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Semi-Hyers–Ulam–Rassias Stability of the Convection Partial Differential Equation via Laplace Transform

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Abstract: In this paper, we study the semi-Hyers–Ulam–Rassias stability and the generalized semi-Hyers–Ulam–Rassias stability of some partial differential equations using Laplace transform. One of them is the convection partial differential equation.

Keywords: semi-Hyers–Ulam–Rassias stability; generalized semi-Hyers–Ulam–Rassias stability; Laplace transform; convection partial differential equation

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1. Introduction

It is well known that the study of Ulam stability began in 1940, with a problem posed by Ulam concerning the stability of homomorphisms [1]. In 1941, Hyers [2] gave a partial answer in the case of the additive Cauchy equation in Banach spaces.

After that, Obloza [3] and Alsina and Ger [4] began the study of the Hyers–Ulam stability of differential equations. The field continued to develop rapidly. Linear differential equations were studied in [5–7], integral equations in [8], delay differential equations in [9], linear difference equations in [10,11], other equations in [12], and systems of differential equations in [13]. A summary of these results can be found in [14].

The Hyers–Ulam stability of linear differential equations was studied using the Laplace transform by H. Rezaei, S. M. Jung, and Th. M. Rassias [15], and by Q. H. Alqifiary and S. M. Jung [16]. This method was also used in [17–19].

The study of the stability of partial differential equations began in 2003, with the paper [20] of A. Prastaro and Th.M. Rassias. The Ulam–Hyers stability of partial differential equations was also studied in [21–26].

In [27], M. N. Qarawani used the Laplace transform to establish the Hyers–Ulam–Rassias–Gavruta stability of initial-boundary value problem for heat equations on a finite rod:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < l.$$

In [28], D.O. Deborah and A. Moyosola studied nonlinear, nonhomogeneous partial differential equations using the Laplace differential transform method:

$$\frac{d^2 w(x,t)}{dt^2} + a_n(x) R w(x,t) + b_n(x) S w(x,t) = f(x,t), t > 0, x > 0, n \in \mathbb{N},$$

where $a_n(x), b_n(x)$ are variable coefficients, $n \in \mathbb{N}$, R is the linear operator, S is the nonlinear operator, and $f(x,t)$ is the source function.

In [29], E. Bicer used the Sumudu transform to study the equation:

$$y_t - k y_{xx} = 0, k \text{ a positive real constant}, (x,t) \in D,\ D = (x_0, x] \times (0, \infty).$$

In [30], the Poisson partial differential equation

$$u_{xx}(x,y) + u_{yy}(x,y) = g(x,y)$$

is studied via the double Laplace transform method (DLTM).

In the following sections, we will study the semi-Hyers–Ulam–Rassias stability and the generalized semi-Hyers–Ulam–Rassias stability of some partial differential equations using Laplace transform. One of them is the convection partial differential equation:

$$\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} = 0,\ a > 0,\ x > 0,\ t > 0, y(0,t) = c,\ y(x,0) = 0. \tag{1}$$

A physical interpretation [31] of these equations is a river of solid goo, since we do not want anything to diffuse. The function $y = y(x,t)$ is the concentration of some toxic substance. The variable x denotes the position where $x = 0$ is the location of a factory spewing the toxic substance into the river. The toxic substance flows into the river so that at $x = 0$, the concentration is always C. We also study the semi-Hyers–Ulam–Rassias stability of the following equation:

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} - x = 0, x > 0, t > 0, y(0,t) = 0, y(x,0) = 0. \tag{2}$$

Our results regarding Equation (1) complete those obtained by S.-M. Jung and K.-S. Lee in [22]. In [22], the following equation:

$$a\frac{\partial y(x,t)}{\partial x} + b\frac{\partial y(x,t)}{\partial t} + cy(x,t) + d = 0,\ a,b \in \mathbb{R},\ b \neq 0,\ c,d \in \mathbb{C}, \text{with } \Re(c) \neq 0, \tag{3}$$

where $\Re(c)$ denotes the real part of c, was studied. In our paper, we consider the case $c = 0$ in Equation (3). Moreover, we also study the generalized stability. The method used in [22] was the method of changing variables.

2. Preliminaries

We first recall some notions and results regarding the Laplace transform.

Let $f : (0,\infty) \to \mathbb{R}$ be a piecewise differentiable and of exponential order, that is $\exists M > 0$ and $\alpha_0 \geq 0$ such that

$$|f(t)| \leq M \cdot e^{\alpha_0 t}, \quad \forall t > 0.$$

We denote by $\mathcal{L}[f]$ the Laplace transform of the function f, defined by

$$\mathcal{L}[f](s) = F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Let

$$u(t) = \begin{cases} 0, & \text{if} \quad t \leq 0 \\ 1, & \text{if} \quad t > 0 \end{cases}$$

be the unit step function of Heaviside. We write $f(0)$ instead of the lateral limit $f(0^+)$. The following properties are used in the paper:

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}},\ s > 0,\ n \in \mathbb{N},$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^n}\right](t) = \frac{t^{n-1}}{(n-1)!}u(t),$$

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0),$$

$$\mathcal{L}[f(t-a)u(t-a)](s) = e^{-as}F(s),\ a > 0,$$

hence,

$$\mathcal{L}^{-1}[e^{-as}F(s)](t) = f(t-a)u(t-a).$$

We now consider the function $y : (0,\infty) \times (0,\infty) \to R, y = y(x,t)$, a piecewise differentiable and of exponential order with respect to t. The Laplace transform of y with respect to t is as follows:

$$\mathcal{L}[y(x,t)] = \int_0^\infty y(x,t)e^{-st}dt,$$

where x is treated as a constant. We also denote the following:

$$\mathcal{L}[y(x,t)] = Y(x,s) = Y(x) = Y.$$

We treat Y as a function of x, leaving s as a parameter. We then have the following:

$$\mathcal{L}\left[\frac{\partial y}{\partial t}\right] = sY(x,s) - y(x,0),$$

$$\mathcal{L}\left[\frac{\partial^2 y}{\partial t^2}\right] = s^2Y(x,s) - sy(x,0) - \frac{\partial y}{\partial t}(x,0).$$

Since we transform with respect to t, we can move $\frac{\partial}{\partial x}$ to the front of the integral; hence, we have:

$$\mathcal{L}\left[\frac{\partial y}{\partial x}\right] = \frac{dY}{dx} = Y'(x).$$

Similarly,

$$\mathcal{L}\left[\frac{\partial^2 y}{\partial x^2}\right] = \int_0^\infty \frac{\partial^2 y}{\partial x^2}e^{-st}dt = \frac{d}{dx^2}\int_0^\infty y(x,t)e^{-st}dt = \frac{dY}{dx^2} = Y''(x).$$

For the Laplace transform properties and applications, see [31,32].

3. Semi-Hyers–Ulam–Rassias Stability of the Convection Partial Differential Equation

Let $\varepsilon > 0$. We also consider the following inequality:

$$\left|\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x}\right| \le \varepsilon, \tag{4}$$

or the equivalent

$$-\varepsilon \le \frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} \le \varepsilon. \tag{5}$$

Analogous to [33], we give the following definition:

Definition 1. *The Equation (1) is called semi-Hyers–Ulam–Rassias stable if there exists a function $\varphi : (0,\infty) \times (0,\infty) \to (0,\infty)$, such that for each solution y of the inequality (4), there exists a solution y_0 for the Equation (1) with*

$$|y(x,t) - y_0(x,t)| \le \varphi(x,t), \quad \forall x > 0, t > 0.$$

Theorem 1. *If a function $y : (0,\infty) \times (0,\infty) \to \mathbb{R}$ satisfies the inequality (4), then there exists a solution $y_0 : (0,\infty) \times (0,\infty) \to \mathbb{R}$ for (1), such that*

$$|y(x,t) - y_0(x,t)| \le \begin{cases} \varepsilon t, \ t < \frac{x}{a} \\ \varepsilon \frac{x}{a}, \ t \ge \frac{x}{a} \end{cases}, \tag{6}$$

that is, the Equation (1) is considered semi-Ulam–Hyers–Rassias stable.

Proof. We apply the Laplace transform with respect to t in (5); thus, we have the following:

$$-\frac{\varepsilon}{s} \leq sY(x) - y(x,0) + aY'(x) \leq \frac{\varepsilon}{s}.$$

Since $y(x,0) = 0$, dividing by a we get the following:

$$-\frac{\varepsilon}{as} \leq Y'(x) + \frac{s}{a}Y(x) \leq \frac{\varepsilon}{as}.$$

We now multiply by $e^{\frac{s}{a}x}$ and we obtain this equation:

$$-\frac{\varepsilon}{as}e^{\frac{s}{a}x} \leq e^{\frac{s}{a}x}Y'(x) + \frac{s}{a}e^{\frac{s}{a}x}Y(x) \leq \frac{\varepsilon}{as}e^{\frac{s}{a}x},$$

hence,

$$-\frac{\varepsilon}{as}e^{\frac{s}{a}x} \leq \frac{d}{dx}\left(e^{\frac{s}{a}x}Y(x)\right) \leq \frac{\varepsilon}{as}e^{\frac{s}{a}x}.$$

Integrating from 0 to x we get the following:

$$-\frac{\varepsilon}{as}\frac{e^{\frac{s}{a}x}}{\frac{s}{a}}\Big|_0^x < e^{\frac{s}{a}x}Y(x)\Big|_0^x \leq \frac{\varepsilon}{as}e^{\frac{s}{a}x}\Big|_0^x,$$

that is,

$$-\varepsilon\left(\frac{e^{\frac{s}{a}x}}{s^2} - \frac{1}{s^2}\right) \leq e^{\frac{s}{a}x}Y(x) - Y(0) \leq \varepsilon\left(\frac{e^{\frac{s}{a}x}}{s^2} - \frac{1}{s^2}\right).$$

But $Y(0) = \mathcal{L}[y(0,t)] = \mathcal{L}[c] = \frac{c}{s}$, so we obtain:

$$-\varepsilon\left(\frac{e^{\frac{s}{a}x}}{s^2} - \frac{1}{s^2}\right) \leq e^{\frac{s}{a}x}Y(x) - \frac{c}{s} \leq \varepsilon\left(\frac{e^{\frac{s}{a}x}}{s^2} - \frac{1}{s^2}\right).$$

We now multiply by $e^{-\frac{s}{a}x}$ and we obtain the equation below:

$$-\varepsilon\left(\frac{1}{s^2} - \frac{e^{-\frac{s}{a}x}}{s^2}\right) \leq Y(x) - c\frac{e^{-\frac{s}{a}x}}{s} \leq \varepsilon\left(\frac{1}{s^2} - \frac{e^{-\frac{s}{a}x}}{s^2}\right).$$

We apply the inverse Laplace transform and we obtain the following:

$$-\varepsilon\left[t - \left(t - \frac{x}{a}\right)u\left(t - \frac{x}{a}\right)\right] \leq y(x,t) - c \cdot u\left(t - \frac{x}{a}\right) \leq \varepsilon\left[t - \left(t - \frac{x}{a}\right)u\left(t - \frac{x}{a}\right)\right],$$

that is,

$$\left|y(x,t) - c \cdot u\left(t - \frac{x}{a}\right)\right| \leq \varepsilon\left[t - \left(t - \frac{x}{a}\right)u\left(t - \frac{x}{a}\right)\right].$$

We then put

$$y_0(x,t) = c \cdot u\left(t - \frac{x}{a}\right) = \begin{cases} 0, \ t < \frac{x}{a} \\ c, \ t \geq \frac{x}{a} \end{cases}.$$

This is the solution of (1) and the equation below:

$$|y(x,t) - y_0(x,t)| \leq \begin{cases} \varepsilon t, \ t < \frac{x}{a} \\ \varepsilon\frac{x}{a}, \ t \geq \frac{x}{a} \end{cases}.$$

□

4. Generalized Semi-Hyers–Ulam–Rassias Stability of the Convection Partial Differential Equation

Let $\phi : (0, \infty) \times \mathbb{R} \rightarrow (0, \infty)$, and $\mathcal{L}[\phi(x, t)] = \Phi(x, s)$. We consider the following inequality:

$$\left| \frac{\partial y}{\partial t} + a \frac{\partial y}{\partial x} \right| \leq \phi(x, t), \tag{7}$$

or the equivalent

$$-\phi(x, t) \leq \frac{\partial y}{\partial t} + a \frac{\partial y}{\partial x} \leq \phi(x, t), \quad \forall x > 0, t > 0. \tag{8}$$

Definition 2. *The Equation (1) is called generalized semi-Hyers–Ulam–Rassias stable if there exists a function $\varphi : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$, such that for each solution y of the inequality (7), there exists a solution y_0 for the Equation (1) with*

$$|y(x, t) - y_0(x, t)| \leq \varphi(x, t), \quad \forall x > 0, t > 0.$$

Theorem 2. *Assume that*

$$\int_0^x e^{\frac{s}{a}x} \Phi(x, s) dx \leq \Phi(x, s), \quad \forall x > 0, s > 0. \tag{9}$$

If a function $y : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ satisfies the inequality (7), then there exists a solution $y_0 : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ for (1), such that

$$|y(x, t) - y_0(x, t)| \leq \frac{1}{a} \phi\left(x, t - \frac{x}{a}\right), \quad \forall x > 0, t > 0,$$

that is, the Equation (1) is considered generalized semi-Hyers–Ulam–Rassias stable.

Proof. We apply the Laplace transform with respect to t in (8), so we have the following:

$$-\Phi(x, s) \leq sY(x) - y(x, 0) + aY'(x) \leq \Phi(x, s).$$

Since $y(x, 0) = 0$, dividing by a we get the equation below:

$$-\frac{1}{a}\Phi(x, s) \leq Y'(x) + \frac{s}{a}Y(x) \leq \frac{1}{a}\Phi(x, s).$$

We now multiply by $e^{\frac{s}{a}x}$ and we obtain the following:

$$-\frac{e^{\frac{s}{a}x}}{a}\Phi(x, s) \leq e^{\frac{s}{a}x}Y'(x) + \frac{s}{a}e^{\frac{s}{a}x}Y(x) \leq \frac{e^{\frac{s}{a}x}}{a}\Phi(x, s),$$

hence,

$$-\frac{e^{\frac{s}{a}x}}{a}\Phi(x, s) \leq \frac{d}{dx}\left(e^{\frac{s}{a}x}Y(x)\right) \leq \frac{e^{\frac{s}{a}x}}{a}\Phi(x, s).$$

Integrating from 0 to x we get the following equation:

$$-\frac{1}{a}\int_0^x e^{\frac{s}{a}x}\Phi(x, s)dx \leq e^{\frac{s}{a}x}Y(x)\Big|_0^x \leq \int_0^x \frac{1}{a}e^{\frac{s}{a}x}\Phi(x, s)dx.$$

Using (9), we have

$$-\frac{1}{a}\Phi(x, s) \leq e^{\frac{s}{a}x}Y(x) - Y(0) \leq \frac{1}{a}\Phi(x, s).$$

But $Y(0) = L[y(0,t)] = L[c] = \frac{c}{s}$, so we obtain

$$-\frac{1}{a}\Phi(x,s) \leq e^{\frac{s}{a}x}Y(x) - \frac{c}{s} \leq \frac{1}{a}\Phi(x,s).$$

We now multiply by $e^{-\frac{s}{a}x}$ and we obtain the following equation:

$$-\frac{1}{a}e^{-\frac{s}{a}x}\Phi(x,s) \leq Y(x) - c\frac{e^{-\frac{s}{a}x}}{s} \leq \frac{1}{a}e^{-\frac{s}{a}x}\Phi(x,s).$$

We apply the inverse Laplace transform and we obtain:

$$-\frac{1}{a}\phi\left(x, t - \frac{x}{a}\right) \leq y(x,t) - c \cdot u\left(t - \frac{x}{a}\right) \leq \frac{1}{a}\phi\left(x, t - \frac{x}{a}\right),$$

that is,

$$\left|y(x,t) - c \cdot u\left(t - \frac{x}{a}\right)\right| \leq \frac{1}{a}\phi\left(x, t - \frac{x}{a}\right).$$

We then put the following:

$$y_0(x,t) = c \cdot u\left(t - \frac{x}{a}\right) = \begin{cases} 0, t < \frac{x}{a} \\ c, t \geq \frac{x}{a} \end{cases}.$$

This is the solution of Equation (1) and the equation below:

$$|y(x,t) - cy_0(x,t)| \leq \frac{1}{a}\phi\left(x, t - \frac{x}{a}\right).$$

□

5. Semi-Hyers–Ulam–Rassias Stability of Equation (2)

Let $\varepsilon > 0$. We also consider the following inequality:

$$\left|\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} - x\right| \leq \varepsilon, \quad (10)$$

or the equivalent

$$-\varepsilon \leq \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} - x \leq \varepsilon. \quad (11)$$

Definition 3. *The Equation (2) is called semi-Hyers–Ulam–Rassias stable if there exists a function $\varphi : (0,\infty) \times (0,\infty) \to (0,\infty)$, such that for each solution y of the inequality (10), there exists a solution y_0 for the Equation (2) with the following:*

$$|y(x,t) - y_0(x,t)| \leq \varphi(x,t), \quad \forall x > 0, t > 0.$$

Theorem 3. *If a function $y : (0,\infty) \times (0,\infty) \to \mathbb{R}$ satisfies the inequality (10), then there exists a solution $y_0 : (0,\infty) \times (0,\infty) \to \mathbb{R}$ for (2), such that*

$$|y(x,t) - y_0(x,t)| \leq \begin{cases} \varepsilon t, \ t < x \\ \varepsilon x, \ t \geq x \end{cases},$$

that is, the Equation (2) is considered semi-Hyers–Ulam–Rassias stable.

Proof. We apply the Laplace transform with respect to t in (11), so we have the equation below:

$$-\frac{\varepsilon}{s} \leq sY(x) - y(x,0) + Y'(x) - x\frac{1}{s} \leq \frac{\varepsilon}{s}.$$

Since $y(x,0) = 0$, we get the following:

$$-\frac{\varepsilon}{s} \leq Y'(x) + sY(x) - x\frac{1}{s} \leq \frac{\varepsilon}{s}.$$

We now multiply by e^{sx} and we obtain the following equation:

$$-\frac{\varepsilon}{s}e^{sx} \leq e^{sx}Y'(x) + se^{sx}Y(x) - x\frac{e^{sx}}{s} \leq \frac{\varepsilon}{s}e^{sx}.$$

hence,

$$-\frac{\varepsilon}{s}e^{sx} \leq \frac{d}{dx}(e^{sx}Y(x)) - x\frac{e^{sx}}{s} \leq \frac{\varepsilon}{s}e^{sx}.$$

Integrating from 0 to x, we get the following:

$$-\frac{\varepsilon}{s}\frac{e^{sx}}{s}\bigg|_0^x \leq e^{sx}Y(x)\bigg|_0^x - \frac{1}{s}\int_0^x xe^{sx}dx \leq \frac{\varepsilon}{s}\frac{e^{sx}}{s}\bigg|_0^x.$$

Integrating by parts, we get the equation below:

$$\int_0^x xe^{sx}dx = \frac{(xs-1)e^{sx}}{s^2} + \frac{1}{s^2},$$

hence,

$$-\varepsilon\left(\frac{e^{sx}}{s^2} - \frac{1}{s^2}\right) \leq e^{sx}Y(x) - Y(0) - \frac{1}{s}\left[\frac{(xs-1)e^{sx}}{s^2} + \frac{1}{s^2}\right] \leq \varepsilon\left(\frac{e^{sx}}{s^2} - \frac{1}{s^2}\right).$$

But $Y(0) = \mathcal{L}[y(0,t)] = 0$, so we obtain the following:

$$-\varepsilon\left(\frac{e^{sx}}{s^2} - \frac{1}{s^2}\right) \leq e^{sx}Y(x) - \frac{1}{s}\left[\frac{(xs-1)e^{sx}}{s^2} + \frac{1}{s^2}\right] \leq \varepsilon\left(\frac{e^{sx}}{s^2} - \frac{1}{s^2}\right).$$

We now multiply by e^{-sx} and we obtain the following:

$$-\varepsilon\left(\frac{1}{s^2} - \frac{e^{-sx}}{s^2}\right) \leq Y(x) - \frac{1}{s}\left[\frac{xs-1}{s^2} + \frac{e^{-sx}}{s^2}\right] \leq \varepsilon\left(\frac{1}{s^2} - \frac{e^{-sx}}{s^2}\right),$$

hence,

$$-\varepsilon\left(\frac{1}{s^2} - \frac{e^{-sx}}{s^2}\right) \leq Y(x) - \frac{x}{s^2} + \frac{1}{s^3} - \frac{e^{-sx}}{s^3} \leq \varepsilon\left(\frac{1}{s^2} - \frac{e^{-sx}}{s^2}\right).$$

We apply the inverse Laplace transform and we obtain the following equation:

$$-\varepsilon[t-(t-x)u(t-x)] \leq y(x,t) - xt + \frac{1}{2}t^2 - \frac{1}{2}(t-x)^2u(t-x) \leq \varepsilon[t-(t-x)u(t-x)].$$

We then put the following:

$$y_0(x,t) = xt - \frac{1}{2}t^2 + \frac{1}{2}(t-x)^2u(t-x) = \begin{cases} xt - \frac{1}{2}t^2, \ t < x \\ \frac{1}{2}x^2, \ t \geq x \end{cases}.$$

This is the solution of (2) and the equation below:

$$|y(x,t) - y_0(x,t)| \leq \begin{cases} \varepsilon t, \ t < x \\ \varepsilon x, \ t \geq x \end{cases}.$$

□

6. Conclusions

In this paper, we studied the semi-Hyers–Ulam–Rassias stability of Equations (1) and (2) and the generalized semi-Hyers–Ulam–Rassias stability of Equation (1) using the Laplace transform. To the best of our knowledge, the Hyers-Ulam-Rassias stability of Equations (1) and (2) has not been discussed in the literature with the use of the Laplace transform method. Our results complete those of Jung and Lee [22]. In [22], the Equation (3) was studied for $\Re(c) \neq 0$. We considered the case $c = 0$ in Equation (3). We can apply our results to the convection equation in the sense that for every solution y of (4), which is called an approximate solution, there exists an exact solution y_0 of (1), such that the relation (6) is satisfied. From a different perspective, the approximate solution can be viewed in relation to the perturbation theory, as any approximate solution of (4) is an exact solution of the perturbed equation $\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} = h(x,t)$, $|h(x,t)| \leq \varepsilon$, $a > 0$, $x > 0$, $t > 0$, $y(0,t) = c$, $y(x,0) = 0$.

We intend to study other partial differential equations as well as other integro-differential equations using this method. We have already applied this method to [34], where we investigated the semi-Hyers–Ulam–Rassias stability of a Volterra integro-differential equation of order I with a convolution-type kernel.

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