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# Implicit Solitary Waves for One of the Generalized Nonlinear Schrödinger Equations

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**Abstract:** Application of transformations for dependent and independent variables is used for finding solitary wave solutions of the generalized Schrödinger equations. This new form of equation can be considered as the model for the description of propagation pulse in a nonlinear optics. The method for finding solutions of equation is given in the general case. Solitary waves of equation are obtained as implicit function taking into account the transformation of variables.

**Keywords:** generalized Schrödinger equation; solitary wave; exact solution; implicit function



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## 1. Introduction

In this paper, we consider the nonlinear partial differential equation

$$i\,q_t + q_{xx} + \alpha\,q + \beta\,|q|^n\,q + \gamma\,|q|^{2n}\,q + \delta\,|q|^{3n}\,q + \lambda\,|q|^{4n}\,q = 0, \quad (1)$$

where $q(x,t)$ is complex function, $x$ is coordinate, $t$ is time, $n$ is rational number and $\alpha$, $\beta$, $\gamma$, $\delta$, $\lambda$ are parameters of Equation (1). It is easy to see that Equation (1) is the generalization of the famous nonlinear Schrödinger equation which follows from Equation (1) at $\beta \neq 0$, $n = 2$, $\alpha = \gamma = \delta = \lambda = 0$. Equation (1) has been presented in recent paper [1] as an equation whose solution can be obtained using the method of transformation for dependent and independent variables. Equation (1) is the generalization of some equations describing propagation pulses in the nonlinear optics (see, for example, [2–19]).

The purpose of this paper is to present the method for finding solutions of Equation (1) and to obtain the implicit solitary wave solutions of Equation (1) using the transformations of variables.

This article is organized as follows. In Section 2, the method of finding solutions of Equation (1) is presented taking into account the traveling wave reduction. In this Section the general approach to finding exact solutions of Equation (1) is described as weel. The implicit solitary waves of Equation (1) in form of kink are given in Section 3. Implicit soliton solutions of Equation (1) are presented in Section 4.

## 2. Method Applied

Let us look for the exact solution of Equation (1) using the the form

$$q(x,t) = y(z)\,e^{i\,(k\,x - \omega\,t)}, \quad (2)$$

where $y(z)$ is a function describing an optical pulse profile, $\omega$ is a frequency and $k$ is a wave number and $z$ is a variable of $x$ and $t$: $z = x - C_0\,t$.

Substituting (2) into Equation (1) and equating expressions for real and imaginary parts yields the overdetermined system of equations for function $y(z)$ in the form

$$(2\,k - C_0)\,y_z = 0, \quad (3)$$

$$y_{zz} + (\omega - k^2)\, y + \alpha\, y + \beta\, y^{n+1} - \gamma\, y^{2n+1} - \delta\, y^{3n+1} + \lambda\, y^{4n+1} = 0. \tag{4}$$

Provided that $C_0 = 2\,k$ we see that Equation (3) is satisfied. Multiplying Equation (4) by $y_z$ and integrating over $z$, we obtain the first integral in the form

$$\begin{gathered} y_z^2 + (\omega + \alpha - k^2)\, y^2 + \tfrac{2\,\beta}{n+2}\, y^{n+2} - \tfrac{\gamma}{n+1}\, y^{2n+2} - \\ \tfrac{2\,\delta}{3n+2}\, y^{3n+2} + \tfrac{\lambda}{2n+1}\, y^{4n+2} = C_1, \end{gathered} \tag{5}$$

where $C_1$ is a constant of integration.

Solution of Equation (5) can be written in the form of quadrature

$$\int \frac{d\xi}{\sqrt{H[y]}} = z - z_0, \tag{6}$$

where

$$\begin{gathered} H[y] = C_1 - (\omega + \alpha - k^2)\, y^2 - \tfrac{2\,\beta}{n+2}\, y^{n+2} + \tfrac{\gamma}{n+1}\, y^{2n+2} + \\ \tfrac{2\,\delta}{3n+2}\, y^{3n+2} - \tfrac{\lambda}{2n+1}\, y^{4n+2}. \end{gathered} \tag{7}$$

However integral (6) cannot be calculated in the general case.

Let us look for solution of Equation (5) in the form

$$y(z) = F(\xi), \qquad \xi_z = F(\xi)^n. \tag{8}$$

Using (8), we have

$$y_z = F_\xi\, \xi_z \; = \; F_\xi\, F(\xi)^n. \tag{9}$$

Substituting (8) and (9) into Equation (5), we obtain the equation

$$\begin{gathered} F_\xi^2 + (\omega + \alpha - k^2)\, F^{2-2n} + \tfrac{2\,\beta}{n+2}\, F^{2-n} - \tfrac{\gamma}{n+1}\, F^2 - \\ \tfrac{2\,\delta}{3n+2}\, F^{n+2} + \tfrac{\lambda}{2n+1}\, F^{2n+2} = 0. \end{gathered} \tag{10}$$

Equation (10) has been previously studied in papers [1–3]. It is important to note that by using the transformation [20–23]

$$F(\xi) = V(\xi)^{-\frac{1}{n}}, \tag{11}$$

Equation (10) can be reduced to the equation with solutions in the form of elliptic function

$$\begin{gathered} V_\xi^2 + (\omega + \alpha - k^2)\, n^2\, V^4 + \frac{2\,n^2\,\beta}{n+2}\, V^3 - \frac{n^2\,\gamma}{n+1}\, V^2 - \\ \frac{2\,n^2\,\delta}{3n+2}\, V + \frac{n^2\,\lambda}{2n+1} \; = 0. \end{gathered} \tag{12}$$

Solution of Equation (12) can be searched for in the form [24–26]

$$V(\xi) = V_1 + \frac{(V_2 - V_1)\, E}{Y^2 + E}, \qquad E = \frac{(V_1 - V_3)}{(V_3 - V_2)}, \tag{13}$$

where $V_1$, $V_2$, $V_3$ and $V_4$ are the roots of the following algebraic equation

$$(\omega + \alpha - k^2)\, V^4 + \frac{2\,\beta}{n+2}\, V^3 - \frac{\gamma}{n+1}\, V^2 - \frac{2\,\delta}{3n+2}\, V + \frac{\lambda}{2n+1} \; = 0 \tag{14}$$

and $Y(\xi)$ is the Jacobi elliptic sine in the form

$$Y(\xi; k) = \mathrm{sn}\left\{\frac{n}{2}\,\sqrt{a\,(V_4 - V_2)(V_1 - V_3)}\,(\xi - \xi_0);\, S\right\}, \tag{15}$$

where $S$ is determined by the formula

$$S^2 = \frac{(V_1 - V_4)\,(V_1 - V_3)}{(V_4 - V_2)\,(V_3 - V_2)}. \tag{16}$$

Taking into account (11), the solution $F(\xi)$ can be expressed by the formula

$$F(\xi) = \left[ \frac{V_1(V_3 - V_2)\mathrm{sn}^2\left\{\frac{n}{2}\sqrt{a(V_4 - V_2)(V_1 - V_3)}(\xi - \xi_0);S\right\} + V_2(V_1 - V_3)}{(V_3 - V_2)\mathrm{sn}^2\left\{\frac{1}{2}\sqrt{d(V_4 - V_2)(V_1 - V_3)}\,(\xi - \xi_0);S\right\} + V_1 - V_3} \right]^{-\frac{1}{n}}. \tag{17}$$

We cannot find the explicit expression for the function $\xi(z)$ using $V(\xi)$ in the general case by means of the formula

$$\int V(\xi)\,d\,\xi = z - z_0. \tag{18}$$

However in the case of solitary wave solutions these solutions of Equation (1) can be found as the implicit functions. To look for these solutions we use the special methods has been developing in the last few years [27–36].

## 3. Implicit Solitary Wave Solutions of the Generalized Nonlinear Schrödinger Equation in Form Kink

Let us look for the solution of Equation (12) using the logistic function. We assume that there exist a solution of Equation (12) in the form [37–46]

$$V(\xi) = A_0 + A_1\,Q(\xi), \tag{19}$$

where $Q(\xi)$ is the logistic function [37]

$$Q(\xi) = \frac{1}{1 + \mathrm{e}^{m\,(\xi - \xi_0)}}. \tag{20}$$

The function $Q(\xi)$ is the solution of the Riccati equation in the form

$$Q_\xi = m\,(Q^2 - Q). \tag{21}$$

The function $Q(\xi)$ satisfies the following second-order differential equation as well

$$Q_{\xi\xi} = m^2\,Q\,(Q - 1)\,(2\,Q - 1). \tag{22}$$

Substituting (19) into Equation (12) and taking Equations (21) and (22) into account, yields the equality

$$\begin{gathered}
\left(n^2{A_1}^4\omega - n^2{A_1}^4k^2 + n^2{A_1}^4\alpha + {A_1}^2m^2\right)Q^4 + \Big(4\,n^2A_0\,{A_1}^3\alpha - \\
2\,{A_1}^2m^2 - 4\,n^2A_0\,{A_1}^3k^2 + 4\,n^2A_0\,{A_1}^3\omega + \tfrac{2\,n^2{A_1}^3\beta}{2+n}\Big)\,Q^3 + \Big({A_1}^2m^2 + \\
\tfrac{6\,n^2A_0\,{A_1}^2\beta}{2+n} + 6\,n^2{A_0}^2{A_1}^2\alpha + 6\,n^2{A_0}^2{A_1}^2\omega - \tfrac{n^2{A_1}^2g}{1+n} - \\
6\,n^2{A_0}^2{A_1}^2k^2\Big)\,Q^2 + \Big(\tfrac{6\,n^2{A_0}^2A_1\,\beta}{2+n} - \tfrac{2\,n^2A_0\,A_1\,g}{1+n} - \tfrac{2\,n^2A_1\,\delta}{2+3\,n} - \\
4\,n^2{A_0}^3A_1\,k^2 + 4\,n^2{A_0}^3A_1\,\alpha + 4\,n^2{A_0}^3A_1\,\omega\Big)\,Q - n^2A0^4k^2 + n^2A0^4\alpha + \\
n^2{A_0}^4\omega + \tfrac{\lambda\,n^2}{1+2\,n} - 2\,\tfrac{n^2A_0\,\delta}{2+3\,n} - \tfrac{n^2{A_0}^2g}{1+n} + 2\,\tfrac{n^2{A_0}^3\beta}{2+n} = 0.
\end{gathered} \tag{23}$$

We have obtained that a polynomial in solutions $Q(z)$ is equal to zero. Such thing is possible if and only if all coefficients are equal to zero. Taking into account this property in (23), we derive the conditions for the parameters of Equation (1). These conditions are the following

$$\alpha = k^2 - \omega - \frac{m^2}{n^2{A_1}^2}, \tag{24}$$

$$\beta = \frac{m^2(2 + n)(A_1 + 2\,A_0)}{n^2{A_1}^2}, \tag{25}$$

$$\gamma = \frac{\left(6\,A_0{}^2 + 6\,A_0\,A_1 + A_1{}^2\right) m^2 (1+n)}{n^2 A_1{}^2}, \tag{26}$$

$$\delta = -\frac{(2+3\,n) m^2 A_0 \left(2\,A_0{}^2 + 3\,A_0\,A_1 + A_1{}^2\right)}{n^2 A_1{}^2}, \tag{27}$$

$$\lambda = -\frac{(1+2\,n) m^2 A_0{}^2 \left(A_0{}^2 + 2\,A_0\,A_1 + A_1{}^2\right)}{n^2 A_1{}^2}. \tag{28}$$

Using solution (19) and definition (18), we get the implicit function $\xi(z)$ in the form

$$(A_0 + A_1)\,\xi - \frac{A_1}{m}\,\log\left(1 + \mathrm{e}^{m\,\xi}\right) = z - z_0. \tag{29}$$

On the other hand taking into account (8) and (11), we obtain

$$\xi = \frac{1}{m}\,\log\left[\frac{(A_0 + A_1)\,y^n - 1}{A_0\,y^n - 1}\right]. \tag{30}$$

Substituting (30) into (29)yields an implicit expression for $y(z)$ in the form

$$\frac{(A_0 + A_1)}{m}\,\log\left[\frac{(A_0 + A_1)\,y^n - 1}{A_0\,y^n - 1}\right] - \frac{A_1}{m}\,\log\left(\frac{A_1\,y^n}{A_0\,y^n - 1}\right) = z - z_0. \tag{31}$$

We have obtained implicit expressions for kinks $y(\xi)$ and $y(z)$, where $A_0$, $A_1$, $m$ and $n$ are arbitrary. These values allow us to calculate the parameters $\alpha$, $\beta$, $\gamma$, $\delta$ and $\lambda$ for Equation (5) using conditions (24)–(28).

Solutions (30) of Equation (10) (on the left) and (31) of (5) (on the right) are demonstrated in Figure 1 at $A_0 = 1.0$, $A_0 = 0.5$, $n = 2$, $m = 0.02$ and $z_0 = 0.0$.

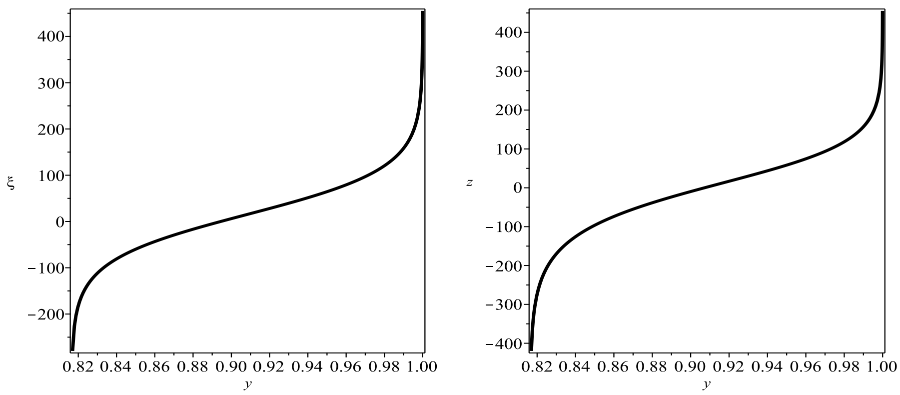


**Figure 1.** Solutions (30) of Equation (10) (**left**) and (31) of (5) (**right**) at $A_0 = 1.0$, $A_0 = 0.5$, $n = 2$, $m = 0.02$ and $z_0 = 0.0$.

## 4. Implicit Optical Solitons of the Generalized Nonlinear Schrödinger Equation

Let us obtain the exact solutions in the form of solitons. We look for the solution of Equation (12) in the form [47–51]

$$V(\xi) = A_0 + A_1\,R(\xi), \tag{32}$$

where the function $R(\xi)$ solves the following equations

$$R_\xi^2 + a\,R^4 + b\,R^3 - c\,R^2 = 0 \tag{33}$$

and

$$R_{\xi\xi} + 2\,a\,R^3 + \frac{3\,b}{2}\,R^2 - c\,R = 0 \tag{34}$$

Solution of Equation (33) is as follows [47]

$$R(\xi) = \frac{4\,c\,\mathrm{e}^{-\xi\,\sqrt{c}}}{4\,a\,c + b^2 + 2\,b\,\mathrm{e}^{-\xi\,\sqrt{c}} + \mathrm{e}^{-2\,\xi\,\sqrt{c}}}. \tag{35}$$

Substituting expression (32) and taking into account (33) and (34) into Equation (12), we obtain the following polynomial

$$\begin{gathered}\left(n^2{A_1}^4\,\alpha - n^2{A_1}^4k^2 + n^2{A_1}^4\omega - {A_1}^2a\right)R^4 + \Big(4\,n^2A_0\,{A_1}^3\alpha - \\ {A_1}^2b - 4\,n^2A_0\,{A_1}^3k^2 + 4\,n^2A_0\,{A_1}^3\omega + \tfrac{2\,n^2{A_1}^3\beta}{2+n}\Big)R^3 + \Big({A_1}^2c + \\ \tfrac{6\,n^2A_0\,{A_1}^2\beta}{2+n} - 6\,n^2{A_0}^2{A_1}^2k^2 + 6\,n^2{A_0}^2{A_1}^2\alpha + 6\,n^2{A_0}^2{A_1}^2\omega - \\ \tfrac{n^2{A_1}^2g}{1+n}\Big)R^2 + \Big(-2\,\tfrac{n^2A_0\,A_1\,g}{1+n} + 6\,\tfrac{n^2{A_0}^2A_1\,\beta}{2+n} - 2\,\tfrac{n^2A_1\,\delta}{2+3\,n} - \\ 4\,n^2{A_0}^3A_1\,k^2 + 4\,n^2{A_0}^3A_1\,\alpha + 4\,n^2{A_0}^3A_1\,\omega\Big)R + \tfrac{\lambda\,n^2}{1+2\,n} - \\ n^2{A_0}^4k^2 + n^2{A_0}^4\alpha + n^2{A_0}^4\omega - 2\,\tfrac{A_0\,\delta\,n^2}{2+3\,n} - \tfrac{{A_0}^2gn^2}{1+n} + 2\,\tfrac{{A_0}^3\beta\,n^2}{2+n} = 0,\end{gathered} \tag{36}$$

Equating the coefficients of polynomial (36) to zero, let us find the following conditions

$$\alpha = \frac{{A_1}^2k^2n^2 - {A_1}^2n^2\omega + a}{{A_1}^2n^2}, \tag{37}$$

$$\beta = -\frac{(2+n)(4\,A_0\,a - A_1\,b)}{2\,{A_1}^2n^2}, \tag{38}$$

$$\gamma = -\frac{\left(6\,{A_0}^2a - 3\,A_0\,A_1\,b - {A_1}^2c\right)(1+n)}{{A_1}^2n^2}, \tag{39}$$

$$\delta = \frac{\left(4\,{A_0}^2a - 3\,A_0\,A_1\,b - 2\,{A_1}^2c\right)A_0\,(2+3\,n)}{2\,{A_1}^2n^2}, \tag{40}$$

$$\lambda = \frac{\left({A_0}^2a - A_0\,A_1\,b - A1^2c\right){A_0}^2(1+2\,n)}{{A_1}^2n^2}. \tag{41}$$

Solution $V(\xi)$ of Equation (12) can be written as the following

$$V(\xi) = A_0 + \frac{4\,A_1\,c\,\mathrm{e}^{-\xi\,\sqrt{c}}}{4\,a\,c + b^2 + 2\,b\,\mathrm{e}^{-\xi\,\sqrt{c}} + \mathrm{e}^{-2\,\xi\,\sqrt{c}}}. \tag{42}$$

At the same time, we find the function $\xi(z)$ from Equation (18)

$$z = A_0\,\xi + \frac{2\,A_1\,\sqrt{c}}{\sqrt{a\,c}}\arctan\left[\frac{(4\,a\,c + b^2)\mathrm{e}^{\xi\,\sqrt{c}} + b}{2\,\sqrt{ac}}\right] + z_0. \tag{43}$$

Solution $V(\xi)$ of Equation (12) is demonstrated in Figure 2 on the left hand side at $A_0 = 5.0$, $A_1 = -2$, $a = 2.0$, $b = 3.0$ and $c = 4.0$. Dependencies $\xi(z)$ are shown on the right hand side of Figure 2 at $A_0 = 5.0$, $A_1 = -2$, $a = 2.0$, $b = 3.0$ and $c = 4.0$ (curve 1), $A_0 = 3.0$, $A_1 = -2$, $a = 2.0$, $b = 3.0$ and $c = 4.0$ (curve 2) and at $A_0 = 1.0$, $A_1 = -2$, $a = 2.0$, $b = 3.0$ and $c = 4.0$ (curve 3).

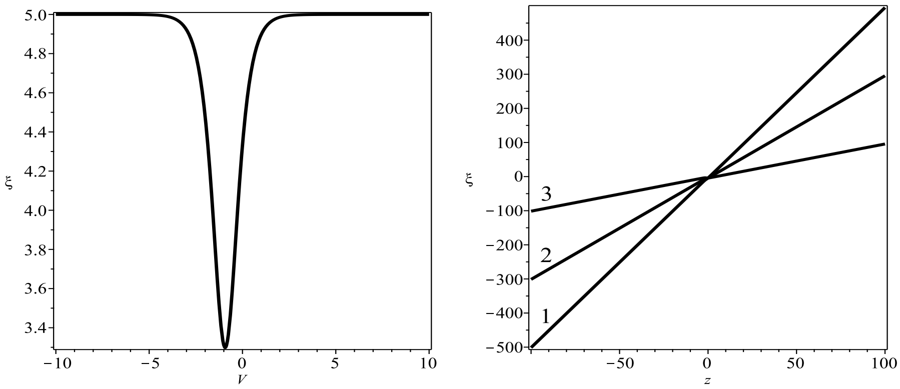


**Figure 2.** Solution (42) of Equation (12) at $A_0 = 5.0$, $A_1 = -2$, $a = 2.0$, $b = 3.0$ and $c = 4.0$ (**left**) and (43) of (18) (**right**) at $A_0 = 5.0$ (curve 1),$A_0 = 3.0$ (curve 2) $A_0 = 1.0$ (curve 3) and at $A_1 = -2.0$, $a = 2.0$, $b = 3.0$ and $c = 4.0$.

Taking into account Equations (8) and (11), we obtain

$$A_0 + \frac{4\,A_1\,c\,\mathrm{e}^{-\xi\,\sqrt{c}}}{4\,a\,c + b^2 + 2\,b\,\mathrm{e}^{-\xi\,\sqrt{c}} + \mathrm{e}^{-2\,\xi\,\sqrt{c}}} - y^{-n}. \tag{44}$$

Solving Equation (44) gives us two expressions for $\xi(y)$

$$\xi_{1,2}(y) = -\frac{1}{\sqrt{c}}\,\log\left[\frac{A_1\,b\,y^n + 2\,A_1\,c\,y^n \mp 2\,\sqrt{P} - b}{1 - A_0\,y}\right], \tag{45}$$

where $P$ is as follows

$$P = \left(A_1^2\,c^2 + A_0\,A_1\,b\,c - A_0^2\,a\,c\right)y^{2n} + (2\,A_0\,a\,c - b\,c)\,y^n - ac. \tag{46}$$

The dependence $\xi(y)$ is the two-valued function. Equating $\xi_1(y)$ and $\xi_2(y)$, we obtain the following formula for $y^*$

$$y^* = \left[\frac{2\,A_0\,a - A_1\,b + \sqrt{4\,a\,c\,{A_1}^2 + {A_1}^2 b^2}}{2\,{A_0}^2 a - 2\,A_0\,A_1\,b - 2\,{A_1}^2 c}\right]^{\frac{1}{n}}. \tag{47}$$

It can be seen that $y^*$ depends on the values of $A_0$, $A_1$, $a$, $b$ and $c$. by substituting $y^*$ into (45) we obtain $\xi^*$. The dependence $\xi(z)$ can be written in the form

$$\xi(y) = \begin{cases} \xi_1(y), & \xi > \xi^*, \\ \xi_2(y), & \xi < \xi^*. \end{cases} \tag{48}$$

Substituting $\xi(y)$ into expression (43), yields the solitary wave in the form

$$z(y) = \begin{cases} A_0\,\xi_1(y) + \frac{2\,A_1\,\sqrt{c}}{\sqrt{a\,c}}\,\arctan\left[\frac{(4\,a\,c + b^2)\mathrm{e}^{\xi_1(y)\,\sqrt{c}} + b}{2\,\sqrt{ac}}\right] + z_0, & z > z^*, \\ A_0\,\xi_2(y) + \frac{2\,A_1\,\sqrt{c}}{\sqrt{a\,c}}\,\arctan\left[\frac{(4\,a\,c + b^2)\mathrm{e}^{\xi_2(y)\,\sqrt{c}} + b}{2\,\sqrt{ac}}\right] + z_0 & z < z^*, \end{cases} \tag{49}$$

where $Z^*$ is found taking into account $y^*$.

Implicit solitary waves solutions $\xi(y)$ of Equation (10) (on the left) and $z(y)$ of Equation (5) are illustrated in Figure 3 at $z_0 = 0.0$, $A_0 = 5.0$, $A_1 = -2.0$, $n = 1$, $a = 2.0$, $b = 3.0$, and $c = 4.0$.

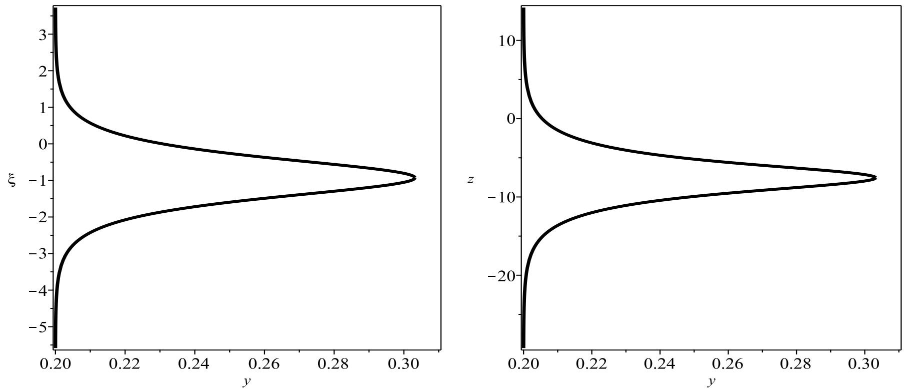


**Figure 3.** Implicit solitary waves solutions $\xi(y)$ of Equation (10) (**left**) and $z(y)$ of Equation (5) (**right**) at $z_0 = 0.0$, $A_0 = 5.0$, $A_1 = -2.0$, $n = 1$, $a = 2.0$, $b = 3.0$, and $c = 4.0$.

## 5. Conclusions

In this paper, Equation (1) has been studied. Equation (1) is the generalization of the famous nonlinear Schrödinger equation and can be used for the description of propagation pulses in optical fiber. Using the transformations for dependent and independent variables we have presented the algorithm for construction of exact solutions of nonlinear differential equations. Exact formulas for solitary waves solutions in the form of kinks and optical solitons are given as the implicit functions. The approach for finding exact solutions can be used for some other nonlinear differential equations.

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