)^i$ to zero; we obtain an algebraic system of equations in terms of A_0, A_1, A_2, A_3, A_4 and B_0, B_1, B_2. Solving this system, one obtains the following solutions:

$$u(x,t) = A_0 + A_1 \mathrm{cn}(\zeta|\omega) + A_2 \mathrm{cn}^2(\zeta|\omega) + A_3 \mathrm{cn}^3(\zeta|\omega) + A_4 \mathrm{cn}^4(\zeta|\omega), \quad (30a)$$
$$v(t,x) = B_0 + B_1 \mathrm{cn}(\zeta|\omega) + B_2 \mathrm{cn}^2(\zeta|\omega) + B_3 \mathrm{cn}^3(\zeta|\omega) + B_4 \mathrm{cn}^4(\zeta|\omega), \quad (30b)$$

$$A_0 = -\frac{1}{821,340,000\,\omega^2 B_2{}^2 k_1{}^7 - 821,340,000\,\omega\,B_2{}^2 k_1{}^7 + 821,340,000\,B_2{}^2 k_1{}^7 - 18,480,150\,B_2{}^2 k_1 k_2{}^2}\Big\{454,896,000\,\omega^3 B_2{}^2 k_1{}^9$$
$$-959,144,716,800\,\omega^3 k_1{}^{11} + 127,088,000\,k_1{}^7\omega^2 B_2{}^4 - 682,344,000\,\omega^2 B_2{}^2 k_1{}^9 + 270,397,180,800\,\omega^2 k_1{}^{11}$$
$$+137,749,507,200\,\omega^3 k_1{}^8 k_2 - 127,088,000\,k_1{}^7\omega\,B_2{}^4 + 682,344,000\,\omega\,B_2{}^2 k_1{}^9 - 59,894,640\,\omega^2 B_2{}^2 k_1{}^6 k_2$$
$$-68,874,753,600\,\omega^2 k_1{}^8 k_2 + 127,088,000\,k_1{}^7 B_2{}^4 - 227,448,000\,B_2{}^2 k_1{}^9 + 3,195,644,400\,\omega\,B_2{}^2 k_1{}^6 k_2$$
$$+4,247,276,472\,\omega^2 k_1{}^5 k_2{}^2 - 943,909,200\,B_2{}^2 k_1{}^6 k_2 - 460,582,200\,\omega\,B_2{}^2 k_1{}^3 k_2{}^2 - 2,859,480\,k_1 B_2{}^4 k_2{}^2$$
$$+230,291,100\,B_2{}^2 k_1{}^3 k_2{}^2 - 12,538,071\,B_2{}^2 k_2{}^3\Big\},$$
$$A_1 = 0,$$
$$A_2 = -\frac{-972\,\omega^2 k_1{}^5 - 520\,\omega\,B_2{}^2 k_1{}^3 + 260\,B_2{}^2 k_1{}^3 - 39\,B_2{}^2 k_2}{1170\,\omega\,k_1{}^3},$$
$$A_3 = 0,$$
$$A_4 = -\frac{B_2{}^2}{3},$$
$$B_0 = -\frac{5832\,\omega^2 k_1{}^5 + 520\,\omega\,B_2{}^2 k_1{}^3 - 260\,B_2{}^2 k_1{}^3 + 39\,B_2{}^2 k_2}{780\,\omega\,B_2 k_1{}^3},$$
$$B_1 = 0,$$
$$a = \frac{26}{3},$$
$$b = \frac{13}{9},$$
$$510,183,360\,\omega^4 k_1{}^{10} - 67,600\,\omega^2 B_2{}^4 k_1{}^6 + 67,600\,\omega\,B_2{}^4 k_1{}^6 - 1,667,952\,\omega^2 B_2{}^2 k_1{}^5 k_2 - 67,600\,B_2{}^4 k_1{}^6 + 1521\,B_2{}^4 k_2{}^2 = 0,$$
$$\zeta = k_1 x + k_2 t + k_3.$$

The term B_2 can be computed from the fourth-degree polynomial in B_2.

4.5. Solutions of (4) Using (17)

Here, we aim to employ the methodology of Section 4.4 and, as a result, one obtains the following solutions:

$$u(x,t) = A_0 + A_1\mathrm{sn}(\zeta|\omega) + A_2\mathrm{sn}^2(\zeta|\omega) + A_3\mathrm{sn}^3(\zeta|\omega) + A_4\mathrm{sn}^4(\zeta|\omega), \quad (31a)$$

$$v(x,t) = B_0 + B_1\mathrm{sn}(\zeta|\omega) + B_2\mathrm{sn}^2(\zeta|\omega) + B_3\mathrm{sn}^3(\zeta|\omega) + B_4\mathrm{sn}^4(\zeta|\omega), \quad (31b)$$

$$\begin{aligned}
A_0 = & -\frac{1}{821,340,000\,\omega^2 {B_2}^2{k_1}^7 - 821,340,000\,\omega\, {B_2}^2{k_1}^7 + 821,340,000\, {B_2}^2{k_1}^7 - 18,480,150\, {B_2}^2 k_1 {k_2}^2}\Big\{ \\
& -227,448,000\,\omega^3 {B_2}^2 {k_1}^9 + 418,350,355,200\,\omega^3 {k_1}^{11} + 35,828,000\,\omega^2 {B_2}^4 {k_1}^7 + 270,397,180,800\,\omega^2 {k_1}^{11} \\
& -68,874,753,600\,\omega^3 {k_1}^8 k_2 - 35,828,000\,\omega\, {B_2}^4 {k_1}^7 - 59,894,640\,\omega^2 {B_2}^2 {k_1}^6 k_2 - 688,74,753,600\,\omega^2 {k_1}^8 k_2 \\
& +35,828,000\, {B_2}^4 {k_1}^7 - 227,448,000\, {B_2}^2 {k_1}^9 - 1,307,826,000\,\omega\, {B_2}^2 {k_1}^6 k_2 + 4,247,276,472\,\omega^2 {k_1}^5 {k_2}^2 \\
& -943,909,200\, {B_2}^2 {k_1}^6 k_2 + 230,291,100\,\omega\, {B_2}^2 {k_1}^3 {k_2}^2 - 806,130\, {B_2}^4 k_1 {k_2}^2 + 230,291,100\, {B_2}^2 {k_1}^3 {k_2}^2 - 12,538,071\, {B_2}^2 {k_2}^3 \Big\}, \\
A_1 = & \ 0, \\
A_2 = & -\frac{972\,\omega^2 {k_1}^5 - 260\,\omega\, {B_2}^2 {k_1}^3 - 260\, {B_2}^2 {k_1}^3 + 39\, {B_2}^2 k_2}{1170\,\omega\, {k_1}^3}, \\
A_3 = & \ 0, \\
A_4 = & -\frac{{B_2}^2}{3}, \\
B_0 = & -\frac{-5832\,\omega^2 {k_1}^5 + 260\,\omega\, {B_2}^2 {k_1}^3 + 260\, {B_2}^2 {k_1}^3 - 39\, {B_2}^2 k_2}{780\,\omega\, B_2 {k_1}^3}, \\
B_1 = & \ 0, \\
a = & \ \frac{26}{3}, \\
b = & \ \frac{13}{9}, \\
& 510,183,360\,\omega^4 {k_1}^{10} - 67,600\,\omega^2 {B_2}^4 {k_1}^6 + 67,600\,\omega\, {B_2}^4 {k_1}^6 - 1,667,952\,\omega^2 {B_2}^2 {k_1}^5 k_2 - 67,600\, {B_2}^4 {k_1}^6 + 1521\, {B_2}^4 {k_2}^2 = 0, \\
\zeta = & \ k_1 x + k_2 t + k_3.
\end{aligned}$$

Note that B_2 can be computed from the fourth-degree polynomial in B_2.

Remark 2. *The functions $cn(\zeta|\omega)$, $sn(\zeta|\omega)$ have the following features:*

(i) *When $\omega \to 1$ converts to $cn(\zeta|\omega) \to sech(\zeta)$, $sn(\zeta|\omega) \to tanh(\zeta)$.*
(ii) *When $\omega \to 0$ transforms to $cn(\zeta|\omega) \to cos(\zeta)$, $sn(\zeta|\omega) \to sin(\zeta)$.*
(iii) $nc(\zeta|\omega) = \frac{1}{cn(\zeta|\omega)}$, $ns(\zeta|\omega) = \frac{1}{sn(\zeta|\omega)}$.

Remark 3. *Various methods to solve partial differential equations have been presented in the literature; there is no unified method. Here, for the first time, the above ansatz method is applied to search solutions for the underlying system. The familiarity of closed-form solutions of nonlinear ordinary and partial differential equations enables numerical solvers and supports stability analysis. Closed-form solutions of nonlinear ordinary and partial differential equations can serve as benchmarks against numerical simulations of the underlying equation. Indeed, the exact solutions offered in this paper relate to homoclinic and heteroclinic orbits in phase space, which are the separatrices of steady and volatile regions.*

5. Conservation Laws

A local conservation law is of the form [13]

$$D_t\Xi^1 + D_x\Xi^2 = 0, \quad (32)$$

which holds for all solutions of Equation (4). Ξ^1 is a conserved density, while Ξ^2 denotes spatial flux. $\Xi^i(i = 1, 2)$ are functions of t, x, u, v and derivatives of u, v. The multipliers $\Gamma_i(i = 1, 2)$ of Equation (4) are determined by invoking the Euler Lagrange operator on Equation (4) and we obtain

$$\frac{\delta}{\delta u}(\Gamma_1)\left(u_t + u_{xxx} + auu_x + bvv_{xxx} + bv_x v_{xx} - bu_x v^2 + auvv_x\right) = 0, \tag{33a}$$

$$\frac{\delta}{\delta v}(\Gamma_2)\left(v_t + v_{xxx} - bv^2 v_x - buv_x - bvu_x\right) = 0. \tag{33b}$$

Solving the above equations prompts this proposition.

Proposition 1. *A generalized (1 + 1) dimensional system (4) admits the multiplier of the form*

$$\Gamma_1 = C_1, \tag{34a}$$

$$\Gamma_2 = -\frac{(a + 2b)v}{b} C_1 + C_2. \tag{34b}$$

Thus, corresponding to the above multiplier, we derive the following conserved vectors:

$$\begin{aligned}
\Xi_1^t &= \frac{2\,ub - v^2 a - 2\,v^2 b}{2b}, \\
\Xi_1^x &= \frac{1}{4b}\Big(v^4 ab + 2\,v^4 b^2 + 4\,uv^2 ab + 4\,uv^2 b^2 + 2\,u^2 ab + 4\,vv_{xx}\,b^2 - 4\,vv_{xx}\,a - 8\,vv_{xx}\,b \\
&\quad + 2\,v_x^2 a + 4\,v_x^2 b + 4\,u_{xx}\,b\Big); \\
\Xi_2^t &= v, \\
\Xi_2^x &= -\frac{1}{3}v^3 b - uvb + v_{xx}.
\end{aligned}$$

Remark 4. *Conservation laws are valuable in the numerical integration of partial differential equations—for example, to control numerical errors. The Korteweg–de Vries equation's conservation laws were the primary point of discovery for many approaches to solving evolutionary equations. Conservation laws play a fundamental role in the theory of non-classical transformations, standard forms, and asymptotic integrability.*

6. Concluding Remarks

Modern group analysis was systematically performed here, pioneering a generalized (1 + 1) dimensional system. The infinitesimal generators consisted of time translation, space translation, and a scaling transformation. Similarity reductions and exact solutions with an ansatz method were derived. Finally, conservation laws were computed.

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