mathematics

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Adjustable Security Proportions in the Fuzzy Portfolio Selection under Guaranteed Return Rates

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Abstract: Based on the concept of high returns as the preference to low returns, this study discusses the adjustable security proportion for excess investment and shortage investment based on the selected guaranteed return rates in a fuzzy environment, in which the return rates for selected securities are characterized by fuzzy variables. We suppose some securities are for excess investment because their return rates are higher than the guaranteed return rates, and the other securities whose return rates are lower than the guaranteed return rates are considered for shortage investment. Then, we solve the proposed expected fuzzy returns by the concept of possibility theory, where fuzzy returns are quantified by possibilistic mean and risks are measured by possibilistic variance, and then we use linear programming model to maximize the expected value of a portfolio's return under investment risk constraints. Finally, we illustrate two numerical examples to show that the expected return rate under a lower guaranteed return rate is better than a higher guaranteed return rates in different levels of investment risks. In shortage investments, the investment proportion for the selected securities are almost zero under higher investment risks, whereas the portfolio is constructed from those securities in excess investments.

Keywords: fuzzy portfolio model; efficient portfolio; guaranteed return rates; excess investment; shortage investment

Citation: Huang, Y.-Y.; Chen, I.-F.; Chiu, C.-L.; Tsaur, R.-C. Adjustable Security Proportions in the Fuzzy Portfolio Selection under Guaranteed Return Rates. *Mathematics* **2021**, *9*, 3026. https://doi.org/10.3390/math9233026

Academic Editor: J. E. Trinidad-Segovia

Received: 13 September 2021
Accepted: 22 November 2021
Published: 25 November 2021

1. Introduction

In 1952, Markowitz [1] proposed the mean-variance (MV) framework to maximize the expected return with certain risk constraints, where returns of financial assets are formulated by a random variable in the Gaussian distribution. Markowitz's proportion provided the fundamental principles for portfolio extensional research, including transaction cost, trading size, and turnover constraints [2–4]. However, the MV model considers high returns as equally undesirable as low returns, and it is incompatible with the axiomatic models of preferences under risk [5]. Consequently, several researchers have explored various risk measures, including the value-at-risk (VaR) to deal with an investment under the possibility of the utmost loss with a known confidence level [6–8], and then conditional value-at-risk (CVaR) is known as mean excess loss, mean shortfall, or tail VaR, in which CVaR is a coherent risk measure having the following properties: transition equivariant, positively homogeneous, convex, monotonic w.r.t. stochastic dominance of order 1, and monotonic w.r.t. monotonic dominance of order 2 [9]. However, owing to the inherent complexity and volatile nature of the investment market, a precise prediction of the security returns using the available historical data is not possible.

A vast majority of researchers have assumed the fuzziness of returns and have utilized possibility measures to select the assets to be included in the investment portfolio to investigate the uncertainty in financial markets. For examples, Tanaka et al. [10] extended traditional probability measures to fuzzy possibilities and derived a fuzzy portfolio model. Carlsson et al. [11] introduced the fuzzy possibilities to portfolio selection, where the highest utility score under the returns of assets are trapezoidal fuzzy numbers. Zhang and Nie [12] proposed that the expected return and risk of securities have admissible errors when reflecting the uncertainty in real investment actions. Bilbao-Terol et al. [13] employed a fuzzy compromise programming model to overcome the fuzzy portfolio selection problem. Some scholars have also focused on the issues of mean variance, mean semi-variance, skewness of a given fuzzy variable, and mean risk curve in the portfolio selection models [14–18]. Tsaur [19] proposed a fuzzy portfolio model with a fuzzy return and fuzzy proportion under incomplete information during a period of depression. Chen and Tsaur [20] used a weighted function to propose a weighted fuzzy portfolio model for collections of sufficient information to derive an efficient portfolio in each stage of the business cycle. Li et al. [21] discussed a dynamic project portfolio selection problem with project divisibility constraints. Tsaur et al. [22] proposed a fuzzy return function and considered excess investment using guaranteed return rates for the selecting securities, and then efficient portfolios could be obtained under different levels of investment risk. Tsaur et al. [23] revised the Chen and Tsaur fuzzy portfolio model [20] for the COVID-19 pandemic, which has greatly influenced the global economy, by using fuzzy goal programming model with different linguistic descriptions for the imprecise goal of expected return, the future stock market, and the optimal portfolio selection that can be solved under different investment risks. Tsaur [24] considered different risk attitudes for investors and extended MV models to assist investors to optimize investment strategies. Berman et al. [25] analyzed the effect of a decision maker's risk attitude towards the median and center problem. Zhou et al. [26] incorporated aggressive–neutral–conservative attitudes alongside VaR constraints and the ε constraint method to solve the fuzzy portfolio selection. Yue et al. [17] constructed a fuzzy portfolio selection model within the lower partial risk framework.

The credibility theory was first introduced by Liu [27], where the credibility measure is consistent with the law of the excluded middle and the law of contradiction [28], which is required in theory and demanded by practitioners. Since then, some researchers have suggested that modeling assets return using credibility measures [29–31]. García et al. [32] proposed a fuzzy multi-objective approach that optimizes the expected return, the expected ESG score, and the downside risk of a given portfolio, subject to real-world constraints. Zhang et al. [33] proposed two credibilistic MV portfolio models and employed a quadratic programming approach to obtain an optimal adjusting strategy. García et al. [34] proposed a mean–semi-variance multi-objective credibilistic portfolio selection model with a price-to-earnings ratio to measure the portfolio performance. Garcia et al. [35] extended the stochastic mean-variance model to a credibilistic multi-objective model in which the semi-variance and the CVaR are used to measure portfolio performance and the non-dominated sorting genetic algorithm II is applied to solve efficient portfolios in the fuzzy return–risk–liquidity trade-off to create an efficient frontier. Mehlawat et al. [36] proposed a multi-objective function with variance and CVaR as risk measures for performance evaluation in the fuzzy portfolio selection models, in which the inherent uncertainty of the investment market is incorporated through trapezoidal fuzzy returns using the credibility theory.

From the above literature reviews, the research on fuzzy portfolio models have made great progress, and most of the existing models deal with risk constraints, economy issues, and investor behaviors. Although some researchers have used possibility or credibility theory to make decision results in portfolio selection, the habitual behaviors of investors may be optimistic or pessimistic, so it is reasonable for them to have different expected return rates which can be defined as the target return in their portfolio. Therefore, in the face of a fuzzy portfolio selection problem, the investor considers high returns as the preference over low returns; in general, we suppose the investment behavior intends to make a shortage investment for those securities with low returns. In contrast, they are more willing to make excess investments in the securities with higher returns, and we suppose that investors can usually realize unexpected returns from those securities. The purpose of this paper is to investigate the fuzzy portfolio model to analyze the proportion of investments for each security with respect to the selected guaranteed rate of return. While many researchers have studied the field of fuzzy portfolio models, little research has been conducted on the portfolio problem where excess investment is required for some securities whose return rates are higher than the threshold of a guaranteed rate of return for that portfolio; otherwise, shortage investments are required for some securities whose return rates are lower than the threshold of a guaranteed rate of return for that portfolio.

We organize this study as follows. In Section 2, the definition of fuzzy numbers, the notions of lower and upper possibilistic means, variances, and covariance of fuzzy numbers are provided. In Section 3, we propose the guaranteed rate of returns to the fuzzy portfolio model and derive the adjustable security proportion. In Section 4, two examples using the proposed model are presented. Finally, the conclusion is discussed in Section 5.

2. Preliminaries

This section reviews some fundamental concepts and theorems of fuzzy numbers, fuzzy expected values, and fuzzy variances for our discussion in this paper.

Definition 1 ([37]). *We denote by F(R) the set of all fuzzy subsets $\tilde{a}$ of R with membership function $\xi_{\tilde{a}}$ satisfying the following conditions:*

(i). $\tilde{a}$ *is normal, i.e., there exists an* $x \in R$ *such that* $\xi_{\tilde{a}}(x) = 1$;
(ii). $\xi_{\tilde{a}}$ *is quasi-concave, i.e.,* $\xi_{\tilde{a}}(\lambda x + (1 - \lambda)y) \geq \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$ *for all* $x, y \in R$ *and* $\lambda \in [0, 1]$;
(iii). $\xi_{\tilde{a}}$ *is upper semicontinuous, i.e.,* $\{x \in R: \xi_{\tilde{a}}(x) \geq \alpha\} = \tilde{a}_\alpha$ *is a closed subset of U for each* $\alpha \in (0, 1]$;
(iv). *the 0-level set* $\tilde{a}_0$ *is a compact subset of R.*

The member $\tilde{a}$ in F(R) is called a fuzzy number.

Therefore, fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ with α-levels, $0 \leq \alpha \leq 1$, can be shown as follows:

$$\widetilde{A}^\alpha = [a_1(\alpha),\ a_2(\alpha)]\ (0 \leq \alpha \leq 1) \text{ and} \tag{1}$$

$$\widetilde{B}^\alpha = [b_1(\alpha),\ b_2(\alpha)](0 \leq \alpha \leq 1), \tag{2}$$

Then, the following additions and multiplications for $\widetilde{A}$ and $\widetilde{B}$ are applied as follows:

$$\widetilde{A}^\alpha + \widetilde{B}^\alpha = [a_1(\alpha) + b_1(\alpha),\ a_2(\alpha) + b_2(\alpha)] \tag{3}$$

$$\widetilde{A}^\alpha - \widetilde{B}^\alpha = [a_1(\alpha) - b_2(\alpha),\ a_2(\alpha) - b_1(\alpha)] \tag{4}$$

$$r \times \widetilde{A}^\alpha = [r \times a_1(\alpha),\ r \times a_2(\alpha)]\ (r > 0) \tag{5}$$

Theorem 1 ([11]). *Suppose that $\widetilde{A}$ is a fuzzy number with differentiable membership function, and α-level set $\widetilde{A}^\alpha = [a_1(\alpha),\ a_2(\alpha)]\ (0 \leq \alpha \leq 1)$. Then the lower possibilistic mean value of $\widetilde{A}$ can be defined as follows:*

$$M_*(\widetilde{A}) = 2\int_0^1 \alpha \cdot a_1(\alpha)d\alpha \tag{6}$$

Theorem 2 ([11]). *Suppose that $\widetilde{A}$ is a fuzzy number with differentiable membership function, and α-level set $\widetilde{A}^\alpha = [a_1(\alpha),\ a_2(\alpha)]\ (0 \leq \alpha \leq 1)$. Then the upper possibilistic mean value of the fuzzy numbers can be defined as follows:*

$$M^*(\widetilde{A}) = 2\int_0^1 \alpha \cdot a_2(\alpha)d\alpha \tag{7}$$

Theorem 3 ([11]). *Suppose that $\widetilde{A}$ is a fuzzy number with differentiable membership function, and α-level set $\widetilde{A}^{\alpha} = [a_1(\alpha),\ a_2(\alpha)]$ $(0 \leq \alpha \leq 1)$. Then its expected value is expressed as follows:*

$$M(\widetilde{A}) = \int_0^1 \alpha \cdot (a_1(\alpha) + a_2(\alpha))d\alpha$$

Then, we derive the following (Carlsson and Fulle, 2001):

$$M_*(\widetilde{A} + \widetilde{B}) = M_*(\widetilde{A}) + M_*(\widetilde{B}) \text{ and} \tag{8}$$

$$M^*(\widetilde{A} + \widetilde{B}) = M^*(\widetilde{A}) + M^*(\widetilde{B}) \tag{9}$$

Thus, the possibilistic mean value of $\widetilde{A} + \widetilde{B}$ can be obtained as

$$M(\widetilde{A} + \widetilde{B}) = \frac{M_*(\widetilde{A} + \widetilde{B}) + M^*(\widetilde{A} + \widetilde{B})}{2}. \tag{10}$$

In addition, the lower and upper possibilistic variances of fuzzy numbers $\widetilde{A}$ are introduced as follows:

Theorem 4 ([38]). *Suppose that $\widetilde{A}$ is a fuzzy number with differentiable membership function, then the lower possibilistic variance of $\widetilde{A}$ can be defined as follows:*

$$Var_*(\widetilde{A}) = 2\int_0^1 \alpha \cdot [M_*(\widetilde{A}) - a_1(\alpha)]^2 d\alpha \tag{11}$$

Theorem 5 ([38]). *Suppose that $\widetilde{A}$ is a fuzzy number with differentiable membership function. Then the upper possibilistic variance of fuzzy numbers $\widetilde{A}$ can be defined as follows:*

$$Var^*(\widetilde{A}) = 2\int_0^1 \alpha \cdot [M^*(\widetilde{A}) - a_2(\alpha)]^2 d\alpha \tag{12}$$

Theorem 6 ([38]). *Suppose that $\widetilde{A}$ is a fuzzy number with differentiable membership function, and α-level set $\widetilde{A}^{\alpha} = [a_1(\alpha),\ a_2(\alpha)]$ $(0 \leq \alpha \leq 1)$. Then its possibilistic variance can be expressed as follows:*

$$Var(\widetilde{A}) = \int_0^1 \alpha \cdot ([M(\widetilde{A}) - a_1(\alpha)]^2 + [a_2(\alpha) - M(\widetilde{A})]^2)d\alpha \tag{13}$$

Theorem 7 ([38]). *Supposed that ρ and μ can be any number. Then the possibilistic variances of the fuzzy number $\rho\widetilde{A} + \mu\widetilde{B}$ can be expressed as follows:*

$$\begin{aligned} &Var(\rho\widetilde{A} + \mu\widetilde{B}) = \\ &\textstyle\int_0^1 \alpha([M(\rho\widetilde{A} + \mu\,\widetilde{B}) - (\rho a_1(\alpha) + \mu b_1(\alpha)]^2 + [(\rho a_2(\alpha) + \mu b_2(\alpha) - M(\rho\widetilde{A} + \mu\widetilde{B})]^2)d\alpha \end{aligned} \tag{14}$$

Ranking fuzzy numbers plays a crucial role in decision making, and Jain [39], Chen [40], Chang and Lee [41], Deng et al. [42], Wang and Mo [43] have proposed a ranking method for fuzzy numbers. However, given most of the ranking methods in the previous research, we find that the ranking procedures cannot discriminate fuzzy quantities, and some are counterintuitive. In order to address the disadvantage of ranking fuzzy numbers, we rank the fuzzy numbers as follows [44]:

Theorem 8 ([44]). *Suppose that $\widetilde{A} = (a, c_1, c_2)$ and $\widetilde{B} = (b, d_1, d_2)$ are fuzzy numbers with a normal, convex, and continuous membership function, where a and b are the central values and c_1, c_2, and d_1, d_2 are the left and right spread values. The circumcenter of a fuzzy number $\widetilde{A}$ is defined* *as* $S_{\widetilde{A}} = (\overline{x}_0, \overline{y}_0) = (\frac{6a+(c_2-c_1)}{6}, \frac{5-c_1c_2}{12})$*, and then we use a ranking function* $R(\widetilde{A})$ *to map the fuzzy number* $\widetilde{A}$ *to a real number* $R(\widetilde{A}) = \sqrt{\overline{x}_0^2 + \overline{y}_0^2}$*. If* $R(\widetilde{A}) > R(\widetilde{B})$*, then* $\widetilde{A} > \widetilde{B}$*. Otherwise,* $\widetilde{A} \leq \widetilde{B}$*.*

3. Fuzzy Portfolio Model for Deriving the Adjustable Security Proportions

In this section, we first introduce the concept of a fuzzy return in the portfolio selection process. Let x_j be the proportion of the total investment devoted to the risk security j, and the fuzzy return rate of security j be a triangular fuzzy number, $\widetilde{r}_j = (r_j,\ c_j,\ d_j)$, $j = 1, \ldots, n$, where r_j is its central value, c_j, d_j are left and right spreads of the triangular fuzzy number. Then, the expected fuzzy returns are defined as $\widetilde{R} = \sum_{j=1}^{n} x_j \widetilde{r}_j$. In the fuzzy environment, the investor usually receives different investment information from different media and then make decisions based on their investment behaviors. The investors may be optimistic or pessimistic, so it is reasonable for them to have different expected values of the uncertain return rate in their portfolio. Therefore, in the face of a fuzzy portfolio selection problem, the investors with different attitudes may have various efficient frontiers. In this study, we tried to revise the overly optimistic and overly pessimistic investment behavior and then propose a neutral investment behavior. Therefore, we assumed that a neutral investor prefers the higher return securities and avoids the lower return securities. Since less risks usually come with lower returns, investors seldom make significant profits from these securities, and thus the investment behavior intends to make shorter investments for these securities. By contrast, higher risk securities usually can realize unexpected returns, and thus we can make excess investment in these securities. The guaranteed rate of returns are the threshold values to evaluate the shortage or excess investment for each security (i.e., the opportunity cost as well as fixed-term deposit interest rate).

For the adjustable security proportion in the fuzzy portfolio selection process, we deal with the collected n securities and rank their fuzzy return rates by $\widetilde{r}_1 < \widetilde{r}_2 < \ldots < \widetilde{r}_n$ (i.e., $R(\widetilde{r}_1) < R(\widetilde{r}_2) < \ldots < R(\widetilde{r}_n)$), and then we select m securities ($m \leq n$) for excess investment based on the fuzzy return rates being higher than the chosen guaranteed return rates. In contrast, the other securities are shortage-investment based on the fuzzy return rates to be lower than the chosen guaranteed return rates. An investor will spend a lot of time collecting the market information for this portfolio and then assume that the selected threshold values for the guaranteed return rates are defined as $\widetilde{p}_k = (p_k,\ e_k,\ f_k)$, where p_k is central value, e_k and f_k are left and right spread values, based on his risk attitude for his investment behavior. As shown in Equation (15), the expected fuzzy returns can be formulated as follows:

$$\widetilde{R} = \sum_{j=1}^{n} x_j \widetilde{r}_j - \sum_{j=1}^{n_1} \sum_{k=1}^{m} x_j \left| \widetilde{r}_j - \widetilde{p}_k \right| + \sum_{j=n_1+1}^{n} \sum_{k=1}^{m} x_j \left| \widetilde{r}_j - \widetilde{p}_k \right| \tag{15}$$

where the first term on the right hand side is the general expected fuzzy returns, and the second term is minus the fuzzy return by allocating shortage investment for security j when the fuzzy return rate $\widetilde{r}_j$ is lower than the guaranteed return rate $\widetilde{p}_k$; the third term is the fuzzy return by allocating excess investments for security j when the fuzzy return rate $\widetilde{r}_j$ is higher than the guaranteed return rate $\widetilde{p}_k$. The guaranteed return rates $\widetilde{p}_k\ \forall k = 1, 2, \ldots, m$ are selected by the decision maker, and the ranking of the guaranteed return rates is assumed as $\widetilde{p}_1 < \widetilde{p}_2 < \ldots < \widetilde{p}_m$ (i.e. $R(\widetilde{p}_1) < R(\widetilde{p}_2) < \ldots < R(\widetilde{p}_m)$). To obtain the expected fuzzy returns of Equation (15), the excess investment is made on security j when the fuzzy return rate $\widetilde{r}_j$ is higher than $\widetilde{p}_k$; otherwise, the shortage investment is made on security j when the fuzzy return rate $\widetilde{r}_j$ is lower than $\widetilde{p}_k$, $j = 1, \ldots, n$. Therefore, we can formulate this concept as follows:

$$|\widetilde{r}_j - \widetilde{p}_k| = \begin{cases} \widetilde{r}_j - \widetilde{p}_k & \text{if } R(\widetilde{r}_j) > R(\widetilde{p}_k) \\ 0 & \text{if } R(\widetilde{r}_j) = R(\widetilde{p}_k) \\ \widetilde{p}_k - \widetilde{r}_j & \text{if } R(\widetilde{r}_j) < R(\widetilde{p}_k) \end{cases} \tag{16}$$

Next, denote excess investment $\widetilde{r}_j - \widetilde{p}_k$ with its lower possibilistic and upper possibilistic mean values as $M_*(\widetilde{r}_j - \widetilde{p}_k)$ and $M^*(\widetilde{r}_j - \widetilde{p}_k)$, $\forall\, k = 1,\, 2, \ldots, m$, as follows:

$$M_*(\widetilde{r}_j - \widetilde{p}_k) = 2\int_0^1 \alpha \times (\widetilde{r}_j - \widetilde{p}_k)_{j1}(\alpha)\, d\alpha = (r_j - p_k) - \frac{1}{3}(c_j + f_k). \tag{17}$$

$$M^*(\widetilde{r}_j - \widetilde{p}_k) = 2\int_0^1 \alpha \times (\widetilde{r}_j - \widetilde{p}_k)_{j2}(\alpha)\, d\alpha = (r_j - p_k) + \frac{1}{3}(d_j + e_k). \tag{18}$$

The α level set of $\widetilde{r}_j - \widetilde{p}_k$ can be obtained as $[\widetilde{r}_j - \widetilde{p}_k]^\alpha = [(\widetilde{r}_j - \widetilde{p}_k)_{j1}(\alpha),\, (\widetilde{r}_j - \widetilde{p}_k)_{j2}(\alpha)]$, $0 \leq \alpha \leq 1$. Then, we can obtain the possibilistic mean value of $M(\widetilde{r}_j - \widetilde{p}_k)$ as follows:

$$M(\widetilde{r}_j - \widetilde{p}_k) = (r_j - p_k) + \frac{1}{6}[(d_j + e_k) - (c_j + f_k)],\ \forall k > 0. \tag{19}$$

Next, the possibilistic mean value of $M(\widetilde{p}_k - \widetilde{r}_j)$ is obtained as follows:

$$M(\widetilde{p}_k - \widetilde{r}_j) = (p_k - r_j) + \frac{1}{6}[(e_k + d_j) - (f_k + c_j)],\ \forall k > 0. \tag{20}$$

In the same way, the lower and upper possibilistic mean values of the fuzzy returns $\widetilde{r}_j, \forall\, j = 1,\, 2, \ldots, n$, can be defined as $M_*(\widetilde{r}_j)$ and $M^*(\widetilde{r}_j)$ as follows:

$$M_*(\widetilde{r}_j) = 2\int_0^1 \alpha \times r_{j1}(\alpha)\, d\alpha = r_j - \frac{1}{3}c_j, \forall k > 0. \tag{21}$$

$$M^*(\widetilde{r}_j) = 2\int_0^1 \alpha \times r_{j2}(\alpha)\, d\alpha = r_j + \frac{1}{3}d_j, \forall k > 0. \tag{22}$$

where $\widetilde{r}_j$ is a fuzzy number whose α level set of $\widetilde{r}_j$ is defined as $[\widetilde{r}_j]^\alpha = [r_{j1}(\alpha),\, r_{j2}(\alpha)]$ for all $\alpha \in [0, 1]$. Then, we can obtain the crisp possibilistic mean value $M(\widetilde{r}_j)$ as follows:

$$M(\widetilde{r}_j) = \frac{M_*(\widetilde{r}_j) + M^*(\widetilde{r}_j)}{2} = r_j + \frac{1}{6}(d_j - c_j),\ \forall k > 0. \tag{23}$$

Based on the above calculations, without loss of generality, we assume that $\widetilde{r}_1 < \widetilde{r}_2 < \ldots < \widetilde{r}_{n_1} < \widetilde{p}_1 < \widetilde{p}_2 < \ldots < \widetilde{p}_m < \widetilde{r}_{n_1+1} < \widetilde{r}_{n_1+2} < \ldots < \widetilde{r}_n$. Then, we can obtain the possibilistic mean value of Formula (15), where some securities return rates are lower and the others are higher than the guaranteed rate of return in the portfolio selection, as follows:

$$\begin{aligned}&M\left[\sum_{j=1}^{n} x_j\widetilde{r}_j-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j(\widetilde{p}_k-\widetilde{r}_j)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j(\widetilde{r}_j-\widetilde{p}_k)\right]\\&=\sum_{j=1}^{n} x_j M(\widetilde{r}_j)-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j M(\widetilde{p}_k-\widetilde{r}_j)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j M(\widetilde{r}_j-\widetilde{p}_k)\\&=\sum_{j=1}^{n} x_j[r_j+\tfrac{1}{6}(d_j-c_j)]-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j[(p_k-r_j)+\tfrac{1}{6}[(e_k+d_j)-(f_k+c_j)]\\&+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j[(r_j-p_k)+\tfrac{1}{6}[(d_j+e_k)-(c_j+f_k)]\end{aligned}\tag{24}$$

Then, the lower and upper possibilistic variances of the fuzzy number $\sum_{j=1}^{n} x_j\widetilde{r}_j-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j(\widetilde{p}_k-\widetilde{r}_j)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j(\widetilde{r}_j-\widetilde{p}_k)$ are derived as follows:

$$\begin{aligned}&Var_*[\sum_{j=1}^{n} x_j\widetilde{r}_j-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j(\widetilde{p}_k-\widetilde{r}_j)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j(\widetilde{r}_j-\widetilde{p}_k)]\\&=\tfrac{1}{36}[\sum_{j=1}^{n} x_jc_j-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j(c_j+f_k)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j(c_j+f_k)]^2,\ \forall k>0.\end{aligned}\tag{25}$$

$$\begin{aligned}&Var_*[\sum_{j=1}^{n} x_j\widetilde{r}_j-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j(\widetilde{p}_k-\widetilde{r}_j)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j(\widetilde{r}_j-\widetilde{p}_k)]\\&=\tfrac{1}{36}[\sum_{j=1}^{n} x_jc_j-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j(c_j+f_k)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j(c_j+f_k)]^2,\ \forall k>0.\end{aligned}\tag{26}$$

Then, the possibilistic standard deviation of $\sum_{j=1}^{n} x_j\widetilde{r}_j-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j(\widetilde{p}_k-\widetilde{r}_j)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j(\widetilde{r}_j-\widetilde{p}_k)$ is obtained as follows:

$$\begin{aligned}&SD[\sum_{j=1}^{n} x_j\widetilde{r}_j-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j(\widetilde{p}_k-\widetilde{r}_j)+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j(\widetilde{r}_j-\widetilde{p}_k)]\\&=\sum_{j=1}^{n} x_j\tfrac{c_j+d_j}{12}-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j\tfrac{(c_j+f_k)+(d_j+e_k)}{12}+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j\tfrac{(c_j+f_k)+(d_j+e_k)}{12}.\end{aligned}\tag{27}$$

Analogous to Markowitz's MV methodology, the possibilistic mean–standard deviation model based on the guaranteed rate of return for considering some security shortages and the other security excess investments is formulated as follows:

$$\begin{aligned}&Max\ \sum_{j=1}^{n} x_j[r_j+\tfrac{1}{6}(d_j-c_j)]-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j[(p_k-r_j)+\tfrac{1}{6}[(e_k+d_j)-(f_k+c_j)]\\&+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j[(r_j-p_k)+\tfrac{1}{6}[(d_j+e_k)-(c_j+f_k)]\\&s.t.\ \sum_{j=1}^{n} x_j\tfrac{c_j+d_j}{12}-\sum_{j=1}^{n_1}\sum_{k=1}^{m} x_j\tfrac{(c_j+f_k)+(d_j+e_k)}{12}+\sum_{j=n_1+1}^{n}\sum_{k=1}^{m} x_j\tfrac{(c_j+f_k)+(d_j+e_k)}{12}\le\sigma\\&\quad\sum_{j=1}^{n} x_j\le 1\\&\quad l_j\le x_j\le u_j,\forall\, j=1,\ 2,\ldots,n.\end{aligned}\tag{28}$$

where l_j and u_j are the lower and upper bounds on proportion x_j, $j = 1, 2, \ldots, n$.

4. Illustrations

In this section, we test two illustrations using the proposed fuzzy portfolio selection. The first illustration is secondary data collected by Zhang [38], and the second illustration is a real case collecting the securities from the Taiwan Stock Exchange.

4.1. Example 1

In this example, five securities obtained from the weekly closed prices from April 2002 to January 2004 were the portfolio sample based on the historical data, the corporations' financial reports, and future information from the Shanghai Stock Exchange [39]. Therefore, the possibility distribution for each security was estimated as the following: $\widetilde{r}_1$ = (0.073, 0.054, 0.087), $\widetilde{r}_2$ = (0.105, 0.075, 0.102), $\widetilde{r}_3$ = (0.138, 0.096, 0.123), $\widetilde{r}_4$ = (0.168, 0.126, 0.162), $\widetilde{r}_5$ = (0.208, 0.168, 0.213). Next, the lower and upper bounds of investment proportion x_j for security j are given by (l_1, l_2, l_3, l_4, l_5) = (0.1, 0.1, 0.1, 0.1, 0.1), and (u_1, u_2, u_3, u_4, u_5) = (0.4, 0.4, 0.4, 0.5, 0.6). However, for the shortage investment security, in this example we relax its lower bound to be 0. The threshold values for the guaranteed return rates are selected as $\widetilde{p}_1$ = (0.1, 0.05, 0.05), higher than $\widetilde{r}_1$ to relax the securities 2, 3, 4, 5 to excess investment; $\widetilde{p}_2$ = (0.15, 0.1, 0.1) is selected based on the average of the highest fuzzy return $\widetilde{r}_1$ and the lowest return $\widetilde{r}_5$ to relax the securities 4, 5 to excess investment; and $\widetilde{p}_3$ = (0.2, 0.1, 0.15) is selected to be lower than $\widetilde{r}_5$ to relax the security 5 to be excess investment, and the securities 1, 2, 3, 4 to be the shortage investment.

4.1.1. Results and Discussions

Step 1: Rank the security fuzzy returns f and selected guaranteed rate of return.

Based on Theorem 8, we defuzzyed all the fuzzy returns and guaranteed return rates as $R(\widetilde{r}_1)$ = 0.4236, $R(\widetilde{r}_2)$ = 0.4302, $R(\widetilde{r}_3)$ = 0.4394, $R(\widetilde{r}_4)$ = 0.4500, and $R(\widetilde{r}_5)$ = 0.4665; $R(\widetilde{p}_1)$ = 0.4283, $R(\widetilde{p}_2)$ = 0.4421, and $R(\widetilde{p}_3)$ = 0.4647. Therefore, the rank of the fuzzy numbers are as follows: $\widetilde{r}_1 < \widetilde{p}_1 < \widetilde{r}_2 < \widetilde{r}_3 < \widetilde{p}_2 < \widetilde{r}_4 < \widetilde{p}_3 < \widetilde{r}_5$.

Step 2: Formulate fuzzy portfolio model in adjustable security proportion.

In this step, we first select the guaranteed return rate $\widetilde{p}_1$ = (0.1, 0.05, 0.05) to deal with the adjustable security proportion investment in the portfolio. Then, security 1 is allocated to shortage investment and the securities 2–5 are excess investments because $R(\widetilde{p}_1)$ is lower than $R(\widetilde{r}_2)$, $R(\widetilde{r}_3)$, $R(\widetilde{r}_4)$, and $R(\widetilde{r}_5)$. In addition, for the shortage investment, we relax the lower bound of investment proportion for security 1 to be zero. Therefore, we obtain the expected fuzzy returns as follows:

$$\begin{aligned}
&\sum_{j=1}^{5} x_j[r_j + \tfrac{1}{6}(d_j - c_j)] - x_1[(p_1 - r_1) + \tfrac{1}{6}[(e_1 + d_1) - (f_1 + c_1)] \\
&+ \sum_{j=2}^{5} x_j[(r_j - p_1) + \tfrac{1}{6}[(d_j + e_1) - (c_j + f_1)] \\
&= 0.111x_1 + 0.119x_2 + 0.185x_3 + 0.248x_4 + 0.331x_5.
\end{aligned} \tag{29}$$

Next, the risk of fuzzy portfolio can be obtained as

$$\begin{aligned}
&\sum_{j=1}^{5} x_j \tfrac{c_j + d_j}{12} - x_1 \tfrac{(c_1+f_1)+(d_1+e_1)}{12} + \sum_{j=2}^{5} x_j \tfrac{(c_j+f_1)+(d_j+e_1)}{12} \\
&= -0.0318x_1 + 0.0378x_2 + 0.0448x_3 + 0.0563x_4 + 0.0718x_5.
\end{aligned} \tag{30}$$

Finally, we can formulate the fuzzy portfolio model as follows:

$$\begin{aligned}
&\mathrm{M}ax 0.111x_1 + 0.119x_2 + 0.185x_3 + 0.248x_4 + 0.331x_5 \\
&s.t. - 0.0318x_1 + 0.0378x_2 + 0.0448x_3 + 0.0563x_4 + 0.0718x_5 \le \sigma \\
&x_1 + x_2 + x_3 + x_4 + x_5 \le 1 \\
&0 \le x_1 \le 0.4,\ 0.1 \le x_2, x_3 \le 0.4;\ 0.1 \le x_4 \le 0.5;\ 0.1 \le x_5 \le 0.6
\end{aligned} \tag{31}$$

Step 3: Fuzzy portfolio selection under adjustable security proportion.

In this step, the investment risks range from 3% to 8%, and we use Model (31) to obtain the portfolio results as shown in Table 1, where the investment proportion for security 1 decreases from 0.3297 to 0. This is because the increase in investment risk makes the investor shortage investment to the security, where return rate is lower than the selected guaranteed return rate $\widetilde{p}_1$. Besides, the investment proportion for security 4 increases from 0.1 to 0.2, and security 5 increases from 0.3703 to 0.6 because the return rates of securities 4 and 5 are higher than the selected guaranteed rate of return $\widetilde{p}_1$ and contributes to the investor making the excess investment. Therefore, with the risk lower than 5%, securities 2–4 are not excess investments but are invested more in security 1. In the other, the investor focuses on excess investment when the investment risk exceeds or equals 6%, and the proportion of securities 4 and 5 increases, as the higher return comes with a higher risk.

Table 1. An efficient portfolio with a guaranteed return rate $\widetilde{p}_1$.

Risk / **Proportion**	**3%**	**4%**	**5%**	**6%**	**7%**	**8%**
x_1	0.3297	0.2333	0.1368	0.0299	0	0
x_2	0.1	0.1	0.1	0.1	0.1	0.1
x_3	0.1	0.1	0.1	0.1	0.1	0.1
x_4	0.1	0.1	0.1	0.1701	0.2	0.2
x_5	0.3703	0.4667	0.5632	0.6	0.6	0.6
Expected Return Rate	0.2144	0.2356	0.2568	0.2745	0.2745	0.2745

Step 4: Sensitivity analysis for the guaranteed return rates.

In this step, we select the guaranteed return rates $\widetilde{p}_2$ and $\widetilde{p}_3$ for portfolio analysis and the results are shown in Tables 2 and 3, respectively. In Table 2, the ranking of the fuzzy numbers are as follows: $\widetilde{r}_1 < \widetilde{r}_2 < \widetilde{r}_3 < \widetilde{p}_2 < \widetilde{r}_4 < \widetilde{r}_5$. Therefore, securities 4 and 5 are assumed as excess investments, and securities 1, 2 and 3 are assumed as shortage investments. Then, we relax the lower bound of shortage investment securities to be 0. We also define the investment risks in this step to range from 3% to 8%, and we find that the investment proportion for security 1 decreases from 0.4 to 0, security 2 decreases from 0.102 to 0, and security 3 is always 0. This is because the increase of investment risks pushes the investor to make a shortage investment to the securities whose return rates are lower than the selected guaranteed rate of return $\widetilde{p}_2$. On the other hand, the investment proportion for security 4 increases from 0.1 to 0.4 and security 5 increases from 0.398 to 0.6 because the return rates of securities 4 and 5 are higher than the selected guaranteed rate of return $\widetilde{p}_2$ and contribute the investor to making an excess investment. Therefore, when the investment risk exceeds or is equal to 8%, implying that the investor focuses on an excess investment, we can suggest that the securities 4 and 5 are the major investment with the proportions 0.4 and 0.6, respectively.

Table 2. An efficient portfolio with a guaranteed return rate $\widetilde{p}_2$.

Risk / **Proportion**	**3%**	**4%**	**5%**	**6%**	**7%**	**8%**
x_1	0.4	0.3988	0.2947	0.1717	0.0488	0
x_2	0.102	0	0	0	0	0
x_3	0	0	0	0	0	0
x_4	0.1	0.1	0.1053	0.2283	0.3512	0.4
x_5	0.398	0.5012	0.6	0.6	0.6	0.6
Expected Return Rate	0.2123	0.2248	0.2369	0.2414	0.2460	0.2478

Table 3. An efficient portfolio in the guaranteed return rate $\widetilde{p}_3$.

Proportion \ Risk	1%	2%	3%	4%	5%	6%
x_1	0.2068	0.1117	0.0167	0	0	0
x_2	0	0	0	0	0	0
x_3	0	0	0	0	0	0
x_4	0.5	0.5	0.5	0.4216	0.4	0.4
x_5	0.2932	0.3883	0.4833	0.5784	0.6	0.6
Expected Return Rate	0.2090	0.2109	0.2128	0.2147	0.2151	0.2151

The guaranteed return rate $\widetilde{p}_3$ is used for the portfolio analysis, while securities 1, 2, 3, and 4 are allowed to be shortage investments and excess investments to security 5 because the ranking of the fuzzy returns are $\widetilde{r}_1 < \widetilde{r}_2 < \widetilde{r}_3 < \widetilde{r}_4 < \widetilde{p}_3 < \widetilde{r}_5$. In Table 3, we define the investment risks to range from 1% to 6% because most of securities are ranked for shortage investments, and the investment proportion for security 1 decreases from 0.2068 to 0, security 4 decreases from 0.5 to 0.4, and securities 2 and 3 are always 0. Next, the investment risk is set to be higher than 5%, the investment proportions for securities 1–3 are all zero, and security 4 also decreases from 0.1 to 0.4; this is because the increase of investment risk makes the investor shortage investment to the securities, whose return rates are lower than the selected guaranteed return rate $\widetilde{p}_3$. By contrast, we find that the investment proportion of security 5 increases from 0.2932 to 0.6 to make an excess investment because the return rates of security 5 are larger than the selected guaranteed return rate $\widetilde{p}_3$. Therefore, we can suggest that the securities 4 and 5 be the major investment with the proportions 0.4 and 0.6, respectively.

From the above analysis, as shown in Figure 1, we believe the expected return rate of a portfolio with a lower guaranteed return rate is higher in different investment risk levels because a higher guaranteed return rate makes some securities shortage investments which thus obtain a lower expected return rate. Therefore, in order to obtain a higher investment return, we suggest that an investor should not select too high a guaranteed rate of return. In addition, the proportion of security 5 is always higher than the other securities when the investment risk is higher, and thus it is the most important security in the investment.

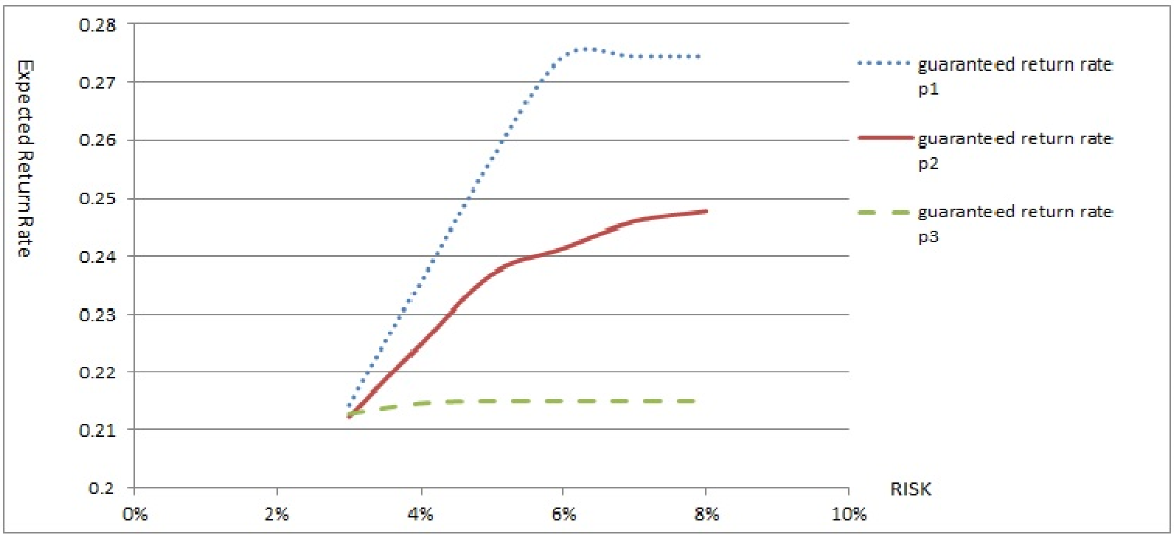

Figure 1. The comparisons between different guaranteed return rates.

4.1.2. The Comparisons among Fuzzy Portfolio Models

Zhang [38] discusses the portfolio selection problem for bounded assets, and the upper possibilistic means and variances method can derive higher expected returns. Chen and Tsaur [20] proposed a fuzzy portfolio model to discuss the investments in a business cycle, where in the stage of recession with downward economics the economics gradually draw closer to the bottom and thus the investment risk is lower. Therefore, with the guaranteed return rates $\widetilde{p}_1$, we compared the proposed model to Zhang [38] and Chen and Tsaur's [20] portfolio models. In Table 4, the comparison results show that the expected return of the proposed fuzzy portfolio model is higher than Zhang [38] and Chen and Tsaur's models [20]. Besides, the excess investment in the proposed model results from the fuzzy expected return rates of some securities being higher than the selected guaranteed rate of returns; by contrast, the shortage investment in the proposed model assumes that the fuzzy return rates of the other securities are lower than the selected guaranteed rate of returns.

Table 4. Comparison among some fuzzy portfolio models.

	σ (%)	M (%)	x_1	x_2	x_3	x_4	x_5	$\sum_{j=1}^{5} x_j$
Chen and Tsaur [20] model in Depression stage	3%	Infeasible	Solution					
	4%							
	5%							
	6%							
	7%	7.87%	0.1	0.1	0.1420	0.1	0.1	0.5420
	8%	9.08%	0.1	0.1	0.2256	0.1	0.1	0.6256
Zhang [38]	3%	17.55%	0.1	0.1	0.4	0.2338	0.1	0.9338
	4%	22.55%	0.1	0.1	0.1611	0.1	0.5389	1
	5%	23.16%	0. 1	0.1	0.1	0.1	0.6	1
	6%	23.16%	0. 1	0.1	0.1	0.1	0.6	1
	7%	23.16%	0. 1	0.1	0.1	0.1	0.6	1
	8%	23.16%	0. 1	0.1	0.1	0.1	0.6	1
Proposed model of a guaranteed return rate $\widetilde{p}_1$	3%	21.44%	0.3297	0.1	0.1	0.1	0.3703	1
	4%	23.56%	0.2333	0.1	0.1	0.1	0.4667	1
	5%	25.68%	0.1368	0.1	0.1	0.1	0.5632	1
	6%	23.56%	0.0299	0.1	0.1	0.1701	0.6	1
	7%	27.45%	0	0.1	0.1	0.2	0.6	1
	8%	27.45%	0	0.1	0.1	0.2	0.6	1

4.2. Example 2

The second example is illustrated through a real-world portfolio selection problem. The top 50 companies in Taiwan with inconsistent data length were excluded. The sample period was from January 2010 to September 2020, and the data frequency was monthly data. Gupta et al. [45] stated that a portfolio should not hold too many assets nor too few. While it is not necessary to hold many assets to achieve a correct diversification, if the number of assets in the portfolio is too reduced it is not possible to correctly manage risks. Therefore, we classified the fifty securities into four industries, and then used Earning per Share to select the top five securities in each industry to create a sample of twenty securities (n = 20) from the Taiwan Stock Exchange (TWSE). The fuzzy return of each selected security can be estimated with the possibility distributions $\widetilde{r}_j = (r_j,\ c_j,\ d_j)$, $j = 1, \ldots, 20$, where r_j is its central value and c_j, d_j are its left and right spread values, respectively. As presented in Table 5, with Theorem 8 we can obtain the rank order of the collected fuzzy returns as $\widetilde{r}_{20} < \widetilde{r}_{19} < \widetilde{r}_{18} < \ldots < \widetilde{r}_1$.

Table 5. The fuzzy return of each selected security.

$\widetilde{r}_1$ = (0.1420, 0.0713, 0.1296)	$\widetilde{r}_2$ = (0.1358, 0.0490, 0.0835)	$\widetilde{r}_3$ = (0.0909, 0.0499, 0.0919)
$\widetilde{r}_4$ = (0.0808, 0.0203, 0.0297)	$\widetilde{r}_5$ = (0.0791, 0.0237, 0.0382)	$\widetilde{r}_6$ = (0.0786, 0.0162, 0.0217)
$\widetilde{r}_7$ = (0.0753, 0.0229, 0.0366)	$\widetilde{r}_8$ = (0.0751, 0.0226, 0.0362)	$\widetilde{r}_9$ = (0.0722, 0.0143, 0.0184)
$\widetilde{r}_{10}$ = (0.0657, 0.0205, 0.0325)	$\widetilde{r}_{11}$ = (0.0654, 0.0217, 0.0352)	$\widetilde{r}_{12}$ = (0.0643, 0.0244, 0.0415)
$\widetilde{r}_{13}$ = (0.0552, 0.0133, 0.0190)	$\widetilde{r}_{14}$ = (0.0438, 0.0079, 0.0102)	$\widetilde{r}_{15}$ = (0.0366, 0.0125, 0.0201)
$\widetilde{r}_{16}$ = (0.0316, 0.0110, 0.0181)	$\widetilde{r}_{17}$ = (0.0309, 0.0109, 0.0182)	$\widetilde{r}_{18}$ = (0.0273, 0.0079, 0.0170)
$\widetilde{r}_{19}$ = (0.0254, 0.0058, 0.0790)	$\widetilde{r}_{20}$ = (0.0212, 0.0051, 0.0075)	

Due to the seriousness of COVID-19, worldwide governments are committed to financially supporting and reducing interest costs for domestic enterprises so that an economic depression can be avoided. Therefore, judging from the stock markets around the world, there was an upward trend in this short period, and the possibility of a decline was relatively small. Therefore, in this study, we set the estimated error of the 95% confidence interval of the average return from January 2010 to September 2020 to be the right spread of fuzzy return and the estimated error of the 90% confidence interval to be the left spread. In the real case data, we selected twenty securities for analysis, and thus we set the lower bounds of investment proportion x_j for security j given by 0.01, and the upper bounds given by 0.3. The threshold values for the guaranteed return rates are selected based on the ranking of the fuzzy return. From the results of the first example, we know lower guaranteed return rates are suggested for selection. We selected the lower guaranteed return rates in the range of first quartile of ranking fuzzy returns. Therefore, we set $\widetilde{p}_1$ = (0.03, 0.01, 0.02) which is higher than $\widetilde{r}_{18}$, $\widetilde{r}_{19}$, $\widetilde{r}_{20}$; the middle guaranteed return rates $\widetilde{p}_2$ = (0.07, 0.03, 0.04) were selected from the second quartile of ranking fuzzy returns, where the ranking is lower than $\widetilde{r}_9$; and the larger guaranteed return rates $\widetilde{p}_3$ = (0.1, 0.04, 0.06) were selected from the third quartile of ranking fuzzy returns. Next, we set the first guaranteed rate of return as $\widetilde{p}_1$ = (0.03, 0.01, 0.02) to relax the securities 1, 2, 3, ... , 17 of the excess investment, and the securities 18, 19, 20 were relaxed to be the shortage investments. The second issue required $\widetilde{p}_2$ = (0.07, 0.03, 0.04) to relax the securities 1–9 to be excess investments, and the securities 10–20 to be shortage investments. The third issue required the guaranteed rate of return as $\widetilde{p}_3$ = (0.1, 0.04, 0.06) to relax the security 1–2 to be an excess investment, and securities 3–20 to be shortage investments.

Results and Discussions

Step 1: Rank the security fuzzy return rates and the selected guaranteed return rates.

Based on Theorem 8, we can map all the fuzzy return and guaranteed return rates to real numbers as $\widetilde{r}_{20} < \widetilde{r}_{19} < \widetilde{r}_{18} < \widetilde{p}_1 < \widetilde{r}_{17} < \widetilde{r}_{16} < \ldots < \widetilde{r}_{10} < \widetilde{p}_2 < \widetilde{r}_9 < \widetilde{r}_8 < \ldots < \widetilde{r}_3 < \widetilde{p}_3 < \widetilde{r}_2 < \widetilde{r}_1$.

Step 2: Formulate the fuzzy portfolio model in adjustable security proportion

In this step, the ranking of guaranteed return rates are $R(\widetilde{p}_1) < R(\widetilde{p}_2) < R(\widetilde{p}_3)$. Therefore, we first selected a guaranteed return rate $\widetilde{p}_1$ = (0.03, 0.01, 0.02) to make adjustable security proportion investment by the possible shortage investment to securities 18, 19, 20 and the excess investment to securities 1–17. In addition, for the shortage investment, we relaxed the lower bound of investment proportion for securities 18, 19, 20 to be zero. After we obtained the expected fuzzy returns and the risk of the fuzzy portfolio as constraints, the fuzzy portfolio model was obtained as follows:

$$\begin{aligned} Max \quad & 0.2718x_1 + 0.2514x_2 + 0.1641x_3 + 0.1331x_4 + 0.1314x_5 + 0.1274x_6 + 0.1235x_7 + 0.1231x_8 \\ & +0.1141x_9 + 0.1037x_{10} + 0.1036x_{11} + 0.1026x_{12} + 0.0806x_{13} + 0.0567x_{14} + 0.0441x_{15} \\ & \quad +0.0339x_{16} + 0.0326x_{17} + 0.0026x_{18} + 0.0151x_{19} + 0.0075x_{20} \end{aligned}$$

$$
\begin{aligned}
s.t. \quad & 0.0360x_1 + 0.0246x_2 + 0.0261x_3 + 0.0108x_4 + 0.0128x_5 + 0.0882x_6 + 0.0124x_7 + 0.0123x_8 \\
& +0.0151x_9 + 0.0113x_{10} + 0.0120x_{11} + 0.0135x_{12} + 0.0079x_{13} + 0.0055x_{14} + 0.0079x_{15} \\
& +0.0074x_{16} + 0.0074x_{17} - 0.0025x_{18} - 0.0025x_{19} - 0.0025x_{20} \leq \sigma \\
& \textstyle\sum_1^{20} x_i \leq 1 \\
& 0.01 \leq x_i \leq 0.3,\ i = 1,\ 2, \ldots, 17;\ 0 \leq x_j \leq 0.3,\ j = 18,\ 19,\ 20
\end{aligned}
$$

Step 3: Fuzzy portfolio selection under adjustable security proportion

The investment risks in this step are defined to range from 0.3% to 4% after the sensitivity analysis for the investment risk, and the portfolio results are shown in Table 6.

Table 6. The efficient portfolio with the guaranteed return rate $\widetilde{p}_1$.

Risk / Proportion	0.3%	0.4%	0.5%	1%	2%	3%	4%
x_1	0.01	0.01	0.01	0.01	0.2184	0.3	0.3
x_2	0.01	0.01	0.01	0.0203	0.3	0.3	0.3
x_3	0.01	0.01	0.01	0.01	0.01	0.179	0.179
x_4	0.01	0.01	0.01	0.1687	0.01	0.01	0.01
x_5	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_6	0.1609	0.2492	0.3	0.3	0.2606	0.01	0.01
x_7	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_8	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_9	0.01	0.01	0.0507	0.3	0.01	0.01	0.01
x_{10}	0.091	0.091	0.091	0.091	0.091	0.091	0.091
x_{11}	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_{12}	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_{13}	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_{14}	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_{15}	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_{16}	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_{17}	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_{18}	0	0	0	0	0	0	0
x_{19}	0.3	0.3	0.3	0	0	0	0
x_{20}	0.2981	0.2098	0.1183	0	0	0	0
Expected Return Rate	0.0544	0.0650	0.0754	0.1221	0.1898	0.2079	0.2079

We found that the investment proportion for securities 19 and 20 decreased from 0.3 to 0 and 0.2981 to 0, respectively. This is because the increase of investment risks makes the security return rate lower than the selected guaranteed rate of return $\widetilde{p}_1$ of the shortage investment. In other cases, the investment proportion for securities 1 and 2 increased from 0.01 to 0.3 and 0.01 to 0.3, respectively, because the return rates of securities 1 and 2 were significantly higher than the selected guaranteed rate of return $\widetilde{p}_1$ and contribute to the investor making an excess investment. Therefore, with a lower risk, the investors do not make excess investments in securities 1 and 2 but invest more in securities 19 and 20, which implies that the investor does not focuses on excess investment but on lower risks. By contrast, when the investment risk exceeds or equals 1%, implying that the investor focuses on excess investment, the proportion of securities 1 and 2 increases, as its higher return comes with higher risk.

Step 4: Sensitivity analysis for the guaranteed return rates

In this step, the portfolios with guaranteed return rates of $\tilde{p}_2$ and $\tilde{p}_3$ are shown in Tables 7 and 8, respectively. In Table 7, the ranking of the fuzzy numbers is as follows: $\tilde{r}_{20} < \tilde{r}_{19} < \tilde{r}_{18} < \tilde{p}_1 < \tilde{r}_{17} < \tilde{r}_{16} < \ldots < \tilde{r}_{10} < \tilde{p}_2 < \tilde{r}_9 < \tilde{r}_8 < \ldots < \tilde{r}_3 < \tilde{r}_2 < \tilde{r}_1$.

Table 7. The efficient portfolio with the guaranteed return rate $\tilde{p}_2$.

Proportion \ Risk	0.3%	0.4%	0.5%	1%	2%	3%	4%
x_1	0.01	0.01	0.01	0.0291	0.3	0.3	0.3
x_2	0.2022	0.2319	0.2615	0.3	0.3	0.3	0.3
x_3	0.01	0.01	0.01	0.01	0.0279	0.3	0.3
x_4	0.01	0.01	0.01	0.01	0.01	0.0298	0.05
x_5	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_6	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_7	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_8	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_9	0.01	0.01	0.01	0.01	0.01	0.01	0.01
x_{10}	0.3	0.3	0.3	0.3	0.3	0.0202	0
x_{11}	0.3	0.3	0.3	0.2379	0.121	0	0
x_{12}	0.1178	0.081	0.0585	0	0	0	0
x_{13}	0	0	0	0	0	0	0
x_{14}	0	0	0	0	0	0	0
x_{15}	0	0	0	0	0	0	0
x_{16}	0	0	0	0	0	0	0
x_{17}	0	0	0	0	0	0	0
x_{18}	0	0	0	0	0	0	0
x_{19}	0	0	0	0	0	0	0
x_{20}	0	0	0	0	0	0	0
Expected Return Rate	0.0962	0.1007	0.1052	0.1249	0.1612	0.1784	0.1790

Therefore, securities 10–20 are assumed to be shortage investments, and securities 1–9 are assumed to be excess investments. We also define the investment risks in this step to range from 0.3% to 4%. We find that the investment proportion for securities 10 and 11 decrease from 0.3 to 0, and security 12 decreases from 0.1178 to 0. This is because the increase of investment risks pushes the investor to decide to make shortage investments to the securities whose return rates are lower than the selected guaranteed return rate $\tilde{p}_2$. In other cases, the investment proportion for securities 1–4 increases from 0.01 to 0.3, 0.2022 to 0.3, 0.01 to 0.3, and 0.01 to 0.05, respectively, because the return rates of securities 1, 2, 3, and 4 are higher than the selected g guaranteed return rate $\tilde{p}_2$ and contribute to the investor making an excess investment. Therefore, when the investment risk exceeds or equals 3%, implying that the investor focuses on excess investment, we can suggest that the securities 1, 2, and 3 be the major investment with the proportions 0.3, respectively.

Next, we set the guaranteed return rate $\tilde{p}_3$ and allowed excess investment to securities 1 and 2, while other securities 3–20 are allowed to be shortage investments. In Table 8, the ranking of the fuzzy returns is as follows: $\tilde{r}_{20} < \tilde{r}_{19} < \tilde{r}_{18} < \tilde{p}_1 < \tilde{r}_{17} < \tilde{r}_{16} < \ldots < \tilde{r}_{10} < \tilde{r}_9 < \tilde{r}_8 < \ldots < \tilde{r}_3 < \tilde{p}_3 < \tilde{r}_2 < \tilde{r}_1$. We define the investment risks in this step to range from 0.3% to 4%. Because most of securities are selected for shortage investments, we found that the investment proportion for securities 4 and 5 decreased from 0.3 to 0.1 and 0.1106 to 0, respectively, and investment proportion for security 3 was 0.091 in each investment risk and the other securities were always 00. This is because the increase of investment risk pushes the investor to make shortage investments in the securities whose return rates are lower than the selected guaranteed return rate $\widetilde{p}_3$. In other cases, the investment proportion for securities 1 and 2 increases from 0.01 to 0.3 and 0.2794 to 0.3 because the return rates of securities 1 and 2 are higher than the selected guaranteed return rate $\widetilde{p}_3$ and contribute to the investor making excess investments. Therefore, with each investment risk that implies that the investor focuses on excess investment, we can suggest that the securities 1 and 2 be the major investment with the proportion 0.3.

Table 8. The efficient portfolio with the guaranteed return rate $\widetilde{p}_3$.

Risk / Proportion	0.3%	0.4%	0.5%	1%	2%	3%	4%
x_1	0.01	0.0141	0.034	0.1337	0.3	0.3	0.3
x_2	0.2794	0.3	0.3	0.3	0.3	0.3	0.3
x_3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
x_4	0.3	0.3	0.3	0.2663	0.1	0.1	0.1
x_5	0.1106	0.0859	0.066	0	0	0	0
x_6	0	0	0	0	0	0	0
x_7	0	0	0	0	0	0	0
x_8	0	0	0	0	0	0	0
x_9	0	0	0	0	0	0	0
x_{10}	0	0	0	0	0	0	0
x_{11}	0	0	0	0	0	0	0
x_{12}	0	0	0	0	0	0	0
x_{13}	0	0	0	0	0	0	0
x_{14}	0	0	0	0	0	0	0
x_{15}	0	0	0	0	0	0	0
x_{16}	0	0	0	0	0	0	0
x_{17}	0	0	0	0	0	0	0
x_{18}	0	0	0	0	0	0	0
x_{19}	0	0	0	0	0	0	0
x_{20}	0	0	0	0	0	0	0
Expected Return Rate	0.1041	0.1071	0.1098	0.1235	0.1460	0.1460	0.1460

From Tables 6–8, as shown in Figure 2 with the increasing investment risk levels, the expected return rate of a portfolio with a lower guaranteed return rate is larger than the one with a higher guaranteed return rate. When we select a higher guaranteed rate of return most of the securities are selected for shortage investments and thus we obtain a smaller expected rate of return. Therefore, an investor should not select too large a guaranteed return rate for obtaining larger investment returns. We suggest that the guaranteed return rate can be decide by investors based on the heuristic experience and they should select smaller values of guaranteed return rates in modelling from the first quartile of the ranking fuzzy returns. In addition, no matter which value of guaranteed return rate we select, the proportion of securities 1 and 2 is always larger than the other securities when the investment risk is at a higher level because their expected returns are higher than the other securities. Therefore, under the guaranteed return rate, the security with the largest fuzzy return rate is the most important investment in that portfolio.

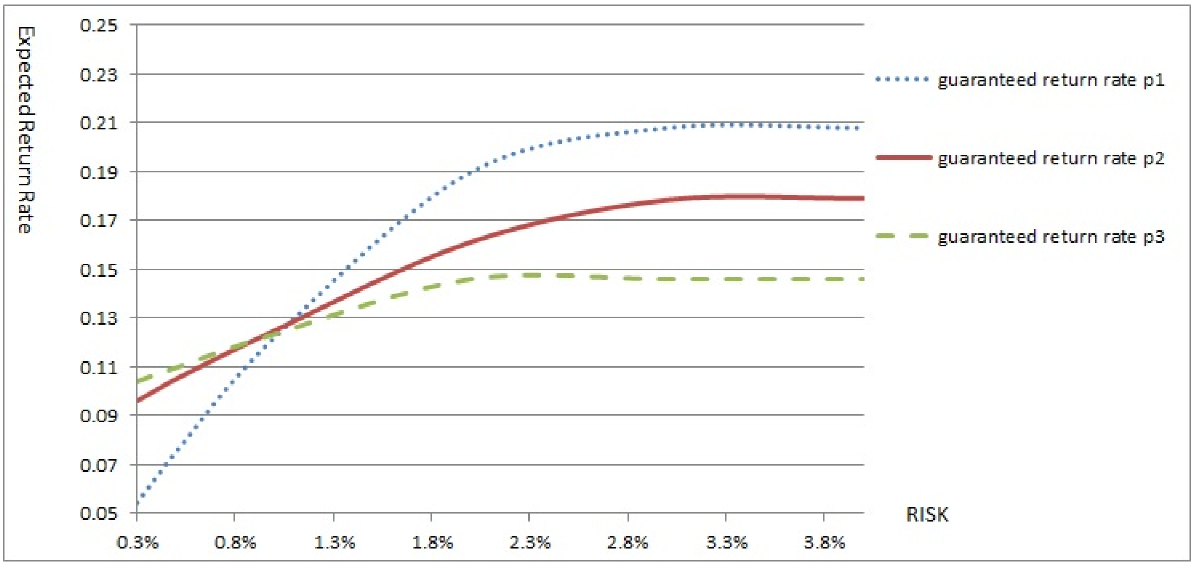

Figure 2. The comparisons between different guaranteed return rates.

5. Conclusions

In this study, we used the concept of guarantee return rates to propose a fuzzy portfolio model for some securities adopting excess investment and others with shortage investments in the fuzzy portfolio analysis. In our proposed model, an investor prefers excess investment in some securities whose investment proportions are larger than those securities with shortage investments. Next, under lower investment risk, the securities with higher fuzzy return rates are usually not selected for excess investment. By contrast, when the investment risk is high, the security with the higher fuzzy return rate will stand as the most important investment in the portfolio. Finally, we suggest not selecting too large of a guaranteed return rate to obtain higher expected returns. We suggest that the guaranteed return rate can be decided by investors based on the heuristic experience, and they should select smaller values of guaranteed return rates in modelling from the first quartile of the ranking fuzzy returns. Most importantly, if the fuzzy portfolio model for a guaranteed return rate meets expectations, this approach will be useful for determining excess investments and shortage investments for the selected securities.

Limitations and Future Research

In this study, the concept of guarantee return rates is used to model fuzzy portfolios in which the selected security return rates are for excess investments because their expected return rates are higher than guarantee return rates, and shortage investments are where the selected securities return rates are lower than guarantee return rates. Because the guaranteed return rates may differ depending on investors, a collaborative discussion involving numerous experts was required to establish objective threshold values. Therefore, only past investment performance for the selected securities was included in the evaluation and selection process for establishing the threshold values. Future research should focus on (1) considering the number of experts for the selected guarantee return rates, (2) establishing comprehensive threshold values and decision criteria according guarantee return rates, (3) incorporating investors' subjective acceptable guaranteed return rates ranges to enhance the usability of the fuzzy portfolio model, (4) making a sensitivity analysis to test the number of securities in a portfolio, and (5) relaxing the constraint to require the sum of security investment ratios to be 1.

Author Contributions: Conceptualization, R.-C.T., I.-F.C. and C.-L.C.; methodology, R.-C.T. and Y.-Y.H.; software, I.-F.C.; validation, R.-C.T., I.-F.C. and C.-L.C.; formal analysis, Y.-Y.H. and I.-F.C.; investigation, Y.-Y.H.; resources, Y.-Y.H.; data curation, Y.-Y.H. and I.-F.C.; writing—original draft

preparation, Y.-Y.H.; writing—review and editing, I.-F.C.; visualization, Y.-Y.H.and I.-F.C.; supervision, R.-C.T. All authors have read and agreed to the published version of the manuscript.

Funding: Ministry of Science and Technology with project No. MOST 110-2221-E-032-033.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors gratefully acknowledge the financial support from Ministry of Science and Technology with project No. MOST 110-2221-E-032-033.

Conflicts of Interest: The authors declare no conflict of interest.

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