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Asymptotic Properties of One Mathematical Model in Food Engineering under Stochastic Perturbations

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Abstract: For the example of one nonlinear mathematical model in food engineering with several equilibria and stochastic perturbations, a simple criterion for determining a stable or unstable equilibrium is reported. The obtained analytical results are illustrated by detailed numerical simulations of solutions of the considered Ito stochastic differential equations. The proposed criterion can be used for a wide class of nonlinear mathematical models in different applications.

Keywords: equilibria; Wiener process; Ito's stochastic differential equation; stabilization by noise; asymptotic mean square stability; stability in probability

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1. Introduction

The aim of this paper is to show a simple criterion to determine a stable or unstable equilibrium for nonlinear systems with several equilibria under stochastic perturbations.

Different types of food models are popular in food engineering research (see, for instance, [1–10] and the references therein). To demonstrate the proposed criterion, the following mathematical model from the food engineering that is described by a nonlinear system with fractional nonlinearity [5]

$$\begin{aligned} \frac{dA(t)}{dt} &= r_1 A(t)(1 - kA(t)) - \frac{\mu A(t)P(t)}{\beta + P(t)}, \\ \frac{dP(t)}{dt} &= r_2 P(t) - \alpha A(t)P(t), \quad t \geq 0, \end{aligned} \tag{1}$$

and the positive initial conditions $A(0) > 0$ and $P(0) > 0$ was chosen. Here, $A(t)$ and $P(t)$ are the concentration of aflatoxins and probiotics in a given food matrix, respectively. All parameters are positive constants and mean the following: r_1 is the intrinsic production rate of aflatoxins, $1/k$ is hte concentration of aflatoxins that can be formed within food matrix, μ is the detoxification ability of probiotics, β is the half-saturation for the association term, r_2 is the rate of the occurrence/application of probiotics, and α is the rate of formation of aflatoxin–probiotics complexes.

Below, all non-negative equilibria of model (1) are considered and the property of stability or instability of each from these equilibria under stochastic perturbations is investigated. The obtained results refine and generalize the results of [5], where model (1) is investigated from the point of view of improving food systems and in the deterministic case only.

The proposed criterion can be used for a wide class of nonlinear mathematical models in different applications.

The remainder of the paper is organized as follows: In Section 2, three possible equilibria of model (1) are described with the necessary and sufficient conditions for the existence of the third equilibrium. In Section 3, a method of stochastic perturbations of model (1) is presented, and the linearization of the considered system of Ito's stochastic differential equations for each from the possible equilibria of model (1) is obtained; in Section 4, some necessary auxiliary definitions and statements from the theory of stability of Ito's stochastic differential equations are included, sufficient conditions of stability or instability for each from three possible equilibria are obtained, and a detail numerical analysis of model (1) under stochastic perturbations is presented via numerical simulations of its solutions. In Section 5, the conclusions are presented, and in Appendix A the standard method used in the paper for the linearization of a nonlinear system is shown.

2. Equilibria

The equilibria of the system (1) are defined by the conditions $A(t) = A = const$ and $P(t) = P = const$, i.e., by the system of two algebraic equations

$$\begin{aligned} \left(r_1(1-kA) - \frac{\mu P}{\beta + P}\right)A = 0, \\ (r_2 - \alpha A)P = 0, \end{aligned} \tag{2}$$

which has the following three solutions:

$$E_0 = (0,0), \qquad E_1 = \left(\frac{1}{k}, 0\right), \qquad E_2 = \left(\frac{r_2}{\alpha}, and \frac{\beta(\alpha - kr_2)}{kr_2 - \alpha(1 - \mu r_1^{-1})}\right). \tag{3}$$

Note that the equilibria E_0 and E_1 always exist and, from (3), it follows that the positive equilibrium E_2 exists if and only if

$$\alpha > kr_2 > \alpha\left(1 - \frac{\mu}{r_1}\right). \tag{4}$$

Remark 1. *Note that in [5], one of the conditions for the existence of equilibrium E_2 is considered in the form $\alpha < kr_2$ and $r_1kr_2 + \mu\alpha < r_1\alpha$, which is equivalent to $\alpha < kr_2 < \alpha\left(1 - \frac{\mu}{r_1}\right)$, which is evidently impossible.*

3. Stochastic Perturbations and Linearization

Let $E^* = (A^*, P^*)$ be one from the equilibria (3) of system (1). Let us suppose that the system (1) is exposed to stochastic perturbations that are of the white noise type and are proportional to the deviation in the system state $(A(t), P(t))$ from the equilibrium $E^* = (A^*, P^*)$. Then, the system (1) transforms to the following system of Ito's stochastic differential equations [11]:

$$\begin{aligned} dA(t) &= \left(r_1 A(t)(1 - kA(t)) - \frac{\mu A(t)P(t)}{\beta + P(t)}\right)dt + \sigma_1(A(t) - A^*)dw_1(t), \\ dP(t) &= (r_2 P(t) - \alpha A(t)P(t))dt + \sigma_2(P(t) - P^*)dw_2(t), \end{aligned} \tag{5}$$

where σ_1 and σ_2 are constants and $w_1(t)$ and $w_2(t)$ are the mutually independent standard Wiener processes.

Note that the equilibrium $E^* = (A^*, P^*)$ of initial system (1) is the solution of the stochastic system (5) too.

Calculating the Jacobian matrix for the system (5), we obtain the linear approximation (see (A3) in the Appendix A) of the nonlinear system (5) in the form

$$\begin{aligned} dz_1(t) &= \left(\left(r_1(1 - 2kA^*) - \frac{\mu P^*}{\beta + P^*}\right)z_1(t) - \frac{\mu\beta A^*}{(\beta + P^*)^2}z_2(t)\right)dt + \sigma_1 z_1(t)dw_1(t), \\ dz_2(t) &= (-\alpha P^* z_1(t) + (r_2 - \alpha A^*)z_2(t))dt + \sigma_2 z_2(t)dw_2(t). \end{aligned} \tag{6}$$

Using (2). rewrite the linear system (6) separately for each equilibrium (3):

- for E_0, the system (6) splits into two separate unrelated equations:

$$\begin{aligned} dz_1(t) &= r_1 z_1(t)dt + \sigma_1 z_1(t)dw_1(t), \\ dz_2(t) &= r_2 z_2(t)dt + \sigma_2 z_2(t)dw_2(t); \end{aligned} \tag{7}$$

- for E_1,

$$\begin{aligned} dz_1(t) &= \left(-r_1 z_1(t) - \frac{\mu}{k\beta} z_2(t)\right)dt + \sigma_1 z_1(t)dw_1(t), \\ dz_2(t) &= -\left(\frac{\alpha}{k} - r_2\right)z_2(t)dt + \sigma_2 z_2(t)dw_2(t); \end{aligned} \tag{8}$$

- for E_2,

$$\begin{aligned} dz_1(t) &= \left(-\frac{kr_1r_2}{\alpha}z_1(t) - \frac{r_1^2 r_2(kr_2 - \alpha(1-\mu r_1^{-1}))^2}{\alpha^3\beta\mu}z_2(t)\right)dt + \sigma_1 z_1(t)dw_1(t); \\ dz_2(t) &= -\frac{r_1\alpha\beta(\alpha - kr_2)}{kr_2 - \alpha(1-\mu r_1^{-1})}z_1(t)dt + \sigma_2 z_2(t)dw_2(t). \end{aligned} \tag{9}$$

4. Stability

Let $\{\Omega, \mathcal{F}, \mathbf{P}\}$ be a complete probability space; $\{\mathcal{F}_t, t \geq 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathcal{F}$, i.e., $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ for $t_1 < t_2$; and $\mathbf{E}$ be the mathematical expectation with respect to the measure $\mathbf{P}$.

Consider the system of two linear stochastic differential equations [12]

$$\begin{aligned} \dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \sigma_1 x_1(t)\dot{w}_1(t), \\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \sigma_2 x_2(t)\dot{w}_2(t), \end{aligned} \tag{10}$$

where a_{ij} and σ_i, $i, j = 1, 2$ are constants; and $w_1(t)$ and $w_2(t)$ are the mutually independent standard Wiener processes.

Definition 1. *The zero solution of the system* (6) *is called stable in probability if for any* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$, *there exists* $\delta > 0$ *such that the solution* $z(t) = (z_1(t), z_2(t))$ *of the system* (6) *satisfies the condition* $\mathbf{P}\{\sup_{t\geq 0} |z(t)| > \varepsilon_1\} < \varepsilon_2$, *provided that* $\mathbf{P}\{|z(0)| < \delta\} = 1$.

Definition 2. *The zero solution of the system* (10) *is called:*

- *mean square stable if for each* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\mathbf{E}|x(t)|^2 < \varepsilon$, $x(t) = (x_1(t), x_2(t))$, *and* $t \geq 0$, *provided that* $|x(0)|^2 < \delta$;
- *asymptotically mean square stable if it is mean square stable and, for each initial value* $x(0)$, *the solution* $x(t)$ *of the system* (10) *satisfies the condition* $\lim_{t\to\infty} \mathbf{E}|x(t)|^2 = 0$.

Remark 2. *Note that the level of nonlinearity of the system* (5) *is higher than one. It is known* [12] *that, in this case, a sufficient condition for asymptotic mean square stability of the zero solution of the linear approximation* (6) *at the same time is a sufficient condition for stability in probability of the appropriate equilibrium of the system* (5)*. Therefore, to obtain the conditions for stability in probability of each from equilibria* (3)*, it is enough to obtain the conditions for asymptotic mean square stability of the zero solution for each from the linear systems* (7)–(9)*. On the other hand, the instability of one of the linear systems* (7)*,* (8)*, or* (9) *means the instability of the corresponding equilibrium of the system* (5)*.*

Remark 3. *In the deterministic case* ($\sigma_1 = \sigma_2 = 0$)*, the zero solution of the system* (10) *is asymptotically stable if and only if* [12]

$$\begin{aligned} Tr(A) &= a_{11} + a_{22} < 0, \\ \det(A) &= a_{11}a_{22} - a_{12}a_{21} > 0. \end{aligned} \tag{11}$$

Lemma 1 ([12])**.** *Put $A = \|a_{ij}\|$, $i, j = 1, 2$, $\delta_i = \frac{1}{2}\sigma_i^2$, $i = 1, 2$. Suppose that the conditions (11) hold, $a_{12} \neq 0$, and*

$$\delta_1 < \frac{|Tr(A)|\det(A)}{A_2}, \qquad \delta_2 < \frac{|Tr(A)|\det(A) - A_2\delta_1}{A_1 - |Tr(A)|\delta_1}, \qquad A_1 = \det(A) + a_{11}^2, \qquad A_2 = \det(A) + a_{22}^2. \tag{12}$$

Then, the zero solution of the system (10) is asymptotically mean square stable.

Remark 4. *The proof of Lemma 1 is based on using the Lyapunov function $v = x'Px$, where $P > 0$ is the positive definite solution of the matrix equation $PA + A'P = -Q$, $Q > 0$, and estimating the negative definite square form $Lv < 0$, where L is the generator [11] of the system (10). (For more details, see [12], p. 48.)*

Corollary 1. *If $a_{12}a_{21} = 0$, then the conditions (11) and (12) take the form*

$$a_{11} < 0, \qquad a_{22} < 0. \qquad \delta_1 < -a_{11}, \qquad \delta_2 < -a_{22}. \tag{13}$$

If $a_{22} = 0$, then the conditions (11) take the form

$$a_{11} < 0, \qquad a_{12}a_{21} < 0. \tag{14}$$

4.1. Equilibrium E_0

From (11), it follows that in the deterministic case ($\sigma_1 = \sigma_2 = 0$), the zero solution of the system (7) is unstable. However, under the conditions

$$\sigma_1^2 > 2r_1, \qquad \sigma_2^2 > 2r_2, \tag{15}$$

the so-called "stabilization by noise" occurs [13,14].

In Figure 1, 25 trajectories of the solution of Equation (5) are shown with $r_1 = 0.1$, $r_2 = 0.15$, $\mu = 0.1$, $k = 2.5$, $\alpha = 0.9$, $\beta = 0.1$, $A(0) = 0.15$, $P(0) = 0.2$, $\sigma_1 = 0.18$, and $\sigma_2 = 0.19$. The equilibrium $E_0(0, 0)$ is unstable, so the trajectories fill the whole space.

In Figure 2, 25 trajectories of the solution of Equation (5) are shown with $A(0) = 0.4$, $P(0) = 0.6$, $\sigma_1 = 1.1$, and $\sigma_2 = 1.2$, and the same values of all other parameters as in Figure 1. The conditions (15) hold, stabilization by noise' occurs, and all trajectories converge to the unstable equilibrium $E_0(0, 0)$.

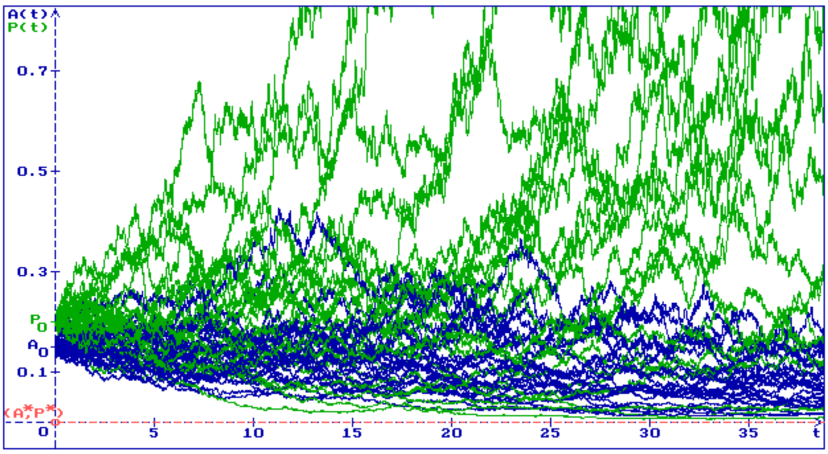

Figure 1. The 25 trajectories of $A(t)$ (blue) and $P(t)$ (green) of a solution of the system (5), with $r_1 = 0.1$, $r_2 = 0.15$, $\mu = 0.1$, $k = 2.5$, $\alpha = 0.9$, $\beta = 0.1$, $A(0) = 0.15$, $P(0) = 0.2$, $\sigma_1 = 0.18$, and $\sigma_2 = 0.19$.

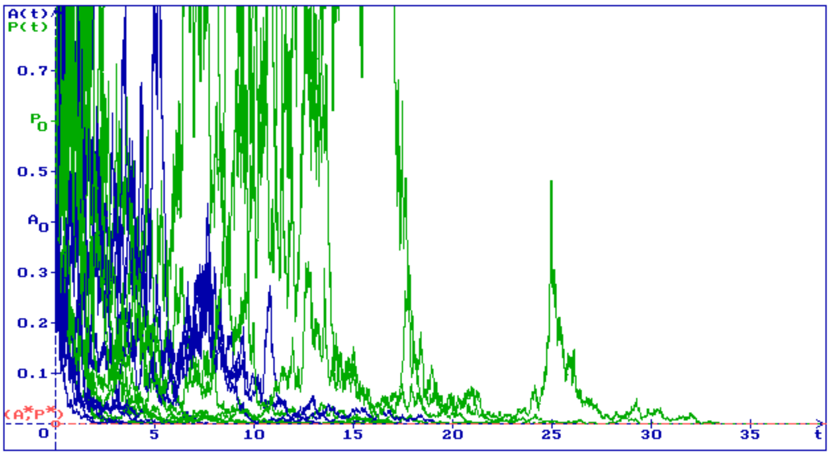

Figure 2. The 25 trajectories $A(t)$ (blue) and $P(t)$ (green) of a solution of the system (5), with $r_1 = 0.1$, $r_2 = 0.15$, $\mu = 0.1$, $k = 2.5$, $\alpha = 0.9$, $\beta = 0.1$, $A(0) = 0.4$, $P(0) = 0.6$, $\sigma_1 = 1.1$, and $\sigma_2 = 1.2$.

4.2. *Equilibrium E_1*

For system (8), the conditions (13) take the form

$$\frac{1}{2}\sigma_1^2 < r_1, \qquad \frac{1}{2}\sigma_2^2 + r_2 < \frac{\alpha}{k}. \tag{16}$$

So, by conditions (16), the equilibrium E_1 is stable in probability.

In Figure 3, three trajectories of the solution of Equation (5) are shown with $r_1 = 0.6$, $r_2 = 0.2$, $\mu = 0.1$, $k = 2.5$, $\alpha = 0.9$, $\beta = 0.1$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.4$, and different initial conditions: $M_1(0.15, 0.4)$, $M_2(0.5, 0.6)$, and $M_3(0.9, 0.5)$. The conditions (16) hold, and all trajectories converge to the stable equilibrium $E_1 = (A^*, P^*)$ with $A^* = 0.4$, and $P^* = 0$.

In Figure 4, 50 trajectories of the solution of Equation (5) are shown with $A(0) = 0.65$, $P(0) = 0.15$, $\sigma_1 = 0.6$, and $\sigma_2 = 0.4$, and the same values of all other parameters as in Figure 3. All trajectories converge to the stable equilibrium $E_1 = (A^*, P^*) = (0.4, 0)$.

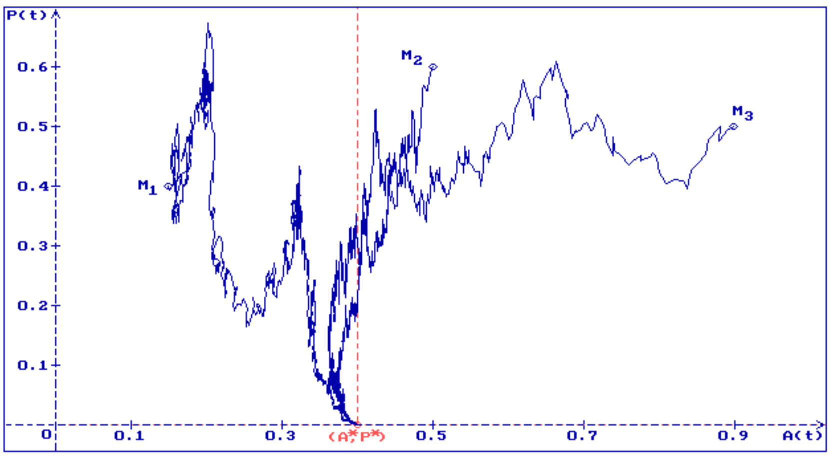

Figure 3. Three trajectories of a solution $(A(t), P(t))$ of the system (5) are shown with $r_1 = 0.6$, $r_2 = 0.2$, $\mu = 0.1$, $k = 2.5$, $\alpha = 0.9$, $\beta = 0.1$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.4$, for different initial conditions: $M_1(0.15, 0.4)$, $M_2(0.5, 0.6)$, and $M_3(0.9, 0.5)$.

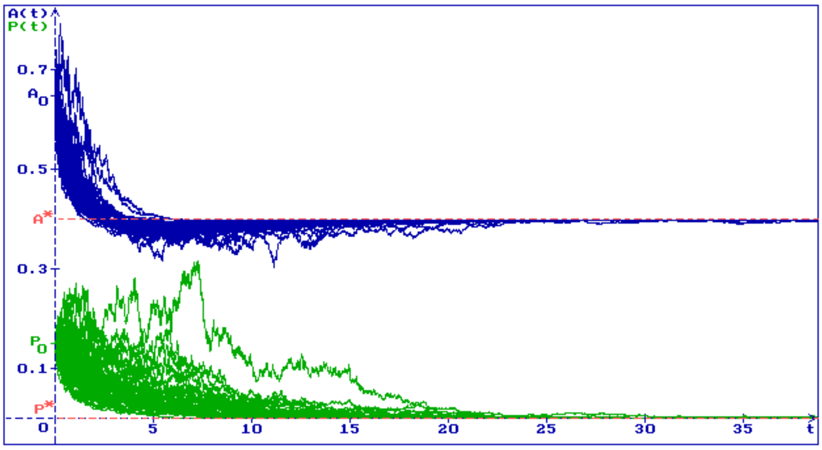

Figure 4. Fifty trajectories $A(t)$ (blue) and $P(t)$ (green) of a solution of the system (5) are shown with $r_1 = 0.6$, $r_2 = 0.2$, $\mu = 0.1$, $k = 2.5$, $\alpha = 0.9$, $\beta = 0.1$, $A(0) = 0.65$, $P(0) = 0.15$, $\sigma_1 = 0.6$, and $\sigma_2 = 0.4$.

4.3. Equilibrium E_2

In the system (9), $a_{22} = 0$ and $a_{12}a_{21} > 0$, i.e., the second condition (14) does not hold. So, in the deterministic case, the equilibrium E_2 is unstable.

In Figure 5, three trajectories of the solution $(A(t), P(t))$ of the system (5) are shown with $r_1 = 0.6$, $r_2 = 0.21$, $\mu = 0.6$, $k = 2.5$, $\alpha = 0.7$, $\beta = 0.1$, $\sigma_1 = 0.12$, $\sigma_2 = 0.03$, $A^* = 0.3$, and $P^* = 0.0333$, and different initial conditions: $M_1(0.28, 0.035)$, $M_2(0.31, 0.035)$, $M_3(0.3, 0.031)$. One can see that the all initial conditions are close enough to the equilibrium (A^*, P^*), but all trajectories move away from the equilibrium, since this equilibrium is unstable.

In Figure 6, 25 trajectories of the solution of Equation (5) are shown with $A(0) = 0.31$, $P(0) = 0.0433$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.1$, and the same values of all other parameters as in Figure 5. All trajectories move away from the equilibrium $E_2 = (A^*, P^*) = (0.3, 0.0333)$, since this equilibrium is unstable.

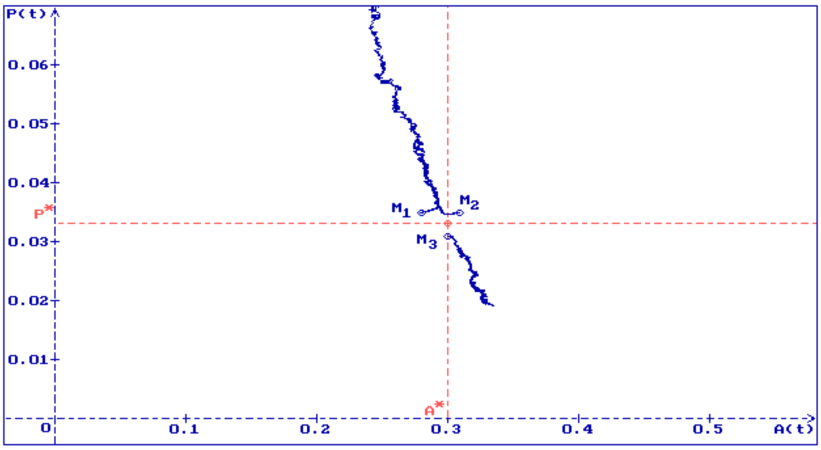

Figure 5. Three trajectories of a solution $(A(t), P(t))$ of the system (5) with $r_1 = 0.6$, $r_2 = 0.21$, $\mu = 0.6$, $k = 2.5$, $\alpha = 0.7$, $\beta = 0.1$, $\sigma_1 = 0.12$, and $\sigma_2 = 0.03$, and initial conditions $M_1(0.28, 0.035)$, $M_2(0.31, 0.035)$, and $M_3(0.3, 0.031)$.

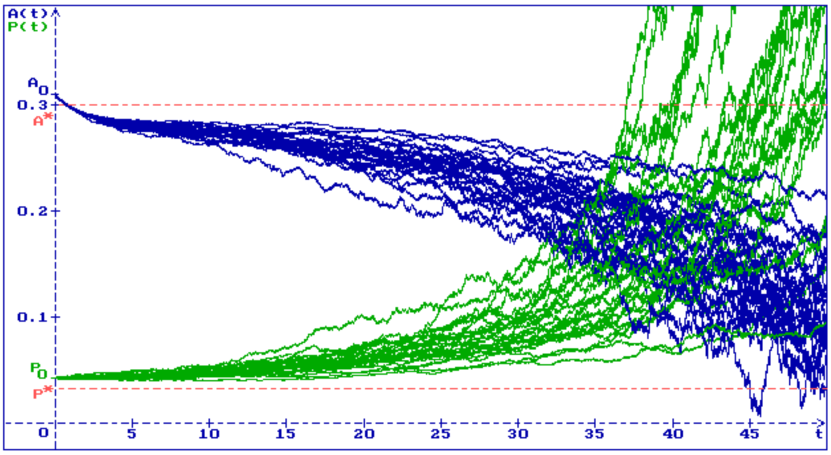

Figure 6. Twenty-five trajectories $A(t)$ (blue) and $P(t)$ (green) of the solution of system (5) with $r_1 = 0.6$, $r_2 = 0.21$, $\mu = 0.6$, $k = 2.5$, $\alpha = 0.7$, $\beta = 0.1$, $A(0) = 0.31$, $P(0) = 0.0433$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.1$.

Remark 5. *Note that to simulate solutions of the considered Ito stochastic differential equations in Figures 1–6, a special algorithm for modeling trajectories of the standard Wiener process was used, as described in [12].*

5. Conclusions

Systems of nonlinear differential equations are used to describe mathematical models in many different applications. As a rule, such models can have several equilibria, each of which can be stable or unstable. A simple criterion was proposed in this paper that can define the stability or instability of each considered equilibrium under the presence of stochastic perturbations around of this equilibrium. Additionally, we showed how the classical "stabilization by noise" can be applied for stabilization of an unstable equilibrium. We also showed how the properties of the nonlinear model equilibria under stochastic perturbations can be demonstrated via numerical simulations of solutions of the considered Ito stochastic differential equations. The obtained results can be applied to many other nonlinear models in different applications.

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Appendix A. Linearization

Consider the nonlinear differential equation

$$\frac{dx(t)}{dt} = F(x(t)), \tag{A1}$$

where $x \in \mathbf{R}^n$ and the equation $F(x) = 0$ has a solution x^* that is an equilibrium of the differential Equation (A1). Using the new variable $y(t) = x(t) - x^*$, Equation (A1) can be represented in the form

$$\frac{dy(t)}{dt} = F(x^* + y(t)). \tag{A2}$$

It is clear that the stability of the zero solution of Equation (A2) is equivalent to stability of the equilibrium x^* of Equation (A1).

Let $J_F = \left\| \dfrac{\partial F_i}{\partial x_j} \right\|$, $i, j = 1, ..., n$, be the Jacobian matrix of the function $F = \{F_1, ..., F_n\}$ and $\lim_{|y| \to 0} \dfrac{|o(y)|}{|y|} = 0$, where $|y|$ is the Euclidean norm in $\mathbf{R}^n$. Using Taylor's expansion in the form $F(x^* + y) = F(x^*) + J_F(x^*)y + o(y)$ and the equality $F(x^*) = 0$, we obtain the linear approximation

$$\frac{dz(t)}{dt} = J_F(x^*)z(t) \tag{A3}$$

of Equation (A2).

So, a condition for asymptotic stability of the zero solution of Equation (A3) is also a condition for the local stability of the equilibrium x^* of the initial Equation (A1).

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