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Accidental Degeneracy of an Elliptic Differential Operator: A Clarification in Terms of Ladder Operators

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Abstract: We consider the linear, second-order elliptic, Schrödinger-type differential operator $\mathcal{L} := -\frac{1}{2}\nabla^2 + \frac{r^2}{2}$. Because of its rotational invariance, that is it does not change under $SO(3)$ transformations, the eigenvalue problem $\left[-\frac{1}{2}\nabla^2 + \frac{r^2}{2}\right] f(x,y,z) = \lambda f(x,y,z)$ can be studied more conveniently in spherical polar coordinates. It is already known that the eigenfunctions of the problem depend on three parameters. The so-called *accidental degeneracy* of $\mathcal{L}$ occurs when the eigenvalues of the problem depend on one of such parameters only. We exploited ladder operators to reformulate accidental degeneracy, so as to provide a new way to describe degeneracy in elliptic PDE problems.

Keywords: degeneracy; elliptic PDE; ladder operator; commuting operator; eigenvalues

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1. Introduction

In this paper, we intend to treat an elliptic PDE (Among the numerous textbooks on elliptic PDEs, we think that Gilbarg and Trudinger's book [1], first published in 1998 and then again in 2001 and 2015, is the main contribution to acquire the necessary knowledge on this fascinating topic. On the other hand, the main notions to tackle the typical mathematical physics problems can be found in [2], for example.) with a special focus on the property of the *degeneracy* of its spectrum.

To begin with, we consider the following elliptic PDE:

$$\left[-\frac{1}{2}\nabla^2 + \frac{r^2}{2}\right] f(x,y,z) = \lambda f(x,y,z), \tag{1}$$

where $r = \sqrt{x^2+y^2+z^2}$, and the function $f(x,y,z)$ belongs to the following Hilbert space:

$$\mathcal{H} = \left\{ f(\cdot) \in L^2(\mathbb{R}^3) \cap C^2(\mathbb{R}^3) \mid \lim_{r\to\infty} f(x,y,z) = 0 \right\}. \tag{2}$$

As is known, ∇^2 is the Laplacian operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The operator $\mathcal{L} := -\frac{1}{2}\nabla^2 + \frac{r^2}{2}$ satisfies the property of rotational invariance, i.e., it is invariant under $SO(3)$ transformations. Addressing the problem (1) in polar coordinates is not difficult, and it is well known that the eigenfuctions in $\mathcal{H}$ depend on three parameters, say, l, m, n, whereas the eigenvalues only depend on n, meaning that $\mathcal{L}$ is a degenerate operator. However, there are different kinds of degeneracy: If the eigenvalues λ_i are independent of m, that is called *natural degeneracy*. If λ_i are independent of l, *accidental* *degeneracy* occurs. Namely, we focus on accidental degeneracy and on its relationship with ladder operators (a similar procedure applied to spherical hydrogen atom eigenfuctions can be found in [3]).

Recent papers in which the various types of degeneracy are treated are [4–6], just to cite a few.

The paper is organized as follows: In Section 2 the main notions and a selection of useful results on invariance and degeneracy are presented. In Section 3, the ladder operators are introduced and summarized. Section 4 intends to describe the accidental degeneracy of the operator $\mathcal{L}$ in detail. Section 5 features a final discussion and the possible future developments of this theory.

2. Invariance and Degeneracy

We took into account only the linear operators having a discrete spectrum. The following definitions and results, which are well known in the literature, are helpful to characterize our setup and to establish the notation that is used.

Definition 1 (**Invariant operator**). *A linear operator $\mathbb{O}$, defined on a Hilbert space, is said to be **invariant** under a linear transformation $\mathbb{U}$, defined on the same Hilbert space, if for any eigenvalue λ of $\mathbb{O}$, the corresponding eigenspace $E_\lambda(\mathbb{O})$ is an invariant subspace, i.e., $\forall\ \mathbf{v} \in E_\lambda(\mathbb{O})$, also $\mathbb{U}\mathbf{v} \in E_\lambda(\mathbb{O})$.*

Definition 2 (**Commuting operators**). *Given two linear operators $\mathbb{O}_1$, $\mathbb{O}_2$ defined on a Hilbert space, they are **commuting** if the commutator is null, that is:*

$$[\mathbb{O}_1,\ \mathbb{O}_2] = \mathbb{O}_1\mathbb{O}_2 - \mathbb{O}_2\mathbb{O}_1 = 0.$$

Since the operator $\mathcal{L}$ is self-adjoint, it is easy to prove an invariance result, which holds for all linear and self-adjoint operators admitting a complete set of eigenvectors generating the Hilbert space.

Theorem 1 (**Invariance theorem**). *The linear operator $\mathbb{O}$ is invariant under a linear transformation $\mathbb{U}$ if and only if $[\mathbb{O},\ \mathbb{U}] = 0$.*

Proof. If the commutator is zero, we have that, for any eigenvector $\mathbf{v}$ of the operator $\mathbb{O}$:

$$\mathbb{O}\mathbb{U} = \mathbb{U}\mathbb{O} \qquad \Longleftrightarrow \qquad \mathbb{U}(\mathbb{O}\mathbf{v}) = \mathbb{O}(\mathbb{U}\mathbf{v}),$$

hence $\mathbb{O}(\mathbb{U}\mathbf{v}) = \lambda\mathbb{U}\mathbf{v}$, meaning that $\mathbb{O}$ is invariant. Conversely, if $\mathbb{O}$ is invariant, this means that, by linearity:

$$\mathbb{O}(\mathbb{U}\mathbf{v}) = \lambda(\mathbb{U}\mathbf{v}) = \mathbb{U}(\lambda\mathbf{v}) = \mathbb{U}(\mathbb{O}\mathbf{v}),$$

implying that $[\mathbb{O},\ \mathbb{U}] = 0$, because the eigenvectors generate the whole Hilbert space, by assumption. □

It is straightforward to note that the operator $\mathcal{L}$ is invariant under the action of three different linear operators, i.e.:

$$\mathbb{M}_1 = i\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right),\ \ \mathbb{M}_2 = i\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right),\ \ \mathbb{M}_3 = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right), \tag{3}$$

where the imaginary unit i is necessary to guarantee that the operators are self-adjoint (we recall that in the framework of quantum mechanics, the operators $\mathbb{M}_1$, $\mathbb{M}_2$, and $\mathbb{M}_3$ are the components of the pseudo-vector "angular momentum", and the invariance of an operator under the action of all three is called *rotational invariance* or *invariance under the rotation group $SO(3)$*).

Given its rotational invariance, it is more convenient to study the eigenvalue problem by employing the spherical polar coordinates, which depend on the original variables through the following relations:

$$r = \sqrt{x^2+y^2+z^2}, \qquad \theta = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right), \qquad \phi = \arctan\left(\frac{y}{x}\right),$$

where $(r,\theta,\phi) \in [0,\ +\infty) \times [0,\ \pi] \times [0,\ 2\pi]$. By using the standard formulas, we also reformulate the Laplacian operator in terms of partial derivatives with respect to the spherical polar coordinates, i.e.:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right), \tag{4}$$

whereas the operator $\mathbb{M}_3$ becomes $\widetilde{\mathbb{M}}_3 = -i\dfrac{\partial}{\partial\phi}$.

Plugging the expression (4) into $\mathcal{L}$ yields the following form for the operator:

$$\widetilde{\mathcal{L}} = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{A(\theta,\phi)}{2r^2} + \frac{r^2}{2}, \tag{5}$$

and consequently, the eigenvalue problem becomes:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{A(\theta,\phi)}{r^2} - r^2 + 2\lambda\right]\psi(r,\theta,\phi) = 0, \tag{6}$$

where $A(\theta,\phi)$ is the following self-adjoint operator (An alternative formulation of the problem (6) takes place when A is a constant, i.e., $A := l(l+q-2)$, where q is the dimension of the space and l is an integer number. This problem is usually solved numerically. Another kind of degeneracy would occur, and although a deep analysis of such a case deserves future research, it is beyond the scope of our paper.).

$$A(\theta,\phi) = -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{7}$$

Based on the change of variables, it is necessary to modify the Hilbert space of the solutions as well:

$$\begin{aligned}\widetilde{\mathcal{H}} = \Big\{\psi : \mathbb{R}^3 \longrightarrow \mathbb{R} \mid \psi \in C^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3),\\ \psi(r,\theta,\phi+2\pi) = \psi(r,\theta,\phi),\ \lim_{r\longrightarrow\infty}\psi(r,\theta,\phi) = 0\Big\}.\end{aligned} \tag{8}$$

The next theorem is very relevant for the subsequent analysis of degeneracy.

Theorem 2. *The following relation holds in spherical polar coordinates:*

$$A(\theta,\phi) = \mathbb{M}_1^2 + \mathbb{M}_2^2 + \mathbb{M}_3^2,$$

where the operators $\mathbb{M}_i$, for $i = 1,2,3$, are defined by (3).

Proof. The sum of the squares of the operators defined in (3) reads as:

$$\begin{aligned}\mathbb{M}_1^2 + \mathbb{M}_2^2 + \mathbb{M}_3^2 &= -\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)^2 - \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)^2 - \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)^2\\ &= \cdots = -x^2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) - z^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\end{aligned}$$

$$+2\left(xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}+xz\frac{\partial}{\partial x}\frac{\partial}{\partial z}+yz\frac{\partial}{\partial y}\frac{\partial}{\partial z}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right).$$

Now, we recall the well-known identities among partial derivatives:

$$\frac{\partial}{\partial x}=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi},$$
$$\frac{\partial}{\partial y}=\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}-\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi},$$
$$\frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}.$$

Applying the above formulas to the latest expression we obtained for the sum of squares yields:

$$\mathbb{M}_1^2+\mathbb{M}_2^2+\mathbb{M}_3^2=\cdots=-\frac{\partial^2}{\partial\theta^2}-\cot\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}=A(\theta,\phi).$$

□

Back to the identification of the solution of (6), we can proceed by the separation of the variables. The eigenvalues of $\widetilde{\mathcal{L}}$ are countable; more precisely for all $n\in\mathbb{N}$, they have the form:

$$\lambda_n=n+\frac{3}{2}.$$

The associated eigenfunctions read as:

$$\psi(\cdot)=\psi_{nlm}(r,\theta,\phi)=R_{n\ell}(r)Y_{\ell m}(\theta,\varphi)=\left[N_1\,r^\ell\,e^{-r^2/2}\,\mathfrak{L}^{(\ell+1/2)}_{n/2-\ell/2}(u)\Big|_{u=r^2}\right][N_2\mathcal{Y}_{\ell m}(\theta,\varphi)], \tag{9}$$

where the terms:

$$R(r)\doteq N_1\,r^\ell\,e^{-r^2/2}\,\mathfrak{L}^{(\ell+1/2)}_{n/2-\ell/2}(u)\Big|_{u=r^2}$$

and

$$Y_{\ell m}(\theta,\varphi)=N_2\,\mathcal{Y}_{\ell m}(\theta,\varphi)$$

are respectively called the *radial part* and the *angular part*.

In the expression (9), we have that:

- the coefficient N_1, N_2 are normalization constants, as in every eigenvalue problem, with respect to the norm of the Hilbert space, that is:

$$\int_0^\infty r^2|R(r)|^2\,dr=1\qquad\text{and}\qquad\int_0^{2\pi}d\varphi\int_0^\pi|Y_{\ell m}(\theta,\varphi)|^2\sin\theta\,d\theta=1;$$

- n is a nonnegative integer number $n=0,1,2,\dots$;
- For any fixed value of n, the parameter ℓ takes all the integer values from zero to n such that n and ℓ are both even numbers or both odd numbers, implying that the difference between two subsequent values of ℓ is two;
- For any fixed value of ℓ, the parameter m assumes all integer values between $-\ell$ and ℓ;
- The functions $\mathfrak{L}^{(\ell+1/2)}_{n/2-\ell/2}(u)$ are the so-called *Laguerre polynomials* whose general expression is:

$$\mathfrak{L}^{(h)}_j(u)=\frac{e^u}{u^h}\frac{d^j}{du^j}\left(u^{j+h}e^{-u}\right);$$

- The functions $\mathcal{Y}_{\ell m}(\theta,\varphi)$ are the *spherical harmonics*:

$$\mathcal{Y}_{\ell m}(\theta,\varphi)=e^{im\varphi}\sin^m\theta\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^{\ell+m}\sin^{2\ell}\theta,$$

for $m > 0$, whereas $\mathcal{Y}^*{}_{\ell,m}(\theta,\varphi) = (-1)^m \mathcal{Y}_{\ell m}(\theta,\varphi)$ for $m < 0$. They are simultaneous eigenfunctions of the operator $A(\theta,\varphi)$ and of the operator $\widetilde{\mathbb{M}}_3$, in compliance with the following equations:

$$A(\theta,\varphi)\mathcal{Y}_{\ell m}(\theta,\varphi) = \ell(\ell+1)\mathcal{Y}_{\ell m}(\theta,\varphi), \tag{10a}$$
$$\widetilde{\mathbb{M}}_3 \mathcal{Y}_{\ell m}(\theta,\varphi) = m\mathcal{Y}_{\ell m}(\theta,\varphi). \tag{10b}$$

It is well known that the "degeneracy" of an eigenvalue λ of a linear operator is the property for which the eigenspace corresponding to λ has dimension greater than one. Under such a circumstance, we can state that the eigenvalue λ is *degenerate* as well. When the spectrum of a linear operator has a *degeneracy*, a problem usually arises: given a degenerate eigenvalue λ, it is not possible to guarantee that a related eigenvector $\mathbf{v}$ is selected unambiguously.

From the secular Equations (10a) and (10b), the three simultaneous secular equations:

$$\widetilde{\mathcal{L}}\psi_{n\ell m}(r,\theta,\varphi) = \lambda_n \psi_{n\ell m}(r,\theta,\varphi), \tag{11a}$$
$$A(\theta,\varphi)\psi_{n\ell m}(r,\theta,\varphi) = \ell(\ell+1)\psi_{n\ell m}(r,\theta,\varphi), \tag{11b}$$
$$\widetilde{\mathbb{M}}_3 \psi_{n\ell m}(r,\theta,\varphi) = m\psi_{n\ell m}(r,\theta,\varphi) \tag{11c}$$

follow, and the degeneracy of the spectrum of the operator $\widetilde{\mathcal{L}}$ given in (5) is, in other words, due to the dependence of the eigenfunctions on the given parameters. Namely, the eigenfunctions $\psi_{n\ell m}(r,\theta,\varphi)$ depend on the three parameters n, ℓ, m, whereas the eigenvalues λ_n of the operator $\widetilde{\mathcal{L}}$ in (11a) depend on n only, being independent of the other two parameters ℓ, m. The next result, whose proof is rather straightforward, describes the *commutation property* of the operators.

Theorem 3 (**Commutation theorem**)**.** *The linear operators $\mathbb{O}_1, \mathbb{O}_2, \ldots, \mathbb{O}_n$ acting on the same Hilbert space are pairwise commuting if and only if there exists a basis of the Hilbert space formed by all simultaneous eigenfunctions of $\mathbb{O}_1, \mathbb{O}_2, \ldots, \mathbb{O}_n$.*

Theorem (3) provides an important connection with the degeneracy of the spectrum of an operator, as the next theorem shows.

Theorem 4 (**Degeneracy theorem**)**.** *If a linear operator $\mathbb{O}$, acting on a Hilbert space, is invariant under at least two linear transformations $\mathbb{U}_1, \mathbb{U}_2$, acting on the same Hilbert space, which are not pairwise commuting, then the spectrum of the operator $\mathbb{O}$ has a degeneracy.*

Proof. By *reductio ad absurdum*, suppose that the spectrum of the operator $\mathbb{O}$ has no degeneracy. Since $\mathbb{O}$ is invariant under the linear transformation $\mathbb{U}_1$, we can apply Theorem 1, from which the commutation relation $[\mathbb{O}, \mathbb{U}_1] = 0$ follows. Hence, there exists a basis of the Hilbert space formed by all simultaneous eigenfunctions $\{y_i^{(1)}\}$ of $\mathbb{O}$ and $\mathbb{U}_1$. For the same reason, there exists a basis of the Hilbert space formed by all simultaneous eigenfunctions $\{y_j^{(2)}\}$ of $\mathbb{O}$ and $\mathbb{U}_2$. Since the spectrum of the operator $\mathbb{O}$ has no degeneracy, it follows that the two sets of eigenfunctions $\{y_i^{(1)}\}$ and $\{y_j^{(2)}\}$ are the same set, but this conclusion is absurd because the operators $\mathbb{U}_1, \mathbb{U}_2$ do not commute with each other, and then, there cannot exist a basis of the Hilbert space formed by all simultaneous eigenfunctions of the non-commuting operators $\mathbb{U}_1, \mathbb{U}_2$. □

At the present stage, based on Theorem 4, we can state that the degeneracy of the spectrum of the operator $\tilde{\mathcal{L}}$ is not surprising, in that this operator is invariant under the action of the three operators $\mathbb{M}_1, \mathbb{M}_2, \mathbb{M}_3$, which fail to be pairwise commuting.

Definition 3 (**Complete set of operators**). *If the linear operators of the set* $\{\mathbb{O}_1, \mathbb{O}_2, \ldots, \mathbb{O}_n\}$ *are all pairwise commuting and there exists no other linear operator commuting with them, except the trivial operators, then the set is called the complete set of operators.*

Each complete set of operators is endowed with the following key property. Provided that an operator has a degenerate spectrum, that is the knowledge of an eigenvalue does not allow selecting its eigenfunction unambiguously in the corresponding eigenspace, such a degeneracy can be removed. Basically, if a certain eigenfunction is also an eigenfunction of all the operators in the complete set with respect to a fixed eigenvalue for every operator simultaneously, then the operator's degeneracy is eliminated. The notion of *ladder operators* is very helpful to outline our procedure.

3. Ladder Operators and the Degeneracy of the Spectrum of Operators

We identify the degeneracy of the spectrum of the operator $\widetilde{\mathcal{L}}$ as a consequence of the existence of a particular kind of operators, called ladder operators. We provide a general definition of ladder operator after proving the following result, which can be indicated as the *shift theorem*.

Theorem 5 (**Shift theorem**). *Let* $\mathbb{O}$ *be an operator acting on a Hilbert space, and let* $\mathbf{v}$ *be an eigenfunction of* $\mathbb{O}$ *having an eigenvalue* λ*. If another operator* $\mathbb{T}$ *satisfies the condition* $[\mathbb{O},\ \mathbb{T}]\mathbf{v} = \mu\mathbb{T}\mathbf{v}$*, where the coefficient* μ *is a real number, then: either* $\mathbb{T}\mathbf{v}$ *is the null function or* $\mathbb{T}\mathbf{v}$ *is another eigenfunction of the operator* $\mathbb{O}$ *with eigenvalue* $\lambda + \mu$*.*

Proof. If such an operator $\mathbb{T}$ exists, we have that, by linearity and and since λ is an eigenvalue of $\mathbb{O}$, the above relation becomes:

$$[\mathbb{O},\ \mathbb{T}]\mathbf{v} = \mu\mathbb{T}\mathbf{v} \qquad \Longleftrightarrow \qquad \mathbb{O}\mathbb{T}\mathbf{v} - \mathbb{T}\mathbb{O}\mathbf{v} = \mathbb{O}\mathbb{T}\mathbf{v} - \lambda\mathbb{T}\mathbf{v} = \mu\mathbb{T}\mathbf{v},$$

implying the new eigenvalue equation:

$$\mathbb{O}\mathbb{T}\mathbf{v} = (\lambda + \mu)\mathbb{T}\mathbf{v},$$

meaning that either $\mathbb{T}\mathbf{v} = \mathbf{0}$ or $\lambda + \mu$ is an eigenvalue of $\mathbb{O}$ associated with the eigenfunction $\mathbb{T}\mathbf{v}$, so the proof is complete. □

Definition 4 (**Ladder operators**). *An operator* $\mathbb{T}$ *satisfying the hypothesis of the shift theorem is called the ladder operator for the operator* $\mathbb{O}$*. In particular,* $\mathbb{T}$ *is a:*

- *Raising operator if* $\mu > 0$*;*
- *Lowering operator if* $\mu < 0$*.*

A very interesting case in which Theorem 5 is applied occurs when there exists a complete set of n self-adjoint operators $\mathbb{O}_1, \mathbb{O}_2, \ldots, \mathbb{O}_n$ acting on a Hilbert space such that, by virtue of Theorem 3, there exists a basis of the space formed by all their simultaneous eigenfunctions $\{y_1, y_2, \ldots, y_n\}$.

If there exists an operator $\mathbb{T}$ commuting with the k operators $\mathbb{O}_{i_1}, \mathbb{O}_{i_2}, \ldots, \mathbb{O}_{i_k}$ and satisfying the $n-k$ relations of the shift theorem with the remaining $n-k$ operators $\mathbb{O}_{j_1}$, $\mathbb{O}_{j_2}, \ldots, \mathbb{O}_{j_{n-k}}$ for some certain eigenfunction $\bar{y}$, the following relations hold:

$$\begin{cases} \mathbb{O}_{i_1}(\mathbb{T}\bar{y}) = \mathbb{T}\mathbb{O}_{i_1}\bar{y} = \mathbb{T}\lambda_{i_1}\bar{y} = \lambda_{i_1}(\mathbb{T}\bar{y}) \\ \mathbb{O}_{i_2}(\mathbb{T}\bar{y}) = \mathbb{T}\mathbb{O}_{i_2}\bar{y} = \mathbb{T}\lambda_{i_2}\bar{y} = \lambda_{i_2}(\mathbb{T}\bar{y}) \\ \qquad \vdots \\ \mathbb{O}_{i_k}(\mathbb{T}\bar{y}) = \mathbb{T}\mathbb{O}_{i_k}\bar{y} = \mathbb{T}\lambda_{i_k}\bar{y} = \lambda_{i_k}(\mathbb{T}\bar{y}) \end{cases} \tag{12a}$$

and:

$$[\mathbb{O}_{j_1},\ \mathbb{T}]\bar{y} = \mu_{j_1}\mathbb{T}\bar{y}, \quad [\mathbb{O}_{j_2},\ \mathbb{T}]\bar{y} = \mu_{j_2}\mathbb{T}\bar{y}, \quad \cdots, \quad [\mathbb{O}_{j_k},\ \mathbb{T}]\bar{y} = \mu_{j_k}\mathbb{T}\bar{y}, \tag{12b}$$

from which we obtain that the function $\mathbb{T}\bar{y}$ is either the null function or a simultaneous eigenfunction of $\mathbb{O}_{i_1}, \mathbb{O}_{i_2}, \ldots, \mathbb{O}_{i_k}$ with respect to the same eigenvalues $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}$, respectively. Therefore, by Theorem 5, that function is a simultaneous eigenfunction of $\mathbb{O}_{j_1}, \ldots, \mathbb{O}_{j_{n-k}}$ with respect to the shifted eigenvalues $\lambda_{j_1} + \mu_{j_1}, \ldots, \lambda_{j_{n-k}} + \mu_{j_{n-k}}$.

Remark 1. *The degeneracy of the spectrum of a given operator $\mathbb{O}$ can be clarified (we precisely use the term 'clarification' if it is viewed in terms of ladder operators). Basically, we can consider the complete set of operators as a necessary tool to eliminate the degeneracy of the spectrum of $\mathbb{O}$ and to identify all the operators $\mathbb{T}_1, \ldots, \mathbb{T}_p$ that satisfy the relations (12a) together with the operator $\mathbb{O}$ and the relations (12b) of the shift theorem with the remaining operators of the complete set.*

The operator $\widetilde{\mathcal{L}}$ has a degenerate spectrum because its eigenvalues λ_n given in (11a) are independent of the parameters ℓ and m. Since the operator $\widetilde{\mathcal{L}}$ belongs to the complete set of operators $\{\widetilde{\mathcal{L}},\ A(\theta,\varphi),\ \widetilde{\mathbb{M}}_3\}$, in order to clarify the whole degeneracy in terms of ladder operators, it is sufficient to find the ladder operators $\mathbb{T}_1$ and $\mathbb{T}_2$ commuting with $\widetilde{\mathcal{L}}$. Besides commuting with $\widetilde{\mathcal{L}}$, such operators also satisfy the relations (12b) with the operators $A(\theta,\varphi)$ and $\widetilde{\mathbb{M}}_3$, in such a way that the functions $\mathbb{T}_1\psi_{n\ell m}(r,\theta,\varphi)$ and $\mathbb{T}_2\psi_{n\ell m}(r,\theta,\varphi)$ are eigenfunctions of $\widetilde{\mathcal{L}}$ associated with the same eigenvalue λ_n and eigenfunctions of $A(\theta,\varphi)$ and of $\widetilde{\mathbb{M}}_3$ associated with a shifted eigenvalue with respect to ℓ and m, respectively.

Natural Degeneracy of a Spectrum

The first ladder operator $\mathbb{T}_1$ of $\widetilde{\mathcal{L}}$ is already well known in the literature. Namely, it can be easily reconstructed as a combination of the three operators $\mathbb{M}_1$, $\mathbb{M}_2$, and $\mathbb{M}_3$. To be more precise, we take into account the two combinations of the operators $\mathbb{M}_1$ and $\mathbb{M}_2$ that we express in Cartesian and in spherical polar coordinates as follows:

$$\begin{aligned}\mathbb{T}_1^{(+)} &:= \mathbb{M}_1 + i\mathbb{M}_2 = i\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) - \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) = e^{i\phi}\left(\frac{\partial}{\partial\theta} + \frac{i\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right),\\ \mathbb{T}_1^{(-)} &:= \mathbb{M}_1 - i\mathbb{M}_2 = i\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) + \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) = e^{-i\phi}\left(\frac{i\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right),\end{aligned} \tag{13}$$

which respectively are the *raising operator* and the *lowering operator*.

Since the two ladder operators $\mathbb{T}_1^{(\pm)}$ satisfy the conditions:

$$[\widetilde{\mathcal{L}},\ \mathbb{T}_1^{(\pm)}] = [A(\theta,\varphi),\ \mathbb{T}_1^{(\pm)}] = 0, \qquad [\widetilde{\mathbb{M}}_3,\ \mathbb{T}_1^{(\pm)}] = \pm\mathbb{T}_1^{(\pm)},$$

we obtain, according the Equations (12a) and (12b), that the functions $\mathbb{T}_1^{(\pm)}\psi_{n,\ell,m}$ are eigenfunctions of the operators $\widetilde{\mathcal{L}}$ and $A(\theta,\varphi)$ with respect to the same eigenvalues $\lambda_n, \ell(\ell+1)$, respectively, and eigenfunctions of $\widetilde{\mathbb{M}}_3$ with respect to the shifted eigenvalue $m \pm 1$.

The action of the ladder operators on the functions $\psi_{n,\ell,m}$ is described by the next result.

Theorem 6. *The functions $\mathbb{T}_1^{(+)}\psi_{n,\ell,\ell}$ and $\mathbb{T}_1^{(-)}\psi_{n,\ell,-\ell}$ are identically zero.*

Proof. If we expand the function $\mathbb{T}_1^{(+)}\psi_{n,\ell,\ell}$, we can note that $(1-u^2)^\ell$ is a polynomial having degree 2ℓ in u. Indicating with the constant $\mathcal{K}$ the 2ℓ-th derivative of the function w.r.t. u, we obtain the following expression:

$$\begin{aligned}\mathbb{T}_1^{(+)}\psi_{n,\ell,\ell} &= e^{i\phi}\left(\frac{\partial}{\partial\theta} + \frac{i\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)\left[e^{i\ell\phi}\sin^l\theta\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{2\ell}\sin^{2\ell}\theta\right]\\ &= e^{i(\ell+1)\phi}\left\{\frac{\partial}{\partial\theta}\left[\sin^\ell\theta\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{2\ell}\sin^{2\ell}\theta\right] - \ell\cos\theta\sin^{\ell-1}\theta\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{2\ell}\sin^{2\ell}\theta\right\}.\end{aligned}$$

Now, if we posit $\cos\theta = u$ and $\sin\theta = \sqrt{1-u^2}$, the latest expression becomes:

$$e^{i(\ell+1)\phi}\left\{-\sqrt{1-u^2}\left[\frac{d}{du}\left((1-u^2)^{\ell/2}\frac{d^{2\ell}}{du^{2\ell}}(1-u^2)^{\ell}\right)\right]\right.$$
$$\left.-\ell u(1-u^2)^{(\ell-1)/2}\frac{d^{2\ell}}{du^{2\ell}}(1-u^2)^{\ell}\right\}$$
$$=\mathcal{K}e^{i(\ell+1)\phi}\left\{-\sqrt{1-u^2}\left[\frac{d}{du}(1-u^2)\ell/2\right]-\ell u(1-u^2)^{(\ell-1)/2}\right\}=0.$$

□

The pair of ladder operators $\mathbb{T}_1^{(\pm)}$ provides a *clarification* of that part of the degeneracy of the spectrum of $\widetilde{\mathcal{L}}$, which is called *natural degeneracy*. As a matter of fact, the operator $\widetilde{\mathcal{L}}$, depending on the Laplacian operator ∇^2 and the norm r of the vector $\mathbf{r}$, only, has a natural and intrinsic invariance under rotations belonging to the proper rotation group $SO(3)$.

The existence of the ladder operators $\mathbb{T}_1^{(\pm)}$ can be easily deduced from such invariance properties. It is also straightforward to capture the notion that the natural degeneracy of the spectrum of the operator $\widetilde{\mathcal{L}}$ is the independence of its eigenvalues λ_n from the parameter m.

More precisely, because we have the following actions:

$$\mathbb{T}_1^{(-)}\psi_{n,-\ell,-\ell}=\mathbb{T}_1^{(+)}\psi_{n\ell\ell}=0,$$

we can iterate the action of the lowering operator $\mathbb{T}_1^{(-)}$ so as to obtain:

$$\mathbb{T}_1^{(-)}\psi_{n\ell\ell}=C_{\ell-1}\psi_{n,\ell,\ell-1},\qquad \mathbb{T}_1^{(-)}\mathbb{T}_1^{(-)}\psi_{n\ell\ell}=C_{\ell-2}\psi_{n,\ell,\ell-2},$$
$$\cdots,\qquad (\mathbb{T}_1^{(-)})^{\ell-1}\psi_{n\ell\ell}=C_{-\ell}\psi_{n,\ell,-\ell},$$

or vice versa, by iterating the action of the raising operator $\mathbb{T}_1^{(+)}$, the sequence:

$$\mathbb{T}_1^{(+)}\psi_{n,\ell,-\ell}=C_{-\ell+1}\psi_{n,\ell,-\ell+1},\qquad \mathbb{T}_1^{(+)}\mathbb{T}_1^{(+)}\psi_{n,\ell,-\ell}=C_{-\ell+2}\psi_{n,\ell,-\ell+2},$$
$$\cdots,\qquad (\mathbb{T}_1^{(+)})^{\ell-1}\psi_{n,\ell,-\ell}=C_{\ell}\psi_{n,\ell,+\ell},$$

where the coefficients C_i are coefficients of normalization, that is the actions of the raising and lowering operators $\mathbb{T}_1^{(\pm)}$ on the eigenfunctions $\psi_{n,\ell,m}$ leave the parameters n and ℓ unchanged and modify the parameter m, only.

4. Accidental Degeneracy of the Spectrum of $\widetilde{\mathcal{L}}$

Here, we illustrate the main result, which is absent in the literature so far, to the best of our knowledge. We intend to determine the suitable ladder operators for the degeneracy with respect to the parameter ℓ, which is denoted as accidental degeneracy.

The so-called *accidental degeneracy* of the spectrum of the operator $\widetilde{\mathcal{L}}$ consists of the independence of the eigenvalues λ_n from the parameter ℓ. We explain also this type of degeneracy with the help of ladder operators, denoted by $\mathbb{T}_2^{(\pm)}$. Such ladder operators map an eigenfunction $\psi_{n,\ell,m}(\mathbf{r})$ associated with the eigenvalue λ_n either to the null function or to another eigenfunction, denoted by:

$$\psi_{n,\ell',m'}(\mathbf{r})=\mathbb{T}_2^{(\pm)}\psi_{n,\ell,m}(\mathbf{r}).$$

The two eigenfunctions belong to the same eigenspace of λ_n, that is the value of n is the same in both of them, whereas the two values of ℓ are different, and the two values of m may be either equal or different.

First of all, we establish the conditions for the functions $\psi_{n,\ell,m}(\mathbf{r})$ and $\mathbb{T}_2^{(\pm)}\psi_{n,\ell,m}(\mathbf{r})$ to be eigenfunctions of $\widetilde{\mathcal{L}}$ associated with the same eigenvalue λ_n. Namely, by virtue of Theorem 1, any ladder operator $\mathbb{T}_2^{(\pm)}$ has to satisfy the following equality:

$$[\widetilde{\mathcal{L}},\, \mathbb{T}_2^{(\pm)}] = 0. \tag{14}$$

If $g(r)$ is any function depending on the polar coordinate r only, and the operator:

$$\widetilde{\mathcal{L}}_g := -\frac{1}{2}\nabla^2 + g(r),$$

is defined on the Hilbert space (8), it follows that every operator $\widetilde{\mathcal{L}}_g$ is endowed with rotational invariance. Moreover, there are only two particular circumstances where the operator $\widetilde{\mathcal{L}}_g \equiv \widetilde{\mathcal{L}}$ has a further invariance, which is then "purely accidental" and is responsible for accidental degeneracy. Such cases occur if either $g(r) = \dfrac{r^2}{2}$ or $g(r) = -\dfrac{1}{r}$. The latter case was extensively treated in [3], so we focus on the former case.

A synthetic explanation may sound as follows: We know that the eigenfunction $\psi_{n,\ell,m}(\mathbf{r})$ is associated with the eigenvalues $\ell(\ell+1)$, with respect to which $\psi_{n,\ell,m}(\mathbf{r})$ is also an eigenfunction of the operator $A(\theta,\varphi)$ in (7). Furthermore, since the variation of ℓ between two consecutive values is two, this implies that the eigenvalue of $A(\theta,\varphi)$, which is subsequent after $\ell(\ell+1)$, is $(\ell+2)(\ell+3)$. Hence, the raising operator $\mathbb{T}_2^{(+)}$, whose expression is to be identified, must induce the shift $(\ell+2)(\ell+3) - \ell(\ell+1) = 4\ell+6$ on the eigenvalues of $A(\theta,\varphi)$.

In order to do that, by virtue of Theorem 5, the raising operator $\mathbb{T}_2^{(+)}$ has to satisfy the following condition:

$$[A(\theta,\varphi),\, \mathbb{T}_2^{(+)}]\psi_{n,\bar{\ell},\bar{m}}(\mathbf{r}) = (4\ell+6)\,[\mathbb{T}_2^{(+)}\psi_{n,\bar{\ell},\bar{m}}(\mathbf{r})], \tag{15}$$

where $\psi_{n,\bar{\ell},\bar{m}}(r)$ is a particular eigenfunction of $A(\theta,\varphi)$. Therefore, we are supposed to identify an operator that verifies both conditions (14) and (15). The underlying degeneracies have different natures. On the one hand, *natural degeneracy* is clarified by the ladder operators $\mathbb{T}_1^{(\pm)}$ given in (13) and obtained as combinations of the angular momentum operators, and this is due to the fact that the ladder operators have to induce a shift of one unit on the parameter m. On the other hand, *accidental degeneracy* has to be clarified by operators $\mathbb{T}_2^{(+)}$, which are obtained from the combinations of the components of a tensor, because such operators have to induce a shift of two units on the parameter ℓ.

The invariance of the operator $\mathcal{L}$ is illustrated by the next result.

Theorem 7. *All the components of the following second-rank tensor:*

$$T_{ij} = -\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j} + r_i r_j, \qquad \textit{for}\ \ i,j = 1,2,3, \tag{16}$$

where $r_1,\ r_2,\ r_3$ are the coordinates of $\mathbf{r}$, satisfy the commutation identity $[\mathcal{L},\, T_{ij}] = 0$, i.e., $\mathcal{L}$ is invariant under the action of all components.

Proof. We can employ the following property of the commutator, which holds for all A, B, and C:

$$[AB,\, C] = A[B,\, C] + [A,\, C]B.$$

Expanding the quantity $[\mathcal{L},\, T_{ij}]$ yields (Some calculations are omitted for the sake of brevity. However, all the calculations are available upon request to the authors.):

$$[\mathcal{L},\, T_{ij}] = \left[-\frac{1}{2}\nabla^2 + \frac{r^2}{2},\, -\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j} + r_i r_j\right] = \left[-\frac{1}{2}\nabla^2,\, r_i r_j\right] + \left[\frac{r^2}{2},\, -\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\right]$$

$$= -\frac{1}{2}\sum_{k=1}^{3}\left\{\left[-\frac{1}{2}\nabla^2,\, r_i r_j\right] + \left[r_k^2,\, -\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\right]\right\}$$

$$= \cdots = -\frac{1}{2}\left(\frac{\partial}{\partial r_j} r_i + \frac{\partial}{\partial r_i} r_j + r_i\frac{\partial}{\partial r_j} + r_j\frac{\partial}{\partial r_i} - r_j\frac{\partial}{\partial r_i} - r_i\frac{\partial}{\partial r_j} - \frac{\partial}{\partial r_i} r_j - \frac{\partial}{\partial r_j} r_i\right) = 0,$$

meaning that $\mathcal{L}$ is invariant under the action of all nine components T_{ij}. □

The components T_{ij} are the further linear operators that accidentally commute with $\mathcal{L}$, in addition to $\mathbb{M}_1$, $\mathbb{M}_2$ and $\mathbb{M}_3$.

Given the above-mentioned T_{ij}, we can consider the following operators:

$$\mathcal{T}_1 = T_{12}, \qquad \mathcal{T}_2 = \frac{T_{22} - T_{11}}{2},$$

so that we are able to define the following ladder operators:

$$\mathbb{T}_2^{(\pm)} := \mathcal{T}_1 \pm i\mathcal{T}_2 = -\frac{\partial}{\partial x}\frac{\partial}{\partial y} + xy \pm \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + y^2 - x^2\right),$$

where $i = \sqrt{-1}$.

Theorem 8. *The ladder operators $\mathbb{T}_2^{(\pm)}$ satisfy the following commutation identity:*

$$[\mathbb{M}_3,\, \mathbb{T}_2^{(\pm)}] = \pm 2\mathbb{T}_2^{(\pm)}. \tag{17a}$$

Proof. If we expand the expression of the commutator in the left-hand side of (17a), we obtain:

$$[\mathbb{M}_3,\, \mathbb{T}_2^{(\pm)}] = i\left[y\frac{\partial}{\partial x} - x,\, \frac{\partial}{\partial y},\, -\frac{\partial}{\partial x}\frac{\partial}{\partial y} + xy \pm \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + y^2 - x^2\right)\right]$$

$$= -i\left[y\frac{\partial}{\partial x},\, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right] + i\left[y\frac{\partial}{\partial x}, xy\right] + i\left[x\frac{\partial}{\partial y},\, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right] - i\left[x\frac{\partial}{\partial y}, xy\right]$$

$$\pm\frac{1}{2}\left\{\left[y\frac{\partial}{\partial x},\, \frac{\partial}{\partial y}\frac{\partial}{\partial y}\right] + \left[y\frac{\partial}{\partial x},\, x^2\right] + \left[x\frac{\partial}{\partial y},\, \frac{\partial}{\partial x}\frac{\partial}{\partial x}\right] + \left[x\frac{\partial}{\partial y},\, y^2\right]\right\}$$

$$= i\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + y^2 - x^2\right) \pm \left(-2\frac{\partial}{\partial x}\frac{\partial}{\partial y} + 2xy\right) = \pm 2\mathbb{T}_2^{(\pm)}.$$

□

Using Relations (12a), (12b), (14), and (17a), we can establish the following actions of the operators $\mathbb{T}_2^{(\pm)}$ on the eigenfunctions $\psi_{n,\ell,m}(\mathbf{r})$:

$$\mathbb{T}_2^{(+)}\psi_{n,\ell,m}(\mathbf{r}) = \sum_{k=0}^{(n-m-2)/2} c_k\psi_{n,n-2k,m+2}(\mathbf{r}), \quad \mathbb{T}_2^{(-)}\psi_{n,\ell,m}(\mathbf{r}) = \sum_{k=0}^{(n-m+2)/2} c_k\psi_{n,n-2k,m-2}(\mathbf{r}).$$

In order to prove that $\mathbb{T}_2^{(\pm)}$ are the ladder operators that give a clarification of accidental degeneracy, we have to determine the commutators $[A(\theta,\varphi),\mathbb{T}_2^{(+)}]$ and $[A(\theta,\varphi),\mathbb{T}_2^{(-)}]$. As in Theorem 8, we also obtain the commutators:

$$
\begin{aligned}
[\mathbb{M}_1,\, \mathbb{T}_2^{(+)}] &= i\left[z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z}\,,\, -\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial y} + xy + \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + y^2 - x^2\right)\right] \\
&= i\left[z\,\frac{\partial}{\partial y}\,,\, xy\right] + i\left[y\,\frac{\partial}{\partial z}\,,\, \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\right] - \frac{1}{2}\left[z\,\frac{\partial}{\partial y}\, y^2\right] - \frac{1}{2}\left[y\,\frac{\partial}{\partial z}\,,\, \frac{\partial^2}{\partial y^2}\right] \\
&= \frac{\partial}{\partial y}\,\frac{\partial}{\partial z} - yz + ixz - i\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial z}, \\
[\mathbb{M}_2,\, \mathbb{T}_2^{(+)}] &= i\left[x\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial x}\,,\, -\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial y} + xy + \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + y^2 - x^2\right)\right] \\
&= i\left[\frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\,,\, x\,\frac{\partial}{\partial z}\right] + i\left[xy,\, z\,\frac{\partial}{\partial x}\right] + \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\,,\, x\,\frac{\partial}{\partial z}\right] + \frac{1}{2}\left[x^2, z\,\frac{\partial}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\,\frac{\partial}{\partial z} - xz - iyz + i\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial z}\,,
\end{aligned}
$$

from which we can prove the following fundamental result.

Theorem 9 (Theorem of accidental degeneracy). *The commutator* $\left[A(\theta,\varphi),\mathbb{T}_2^{(+)}\right]$ *is the operator:*

$$
\left[A(\theta,\varphi),\, \mathbb{T}_2^{(+)}\right] = 4\mathbb{T}_2^{(+)}\mathbb{M}_3 + 6\mathbb{T}_2^{(+)} + \left(-\,2i\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial z} + 2\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial z} + 2ixz - 2yz\right)\mathbb{T}_1^{(+)},
$$

where $\mathbb{T}_1^{(+)}$ *is the raising operator of the natural degeneracy in (13).*

Proof. Using the relation $A(\theta,\varphi) \equiv \mathbb{M}_1^2 + \mathbb{M}_2^2 + \mathbb{M}_3^2$ and expanding the left-hand side, we have:

$$
\begin{aligned}
\left[A(\theta,\varphi),\, \mathbb{T}_2^{(+)}\right] &= \left[\mathbb{M}_1^2 + \mathbb{M}_2^2 + \mathbb{M}_3^2,\, \mathbb{T}_2^{(+)}\right] \\
= \mathbb{M}_1\left[\mathbb{M}_1,\, \mathbb{T}_2^{(+)}\right] &+ \left[\mathbb{M}_1,\, \mathbb{T}_2^{(+)}\right]\mathbb{M}_1 + \mathbb{M}_2\left[\mathbb{M}_2,\, \mathbb{T}_2^{(+)}\right] + \left[\mathbb{M}_2,\, \mathbb{T}_2^{(+)}\right]\mathbb{M}_2 \\
&+\mathbb{M}_3\left[\mathbb{M}_3,\, \mathbb{T}_2^{(+)}\right] + \left[\mathbb{M}_3,\, \mathbb{T}_2^{(+)}\right]\mathbb{M}_3 \\
&= i\left(z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial y}\,\frac{\partial}{\partial z} - yz + ixz - i\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial z}\right) \\
&+ i\left(\frac{\partial}{\partial y}\,\frac{\partial}{\partial z} - yz + ixz - i\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial z}\right)\left(z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z}\right) \\
&+ i\left(x\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\,\frac{\partial}{\partial z} - xz - iyz + i\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial z}\right) \\
&+ i\left(\frac{\partial}{\partial x}\,\frac{\partial}{\partial z} - xz - iyz + i\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial z}\right)\left(x\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial x}\right) \\
+ 2i\left(y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}\right)&\left[-\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial y} + xy + \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + y^2 - x^2\right)\right] \\
+ 2i\left[-\,\frac{\partial}{\partial x}\,\frac{\partial}{\partial y} + xy \right.&\left. + \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + y^2 - x^2\right)\right]\left(y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}\right).
\end{aligned}
$$

Expanding the right-hand side yields:

$$4\mathbb{T}_2^{(+)}\mathbb{M}_3+6\mathbb{T}_2^{(+)}+\left(-2i\,\frac{\partial}{\partial x}\frac{\partial}{\partial z}+2\,\frac{\partial}{\partial y}\frac{\partial}{\partial z}+2ixz-2yz\right)\mathbb{T}_1^{(+)}$$

$$\begin{aligned}&=4i\left(-\frac{\partial}{\partial x}\frac{\partial}{\partial y}+xy+\frac{i}{2}\frac{\partial^2}{\partial x^2}-\frac{i}{2}\frac{\partial^2}{\partial y^2}+\frac{iy^2}{2}-\frac{ix^2}{2}\right)\left(y\,\frac{\partial}{\partial x}-x\,\frac{\partial}{\partial y}\right)\\&\quad+6\left(-\frac{\partial}{\partial x}\frac{\partial}{\partial y}+xy+\frac{i}{2}\frac{\partial^2}{\partial x^2}-\frac{i}{2}\frac{\partial^2}{\partial y^2}+\frac{iy^2}{2}-\frac{ix^2}{2}\right)\\&\quad+\left(-2i\,\frac{\partial}{\partial x}\frac{\partial}{\partial z}+2\,\frac{\partial}{\partial y}\frac{\partial}{\partial z}+2ixz-2yz\right)\left(iz\,\frac{\partial}{\partial y}-iy\,\frac{\partial}{\partial z}-x\,\frac{\partial}{\partial z}+z\,\frac{\partial}{\partial z}\right).\end{aligned}$$

With the help of some algebra, we can recognize that the two expansions are equal; hence, the proof is complete. □

Theorem 9 and the identity $\mathbb{T}_1^{(+)}\psi_{n\ell\ell}=0$ lead to establishing the action of the commutator $[A(\theta,\varphi),\ \mathbb{T}_2^{(+)}]$ on the eigenfunction $\psi_{n,\ell,\ell}$, obtained when positing $m=\ell$.

$$\begin{aligned}&\left[A(\theta,\varphi),\ \mathbb{T}_2^{(+)}\right]\psi_{n,\ell,\ell}\\&=\left[4\mathbb{T}_2^{(+)}\mathbb{M}_3+6\mathbb{T}_2^{(+)}+\left(-2i\,\frac{\partial}{\partial x}\frac{\partial}{\partial z}+2\,\frac{\partial}{\partial y}\frac{\partial}{\partial z}+2ixz-2yz\right)\mathbb{T}_1^{(+)}\right]\psi_{n,\ell,\ell}\\&=\left[4\mathbb{T}_2^{(+)}\mathbb{M}_3+6\mathbb{T}_2^{(+)}\right]\psi_{n,\ell,\ell}+\left(-2i\,\frac{\partial}{\partial x}\frac{\partial}{\partial z}+2\,\frac{\partial}{\partial y}\frac{\partial}{\partial z}+2ixz-2yz\right)\left[\mathbb{T}_1^{(+)}\psi_{n,\ell,\ell}\right]\\&=(4\ell+6)\left[\mathbb{T}_2^{(+)}\psi_{n,\ell,\ell}\right],\end{aligned}$$

that is we have found the fundamental commutator:

$$\left[A(\theta,\varphi),\mathbb{T}_2^{(+)}\right]\psi_{n,\ell,\ell}=(4\ell+6)\left[\mathbb{T}_2^{(+)}\psi_{n,\ell,\ell}\right].\tag{17b}$$

The function $\mathbb{T}_2^{(+)}\psi_{n,\ell,\ell}$ is either the null function or a simultaneous eigenfunction of the operator $\mathcal{L}$, with respect to the same eigenvalue λ_n as $\psi_{n,\ell,\ell}$, and of the operators $A(\theta,\varphi)$, $\mathbb{M}_3$, with respect to the eigenvalues $\ell(\ell+1)+4\ell+6\equiv(\ell+2)(\ell+3)$ and $\ell+2$, respectively, that is:

$$\mathbb{T}_2^{(+)}\psi_{n,\ell,\ell}=C_{\ell+2}\,\psi_{n,\ell+2,\ell+2}\,.$$

Furthermore, the raising operator $\mathbb{T}_2^{(+)}$ provides a clarification of the accidental degeneracy of the spectrum of the operator $\widetilde{\mathcal{L}}$ because its iterated action on the eigenfunction $\psi_{n,0,0}$, where n is even, gives:

$$\mathbb{T}_2^{(+)}\psi_{n,0,0}=C_2\,\psi_{n,2,2}\,,\quad\mathbb{T}_2^{(+)}\psi_{n,2,2}=C_4\,\psi_{n,4,4}\,,\qquad\ldots,\quad\mathbb{T}_2^{(+)}\psi_{n,n-2,n-2}=C_n\,\psi_{n,n,n}$$

and analogously with n odd:

$$\mathbb{T}_2^{(+)}\psi_{n,1,1}=C_3\,\psi_{n,3,3}\,,\quad\mathbb{T}_2^{(+)}\psi_{n,3,3}=C_5\,\psi_{n,5,5}\,,\qquad\ldots,\quad\mathbb{T}_2^{(+)}\psi_{n,n-2,n-2}=C_n\,\psi_{n,n,n}\,,$$

where the coefficients C_i are coefficients of normalization, that is the action of the raising operator $\mathbb{T}_2^{(+)}$ on the eigenfunctions $\psi_{n,\ell,\ell}$ leaves the parameters n unchanged and modifies the parameters ℓ, m, only.

Regarding the operator $\mathbb{T}_2^{(-)}$, we have the relation (it can be proven by means of the same strategy as in Theorem 9):

$$\left[A(\theta,\varphi),\, \mathbb{T}_2^{(-)}\right] = -4\mathbb{T}_2^{(-)}\mathbb{M}_3 + 6\mathbb{T}_2^{(-)} + \left(-2i\,\frac{\partial}{\partial x}\frac{\partial}{\partial z} - 2\,\frac{\partial}{\partial y}\frac{\partial}{\partial z} + 2ixz + 2yz\right)\mathbb{T}_1^{(-)},$$

where $\mathbb{T}_1^{(-)}$ is the lowering operator of the natural degeneracy in (13), from which we obtain the action:

$$\left[A(\theta,\varphi),\, \mathbb{T}_2^{(-)}\right]\psi_{n,\ell,-\ell} = (4\ell+6)\left[\mathbb{T}_2^{(-)}\psi_{n,\ell,-\ell}\right]. \tag{18}$$

Again, for the above reasons, the function $\mathbb{T}_2^{(-)}\psi_{n,\ell,-\ell}$ is either the null function or a simultaneous eigenfunction of the operator $\mathcal{L}$, with respect to the same eigenvalue λ_n as $\psi_{n,\ell,-\ell}$, and of the operators $A(\theta,\varphi)$, $\mathbb{M}_3$, with respect to the eigenvalues $\ell(\ell+1)+4\ell+6 \equiv (\ell+2)(\ell+3)$ and $-\ell-2$, respectively, that is:

$$\mathbb{T}_2^{(-)}\psi_{n,\ell,-\ell} = \widetilde{C}_{\ell+2}\,\psi_{n,\ell+2,-\ell-2}\,.$$

Since the action of the operator $\mathbb{T}_2^{(-)}$ on the eigenfunctions $\psi_{n,\ell,-\ell}$ raises the parameter ℓ by two units, as the operator $\mathbb{T}_2^{(+)}$, we can conclude that there is no lowering operator for the parameter ℓ, but this is not surprising because, by virtue of (17a), the operator $\mathbb{T}_2^{(-)}$ lowers the parameter m of the eigenfunctions $\psi_{n,\ell,-\ell}$ from $-\ell$ to $-\ell-2$. This means that the parameter ℓ cannot change from ℓ to $\ell-2$ because otherwise, we would have that:

$$|m| = |-\ell-2| > \ell-2,$$

which is absurd, due to the constraint $|m| \leqslant \ell$.

Furthermore, the operator $\mathbb{T}_2^{(-)}$ provides a clarification of the accidental degeneracy of the spectrum of the operator $\widetilde{\mathcal{L}}$ because its iterated action on the eigenfunction $\psi_{n,0,0}$, where n is even, gives:

$$\mathbb{T}_2^{(-)}\psi_{n,0,0} = \widetilde{C}_2\,\psi_{n,2,-2}\,,\;\; \mathbb{T}_2^{(-)}\psi_{n,2,-2} = \widetilde{C}_4\,\psi_{n,4,-4}\,,\;\; \ldots,\;\; \mathbb{T}_2^{(-)}\psi_{n,n-2,-n+2} = \widetilde{C}_n\,\psi_{n,n,-n},$$

and also, if n is odd:

$$\mathbb{T}_2^{(-)}\psi_{n,1,-1} = \widetilde{C}_3\,\psi_{n,3,-3}\,,\;\; \mathbb{T}_2^{(-)}\psi_{n,3,-3} = \widetilde{C}_5\,\psi_{n,5,-5}\,,\;\; \ldots,\;\; \mathbb{T}_2^{(-)}\psi_{n,n-2,-n+2} = \widetilde{C}_n\,\psi_{n,n,-n}\,,$$

where the coefficients $\widetilde{C}_i$ are coefficients of normalization, i.e., the action of the operator $\mathbb{T}_2^{(-)}$ on the eigenfunctions $\psi_{n,\ell,-\ell}$ leaves the parameters n unchanged and modifies the parameters ℓ, m, only.

5. Discussion

In this paper, we focused on the accidental degeneracy of a second-order, Schrödinger-type differential operator, acting on a Hilbert space. Typically, in the theory of PDEs, the concept of degeneracy is connected to the number of parameters on which the eigenvalues depend. Natural degeneracy and accidental degeneracy were reformulated and characterized by using the ladder operators. Such a useful tool can be further employed to provide a new way to describe degeneracy in eigenvalue problems with elliptic operators.

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