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Optimality Conditions and Duality for a Class of Generalized Convex Interval-Valued Optimization Problems

Yating Guo ¹, Guoju Ye ¹, Wei Liu ^{1,*}, Dafang Zhao ² and Savin Treanță ³ 

¹ College of Science, Hohai University, Nanjing 210098, China; yatingguo@hhu.edu.cn (Y.G.); yegj@hhu.edu.cn (G.Y.)

² School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China; dafangzhao@163.com or dafangzhao@hbnu.edu.cn

³ Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania; savin.treanta@upb.ro

* Correspondence: liuw@hhu.edu.cn or liuw626@hhu.edu.cn; Tel.: +86-182-6005-7068

Abstract: This paper is devoted to derive optimality conditions and duality theorems for interval-valued optimization problems based on gH-symmetrically derivative. Further, the concepts of symmetric pseudo-convexity and symmetric quasi-convexity for interval-valued functions are proposed to extend above optimization conditions. Examples are also presented to illustrate corresponding results.

Keywords: gH-symmetrically derivative; optimality conditions; wolfe duality; symmetric pseudo-convexity; symmetric quasi-convexity



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1. Introduction

Due to the complexity of the environment and the inherent ambiguity of human cognition, the information data in real world optimization problems are usually uncertain. More often, we can not ignore the fact that small uncertainties in data may lead to a completely meaningless of the usual optimal solutions from a practical viewpoint. Therefore, much interest has been paid to the uncertain optimization problems, see [1–4].

There are various approaches used to tackle the optimization problems with uncertainty, such as stochastic process [5], fuzzy set theory [6] and interval analysis [7]. Among them, the method of interval analysis is to express an uncertain variable as a real interval or an interval-valued function (IVF), which has been applied to many fields, such as, the models involving inexact linear programming problems [8], data envelopment analysis [9], optimal control [10], goal programming [11], minimax regret solutions [12] and multi-period portfolio selection problems [13] etc. Up to now, we can find many works involving interval-valued optimization problems (IVOPs) (see [14,15]).

In classical optimization theory, the derivative is the most frequently used one. It plays an important role in the study of optimality conditions and duality theorems in constrained optimization problems. To date, various notions of IVF's derivative have been proposed, see [16–23]. One famous concept is H-derivative defined in [16]. However, the H-derivative is restrictive. In 2009, Stefanini and Bede presented the gH-derivative [23] to overcome the disadvantages of H-derivative. Furthermore, in [24], Guo et al. proposed the gH-symmetrically derivative which is more general than gH-derivative. Researchers of optimal problems have largely used these derivatives of IVFs. For instance, Wu [25] considered the Karush–Kuhn–Tucker (KKT) conditions for nonlinear IVOPs using H-derivative. In [26,27], Wolfe type dual problems of IVOPs were investigated. Later, more general KKT optimality conditions has been proposed by Chalco-Cano et al. [28,29] based on gH-derivative. Besides, Jayswal et al. [30] extended optimality conditions and duality theorems for IVOPs with the generalized convexity. Antczak et al. [31] studied the optimality conditions and duality results for nonsmooth vector optimization problems with multiple IVFs [32]. In

2019, Ghosh [33] have extended the KKT condition for constrained IVOPs. In addition, Van [34] investigated the duality results for interval-valued pseudoconvex optimization problems with equilibrium constraints.

Based on the fact that the IVOPs have been extensively studied on optimality condition and duality by many researchers in recent years, in this paper, we continue to study and develop these results on optimality conditions and Wolfe duality of IVOPs on the basis of the gH-symmetrically derivative. In addition, we are going to introduce more appropriate concepts of symmetric pseudo-convexity and symmetric quasi-convexity to weak the convexity hypothesis.

The remaining of the paper is as follows: In Section 2, we give preliminaries and recall some main concepts. In Section 3, we propose the directional gH-symmetrically derivative and more appropriate concepts of generalized convexity. Section 4 establishes the necessary optimality conditions and Wolfe duality theorems. In Section 5, we apply the generalized convexities to investigate the content in Section 4. Our results are properly wider than the results in [28–30].

2. Preliminaries

Theorem 1 ([35]). Suppose that $f : M \rightarrow \mathbb{R}$ is symmetrically differentiable on M and N is an open convex subset of M . Then f is convex on N if and only if

$$f(t) - f(t^*) \geq f^s(t^*)(t - t^*), \text{ for all } t, t^* \in N. \quad (1)$$

Theorem 2 ([36]). Let A be a $m \times n$ real matrix and let $c \in \mathbb{R}^n$ be a column vector. Then the implication

$$At \leq 0 \Rightarrow c^T t \leq 0 \quad (2)$$

holds for all $t \in \mathbb{R}^n$ if and only if

$$\exists u \geq 0 : u^T A = c^T, \quad (3)$$

where $u \in \mathbb{R}^m$.

Let \mathbb{I} be the set of all bounded and closed intervals in \mathbb{R} , i.e.,

$$\mathbb{I} = \{a = [\underline{a}, \bar{a}] | \underline{a}, \bar{a} \in \mathbb{R} \text{ and } \underline{a} \leq \bar{a}\}.$$

For $a = [\underline{a}, \bar{a}]$, $b = [\underline{b}, \bar{b}]$, $c = [\underline{c}, \bar{c}] \in \mathbb{I}$ and $k \in \mathbb{R}$, we have

$$a + b = [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}],$$

$$k \cdot a = k \cdot [\underline{a}, \bar{a}] = \begin{cases} [k\underline{a}, k\bar{a}], & \text{if } k > 0; \\ [k\bar{a}, k\underline{a}], & \text{if } k \leq 0. \end{cases}$$

In [23], Stefanini and Bede presented the gH-difference:

$$a \ominus_g b = c \Leftrightarrow \begin{cases} a = b + c; \\ \text{or } b = a + (-1)c. \end{cases}$$

In addition, this difference between two intervals always exists, i.e.,

$$a \ominus_g b = [\min\{\underline{a} - \underline{b}, \bar{a} - \bar{b}\}, \max\{\underline{a} - \bar{b}, \bar{a} - \underline{b}\}].$$

Furthermore, the partial order relation " \preceq_{LU} " on \mathbb{I} is determined as follows:

$$[\underline{a}, \bar{a}] \preceq_{LU} [\underline{b}, \bar{b}] \Leftrightarrow \underline{a} \leq \underline{b} \text{ and } \bar{a} \leq \bar{b},$$

$$[\underline{a}, \bar{a}] \prec_{LU} [\underline{b}, \bar{b}] \Leftrightarrow [\underline{a}, \bar{a}] \preceq_{LU} [\underline{b}, \bar{b}] \text{ and } [\underline{a}, \bar{a}] \neq [\underline{b}, \bar{b}],$$

a and b are said to be comparable if and only if $a \preceq_{LU} b$ or $a \succeq_{LU} b$.

Let \mathbb{R}^n be the n -dimensional Euclidean space, and $T \subset \mathbb{R}^n$ is an open set. We call the function $F : T \rightarrow \mathbb{I}$ an IVF, i.e., $F(t)$ is a closed interval in \mathbb{R} for every $t \in T$. The IVF F can also be denoted as $F = [\underline{F}, \overline{F}]$, where \underline{F} and \overline{F} are real functions and $\underline{F} \leq \overline{F}$ on T . Moreover, $\underline{F}, \overline{F}$ are called the endpoint functions of F .

Definition 1 ([24]). Let $F : T \rightarrow \mathbb{I}$. Then F is said to be gH -symmetrically differentiable at $t^0 \in T$ if there exists $F^s(t^0) \in \mathbb{I}$ such that:

$$\lim_{||h|| \rightarrow 0} \frac{F(t^0 + h) \ominus_g F(t^0 - h)}{||h||} = F^s(t^0). \quad (4)$$

Definition 2 ([24]). Let $F : T \rightarrow \mathbb{I}$ and $t^0 \in T$. If the IVF $\varphi(t_i) = F(t_1^0, \dots, t_{i-1}^0, t_i, t_{i+1}^0, \dots, t_n^0)$ is gH -symmetrically differentiable at t_i^0 , then we say that F has the i th partial gH -symmetrically derivative $(\frac{\partial^s F}{\partial t_i})_g(t^0)$ at t^0 and

$$(\frac{\partial^s F}{\partial t_i})_g(t^0) = \varphi^s(t_i^0).$$

Definition 3 ([24]). Let $F : T \rightarrow \mathbb{I}$ be an IVF, and $\partial_{t_i}^s F$ stands for the partial gH -symmetrically derivative with respect to the i th variable t_i . If $\partial_{t_i}^s F(t^0)$ ($i = 1, \dots, n$) exist on some neighborhoods of t^0 and are continuous at t^0 , then F is said to be gH -symmetrically differentiable at $t^0 \in T$. Moreover, we denote by

$$\nabla^s F(t^0) = (\partial_{t_1}^s F(t^0), \dots, \partial_{t_n}^s F(t^0))$$

the symmetric gradient of F at t^0 .

Theorem 3 ([24]). Let the IVF $F : T \rightarrow \mathbb{I}$ be continuous in $(t^0 - \delta, t^0 + \delta)$ for some $\delta > 0$. Then F is gH -symmetrically differentiable at $t^0 \in T$ if and only if \underline{F} and \overline{F} are symmetrically differentiable at t^0 .

Definition 4 ([28]). Let $F = [\underline{F}, \overline{F}]$ be an IVF defined on T . We say that F is LU -convex at t^* if

$$F(\theta t^* + (1 - \theta)t) \preceq_{LU} \theta F(t^*) + (1 - \theta)F(t)$$

for every $\theta \in [0, 1]$ and $t \in T$.

Now, we introduce the following IVOP :

$$\begin{aligned} & \min F(t) \\ & \text{subject to } g_i(t) \leq 0, \quad i = 1, \dots, m, \end{aligned} \quad (5)$$

where $F : M \rightarrow \mathbb{I}$, $g_i : M \rightarrow \mathbb{R}$ ($i = 1, \dots, m$), and $M \subset \mathbb{R}^n$ is an open and convex set. Let

$$\mathcal{X} = \{t \in \mathbb{R}^n : t \in M \text{ and } g_i(t) \leq 0, i = 1, \dots, m\}$$

be the collection of feasible points of Problem (5), and the set of objective values of primal Problem (5) is indicated by:

$$O_P(F, \mathcal{X}) = \{F(t) : t \in \mathcal{X}\}. \quad (6)$$

Moreover, we review the definition of non-dominated solution to the Problem (5):

Definition 5 ([27]). Let t^* be a feasible solution of Problem (5), i.e., $t^* \in \mathcal{X}$. Then t^* is said to be a non-dominated solution of Problem (5) if there exists no $t \in \mathcal{X} \setminus \{t^*\}$ such that: $F(t) \prec_{LU} F(t^*)$.

The KKT sufficient optimality conditions of Problem (5) have been obtained in [24]:

Theorem 4 ([24], Sufficient optimality condition). Assume that $F : M \rightarrow \mathbb{I}$ is LU-convex and gH-symmetrically differentiable at t^* , $g : M \rightarrow \mathbb{R}^n$ is convex and symmetrically differentiable at t^* . If there exist (Lagrange) multipliers $0 \leq \mu_i \in \mathbb{R}$, $i = 1, \dots, m$ such that

$$\begin{aligned} \nabla^s F(t^*) + \nabla^s \bar{F}(t^*) + \sum_{i=1}^m \mu_i \nabla^s g_i(t^*) &= 0; \\ \sum_{i=1}^m \mu_i g_i(t^*) &= 0, \text{ where } \mu = (\mu_1, \dots, \mu_m)^T, \end{aligned} \quad (7)$$

then t^* is a non-dominated solution to Problem (5).

Example 1. Consider the IVOP as below:

$$\begin{aligned} \min \quad & F(t) \\ \text{subject to} \quad & g_1(t) \leq 0, \\ & g_2(t) \leq 0, \end{aligned} \quad (8)$$

where

$$F(t) = \begin{cases} [4t^2 + 2t - 3, 3t^2 + 3t], & \text{if } t \in (-1, 0); \\ [3t - 3, 3t], & \text{if } t \in [0, 1), \end{cases}$$

and

$$g_1(t) = -t; \quad g_2(t) = t - 1.$$

By simple calculation, F is LU-convex and gH-symmetrically differentiable at $t = 0$ and

$$\nabla^s F(0) = [\frac{5}{2}, 3], \quad g_1^s(0) = -1, \text{ and } g_2^s(0) = 1.$$

The condition (7) in Theorem 4 is satisfied at $t = 0$ when $\mu_1 = \frac{11}{2}$, and $\mu_2 = 0$.

On the other hand, it can be easily verified that $t = 0$ is a non-dominated solution of Problem (8). Hence, Theorem 4 is verified.

Noted that F is not gH-differentiable at $t = 0$, the sufficient conditions in [24] are properly wider than those in [28].

3. Generalized Convexity of gH-Symmetrically Differentiable IVFs

The LU-convexity assumption in [28] may be restrictive. For example, the IVF

$$F(t) = \begin{cases} [t, 2t], & \text{if } t \geq 0; \\ [2t, t], & \text{if } t < 0, \end{cases}$$

is not LU-convex at $t = 0$. Inspired by this, we introduce the directional gH-symmetrically derivative and the concepts of generalized convexity for IVFs which will be used in Section 4.

Definition 6. Let $F : T \rightarrow \mathbb{I}$ be an IVF and $h \in \mathbb{R}^n$. Then F is called directional gH-symmetrically differentiable at t^0 in the direction h if $D^s F(t^0 : h) \in \mathbb{I}$ exists and

$$D^s F(t^0 : h) = \lim_{\alpha \rightarrow 0^+} \frac{F(t^0 + \alpha h) \ominus_g F(t^0 - \alpha h)}{2\alpha}. \quad (9)$$

If $t = (t_1, \dots, t_n)^T$ and $e_i = (0, \dots, \overset{i}{1}, \dots, 0)$, then $D^s F(t : e^i)$ is the partial gH-symmetrically derivative of F with respect to t_i at t .

Theorem 5. If $F : T \rightarrow \mathbb{I}$ is gH -symmetrically differentiable at $t \in T$ and $h \in \mathbb{R}^n$, then the directional gH -symmetrically derivative exists and

$$D^s F(t : h) = F^s(t)^T h.$$

Proof. Since, by hypothesis, F is gH -symmetrically differentiable at t , then there exists $F^s(t) \in \mathbb{I}$ such that:

$$\lim_{\alpha h \rightarrow 0} \frac{F(t + \alpha h) \ominus_g F(t - \alpha h)}{2\alpha h} = F^s(t).$$

Then, we have:

$$\lim_{\alpha \rightarrow 0} D\left(\frac{F(t + \alpha h) \ominus_g F(t - \alpha h)}{2\alpha}, F^s(t)h\right) = 0.$$

i.e.,

$$D^s F(t : h) = F^s(t)h.$$

Thus, we complete the proof. \square

Definition 7. The IVF $F : T \rightarrow \mathbb{I}$ is called symmetric pseudo-convex (SP-convex) at $t^0 \in T$, if F is gH -symmetrically differentiable at t^0 and

$$F^s(t^0)(t - t^0) \succeq_{LU} 0 \text{ implies } F(t) \succeq_{LU} F(t^0),$$

for all $t \in T$.

F is said to be symmetric pseudo-concave (SP-concave) at t^0 if $-F$ is SP-convex at t^0 .

Definition 8. The IVF $F : T \rightarrow \mathbb{I}$ is called symmetric quasi-convex (SQ-convex) at $t^0 \in T$, if F is gH -symmetrically differentiable at t^0 and

$$F(t) \preceq_{LU} F(t^0) \text{ implies } F^s(t^0)(t - t^0) \preceq_{LU} 0,$$

for all $t \in T$.

F is said to be symmetric quasi-concave (SQ-concave) at t^0 if $-F$ is SQ-convex at t^0 .

Remark 1. When $\underline{F} = \bar{F}$, i.e., F degenerates to a real function, the concepts of SQ-convexity and SP-convexity will degenerate to s -quasiconvexity and s -pseudoconvexity in [35].

4. KKT Necessary Conditions

The necessary optimality conditions are an important part of the optimization theory, because these conditions can be used to exclude all the feasible solutions which are not optimal solutions, i.e., they can identify all options for solving the problem. From this point, using gH -symmetrically derivative, we establish a KKT necessary optimality condition which is more general than [28,29].

In order to obtain the necessary condition of Problem (5), we shall use the Slater's constraint qualification [37]. Such condition is:

$$\exists t^0 \in \mathcal{X} \text{ such that } g_i(t^0) < 0, i = 1, \dots, m. \quad (10)$$

Theorem 6 (Necessary optimality condition). Assume that $F : M \rightarrow \mathbb{I}$ is LU-convex and gH -symmetrically differentiable, $g_i : M \rightarrow \mathbb{R} (i = 1, \dots, m)$ are symmetrically differentiable and convex on M . Suppose $H = \{i : g_i(t^*) = 0\}$. If t^* is a non-dominated solution to Problem (5) and the following conditions are satisfied:

(A1) For every $i \in H$ and for all $y \in \mathbb{R}^n$, there exist some positive real numbers ξ_i , when $0 < \xi < \xi_i$ and $\nabla^s g_i(t^*)^T y < 0$, we have:

$$\nabla^s g_i(t^* + \xi y)^T y < 0;$$

(A2) The set \mathcal{X} satisfies the Slater's constraint qualification. For $i \in H$ and for all $h \in \mathbb{R}^n$, $D^+ \underline{F}(t^* : h) \geq 0$ implies that $D^s \underline{F}(t^* : h) \geq 0$ or $D^+ \bar{F}(t^* : h) \geq 0$ implies that $D^s \bar{F}(t^* : h) \geq 0$;

where $D^+ \underline{F}$ and $D^- \underline{F}$ ($D^+ \bar{F}$ and $D^- \bar{F}$) are the right-sided and left-sided directional derivative of \underline{F} (\bar{F}). Then, there exists $u^* \in \mathbb{R}_+^m$ such that condition (7) in Theorem 4 holds.

Proof. Suppose the above conditions are satisfied. Assume there exists $w \in \mathbb{R}^n$ such that:

$$\begin{aligned} w^T \nabla^s g_i(t^*) &\leq 0, \\ \text{and } w^T \nabla^s \underline{F}(t^*) &< 0, \quad w^T \nabla^s \bar{F}(t^*) < 0, \quad (\forall i \in H). \end{aligned} \quad (11)$$

Since \mathcal{X} satisfies the Slater's constraint qualification, by Equation (10), there exists $t^0 \in \mathcal{X}$ such that $g_i(t^0) < 0$ ($i = 1, \dots, m$). Then we have:

$$g_i(t^0) - g_i(t^*) < 0, \quad (\forall i \in H),$$

Combining Theorem 1 and the convexity of g_i , we have

$$\nabla^s g_i(t^*)(t^0 - t^*) < 0, \quad (\forall i \in H).$$

by inequality (11), we get

$$\nabla^s g_i(t^*)[w + \rho(t^0 - t^*)] < 0, \quad (\forall i \in H)$$

for all $\rho > 0$. By hypothesis in (A1), there exists $\xi_i > 0$ such that

$$g_i(t^* + \xi[w + \rho(t^0 - t^*)]) < 0, \quad (\forall i \in H)$$

for $0 < \xi < \xi_i$. Therefore, we have: $t^* + \xi[w + \rho(t^0 - t^*)] \in \mathcal{X}$.

Since t^* is a non-dominated solution to Problem (5), there exists no feasible solution t such that: $F(t) \prec F(t^*)$, i.e.,

$$\begin{aligned} \underline{F}(t^* + \xi[w + \rho(t^0 - t^*)]) &\geq \underline{F}(t^*), \\ \text{or } \bar{F}(t^* + \xi[w + \rho(t^0 - t^*)]) &\geq \bar{F}(t^*). \end{aligned}$$

By hypothesis (A2), we have

$$\begin{aligned} [w + \rho(t^0 - t^*)]^T \nabla^s \underline{F}(t^*) &\geq 0, \\ \text{or } [w + \rho(t^0 - t^*)]^T \nabla^s \bar{F}(t^*) &\geq 0, \end{aligned}$$

for all $\rho > 0$. When $\rho \rightarrow 0^+$, we obtain

$$w^T \nabla^s \underline{F}(t^*) \geq 0, \text{ or } w^T \nabla^s \bar{F}(t^*) \geq 0, \quad (12)$$

which contradicts to the inequality (11).

Thus, inequality (11) has no solution. By Theorem 2, there exists $0 \leq \mu_i^* \in \mathbb{R}$ such that

$$\nabla^s \underline{F}(t^*) + \nabla^s \bar{F}(t^*) + \sum_{i=1}^m \mu_i^* \nabla^s g_i(t^*) = 0.$$

For $i \notin H$, let $\mu_i = 0$, then we have $\sum_{i=1}^m \mu_i g_i(t^*) = 0$. The proof is complete. \square

Example 2. Continued from Example 1, note that $g_1(0) = 0$ and $g_1^s(t) \equiv -1$. Moreover, M satisfies the Slater's condition. For $h \in \mathbb{R}^n$ we have:

$$D^+ \underline{F}(0 : h) = \lim_{\alpha \rightarrow 0^+} \frac{\underline{F}(0 + \alpha h) - \underline{F}(0)}{\alpha} = \begin{cases} 3h, & h > 0; \\ 2h, & h \leq 0. \end{cases}$$

$$D^- \underline{F}(0 : h) = \lim_{\alpha \rightarrow 0^-} \frac{\underline{F}(0 + \alpha h) - \underline{F}(0)}{\alpha} = 3h.$$

Obviously, $D^+ \underline{F}(t^* : h) \geq 0$ implies that

$$D^+ \underline{F}(t^* : h) + D^- \underline{F}(t^* : h) \geq 0.$$

Thus, the conditions in Theorem 6 hold at $t = 0$.

On the other hand, we have:

$$\begin{aligned} & \nabla^s \underline{F}(0) + \nabla^s \bar{F}(0) + \sum_{i \in H} \mu_i^* \nabla^s g_i(0) \\ &= \frac{5}{2} + 3 + \mu_1 \cdot (-1) + \mu_2 \cdot 1 = 0 \end{aligned} \quad (13)$$

when $\mu_1 = \frac{11}{2}$, $\mu_2 = 0$. Hence, Theorem 6 is verified.

5. Wolfe Type Duality

In this section, we consider the Wolfe dual Problem (14) of Problem (5) as follows:

$$\begin{aligned} & \max \quad F(t) + \sum_{i=1}^m \mu_i g_i(t) \\ & \text{subject to} \quad \nabla^s \underline{F}(t) + \nabla^s \bar{F}(t) + \sum_{i=1}^m \mu_i \nabla^s g_i(t) = 0, \\ & \quad \mu = (\mu_1, \dots, \mu_m) \geq 0. \end{aligned} \quad (14)$$

For convenience, we write:

$$L(t, \mu) = F(t) + \sum_{i=1}^m \mu_i g_i(t). \quad (15)$$

We denote by

$$\mathcal{Y} = \{(t, \mu) \in \mathbb{R}^n \times \mathbb{R}^m : \nabla^s \underline{F}(t) + \nabla^s \bar{F}(t) + \sum_{i=1}^m \mu_i \nabla^s g_i(t) = 0\} \quad (16)$$

the feasible set of dual Problem (14) and

$$O_D(L, \mathcal{Y}) = \{L(t, \mu) : (t, \mu) \in \mathcal{Y}\} \quad (17)$$

the set of all objective values of Problem (14).

Definition 9. Let (t^*, μ^*) be a feasible solution to Problem (14), i.e., $(t^*, \mu^*) \in \mathcal{Y}$. (t^*, μ^*) is said to be a non-dominated solution to Problem (14), if there is no $(t, \mu) \in \mathcal{Y}$ such that $L(t^*, \mu^*) \prec_{LU} L(t, \mu)$.

Next, we discuss the solvability for Wolfe primal and dual problems.

Lemma 1. Assume that $F : M \rightarrow \mathbb{I}$ is LU-convex and gH -symmetrically differentiable, $g_i : M \rightarrow \mathbb{R} (i = 1, \dots, m)$ are symmetrically differentiable and convex on M . Furthermore, $H = \{i : g_i(t^*) = 0\}$. If $\hat{t}, (t, \mu)$ are feasible solutions to Problems (5) and (14), respectively, then the following statements hold true:

(B1) If $\underline{F}(t) \geq \underline{F}(\hat{t})$, then $\bar{F}(\hat{t}) \geq \bar{L}(t, \mu)$;

(B2) If $\bar{F}(t) \geq \bar{F}(\hat{t})$, then $\underline{F}(\hat{t}) \geq \underline{L}(t, \mu)$.

Moreover, the statements still hold true under strict inequality.

Proof. Suppose $\hat{t}, (t, \mu)$ are feasible solutions to Problem (5) and (14), respectively. Since F is LU-convex, we have:

$$\begin{aligned}\bar{F}(\hat{t}) &\geq \bar{F}(t) + \nabla^s \bar{F}(t)(\hat{t} - t) \\ &= \bar{F}(t) - \nabla^s \underline{F}(t)(\hat{t} - t) - \sum_{i=1}^m \nabla^s g_i(t)(\hat{t} - t) \\ &\geq \bar{F}(t) + \underline{F}(t) - \underline{F}(\hat{t}) + \sum_{i=1}^m [g_i(t) - g_i(\hat{t})].\end{aligned}$$

If $\underline{F}(t) - \underline{F}(\hat{t}) \geq 0$, it follows that

$$\bar{F}(\hat{t}) \geq \bar{F}(t) + \sum_{i=1}^m g_i(t) = \bar{L}(t, \mu).$$

Thus, the statement (B1) holds true. On the other hand, if $\underline{F}(t) - \underline{F}(\hat{t}) > 0$, then

$$\bar{F}(\hat{t}) > \bar{F}(t) + \sum_{i=1}^m g_i(t) = \bar{L}(t, \mu).$$

The other statements can also be proof by using similar arguments. \square

Lemma 2. Under the same assumption to Lemma 1, if $\hat{t}, (t, \mu)$ are feasible solutions to Problems (5) and (14), respectively, then the following statements hold true:

(C1) If $\bar{F}(t) \leq \bar{F}(\hat{t})$, then $\bar{F}(\hat{t}) \geq \bar{L}(t, \mu)$;

(C2) If $\underline{F}(t) \leq \underline{F}(\hat{t})$, then $\underline{F}(\hat{t}) \geq \underline{L}(t, \mu)$.

Moreover, the statements still hold true under strict inequality.

Proof. Suppose $\bar{F}(t) \leq \bar{F}(\hat{t})$, then we have:

$$\begin{aligned}&\bar{F}(\hat{t}) - \bar{L}(t, \mu) \\ &= \bar{F}(\hat{t}) - \bar{F}(t) - \sum_{i=1}^m \mu_i g_i(t) \\ &\geq \bar{F}^s(t)(\hat{t} - t) + [-\sum_{i=1}^m \mu_i g_i(\hat{t}) + \sum_{i=1}^m \mu_i g_i(\hat{t}) - \sum_{i=1}^m \mu_i g_i(t)] \\ &\geq \bar{F}^s(t)(\hat{t} - t) + [-\sum_{i=1}^m \mu_i g_i(\hat{t}) + \sum_{i=1}^m \mu_i g_i^s(t)(\hat{t} - t)] \\ &= [\bar{F}^s(t) + \sum_{i=1}^m \mu_i g_i^s(t)](\hat{t} - t) - \sum_{i=1}^m \mu_i g_i(\hat{t}) \\ &= -\underline{F}^s(\hat{t} - t) - \sum_{i=1}^m \mu_i g_i(\hat{t}) \\ &\geq \underline{F}(t) - \underline{F}(\hat{t}) - \sum_{i=1}^m \mu_i g_i(\hat{t})\end{aligned}$$

$$\begin{aligned}
 &= \underline{F}(t) - \underline{L}(\hat{t}, \mu) \\
 &\geq 0.
 \end{aligned}$$

Thus, the statement (C1) holds true. On the other hand, if $\bar{F}(t) < \bar{F}(\hat{t})$, then:

$$\bar{F}(\hat{t}) > \bar{L}(t, \mu).$$

The proof of (C2) is similar to (C1), so we omit it. \square

Theorem 7. (Weak duality) Under the same assumption of Lemma 1, if $\hat{t}, (t, \mu)$ are feasible solutions to Problems (5) and (14), respectively, then the following statements hold true:

(D1) If $F(t)$ and $F(\hat{t})$ are comparable, then $F(\hat{t}) \succeq L(t, \mu)$.

(D2) If $F(t)$ and $F(\hat{t})$ are not comparable, then $\underline{F}(\hat{t}) > \underline{L}(t, \mu)$ or $\bar{F}(\hat{t}) > \bar{L}(t, \mu)$.

Proof. If $F(t)$ and $F(\hat{t})$ are comparable, by Lemmas 1 and 2, we can obtain the statement (D1); If $F(t), F(\hat{t})$ are not comparable, then we have:

$$F(\hat{t}) \subset F(t), \text{ or } F(\hat{t}) \supset F(t).$$

By Lemmas 1 and 2, we obtain that:

$$\underline{F}(\hat{t}) > \underline{L}(t, \mu), \text{ or } \bar{F}(\hat{t}) > \bar{L}(t, \mu).$$

The proof is complete. \square

Example 3. Consider the optimization problem in Example 1. The corresponding Wolfe duality problem is:

$$\begin{aligned}
 \max \quad & F(t) + \mu_1 g_1(t) + \mu_2 g_2(t) \\
 \text{subject to} \quad & \nabla^s \underline{F}(t) + \nabla^s \bar{F}(t) + \mu_1 \nabla^s g_1(t) + \mu_2 \nabla^s g_2(t) = 0, \\
 & \mu = (\mu_1, \mu_2) \geq 0.
 \end{aligned} \tag{18}$$

Clearly, $\hat{t} = 0$ is a feasible solution of the Problem (8) and the objective value is $[-3, 0]$. Moreover, $(t, \mu_1, \mu_2) = (-\frac{1}{2}, 0, 2)$ is a feasible solution to the Problem (18), and objective value is $[-6, -\frac{15}{4}]$.

We observe that

$$F(0) \succ L(-\frac{1}{2}, 0, 2). \tag{19}$$

Hence, Theorem 7 is verified.

Theorem 8. (Solvability) Under the same assumption of Lemma 1, if $(t^*, \mu^*) \in \mathcal{Y}$ and $L(t^*, \mu^*) \in O_P(F, \mathcal{X})$, then (t^*, μ^*) solves the Problem (14).

Proof. Suppose (t^*, μ^*) is not a non-dominated solution to Problem (14), then there exists $(t, \mu) \in \mathcal{Y}$ so that:

$$L(t^*, \mu^*) \prec L(t, \mu).$$

Since $L(t^*, \mu^*) \in O_P(F, \mathcal{X})$, there exists $\hat{t} \in \mathcal{X}$ such that:

$$F(\hat{t}) = L(t^*, \mu^*) \prec L(t, \mu). \tag{20}$$

According to Theorem 7, if $F(t), F(\hat{t})$ are comparable, then we have

$$F(\hat{t}) \succeq L(t, \mu).$$

If $F(t)$, $F(\hat{t})$ are not comparable, then:

$$\underline{F}(\hat{t}) > \underline{L}(t, \mu), \text{ or } \bar{F}(\hat{t}) > \bar{L}(t, \mu).$$

These two results are contradict to Equation (20). Thus, we complete the proof. \square

Theorem 9. (Solvability) Under the same assumption of Lemma 1, if $\hat{t} \in \mathcal{X}$ is a feasible solution to Problem (5) and $F(\hat{t}) \in O_D(L, \mathcal{Y})$, then \hat{t} solves the Problem (5).

Proof. The proof is similar to Theorem 8, so we omit it. \square

Corollary 1. Under the same assumption of Lemma 1, if \hat{t} , (t^*, μ^*) are feasible solutions to Problems (5) and (14), respectively, moreover, if $F(\hat{t}) = L(t^*, \mu^*)$, then \hat{t} solves Problem (5) and (t^*, μ^*) solves the Problem (14).

Proof. The proof follows Theorem 8 and Theorem 9. \square

Theorem 10. (Strong duality) Under the same assumption of Lemma 1, if F , g_i ($i = 1, \dots, m$) satisfy the conditions (A1) and (A2) at t^* , then there exists $\mu^* \in \mathbb{R}_+^m$ such that (t^*, μ^*) is a solution of Problem (14) and

$$L(t^*, \mu^*) = F(t^*).$$

Proof. By Theorem 6, there exists $\mu^* \in \mathbb{R}_+^m$ such that:

$$\nabla^s \underline{F}(t^*) + \nabla^s \bar{F}(t^*) + \sum_{i=1}^m \mu_i^* \nabla^s g_i(t^*) = 0, \quad (21)$$

and $\sum_{i=1}^m g_i(t^*) = 0$. It can be shown that $L(t^*, \mu^*) \in O_D(L, \mathcal{Y})$ and

$$L(t^*, \mu^*) = F(t^*).$$

By Corollary 1, there exists $\mu^* \in \mathbb{R}_+^m$ such that (t^*, μ^*) is a solution to Problem (14). The proof is complete. \square

Example 4. Continued from Example 2, after calculation, the non-dominated solution to Problem (18) is $(0, \frac{11}{2}, 0)$ and the objective value is $[-6, 0]$; While $t = 0$ is also a non-dominated solution to Problem (8) and the objective value is $[-6, 0]$. Then we have:

$$L(0, \frac{7}{2}, 0) = F(0).$$

On the other hand, the IVF F in Example 2 satisfies the conditions (A1) and (A2), which verifies Theorem 10.

6. The optimality Conditions with Generalized Convexity

In this section, we use the concepts of SP-convexity and SQ-convexity which are less restrictive than LU-convexity to obtain some generalized optimality theorems of Problem (5).

Theorem 11. (Sufficient condition) Suppose F is SP-convex and g_i is s -quasiconvex at t^* for $i \in H$. If $t^* \in \mathcal{X}$, and for some $\mu^* \in \mathbb{R}_+^n$ condition (7) in Theorem 4 holds, then t^* is a non-dominated solution to Problem (5).

Proof. Assume for some $\mu^* \geq 0$, condition (7) in Theorem 4 holds. We have $\sum_{i=1}^m \mu_i^* g_i(t^*) = 0$, where $\mu_i^* = 0$ when $i \notin H$. Since $g_i(t) \leq g_i(t^*)$ and g_i is s-quasiconvex at t^* for $i \in H$, we obtain $g_i^s(t^*)(t - t^*) \leq 0$. Thus:

$$\sum_{i=1}^m \mu_i^* g_i^s(t^*)(t - t^*) \leq 0, \quad \text{for all } t \in \mathcal{X},$$

which implies:

$$\nabla^s(\underline{F}(t^*) + \bar{F}(t^*))(t - t^*) \geq 0 \quad \text{for all } t \in \mathcal{X}.$$

Thanks to the SP-convexity of F , we have:

$$\underline{F}(t) + \bar{F}(t) \geq \underline{F}(t^*) + \bar{F}(t^*) \quad \text{for all } t \in \mathcal{X}. \quad (22)$$

Then t^* is an optimal solution to the real-valued objective function $\underline{F} + \bar{F}$ subject to the same constraints of Problem (5). Suppose t^* is not a non-dominated solution of Problem (5), there exists $t \in \mathcal{X}$ such that:

$$F(t) \prec F(t^*)$$

which contradicts Equation (22). The proof is complete. \square

Example 5. Consider the following optimization:

$$\begin{aligned} \min \quad & F(t) \\ \text{subject to} \quad & g_1(t) \leq 0, \\ & g_2(t) \leq 0. \end{aligned} \quad (23)$$

where:

$$F(t) = \begin{cases} [t^3 + t, 2t^3 + t], & \text{if } t \geq 0; \\ [2t, 1.5t], & \text{if } t < 0, \end{cases}$$

and $g_1(t) = -t$, $g_2(t) = t - 1$.

We observe that F is not gH-differentiable at $t = 0$, and F is not LU-convex at $t = 0$ with:

$$F(0) \not\leq \frac{2}{3}F\left(\frac{1}{4}\right) + \frac{1}{3}F\left(-\frac{1}{2}\right).$$

However, F is SP-convex at $t = 0$ and g_i is s-quasiconvex at $t = 0$ for $i \in H$. Furthermore, F is gH-symmetrically differentiable at $t = 0$ with

$$F^s(0) = \left[\frac{5}{4}, \frac{3}{2}\right].$$

Moreover, we have:

$$\begin{aligned} \nabla^s \underline{F}(0) + \nabla^s \bar{F}(0) + \sum_{i=1}^m \mu_i \nabla^s g_i(0) &= 0; \\ \sum_{i=1}^m \mu_i g_i(0) &= 0, \text{ where } \mu = \left(\frac{11}{4}, 0\right)^T. \end{aligned} \quad (24)$$

On the other hand, $t = 0$ is a non-dominated solution to Problem (23), which verifies Theorem 11.

Theorem 12. (Necessary condition) Suppose F is SQ-concave at t^* and g_i is s-pseudoconcave at t^* for $i \in H$. If t^* is a non-dominated solution to Problem (5) and g_i is lower semicontinuous on M for all $i \notin H$, then (t^*, μ^*) satisfies condition (7) in Theorem 4 with some $\mu^* \geq 0$.

Proof. Assume $\mathcal{X}_1 = \{t \in \mathcal{X} : g_i(t) < 0 \text{ for all } i \notin H\}$. The set \mathcal{X}_1 is relatively open since g_i is lower semicontinuous on M for each $i \notin H$. Since $t^* \in \mathcal{X}_1$, there is some α_0 such that for any $y \in E^n$, $t^* + \alpha y \in \mathcal{X}_1$ when: $0 < \alpha < \alpha_0$.

Suppose $0 < \alpha < \alpha_0$ and for $i \in H$ we have $g_i^s(t^*)^T y \leq 0$, then $g_i^s(t^*)^T \alpha y \leq 0$ for $i \in H$. According to the s-pseudoconcavity of g_i at t^* , we have:

$$g_i(t^* + \alpha y) \leq g_i(t^*).$$

Since t^* solves Problem (5), we have: $F(t^*) \preceq_{LU} F(t^* + \alpha y)$. The SQ-concavity of F at t^* implies that

$$(\nabla^s \underline{F}(t^*) + \nabla^s \bar{F}(t^*))(\alpha y) \geq 0.$$

Thus:

$$g_i^s(t^*)^T y \leq 0, (\nabla^s \underline{F}(t^*) + \nabla^s \bar{F}(t^*))y < 0$$

has no solution y in \mathbb{R}^n . Hence, by Farkas' lemma, there exist $\mu_i^* \geq 0$ such that:

$$\nabla^s \underline{F}(t^*) + \nabla^s \bar{F}(t^*) + \sum_{i=1}^m \mu_i^* \nabla^s g_i(t^*) = 0.$$

□

Example 6. Note that in Example 5, $t = 0$ is a non-dominated solution. F is SQ-concave at $t = 0$, and $g_1(t) = -t$ is s-pseudoconcave at $t = 0$, $g_2(t) = t - 1$ is lower semicontinuous on \mathbb{R} .

On the other hand, for $\mu = (\frac{11}{4}, 0)$, the condition (7) is satisfied at $t = 0$ which verifies Theorem 12.

Theorem 13. (Weak duality) Suppose for each μ such that $(t, \mu) \in \mathbb{R}$, $L(\cdot, \mu)$ is SP-convex on \mathcal{X} . Then for all $\hat{t} \in \mathcal{X}$ and $(t, \mu) \in \mathcal{Y}$, $L(t, \mu) \preceq_{LU} F(\hat{t})$.

Proof. Consider $\hat{t} \in \mathcal{X}$ and $(t, \mu) \in \mathcal{Y}$. Then we have: $L_t^s(t, \mu) = 0$. Since $L(\cdot, \mu)$ is SP-convex on \mathcal{X} , we obtain $L(\hat{t}, \mu) \succeq L(t, \mu)$. Therefore,

$$F(\hat{t}) + \sum_{i=1}^m u_i g_i(\hat{t}) \succeq L(t, \mu).$$

The proof is complete. □

Example 7. Continued the problem of Example 5, $t = 0$ is a feasible solution to Problem (23) and the objective value is $F(0) = 0$.

Moreover, $(t, \mu) = (1, 11, 0)$ is a feasible solution to the Wolfe problem of Problem (23) and the objective value is $[-9, -8]$. Furthermore, we have

$$F(0) \succ L(1, 11, 0),$$

which verifies Theorem 13.

Theorem 14. (Strong duality) Suppose F , g_i ($i = 1, \dots, m$) and t^* satisfy the conditions of Theorem 12. Furthermore, for each μ such that $(t, \mu) \in \mathbb{R}$, $L(\cdot, \mu)$ is SP-convex on \mathcal{X} . Then there exists a $\mu^* \geq 0$ such that (t^*, μ^*) solves Problem (14) and $L(t^*, \mu^*) = F(t^*)$.

Proof. The proof is similar to the proof of Theorem 10. □

Example 8. Continued from Example 5, the non-dominated solution to Wolfe dual of Problem (23) is $(0, \frac{11}{4}, 0)$ and the objective value is $L(0, \frac{11}{4}, 0) = 0$.

While $t = 0$ is also a non-dominated solution of Problem (23) and the objective value is $F(0) = 0$. Then we have:

$$L(0, \frac{11}{4}, 0) = F(0).$$

On the other hand, the IVF F in Example 5 satisfies the conditions of Theorem 14, which verifies Theorem 14.

7. Conclusions

The IVOP is an interesting topic with many real world applications. The nondifferentiable counterpart of this problem is an interesting topic too. In this work, we newly investigate a topic on gH-symmetrically differentiable IVOPs and obtain the KKT conditions and duality theorems which are properly wider than those in [28]. Additionally, more appropriate concepts of generalized convexity are introduced to extend the optimality conditions in [24]. Some developments of the results presented in this paper, which will be investigated in future papers, are given by the study of the saddle-point optimality criteria for the considered class of IVOPs.

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