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Existence of Positive Solutions for a Higher-Order Fractional Differential Equation with Multi-Term Lower-Order Derivatives

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Abstract: This paper deals with the study of the existence of positive solutions for a class of nonlinear higher-order fractional differential equations in which the nonlinear term contains multi-term lower-order derivatives. By reducing the order of the highest derivative, the higher-order fractional differential equation is transformed into a lower-order fractional differential equation. Then, combining with the properties of left-sided Riemann–Liouville integral operators, we obtain the existence of the positive solutions of fractional differential equations utilizing some weaker conditions. Furthermore, some examples are given to demonstrate the validity of our main results.

Keywords: fractional differential equation; boundary value problems; positive solution

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1. Introduction

The fractional differential equation has a broad application background, so it has received attention and interest from the majority of researchers and has been rapidly developed. At the same time, its applications in science and engineering are gradually expanding, and in recent decades, its application fields have come to include fluid mechanics, genetic epidemiology [1], viscoelastic mechanics [2], neural fractional models [3], and electrochemistry [4]. For other applications of fractional differential equations, we refer the reader to [5–12]. For example, in [8], there is a fractional derivative relaxation–vibration equation: $\frac{d^p u(t)}{dt^p} + Bu(t) = f(t)(1 < p \leq 2)$. This model can be used to describe the slow stress relaxation of memorization instead of Newton's sticky pot of integral derivative in the standard model because the fractional derivative contains the time convolution integral that can describe the memorization process.

Many studies of fractional calculus and fractional differential equations have involved different derivatives such as Riemann–Liouville, Erdelyi–Kober, Weyl–Riesz, Caputo, Hadamard and Grunwald–Letnikov. In the realm of fractional differential equations, the Caputo derivative and Riemann–Liouville ones are used the most. Fractional differential equations with various initial or boundary value conditions have been widely discussed, meanwhile, a variety of techniques have been applied to obtain the existence of solutions, uniqueness, multiplicity, etc. In all studies of higher-order differential equations that depend on lower-order derivatives of either integer or fractional order, there is a limitation, i.e., the difference between the highest derivative and adjacent lower-order derivative is greater than or equal to 1 (see [13–24]).

Due to the inherent difficulties in the fractional calculus, to the best of our knowledge, if only the left (or right) Riemann–Liouville fractional derivatives are involved, the most feasible approach to study the existence of solutions of a boundary value problem is to convert it into a fixed point problem for an appropriate operator. This idea has been widely used by many researchers, for a small sample of such work, as can be seen in [25–31] and the references therein for more comments and citations.

In the study of higher-order differential equations of integer order, it has been possible to make this difference between the order of the differential equation and adjacent lower-order derivative equal to 1, as can be seen in [32–35]. For example, in [32], the author studied the singular boundary value problems of the nth-order ordinary differential equation with all derivative terms:

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), x''(t), ..., x^{(i)}(t), ..., x^{(n-1)}(t)) = 0, t \in (0,1), \\ x^{(i)}(0) = 0, i = 0,1,2,...,n-2, x^{(n-1)}(1) = 0, \end{cases}$$

where $f : (0,1) \times (0,\infty)^n \to \mathbb{R}^+$ is continuous with $\mathbb{R}^+ = [0,\infty)$. A necessary and sufficient condition for the existence of $C^{n-1}[0,1]$ positive solutions was given by constructing lower and upper solutions and with the comparison theorem.

In this respect, Liouville–Caputo-type fractional differential equations have also made progress similar to that of integral order ordinary differential equations, as can be seen in [13–16]. Yang in [13] investigated the nonlinear differential equation of fractional order:

$$\begin{cases} {}^{C}D^{\alpha}_{0+}u(t) = f(t, u(t), u'(t), t \in (0,1), 1 < \alpha \leq 2, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0. \end{cases}$$

By means of Schauder's fixed point theorem and an extension of Krasnoselskii's fixed point theorem in a cone, some results on the existence of positive solutions were obtained.

As for Riemann–Liouville-type fractional differential equations, the current result is that the difference between the highest derivative and the adjacent lower derivative is greater than or equal to 1, as can be seen in [17–24]. For example, in [17], applying Schauder's fixed point theorem and upper and lower solutions method, Zhang established the existence of positive solutions to a singular higher-order fractional differential equation involving fractional derivatives:

$$\begin{cases} -D^{\alpha}x(t) = \lambda f(x(t), D^{\mu_1}x(t), D^{\mu_2}x(t), \cdots, D^{\mu_{n-1}}x(t)), 0 < t < 1, \\ x(0) = 0, D^{\mu_i}x(0) = 0, D^{\mu}x(1) = \sum_{j=1}^{p-2} a_j D^{\mu}x(\xi_j), 1 \leq i \leq n-1, \end{cases}$$

where D^{α} is the standard Riemann–Liouville derivative, $n \geq 3, n \in \mathbb{N}, n-1 < \alpha \leq n, n-l-1 < \alpha - \mu_l < n-l \ (l = 1,2,\cdots,n-2), \mu - \mu_{n-1} > 0, 1 < \alpha - \mu_{n-1} \leq 2, \alpha - \mu > 1, a_j \in [0,+\infty), 0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1, 0 < \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1} < 1, f : (0,+\infty)^n \to \mathbb{R}^+$ is continuous.

By using the properties of the Green function, the fixed point index theory and the Banach contraction mapping principle based on some available operators, in [18], Wang et al. obtained the existence of positive solutions and a unique positive solution of the fractional differential equation:

$$\begin{cases} D^{\gamma}_{0+}x(t) + f(t, x(t), D^{\alpha}_{0+}x(t), D^{\beta}_{0+}x(t)) = 0, 0 < t < 1, \\ x^{(j)}(0) = 0, D^{\beta}_{0+}x(0) = 0, j = 0,1,\cdots,n-3, \\ D^{\beta}_{0+}x(1) = a_1 \int_0^1 p_1(s) D^{\beta}_{0+}x(s) dA_1(s) + a_2 \int_0^{\eta} p_2(s) D^{\beta}_{0+}x(s) dA_2(s) + a_3 \sum_{i=1}^{\infty} \mu_i D^{\beta}_{0+}x(\zeta_i), \end{cases}$$

where $D^{\gamma}, D^{\alpha}, D^{\beta}$ denote the Riemann–Liouville fractional derivative, $0 < \alpha < n-2 \leq \beta < n-1, \gamma - \beta > 1, \beta - \alpha \geq 1, a_j, \mu_i \geq 0, 0 < \eta < \zeta_1 < \zeta_2 < \cdots < 1, 1 - a_3 \sum_{i=1}^{\infty} \mu_i \zeta_i^{\delta-1} > 0$, $p_1, p_2 \in C(0,1) \cap L^1(0,1)$ are nonnegative, $\int_0^1 p_1(s)u(s)dA_1(s)$, $\int_0^1 p_2(s)u(s)dA_2(s)$ denote the Riemann–Stieltjes integrals, $A_1, A_2 : [0,1] \to R$ are the function of bounded variation, $f : [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

By means of the Guo–Krasnoselskii fixed point theorem, under sublinearity conditions, Ref. [19] investigated the existence of at least one positive solution of the following initial value problem with the higher-order Riemann–Liouville-type fractional differential equation:

$$\begin{cases} D^{q}u(t) = f(t, u(t), D^{q-1}u(t)) = 0, (q > 2), \\ u(0) = 0, D^{q-i}u(0) = 0, i = 1,2,\cdots,n-1, \end{cases}$$

where D^q represents the standard Riemann–Liouville fractional derivative, $f(t,x,y) = g(t)f_1(x,y)$, $g \in L^1([0,1],\mathbb{R}_+^*)$, $f_1(0,0) \neq 0$, $f_1 \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$ and f_1 is a convex, nonnegative function, decreasing according to each of its variables.

Motivated by the previously mentioned works, we will establish the following higher-order fractional boundary value problems:

$$\begin{cases} D_{0+}^{\delta}u(t) + f(t,u(t),D_{0+}^{\delta_1}u(t),D_{0+}^{\delta_2}u(t),\cdots,D_{0+}^{\delta_{n-1}}u(t),D_{0+}^{\delta_n}u(t)) = 0,\ 0 < t < 1, \\ u(0) = D_{0+}^{\alpha_1}u(0) = \cdots = D_{0+}^{\alpha_{n-2}}u(0) = 0, \\ D_{0+}^{\alpha_{n-1}}u(1) = \varphi[D_{0+}^{\alpha_{n-2}}u(t)], \end{cases} \tag{1}$$

where $D_{0+}^{\delta}, D_{0+}^{\delta_i}$ $(i = 1,2,...,n), D_{0+}^{\alpha_j}$ $(j = 1,2,..,n-1)$ represent the standard Riemann–Liouville fractional derivatives. $f : (0,1] \times (\mathbb{R}^+)^n \times \mathbb{R} \to \mathbb{R}^+$ with $\mathbb{R}^+ = [0,\infty)$. $n-1 < \delta \leq n$, n is a positive integer ($n \geq 3$). The parameters and function φ of the problem satisfy the following conditions:

($\mathbf{C}_1$). $0 = \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_{n-2} \leq \alpha_{n-2} \leq n-2 < \delta_{n-1} \leq n-1 < \delta_n < \delta \leq n$, $n-3-i < \alpha_{n-2} - \alpha_i \leq n-2-i$ $(i = 1,2,...,n-3), 0 < \delta - \alpha_{n-1} \leq 1 < \delta - \delta_{n-1} < \delta - \alpha_{n-2} \leq 2, \triangle = \Gamma(\delta - \alpha_{n-2}) - \Gamma(\delta - \alpha_{n-1})\varphi[t^{\delta-\alpha_{n-2}-1}] > 0$, $\varphi : C[0,1) \to \mathbb{R}$ is a nondecreasing linear function and $\varphi[0] \geq 0$.

There are various forms of nonlocal boundary conditions, such as multi-point boundary value, infinite point boundary value and integral boundary value:

$$\varphi[u] = \sum_{i=1}^{m} \beta_i u(\eta_i), \beta_i > 0, i = 1,2,...,m, 0 \leq \eta_1 < \eta_2 < \cdots < \eta_m \leq 1,$$

$$\varphi[u] = \sum_{i=1}^{\infty} \rho_i u(\xi_i), \rho_i > 0, i = 1,2,...,0 \leq \xi_1 < \xi_2 < \cdots \leq 1,$$

or:

$$\varphi[u] = \int_0^1 \psi(s)u(s)ds, \psi \geq 0.$$

Apart from these conditions, the function φ of the BVP (1) represents a wider range of other conditions.

To simplify our statement, in the sequel, we refer to $C := C[0,1]$ the classical space of continuous functions defined on $[0,1]$, endowed with the classical uniform norm $\|x\|_0 = \max_{t\in[0,1]} |x(t)|$ and to $C_\varepsilon := C_\varepsilon[0,1]$ the set of continuous functions x on $(0,1]$ such that $t \to t^{1-\varepsilon}x(t)$ is continuous on [0,1], endowed with the norm $\|x\|_\varepsilon = \max_{t\in[0,1]} |t^{1-\varepsilon}x(t)| (0 < \varepsilon < 1)$. The set $L^m := L^m[0,1]$ is the classical Lebesgue space of m-integrable functions on [0,1], endowed with its usual norm $\|x\|_{L^m} = (\int_0^1 |x(t)|^m dt)^{\frac{1}{m}}, (1 \leq m < \infty)$. $L_\varepsilon^m := L_\varepsilon^m[0,1]$ is the set of functions x on $(0,1]$ such that $t \to t^{1-\varepsilon}x(t)$ belongs to L^m, and its norm is expressed by $\|x\|_{L_\varepsilon^m} = (\int_0^1 |t^{1-\varepsilon}x(t)|^m dt)^{\frac{1}{m}}, (1 \leq m < \infty, 0 < \varepsilon < 1)$. In particular, function spaces $L^p, L_{\delta-\delta_n}^p$ will be used in this article, where p satisfies:

($\mathbf{C}_2$). $0 < \frac{1}{p} < \min\{\delta - \delta_{n-1} - 1, \delta - \delta_n\}, \frac{1}{p} + \frac{1}{q} = 1$.

Comparing the problem (1) to the aforementioned papers, the highlights of our results lie in the following aspects. First, the nonlinear term f contains a series of lower derivatives, especially the lower derivative $D_{0+}^{\delta_n}$ which satisfies $0 < \delta - \delta_n < 1$. In previous studies, there was only a relationship such as $\delta - \delta_{n-1} \geq 1$. However, we allow $n-1 < \delta_n < \delta \leq n$, which fills in a gap in previous research. Second, some of the lower derivatives in the nonlinear term simply require that $0 < \delta_1 < \delta_2 < \cdots < \delta_{n-2} \leq n-2$, however, previous studies have required that $n-k-1 < \delta - \delta_k \leq n-k$ $(k = 1,2,...,n-2)$. Hence, our equation is more extensive. Third, the boundary condition is more general, as it can be not only a multi-point/infinite-point boundary value, but also an integral boundary value, etc. Fourth, in most of the previous literature, the integral operator maps the continuous function space C, or the L^p space to the continuous function space. A better result was obtained in this paper, i.e., the integral operator can map space $L^p_{\delta-\delta_n}$ to the space of weighted continuous functions. This property help us to obtain the existence of solutions under weaker conditions.

The goal of our research is to propose new existence criteria for the positive solutions of the BVP (1) under weaker conditions. In addition, we studied the uniqueness result for (1).

The remaining part of the paper is organized as follows. In Section 2, we recall some basic properties and introduce some new lemmas which will be used later. Properties of Green's function are obtained in Section 3. The main results are presented in Section 4. In Section 5, some examples are given to demonstrate the application of our main result. Section 6 is our conclusions section.

2. Preliminaries

We begin this section with some fundamental facts of the fractional calculus theory, which are used throughout the paper:

Lemma 1 ([9,36])**.** *Let $\alpha \geq \beta > 0$, supposing that $D^{\alpha}_{0+}u(t)$ is integrable on [0,1], then:*

$$I^{\beta}_{0+}D^{\alpha}_{0+}u(t) = D^{\alpha-\beta}_{0+}u(t) - \sum_{i=1}^{n} \frac{D^{\alpha-i}_{0+}u(t)|_{t=0}}{\Gamma(1+\beta-i)} t^{\beta-i}, t \in [0,1],\ 0 \leq n-1 \leq \alpha < n.$$

Lemma 2 ([9,36])**.** *When $\beta > \alpha > 0, h \in L[0,1]$, then:*

(1) $I^{\alpha}_{0+}I^{\beta}_{0+}h(t) = I^{\beta}_{0+}I^{\alpha}_{0+}h(t) = I^{\alpha+\beta}_{0+}h(t)$;

(2) $D^{\alpha}_{0+}I^{\beta}_{0+}h(t) = I^{\beta-\alpha}_{0+}h(t)$;

(3) $D^{\alpha}_{0+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}, \beta > -1,\ \beta > \alpha-1,\ t > 0.$

Now, we define a modified problem of problem (1) as follows:

$$\begin{cases} -D^{\delta-\alpha_{n-2}}_{0+}v(t) = f(t, I^{\alpha_{n-2}}_{0+}v(t), \cdots, I^{\alpha_{n-2}-\delta_{n-2}}_{0+}v(t), D^{\delta_{n-1}-\alpha_{n-2}}_{0+}v(t), D^{\delta_n-\alpha_{n-2}}_{0+}v(t)), \\ v(0) = 0, D^{\alpha_{n-1}-\alpha_{n-2}}_{0+}v(1) = \varphi[v(t)], \end{cases} \tag{2}$$

Lemma 3. *Let $v(t) = D^{\alpha_{n-2}}_{0+}u(t)$, then we can transform problem (1) into problem (2). In other words, if $v \in C[0,1]$ is a positive solution of problem (2), then the function $u(t) = I^{\alpha_{n-2}}_{0+}v(t)$ is the positive solution of problem (1).*

Proof of Lemma 3. By the definition of the Riemann–Liouville fractional derivative and Lemma 1, we obtain $u(t) = I^{\alpha_{n-2}}_{0+}v(t) + d_1t^{\alpha_{n-2}-1} + \cdots + d_{n-2}t^{\alpha_{n-2}-n+2}$. Combining with the condition $u(0) = D^{\alpha_1}_{0+}u(0) = \cdots = D^{\alpha_{n-3}}_{0+}u(0) = 0$ and $n-3-i < \alpha_{n-2} - \alpha_i \leq n - 2 - i\ (i = 1,2,...,n-3)$, we have $d_i = 0\ (i = 1,2,...,n-2)$, then we obtain $u(t) = I^{\alpha_{n-2}}_{0+}v(t)$. Again, by Lemma 2, we have:

$$D^{\delta}_{0+}u(t) = \frac{d^n}{dt^n}I^{n-\delta}_{0+}u(t) = \frac{d^n}{dt^n}I^{n-\delta}_{0+}I^{\alpha_{n-2}}_{0+}v(t) = \frac{d^n}{dt^n}I^{n-\delta+\alpha_{n-2}}_{0+}v(t) = D^{\delta-\alpha_{n-2}}_{0+}v(t), \tag{3}$$

$$D^{\delta_i}_{0+}u(t) = D^{\delta_i}_{0+}I^{\alpha_{n-2}}_{0+}v(t) = I^{\alpha_{n-2}-\delta_i}_{0+}v(t), i = 1,2,...,n-2, \tag{4}$$

$$D^{\delta_{n-1}}_{0+}u(t) = \frac{d^{n-1}}{dt^{n-1}}I^{n-1-\delta_{n-1}}_{0+}I^{\alpha_{n-2}}_{0+}v(t) = \frac{d^{n-1}}{dt^{n-1}}I^{n-1-(\delta_{n-1}-\alpha_{n-2})}_{0+}v(t) = D^{\delta_{n-1}-\alpha_{n-2}}_{0+}v(t), \tag{5}$$

$$D^{\delta_n}_{0+}u(t) = \frac{d^n}{dt^n}I^{n-\delta_n}_{0+}I^{\alpha_{n-2}}_{0+}v(t) = \frac{d^n}{dt^n}I^{n-(\delta_n-\alpha_{n-2})}_{0+}v(t) = D^{\delta_n-\alpha_{n-2}}_{0+}v(t), \tag{6}$$

$$D^{\alpha_{n-1}}_{0+}u(t) = D^{\alpha_{n-1}}_{0+}I^{\alpha_{n-2}}_{0+}v(t) = D^{\alpha_{n-1}-\alpha_{n-2}}_{0+}v(t). \tag{7}$$

It is obvious that the conditions $v(0) = D_{0+}^{\alpha_{n-2}}u(0) = 0$ as well as $D_{0+}^{\alpha_{n-1}-\alpha_{n-2}}v(1) = \varphi[v(t)]$ hold. Hence, problem (1) is transformed into problem (2).

Now, suppose $v \in C[0,1]$ is a positive solution of problem (2), let $u(t) = I_{0+}^{\alpha_{n-2}}v(t)$, then $u(t) \geq 0, t \in [0,1]$. From (3)–(6), we have:

$$\begin{aligned}&D_{0+}^{\delta}u(t) + f(t, u(t), D_{0+}^{\delta_1}u(t), D_{0+}^{\delta_2}u(t), \cdots, D_{0+}^{\delta_n}u(t))\\ =&D_{0+}^{\delta-\alpha_{n-2}}v(t) + f(t, I_{0+}^{\alpha_{n-2}}v(t), \cdots, I_{0+}^{\alpha_{n-2}-\delta_{n-2}}v(t), D_{0+}^{\delta_{n-1}-\alpha_{n-2}}v(t), D_{0+}^{\delta_n-\alpha_{n-2}}v(t)) = 0,\end{aligned}$$

we also obtain $u(0) = D_{0+}^{\alpha_1}u(0) = \cdots = D_{0+}^{\alpha_{n-2}}u(0) = 0$ from the representation $u(t) = I_{0+}^{\alpha_{n-2}}v(t)$ and the condition $v(0) = 0$ in problem (2). Combining with the condition $D_{0+}^{\alpha_{n-1}-\alpha_{n-2}}v(1) = \varphi[v(t)]$ and (7), we know that $D_{0+}^{\alpha_{n-1}}u(1) = \varphi[D_{0+}^{\alpha_{n-2}}u(t)]$, which means that the function $u(t) = I_{0+}^{\alpha_{n-2}}v(t)$ is a solution of problem (1). □

We then introduce the properties of integral operators which are going to play a very important role in the subsequent proofs of the main results.

Lemma 4. *Suppose that* $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$ *hold, integral operators* $I_{0+}^{\delta-\alpha_{n-2}} : L_{\delta-\delta_n}^{p} \to C$ *and* $I_{0+}^{\delta-\delta_{n-1}} : L_{\delta-\delta_n}^{p} \to C$ *are continuous.*

Proof of Lemma 4. For all $h \in L_{\delta-\delta_n}^{p}$, we will show that $I_{0+}^{\delta-\alpha_{n-2}}h \in C$. Let $0 \leq t_1 < t_2 \leq 1:$

$$\begin{aligned}&|I_{0+}^{\delta-\alpha_{n-2}}h(t_2) - I_{0+}^{\delta-\alpha_{n-2}}h(t_1)|\\ \leq&\frac{1}{\Gamma(\delta-\alpha_{n-2})}\left[\int_0^{t_1}((t_2-s)^{\delta-\alpha_{n-2}-1} - (t_1-s)^{\delta-\alpha_{n-2}-1})|h(s)|ds\right.\\ &\left.+\int_{t_1}^{t_2}(t_2-s)^{\delta-\alpha_{n-2}-1}|h(s)|ds\right]\\ \leq&\frac{\|h\|_{L_{\delta-\delta_n}^{p}}}{\Gamma(\delta-\alpha_{n-2})}\left[\left(\int_0^{t_1}((t_2-s)^{(\delta-\alpha_{n-2}-1)q} - (t_1-s)^{(\delta-\alpha_{n-2}-1)q})s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{t_1}^{t_2}(t_2-s)^{(\delta-\alpha_{n-2}-1)q}s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\right](q=\frac{p}{p-1}).\end{aligned} \tag{8}$$

Since $0 < \frac{1}{p} < \delta - \delta_n < 1$, we have $0 < (\delta - \delta_n - 1)q + 1 < 1$, then the second right-hand side term in the above inequality (8) has the following estimate:

$$\begin{aligned}&\left(\int_{t_1}^{t_2}(t_2-s)^{(\delta-\alpha_{n-2}-1)q}s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\\ \leq&\left(\int_{t_1}^{t_2}(t_2-s)^{(\delta-\alpha_{n-2}-1)q}(s-t_1)^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\\ =&[(t_2-t_1)^{((\delta-\alpha_{n-2}-1+\delta-\delta_n-1)q+1)\frac{1}{q}}\mathcal{B}((\delta-\alpha_{n-2}-1)q+1, (\delta-\delta_n-1)q+1)]^{\frac{1}{q}}\\ =&\kappa_0(t_2-t_1)^{\delta-\alpha_{n-2}-1-\frac{1}{p}+(\delta-\delta_n)},\end{aligned} \tag{9}$$

where $\mathcal{B}(\cdot,\cdot)$ denotes the usual Beta function and:

$$\kappa_0 = [\mathcal{B}((\delta-\alpha_{n-2}-1)q+1, (\delta-\delta_n-1)q+1)]^{\frac{1}{q}}. \tag{10}$$

We will then estimate the first right-hand side term in the above inequality (8) in two cases.

(i) If $0 < (\delta - \alpha_{n-2} - 1)q \leq 1:$

$$\begin{aligned}
&\left(\int_0^{t_1}\left((t_2-s)^{(\delta-\alpha_{n-2}-1)q}-(t_1-s)^{(\delta-\alpha_{n-2}-1)q}\right)s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\\
\le&\left(\int_0^{t_1}(t_2-t_1)^{(\delta-\alpha_{n-2}-1)q}s^{(\delta-\delta_n-1)q}\right)^{\frac{1}{q}}\\
=&(t_2-t_1)^{\delta-\alpha_{n-2}-1}\left(\frac{t_1^{(\delta-\delta_n-1)q+1}}{(\delta-\delta_n-1)q+1}\right)^{\frac{1}{q}}.
\end{aligned}\tag{11}$$

(ii) If $(\delta-\alpha_{n-2}-1)q>1$; owing to the Lagrange's mean value theorem, we choose $\xi\in(t_1-s,t_2-s)$ such that:

$$\begin{aligned}
&\left(\int_0^{t_1}\left((t_2-s)^{(\delta-\alpha_{n-2}-1)q}-(t_1-s)^{(\delta-\alpha_{n-2}-1)q}\right)s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\\
=&\left((t_2-t_1)(\delta-\alpha_{n-2}-1)q\int_0^{t_1}\xi^{(\delta-\alpha_{n-2}-1)q-1}s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\\
\le&\left((t_2-t_1)(\delta-\alpha_{n-2}-1)q\frac{t_1^{(\delta-\delta_n-1)q+1}}{(\delta-\delta_n-1)q+1}\right)^{\frac{1}{q}}\\
\le&\left(\frac{(\delta-\alpha_{n-2}-1)q}{(\delta-\delta_n-1)q+1}\right)^{\frac{1}{q}}(t_2-t_1)^{\frac{1}{q}}.
\end{aligned}\tag{12}$$

Therefore, from (9), (11) and (12), as $t_2\to t_1$, we obtain $|I_{0+}^{\delta-\alpha_{n-2}}h(t_2)-I_{0+}^{\delta-\alpha_{n-2}}h(t_1)|\to 0$, which means that $I_{0+}^{\delta-\alpha_{n-2}}h$ is continuous on [0,1].

Let $h_j\to h$ in $L^p_{\delta-\delta_n}$, we need to prove that $\lim\limits_{j\to\infty}\|I_{0+}^{\delta-\alpha_{n-2}}h_j-I_{0+}^{\delta-\alpha_{n-2}}h\|_0=0$. In fact:

$$\begin{aligned}
\|I_{0+}^{\delta-\alpha_{n-2}}h_j-I_{0+}^{\delta-\alpha_{n-2}}h\|_0\le&\frac{1}{\Gamma(\delta-\alpha_{n-2})}\max_{0\le t\le 1}\int_0^t(t-s)^{\delta-\alpha_{n-2}-1}|h_j(s)-h(s)|ds\\
\le&\frac{\|h_j-h\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\alpha_{n-2})}\max_{0\le t\le 1}\left(\int_0^t(t-s)^{(\delta-\alpha_{n-2}-1)q}s^{(\delta-\delta_n-1)q}\right)^{\frac{1}{q}}\\
=&\frac{\|h_j-h\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\alpha_{n-2})}\kappa_0,
\end{aligned}$$

where κ_0 is defined in (10). Obviously, in view of the estimate, we naturally have $\|I_{0+}^{\delta-\alpha_{n-2}}h_j-I_{0+}^{\delta-\alpha_{n-2}}h\|_0\to 0$, as $j\to\infty$. Since $1<\delta-\delta_{n-1}<\delta-\alpha_{n-2}$, following the same procedure as above, we will naturally come to the continuity of $I_{0+}^{\delta-\delta_{n-1}}$. □

Lemma 5. *Suppose that* $(C_1),(C_2)$ *hold, then* $I_{0+}^{\delta-\delta_n}:L^p_{\delta-\delta_n}\to C_{\delta-\delta_n}$ *is continuous.*

Proof of Lemma 5. For all $h\in L^p_{\delta-\delta_n}$, we need to show that $I_{0+}^{\delta-\delta_n}h\in C_{\delta-\delta_n}$. For convenience, denote $H(t)=t^{1+\delta_n-\delta}I_{0+}^{\delta-\delta_n}h(t)$. Then, for $t\in(0,1]$:

$$\begin{aligned}
|H(t)|=&\frac{t^{1+\delta_n-\delta}}{\Gamma(\delta-\delta_n)}\left|\int_0^t(t-s)^{\delta-\delta_n-1}h(s)ds\right|\\
\le&\frac{t^{1+\delta_n-\delta}}{\Gamma(\delta-\delta_n)}\|h\|_{L^p_{\delta-\delta_n}}\left(\int_0^t(t-s)^{(\delta-\delta_n-1)q}s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\\
=&\frac{\|h\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\delta_n)}t^{1+\delta_n-\delta}t^{\delta-\delta_n-1+\delta-\delta_n-\frac{1}{p}}(\mathcal{B}((\delta-\delta_n-1)q+1,(\delta-\delta_n-1)q+1))^{\frac{1}{q}}\\
=&\frac{\|h\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\delta_n)}\kappa_1t^{\delta-\delta_n-\frac{1}{p}},
\end{aligned}\tag{13}$$

where:

$$\kappa_1 = (\mathcal{B}((\delta-\delta_n-1)q+1,(\delta-\delta_n-1)q+1))^{\frac{1}{q}}. \tag{14}$$

By (13), we notice that $\lim_{t\to 0} H(t) = 0$. As a consequence, we supplement the definition of H on $t = 0$; thus, H is continuous on $t = 0$.

For $0 < t_1 \leq t_2 \leq 1$, we have:

$$\begin{aligned}
&|t_1^{1+\delta_n-\delta} I_{0+}^{\delta-\delta_n} h(t_1) - t_2^{1+\delta_n-\delta} I_{0+}^{\delta-\delta_n} h(t_2)| \\
\leq& \frac{1}{\Gamma(\delta-\delta_n)}\bigg[t_1^{1+\delta_n-\delta}\int_0^{t_1}((t_1-s)^{\delta-\delta_n-1}-(t_2-s)^{\delta-\delta_n-1})|h(s)|ds \\
&+t_1^{1+\delta_n-\delta}\int_{t_1}^{t_2}(t_2-s)^{\delta-\delta_n-1}|h(s)|ds + (t_2^{1+\delta_n-\delta}-t_1^{1+\delta_n-\delta})\int_0^{t_2}(t_2-s)^{\delta-\delta_n-1}|h(s)|ds\bigg] \\
\leq& \frac{\|h\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\delta_n)}\bigg[t_1^{1+\delta_n-\delta}\bigg(\int_0^{t_1}((t_1-s)^{\delta-\delta_n-1}-(t_2-s)^{\delta-\delta_n-1})^q s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} && (15)\\
&+t_1^{1+\delta_n-\delta}\bigg(\int_{t_1}^{t_2}(t_2-s)^{(\delta-\delta_n-1)q}s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} && (16)\\
&+(t_2^{1+\delta_n-\delta}-t_1^{1+\delta_n-\delta})\bigg(\int_0^{t_2}(t_2-s)^{(\delta-\delta_n-1)q}s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}}\bigg]. && (17)
\end{aligned}$$

We will now evaluate each of these expressions (15)–(17) separately.

(a) We choose a constant $0 < \gamma < \min\{1+\delta_n-\delta, \delta-\delta_n-\frac{1}{p}\}$:

$$\begin{aligned}
&t_1^{1+\delta_n-\delta}\bigg(\int_0^{t_1}((t_1-s)^{\delta-\delta_n-1}-(t_2-s)^{\delta-\delta_n-1})^q s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} \\
\leq& t_1^{1+\delta_n-\delta}\bigg(\int_0^{t_1}(\frac{1}{t_1-s}-\frac{1}{t_2-s})^{(1+\delta_n-\delta)q}s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} (\because 0<(1+\delta_n-\delta)q<1) \\
=& t_1^{1+\delta_n-\delta}\bigg(\int_0^{t_1}(\frac{1}{t_1-s}-\frac{1}{t_2-s})^{(1+\delta_n-\delta-\gamma)q}(\frac{t_2-t_1}{(t_1-s)(t_2-s)})^{\gamma q}s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} \\
\leq& t_1^{1+\delta_n-\delta}(t_2-t_1)^{\gamma}\bigg(\int_0^{t_1}(t_1-s)^{(\delta-\delta_n-1+\gamma)q}(t_1-s)^{-\gamma q}(t_2-s)^{-\gamma q}s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} \\
\leq& t_1^{1+\delta_n-\delta}(t_2-t_1)^{\gamma}\bigg(\int_0^{t_1}(t_1-s)^{(\delta-\delta_n-1-\gamma)q}s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} \\
=& (t_2-t_1)^{\gamma}t_1^{1+\delta_n-\delta}t_1^{\delta-\delta_n-1-\gamma+\delta-\delta_n-\frac{1}{p}}(\mathcal{B}((\delta-\delta_n-1-\gamma)q+1,(\delta-\delta_n-1)q+1))^{\frac{1}{q}} \\
\leq& (t_2-t_1)^{\gamma}(\mathcal{B}((\delta-\delta_n-1-\gamma)q+1,(\delta-\delta_n-1)q+1))^{\frac{1}{q}}.
\end{aligned}$$

(b) Since $1+\delta_n-\delta > 0$, we have $t_1^{(1+\delta_n-\delta)q}s^{(\delta-\delta_n-1)q} \leq 1$ for all $0 < t_1 \leq s$. Hence:

$$\begin{aligned}
t_1^{1+\delta_n-\delta}\bigg(\int_{t_1}^{t_2}(t_2-s)^{(\delta-\delta_n-1)q}s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} \leq& \bigg(\int_{t_1}^{t_2}(t_2-s)^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} \\
=& \frac{(t_2-t_1)^{\delta-\delta_n-\frac{1}{p}}}{((\delta-\delta_n-1)q+1)^{\frac{1}{q}}}.
\end{aligned}$$

(c) As for (17), we have the following estimate:

$$\begin{aligned}
&(t_2^{1+\delta_n-\delta}-t_1^{1+\delta_n-\delta})\bigg(\int_0^{t_2}(t_2-s)^{(\delta-\delta_n-1)q}s^{(\delta-\delta_n-1)q}ds\bigg)^{\frac{1}{q}} \\
=& (t_2^{1+\delta_n-\delta}-t_1^{1+\delta_n-\delta})t_2^{\delta-\delta_n-1+\delta-\delta_n-\frac{1}{p}}(\mathcal{B}((\delta-\delta_n-1)q+1,(\delta-\delta_n-1)q+1))^{\frac{1}{q}} \\
=& \kappa_1 t_2^{\delta-\delta_n-\frac{1}{p}}(1-t_1^{1+\delta_n-\delta}t_2^{\delta-\delta_n-1}),
\end{aligned}$$

where κ_1 is defined in (14).

Gathering these estimates $(a)-(c)$, we conclude that:

$$\lim_{t_2\to t_1}|t_1^{1+\delta_n-\delta}I_{0+}^{\delta-\delta_n}h(t_1)-t_2^{1+\delta_n-\delta}I_{0+}^{\delta-\delta_n}h(t_2)|=0.$$

We will now prove that the integral operator $I_{0+}^{\delta-\delta_n}$ is continuous. Suppose $\{h_j\}\subseteq L^p_{\delta-\delta_n}$ and $\lim\limits_{j\to\infty}\|h_j-h\|_{L^p_{\delta-\delta_n}}=0$, we need to prove $\lim\limits_{j\to\infty}\|I_{0+}^{\delta-\delta_n}h_j-I_{0+}^{\delta-\delta_n}h\|_{C_{\delta-\delta_n}}=0$. In fact:

$$\begin{aligned}
&\|I_{0+}^{\delta-\delta_n}h_j-I_{0+}^{\delta-\delta_n}h\|_{C_{\delta-\delta_n}}\\
=&\max_{0\le t\le 1}|t^{1+\delta_n-\delta}(I_{0+}^{\delta-\delta_n}h_j(t)-I_{0+}^{\delta-\delta_n}h(t))|\\
\le&\frac{\|h_j-h\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\delta_n)}\max_{0\le t\le 1}t^{1+\delta_n-\delta}\left(\int_0^t(t-s)^{(\delta-\delta_n-1)q}s^{(\delta-\delta_n-1)q}\right)^{\frac{1}{q}}\\
=&\frac{\|h_j-h\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\delta_n)}\max_{0\le t\le 1}t^{\delta-\delta_n-\frac{1}{p}}[\mathcal{B}((\delta-\delta_n-1)q+1,(\delta-\delta_n-1)q+1)]^{\frac{1}{q}}\\
=&\frac{\|h_j-h\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\delta_n)}\kappa_1,
\end{aligned}$$

and let $j\to\infty$—this finally completes the demonstration. □

Lemma 6. *Suppose that* $(\boldsymbol{C}_1),(\boldsymbol{C}_2)$ *hold, for any* $h\in L^p_{\delta-\delta_n}$, $\int_0^1(1-s)^{\delta-\alpha_{n-1}-1}h(s)ds<\infty$.

Proof of Lemma 6. By means of the Hölder inequality, we immediately infer the conclusion. □

To prove the existence of at least one positive solution of (1), we state the following Guo–Krasnoselskii fixed point theorem [25] and Schauder fixed point theorem [26].

Theorem 1. *Let E be a Banach space, $P\subseteq E$ a cone, and Ω_1,Ω_2 are two bounded open balls of E centered at the origin with $\overline{\Omega_1}\subset\Omega_2$. Suppose that $A: P\bigcap(\overline{\Omega_2}\backslash\Omega_1)\to P$ is a completely continuous operator such that either:*

(i) $\|Ax|\le\|x\|, x\in P\cap\partial\Omega_1$ *and* $\|Ax|\ge\|x\|, x\in P\cap\partial\Omega_2$; *or*
(ii) $\|Ax|\ge\|x\|, x\in P\cap\partial\Omega_1$ *and* $\|Ax|\le\|x\|, x\in P\cap\partial\Omega_2$ *holds. Then, A has a fixed point in* $P\bigcap(\overline{\Omega_2}\backslash\Omega_1)$.

Theorem 2. *Let Ω be a convex and closed subset of a Banach space E. Then, any continuous and compact map $F:\Omega\to\Omega$ has a fixed point.*

3. Properties of Green's Function

Lemma 7. *Assume that* $(\boldsymbol{C}_1),(\boldsymbol{C}_2)$ *hold and* $h\in L^p_{\delta-\delta_n}[0,1]$, *if* $\delta-\alpha_{n-1}<1$, *then the following boundary value problem:*

$$\begin{cases}-D_{0+}^{\delta-\alpha_{n-2}}v(t)=h(t), & 0<t<1,\\ v(0)=0, D_{0+}^{\alpha_{n-1}-\alpha_{n-2}}v(1)=\varphi[v(t)]\end{cases}\tag{18}$$

has a unique solution:

$$v(t)=\int_0^1K(t,s)h(s)ds+\frac{t^{\delta-\alpha_{n-2}-1}\Gamma(\delta-\alpha_{n-1})}{\triangle}\int_0^1\varphi[K(\cdot,s)]h(s)ds,$$

where:

$$K(t,s) = \frac{1}{\Gamma(\delta-\alpha_{n-2})}\begin{cases} t^{\delta-\alpha_{n-2}-1}(1-s)^{\delta-\alpha_{n-1}-1} - (t-s)^{\delta-\alpha_{n-2}-1}, & 0 \le s \le t \le 1, \\ t^{\delta-\alpha_{n-2}-1}(1-s)^{\delta-\alpha_{n-1}-1}, & 0 \le t \le s < 1. \end{cases} \quad (19)$$

Proof of Lemma 7. Applying the Riemann–Liouville operator $I_{0+}^{\delta-\alpha_{n-2}}$ on both sides of the equation and using Lemma 1, we obtain:

$$v(t) = -I_{0+}^{\delta-\alpha_{n-2}}h(t) + C_1 t^{\delta-\alpha_{n-2}-1} + C_2 t^{\delta-\alpha_{n-2}-2},$$

where $C_1, C_2 \in R$ are arbitrary constants. At the same time, the boundary condition $v(0) = 0$ and $1 < \delta - \alpha_{n-2} \le 2$ imply that $C_2 = 0$. Consequently, the solution of (18) is:

$$v(t) = -I_{0+}^{\delta-\alpha_{n-2}}h(t) + C_1 t^{\delta-\alpha_{n-2}-1}.$$

By Lemma 2 and the boundary condition $D_{0+}^{\alpha_{n-1}-\alpha_{n-2}}v(1) = \varphi[v(t)]$, we obtain:

$$-I_{0+}^{\delta-\alpha_{n-1}}h(1) + C_1\frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\alpha_{n-1})} = \varphi[v(t)],$$

hence, $C_1 = \frac{\Gamma(\delta-\alpha_{n-1})}{\Gamma(\delta-\alpha_{n-2})}(I_{0+}^{\delta-\alpha_{n-1}}h(1) + \varphi[v(t)])$. Therefore, the unique solution of the problem (18) is given by

$$\begin{aligned} v(t) &= -\frac{1}{\Gamma(\delta-\alpha_{n-2})}\int_0^t (t-s)^{\delta-\alpha_{n-2}-1}h(s)ds + \frac{t^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}h(s)ds \\ &\quad + t^{\delta-\alpha_{n-2}-1}\frac{\Gamma(\delta-\alpha_{n-1})}{\Gamma(\delta-\alpha_{n-2})}\varphi[v(t)] \\ &= \int_0^1 K(t,s)h(s)ds + t^{\delta-\alpha_{n-2}-1}\frac{\Gamma(\delta-\alpha_{n-1})}{\Gamma(\delta-\alpha_{n-2})}\varphi[v(t)]. \end{aligned} \quad (20)$$

Since φ is linear, applying φ to both sides of (20) gives:

$$\varphi[v(t)] = \int_0^1 \varphi[K(\cdot,s)]h(s)ds + \varphi[t^{\delta-\alpha_{n-2}-1}]\frac{\Gamma(\delta-\alpha_{n-1})}{\Gamma(\delta-\alpha_{n-2})}\varphi[v(t)],$$

then, $\varphi[v(t)] = \triangle^{-1}\Gamma(\delta-\alpha_{n-2})\int_0^1 \varphi[K(\cdot,s)]h(s)ds$. Therefore:

$$v(t) = \int_0^1 K(t,s)h(s)ds + \frac{t^{\delta-\alpha_{n-2}-1}\Gamma(\delta-\alpha_{n-1})}{\triangle}\int_0^1 \varphi[K(\cdot,s)]h(s)ds.$$

□

Corollary 1. *Assume that* $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$ *hold and* $h \in L^p_{\delta-\delta_n}[0,1]$*, if* $\delta - \alpha_{n-1} = 1$*, then BVP (18) has a unique solution* $v(t)$ *which can be represented by Lemma 7:*

$$v(t) = \int_0^1 \widetilde{K}(t,s)h(s)ds + \frac{t^{\delta-\alpha_{n-2}-1}}{\triangle}\int_0^1 \varphi[\widetilde{K}(\cdot,s)]h(s)ds,$$

where $\widetilde{K}(t,s)$ *defined in* $[0,1]\times[0,1]$ *can be expressed by*

$$\widetilde{K}(t,s) = \frac{1}{\Gamma(\delta-\alpha_{n-2})}\begin{cases} t^{\delta-\alpha_{n-2}-1} - (t-s)^{\delta-\alpha_{n-2}-1}, & 0 \le s \le t \le 1, \\ t^{\delta-\alpha_{n-2}-1}, & 0 \le t \le s \le 1. \end{cases} \quad (21)$$

Lemma 8. *If* $\delta - \alpha_{n-1} < 1$*, the Green's function* $K(t,s)$ *defined in (19) has the following properties:*

(i) $K(t,s)$ *is continuous for* $t,s \in [0,1]\times[0,1)$*;*
(ii) $K(0,s) = K(t,0) = 0$*, for* $t \in [0,1], s \in [0,1)$*;*

(iii) $\lim_{s\to 1-} K(t,s) = +\infty, t \in [0,1)$;

(iv) *For any* $(t,s) \in [0,1] \times [0,1)$:

$$0 \le mt^{\delta-\alpha_{n-2}-1}s(1-s)^{\delta-\alpha_{n-1}-1} \le K(t,s) \le Mt^{\delta-\alpha_{n-2}-1}(1-s)^{\delta-\alpha_{n-1}-1},$$

where m, M *are in the definition (22) below.*

Proof of Lemma 8. From the expression of K, it can be readily seen that statements (i)–(iii) hold. Let us then turn to prove statement (iv). Define:

$$m := \frac{1}{\Gamma(\delta-\alpha_{n-2})}\min\{1, \alpha_{n-1}-\alpha_{n-2}\}, M = \frac{1}{\Gamma(\delta-\alpha_{n-2})}. \tag{22}$$

From the representation of K, it is obvious that $K(t,s) \le Mt^{\delta-\alpha_{n-2}-1}(1-s)^{\delta-\alpha_{n-1}-1}$, $(t,s) \in [0,1] \times [0,1)$. On the other hand, for $0 \le s \le t \le 1$:

$$\begin{aligned}\Gamma(\delta-\alpha_{n-2})K(t,s) \ge& t^{\delta-\alpha_{n-2}-1}[(1-s)^{\delta-\alpha_{n-1}-1} - (1-s)^{\delta-\alpha_{n-2}-1}]\\ =& t^{\delta-\alpha_{n-2}-1}(1-s)^{\delta-\alpha_{n-1}-1}[1-(1-s)^{\alpha_{n-1}-\alpha_{n-2}}]\\ \ge& t^{\delta-\alpha_{n-2}-1}(1-s)^{\delta-\alpha_{n-1}-1}\min\{1,\alpha_{n-1}-\alpha_{n-2}\}(1-(1-s))\\ =& \min\{1,\alpha_{n-1}-\alpha_{n-2}\}t^{\delta-\alpha_{n-2}-1}s(1-s)^{\delta-\alpha_{n-1}-1}.\end{aligned}$$

For $0 \le t \le s < 1$:

$$\Gamma(\delta-\alpha_{n-2})K(t,s) = t^{\delta-\alpha_{n-2}-1}(1-s)^{\delta-\alpha_{n-1}-1} \ge \min\{1,\alpha_{n-1}-\alpha_{n-2}\}t^{\delta-\alpha_{n-2}-1}s(1-s)^{\delta-\alpha_{n-1}-1}.$$

□

Corollary 2. *If* $\delta - \alpha_{n-1} = 1$, *according to Corollary 1, we know the properties (i), (ii), and (iv) of* $\widetilde{K}$ *defined in* $[0,1] \times [0,1]$ *are also satisfied, that is:*

(i) $\widetilde{K}(t,s)$ *is continuous for* $t,s \in [0,1] \times [0,1]$;

(ii) $\widetilde{K}(0,s) = \widetilde{K}(t,0) = 0$, *for* $t \in [0,1], s \in [0,1]$;

(iii) *For any* $(t,s) \in [0,1] \times [0,1)$:

$$0 \le mt^{\delta-\alpha_{n-2}-1}s \le \widetilde{K}(t,s) \le Mt^{\delta-\alpha_{n-2}-1},$$

where m, M *are in the definition (22) above.*

In light of Lemma 2, it follows that:

$$D_{0+}^{\delta_{n-1}-\alpha_{n-2}}t^{\delta-\alpha_{n-2}-1} = \frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_{n-1})}t^{\delta-\delta_{n-1}-1}, D_{0+}^{\delta_n-\alpha_{n-2}}t^{\delta-\alpha_{n-2}-1} = \frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_n)}t^{\delta-\delta_n-1}.$$

If $\delta - \alpha_{n-1} < 1$, denoting $K_1(t,s) = D_{0+}^{\delta_{n-1}-\alpha_{n-2}}K(t,s), K_2(t,s) = D_{0+}^{\delta_n-\alpha_{n-2}}K(t,s)$, we obtain:

$$K_1(t,s) = \frac{1}{\Gamma(\delta-\delta_{n-1})}\begin{cases} t^{\delta-\delta_{n-1}-1}(1-s)^{\delta-\alpha_{n-1}-1} - (t-s)^{\delta-\delta_{n-1}-1}, & 0 \le s \le t \le 1,\\ t^{\delta-\delta_{n-1}-1}(1-s)^{\delta-\alpha_{n-1}-1}, & 0 \le t \le s < 1.\end{cases} \tag{23}$$

$$K_2(t,s) = \frac{1}{\Gamma(\delta-\delta_n)}\begin{cases} t^{\delta-\delta_n-1}(1-s)^{\delta-\alpha_{n-1}-1} - (t-s)^{\delta-\delta_n-1}, & 0 \le s < t \le 1,\\ t^{\delta-\delta_n-1}(1-s)^{\delta-\alpha_{n-1}-1}, & 0 < t \le s < 1.\end{cases} \tag{24}$$

If $\delta - \alpha_{n-1} = 1$, let $\widetilde{K}_1(t,s) = D_{0+}^{\delta_{n-1}-\alpha_{n-2}}\widetilde{K}(t,s), \widetilde{K}_2(t,s) = D_{0+}^{\delta_n-\alpha_{n-2}}\widetilde{K}(t,s)$.

Lemma 9. *If $\delta - \alpha_{n-1} < 1, K_i(t,s)(i = 1,2)$ defined in (23) and (24) has the following properties:*

(i) $K_1(t,s)$ *is continuous on* $[0,1] \times [0,1)$;

(ii) $0 \le m_1 t^{\delta-\delta_{n-1}-1} s(1-s)^{\delta-\alpha_{n-1}-1} \le K_1(t,s) \le M_1 t^{\delta-\delta_{n-1}-1}(1-s)^{\delta-\alpha_{n-1}-1}, (t,s) \in [0,1] \times [0,1)$, *where* m_1, M_1 *are defined in (25) below;*

(iii) $K_2(t,s)$ *is continuous on* $\Sigma_1 \cup \Sigma_2, K_2(t,s) \le 0$ *for* $(t,s) \in \Sigma_1$ *and* $K_2(t,s) > 0$ *for* $(t,s) \in \Sigma_2$, *where* $\Sigma_1 = \{(t,s)|0 \le s < t \le 1\}, \Sigma_2 = \{(t,s)|0 < t \le s < 1\}$.

Proof of Lemma 9. We only prove statement (ii), since (i) and (iii) are obvious. Following the proof of Lemma 8 (iv), we denote:

$$m_1 = \frac{1}{\Gamma(\delta - \delta_{n-1})} \min\{1, \alpha_{n-1} - \delta_{n-1}\}, M_1 = \frac{1}{\Gamma(\delta - \delta_{n-1})}, \tag{25}$$

analogously, we naturally obtained statement (ii). □

Corollary 3. *If $\delta - \alpha_{n-1} = 1$,*

(i) $\widetilde{K}_1(t,s)$ *is continuous on* $[0,1] \times [0,1]$;

(ii) $0 \le m_1 t^{\delta-\delta_{n-1}-1} s \le \widetilde{K}_1(t,s) \le M_1 t^{\delta-\delta_{n-1}-1}, (t,s) \in [0,1] \times [0,1]$, *where* m_1, M_1 *are defined in (25) above;*

(iii) $\widetilde{K}_2(t,s) \le 0$ *for* $(t,s) \in \{(t,s)|0 \le s < t \le 1\}$ *and* $\widetilde{K}_2(t,s) > 0$ *for* $(t,s) \in \{(t,s)|0 < t \le s \le 1\}$.

Remark 1. *In view of Corollaries 2 and 3, we know that K, K_i have properties almost identical to those of $\widetilde{K}, \widetilde{K}_i$. In combination with (19) and Corollary 1, we can unify the expressions of K, K_i and $\widetilde{K}, \widetilde{K}_i$, and we will uniformly write them as $K, K_i(i = 1,2)$ below.*

4. Result of Existence and Uniqueness

Let the Banach space $E = \{v : [0,1] \to R : v \in C, D_{0+}^{\delta_{n-1}-\alpha_{n-2}} v \in C, D_{0+}^{\delta_n-\alpha_{n-2}} v \in C_{\delta-\delta_n}\}$ endowed with the norm $\|v\|_E = \max\{\|v\|_0, \|v\|_1, \|v\|_2\}$, where:

$$\|v\|_1 = \|D_{0+}^{\delta_{n-1}-\alpha_{n-2}} v\|_0, \|v\|_2 = \|D_{0+}^{\delta_n-\alpha_{n-2}} v\|_{\delta-\delta_n}.$$

Let:

$$P = \{v \in E : v(t) \ge 0, D_{0+}^{\delta_{n-1}-\alpha_{n-2}} v(t) \ge 0, t \in [0,1]\},$$

then P is a cone of E.

We now consider the operator $F : P \to L^p_{\delta-\delta_n}$, for any $v \in E$, defining the function F_v by $F_v(t) = f\Big(t, I_{0+}^{\alpha_{n-2}} v(t), I_{0+}^{\alpha_{n-2}-\delta_1} v(t), \cdots, I_{0+}^{\alpha_{n-2}-\delta_{n-2}} v(t), D_{0+}^{\delta_{n-1}-\alpha_{n-2}} v(t), D_{0+}^{\delta_n-\alpha_{n-2}} v(t)\Big)$. For the forthcoming analysis, we need the following assumptions:

Hypothesis 1. $f : (0,1] \times (\mathbb{R}^+)^n \times \mathbb{R} \to \mathbb{R}^+$ *satisfies the Carathéodory condition, meaning:*

(i) $f(\cdot, x_0, x_1, \cdots, x_n) : (0,1] \to \mathbb{R}^+$ *is measurable for all* $(x_0, x_1, \cdots, x_n) \in (\mathbb{R}^+)^n \times \mathbb{R}$;

(ii) $f(t, \cdot, \cdot, \cdots, \cdot) : (\mathbb{R}^+)^n \times \mathbb{R} \to \mathbb{R}^+$ *is continuous for a.e.* $t \in (0,1]$.

Hypothesis 2. *There exist some constants $0 \le \rho_i < 1$ $(i = 0,1,...,n)$ and nonnegative functions $b(t), b_0(t), ..., b_{n-1}(t) \in L^p_{\delta-\delta_n}, t^{(1+\delta_n-\delta)(1-\rho_n)} b_n(t) \in L^p$ such that for any $t \in (0,1]$ and $(x_0, x_1, \cdots, x_n) \in (\mathbb{R}^+)^n \times \mathbb{R}$:*

$$|f(t, x_0, \cdots, x_n)| \le b(t) + \sum_{i=0}^{n} b_i(t)|x_i|^{\rho_i}. \tag{26}$$

Hypothesis 3. *There exist some constants $\rho_i \geq 1$ $(i = 0, 1, ..., n)$ (there is at least one constant $\rho_i > 1$) and nonnegative functions $b(t), b_0(t), ..., b_{n-1}(t) \in L^p_{\delta-\delta_n}, t^{(\delta-\delta_n-1)(\rho_n-1)}b_n(t) \in L^p$ such that for any $t \in (0, 1]$ and $(x_0, x_1, \cdots, x_n) \in (\mathbb{R}^+)^n \times \mathbb{R}$:*

$$|f(t, x_0, \cdots, x_n)| \leq b(t) + \sum_{i=0}^{n} b_i(t)|x_i|^{\rho_i}. \tag{27}$$

Hypothesis 4. *There is a nonnegative function $a(t)$ with $\int_0^1 s(1-s)^{\delta-\alpha_{n-1}-1}a(s)ds > 0$ and a constant $r_0 > 0$ such that for any $t \in (0, 1], 0 \leq x_i \leq \frac{r_0}{\Gamma(\alpha_{n-2}-\delta_i+1)}(i = 0, 1, ..., n-2), 0 \leq x_{n-1} \leq r_0, t^{1+\delta_n-\delta}|x_n| \leq r_0$:*

$$f(t, x_0, ..., x_n) \geq a(t).$$

Hypothesis 5. *There are positive numbers χ_1, χ_2 satisfying $\chi_1 < \chi_2, \overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum\limits_{i=0}^{n} B_i\chi_2^{\rho_i}) \leq \chi_2$ such that:*

$$\int_0^1 s(1-s)^{\delta-\alpha_{n-1}-1}[\inf_{(x_0,x_1,...,x_n)\in\Xi} f(s, x_0, ..., x_n)]ds \geq \frac{\chi_1}{m(1+\varphi_0)},$$

where $\Xi = \{(x_0, x_1, ..., x_n) : x_i \in [0, \frac{\chi_2}{\Gamma(\alpha_{n-2}-\delta_i+1)}], i = 0, 1, ..., n-2, x_{n-1}, t^{1+\delta_n-\delta}|x_n| \in [0, \chi_2]\}$, $\overline{M}, B_i(i = 0, 1, ..., n)$ are defined in (43), m is in (22) and φ_0 is in (35), respectively.

Hypothesis 6. *There exist nonnegative functions $a_0(t), ..., a_n(t) \in L^p$, such that for any $t \in (0, 1]$ and $(x_0, x_1, \cdots, x_n), (y_0, y_1, \cdots, y_n) \in (\mathbb{R}^+)^n \times \mathbb{R}$:*

$$|f(t, x_0, \cdots, x_n) - f(t, y_0, \cdots, y_n)| \leq \sum_{i=0}^{n} a_i(t)|x_i - y_i|.$$

Lemma 10. *Assume that $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$ and Hypotheses 1 and 2 hold, then $F : P \to L^p_{\delta-\delta_n}$ is continuous.*

Proof of Lemma 10. First, we will show that F is well defined, that is for any $v \in P$, $F_v \in L^p_{\delta-\delta_n}$. In fact, for $t \in (0, 1]$, by Hypothesis 2:

$$\begin{aligned}
&|t^{1+\delta_n-\delta}F_v(t)| \\
\leq& t^{1+\delta_n-\delta}\left(b(t) + \sum_{i=0}^{n-2} b_i(t)|I_{0+}^{\alpha_{n-2}-\delta_i}v(t)|^{\rho_i}\right. \\
&\left. + b_{n-1}(t)|D_{0+}^{\delta_{n-1}-\alpha_{n-2}}v(t)|^{\rho_{n-1}} + b_n(t)|D_{0+}^{\delta_n-\alpha_{n-2}}v(t)|^{\rho_n}\right) \\
\leq& t^{1+\delta_n-\delta}b(t) + \sum_{i=0}^{n-2} \frac{t^{1+\delta_n-\delta}b_i(t)(\|v\|_0)^{\rho_i}}{(\Gamma(\alpha_{n-2}-\delta_i+1))^{\rho_i}} + t^{1+\delta_n-\delta}b_{n-1}(t)\|v\|_1^{\rho_{n-1}} \\
&+ t^{(1+\delta_n-\delta)(1-\rho_n)}b_n(t)\|v\|_2^{\rho_n}.
\end{aligned} \tag{28}$$

As a consequence, we obtain $t^{1+\delta_n-\delta}F_v(t) \in L^p$, i.e., $F_v \in L^p_{\delta-\delta_n}$.

We now turn to prove the continuity of F. Let $v_0 \in P$ be fixed and let $\{v_k\} \subseteq P$ be the sequence converging to v_0 as $k \to \infty$. Then, for any $t \in [0, 1]$, we have $v_k(t) \to v_0(t), D_{0+}^{\delta_{n-1}-\alpha_{n-2}}v_k(t) \to D_{0+}^{\delta_{n-1}-\alpha_{n-2}}v_0(t), t^{1+\delta_n-\delta}D_{0+}^{\delta_n-\alpha_{n-2}}v_k(t) \to t^{1+\delta_n-\delta}D_{0+}^{\delta_n-\alpha_{n-2}}v_0(t)$, as $k \to \infty$. There exists a positive number $\mathbb{M}$ such that $\|v_k\|_E \leq \mathbb{M}(k = 0, 1, ...)$. We need to prove $F_{v_k} \to F_{v_0}$ in $L^p_{\delta-\delta_n}$ as $k \to \infty$. Since f satisfies the Carathéodory condition, we know for any $t \in (0, 1]$, we obtain the conclusion $F_{v_k}(t) \to F_{v_0}(t), k \to \infty$. According to (28), we can easily deduce that:

$$t^{1+\delta_n-\delta}|F_{v_k}(t)| \leq t^{1+\delta_n-\delta}b(t) + \sum_{i=0}^{n-2} \frac{t^{1+\delta_n-\delta}b_i(t)}{(\Gamma(\alpha_{n-2}-\delta_i+1))^{\rho_i}}\mathbb{M}^{\rho_i} + t^{1+\delta_n-\delta}b_{n-1}(t)\mathbb{M}^{\rho_{n-1}}$$
$$+ t^{(1+\delta_n-\delta)(1-\rho_n)}b_n(t)\mathbb{M}^{\rho_n}.$$

By utilizing the Lebesgue-dominated convergence theorem, we obtain:

$$\lim_{k\to\infty} \|F_{v_k} - F_{v_0}\|^p_{L^p_{\delta-\delta_n}} = \lim_{k\to\infty} \int_0^1 t^{(1+\delta_n-\delta)p}|F_{v_k}(t) - F_{v_0}(t)|^p dt = 0.$$

□

Corollary 4. *Assuming that* $(C_1), (C_2)$*, Hypotheses 1 and 3 hold, then* $F : P \to L^p_{\delta-\delta_n}$ *is continuous.*

The proof is similar to that of Lemma 10, so it is omitted.
Now, we define an operator T by

$$\begin{aligned} Tv(t) =& - I_{0+}^{\delta-\alpha_{n-2}}F_v(t) + \frac{t^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\ &+ \frac{t^{\delta-\alpha_{n-2}-1}\Gamma(\delta-\alpha_{n-1})}{\triangle}\int_0^1 \varphi[K(t,s)]F_v(s)ds \\ =& \int_0^1 K(t,s)F_v(s)ds + \frac{t^{\delta-\alpha_{n-2}-1}\Gamma(\delta-\alpha_{n-1})}{\triangle}\int_0^1 \varphi[K(\cdot,s)]F_v(s)ds. \end{aligned} \tag{29}$$

By simple calculation, we can deduce that:

$$\begin{aligned} &D_{0+}^{\delta_{n-1}-\alpha_{n-2}}Tv(t) \\ =& - I_{0+}^{\delta-\delta_{n-1}}F_v(t) + \frac{t^{\delta-\delta_{n-1}-1}}{\Gamma(\delta-\delta_{n-1})}\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\ &+ \frac{t^{\delta-\delta_{n-1}-1}\Gamma(\delta-\alpha_{n-2})\Gamma(\delta-\alpha_{n-1})}{\Gamma(\delta-\delta_{n-1})\triangle}\int_0^1 \varphi[K(\cdot,s)]F_v(s)ds \\ =& \int_0^1 K_1(t,s)F_v(s)ds + \frac{\Gamma(\delta-\alpha_{n-1})\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_{n-1})\triangle}t^{\delta-\delta_{n-1}-1}\int_0^1 \varphi[K(\cdot,s)]F_v(s)ds. \end{aligned} \tag{30}$$

Similarly, we also obtain:

$$\begin{aligned} &D_{0+}^{\delta_n-\alpha_{n-2}}Tv(t) \\ =& - I_{0+}^{\delta-\delta_n}F_v(t) + \frac{t^{\delta-\delta_n-1}}{\Gamma(\delta-\delta_n)}\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\ &+ \frac{t^{\delta-\delta_n-1}\Gamma(\delta-\alpha_{n-2})\Gamma(\delta-\alpha_{n-1})}{\Gamma(\delta-\delta_n)\triangle}\int_0^1 \varphi[K(\cdot,s)]F_v(s)ds \\ =& \int_0^1 K_2(t,s)F_v(s)ds + \frac{\Gamma(\delta-\alpha_{n-1})\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_n)\triangle}t^{\delta-\delta_n-1}\int_0^1 \varphi[K(\cdot,s)]F_v(s)ds. \end{aligned} \tag{31}$$

Further calculation yields:

$$\begin{aligned} &t^{1+\delta_n-\delta}D_{0+}^{\delta_n-\alpha_{n-2}}Tv(t) \\ =& - t^{1+\delta_n-\delta}I_{0+}^{\delta-\delta_n}F_v(t) + \frac{1}{\Gamma(\delta-\delta_n)}\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\ &+ \frac{\Gamma(\delta-\alpha_{n-1})\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_n)\triangle}\int_0^1 \varphi[K(\cdot,s)]F_v(s)ds. \end{aligned} \tag{32}$$

Lemma 11. *Suppose that* $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$ *and Hypotheses 1 and 2 hold. Then,* $T : P \to P$ *is completely continuous.*

Proof of Lemma 11. First, we will show that T is well defined. For any $v \in P$, in view of Lemma 10, we observe that $F_v \in L^p_{\delta-\delta_n}$. With this conclusion, on the basis of Lemmas 4 and 5, we know $I^{\delta-\alpha_{n-2}}_{0+}F_v, I^{\delta-\delta_{n-1}}_{0+}F_v \in C, I^{\delta-\delta_n}_{0+}F_v \in C_{\delta-\delta_n}$ and from Lemma 6, we obtain

$$\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds < \infty.$$

Furthermore, according to Lemma 8 and Corollary 2, we conclude that:

$$\begin{aligned}\int_0^1 \varphi[K(\cdot,s)]F_v(s)ds \leq & M\varphi[t^{\delta-\alpha_{n-2}-1}]\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\ \leq & \kappa_2 M\varphi[t^{\delta-\alpha_{n-2}-1}]\|F_v\|_{L^p_{\delta-\delta_n}} < \infty,\end{aligned}$$

where:

$$\kappa_2 = [\mathcal{B}((\delta-\alpha_{n-1}-1)q+1, (\delta-\delta_n-1)q+1)]^{\frac{1}{q}}. \tag{33}$$

With these estimates, from (29)–(31), we have $Tv, D^{\delta_{n-1}-\alpha_{n-2}}_{0+}Tv \in C$ and $D^{\delta_n-\alpha_{n-2}}_{0+}Tv \in C_{\delta-\delta_n}$. Moreover, $Tv(t) \geq 0, D^{\delta_{n-1}-\alpha_{n-2}}_{0+}Tv(t) \geq 0, t \in [0,1]$, then $Tv \in P$.

Then, we will show that $T : P \to P$ is uniformly bounded. For any bounded subset $B \subset P$, from Lemma 10, we deduce that $\|F_v\|_{L^p_{\delta-\delta_n}} < \infty, \forall v \in B$. In light of Lemma 8 and Corollary 2, by the Hölder inequality, we obtain:

$$\begin{aligned}|Tv(t)| =& \left|\int_0^1 K(t,s)F_v(s)ds + \frac{t^{\delta-\alpha_{n-2}-1}\Gamma(\delta-\alpha_{n-1})}{\triangle}\int_0^1 \varphi[K(\cdot,s)]F_v(s)ds\right| \\ \leq & \left(M + \frac{\Gamma(\delta-\alpha_{n-1})}{\triangle}M\varphi[t^{\delta-\alpha_{n-2}-1}]\right)\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}|F_v(s)|ds \\ \leq & (M+M\varphi_0)\left(\int_0^1 (1-s)^{(\delta-\alpha_{n-1}-1)q}s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}}\|F_v\|_{L^p_{\delta-\delta_n}} \\ =& M(1+\varphi_0)\kappa_2\|F_v\|_{L^p_{\delta-\delta_n}},\end{aligned} \tag{34}$$

where κ_2 is defined in (33) and:

$$\varphi_0 := \frac{\Gamma(\delta-\alpha_{n-1})}{\triangle}\varphi[t^{\delta-\alpha_{n-2}-1}]. \tag{35}$$

By $(\mathbf{C}_1)$, we know that $0 \leq \varphi_0 < \infty$. It follows from (34) that:

$$\|Tv\|_0 \leq M(1+\varphi_0)\kappa_2\|F_v\|_{L^p_{\delta-\delta_n}} < \infty. \tag{36}$$

Similarly, from (30) and Lemma 9 and Corollary 3, we also obtain:

$$\begin{aligned}\|Tv\|_1 =& \max_{0\leq t\leq 1}\left|D^{\delta_{n-1}-\alpha_{n-2}}_{0+}Tv(t)\right| \\ \leq & \left\{M_1 + \frac{\Gamma(\delta-\alpha_{n-1})\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_{n-1})\triangle}M\varphi[t^{\delta-\alpha_{n-2}-1}]\right\}\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\ \leq & \left[M_1 + \frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_{n-1})}M\varphi_0\right]\kappa_2\|F_v\|_{L^p_{\delta-\delta_n}} < \infty.\end{aligned} \tag{37}$$

Moreover, in view of (32) we have:

$$\begin{aligned}
\|Tv\|_2 &= \max_{0\le t\le 1}\left|t^{1+\delta_n-\delta}D_{0+}^{\delta_n-\alpha_{n-2}}Tv(t)\right| \\
&\le \max_{0\le t\le 1}|t^{1+\delta_n-\delta}I_{0+}^{\delta-\delta_n}F_v(t)| + \frac{1}{\Gamma(\delta-\delta_n)}\int_0^1(1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\
&\quad + \frac{\Gamma(\delta-\alpha_{n-1})\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_n)\triangle}M\varphi[t^{\delta-\alpha_{n-2}-1}]\int_0^1(1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\
&\le\left[\max_{0\le t\le 1}\frac{t^{1+\delta_n-\delta}}{\Gamma(\delta-\delta_n)}t^{\delta-\delta_n-1+\delta-\delta_n-\frac{1}{p}}\kappa_1 + \frac{\kappa_2}{\Gamma(\delta-\delta_n)} + \frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_n)}\kappa_2 M\varphi_0\right]\|F_v\|_{L^p_{\delta-\delta_n}} \\
&= \left[\frac{\kappa_1+\kappa_2}{\Gamma(\delta-\delta_n)} + \frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_n)}M\varphi_0\kappa_2\right]\|F_v\|_{L^p_{\delta-\delta_n}} < \infty,
\end{aligned} \tag{38}$$

where κ_1 is defined in (14). Gathering together these conclusions (36)–(38), we infer that $T(B)$ is uniformly bounded.

Now, we need to show that $T(B)$ is equicontinuous. For $v \in B, 0 \le t_1 < t_2 \le 1$, we have:

$$\begin{aligned}
&|(Tv)(t_2) - (Tv)(t_1)| \\
&\le \int_0^{t_1}\frac{(t_2-s)^{\delta-\alpha_{n-2}-1}-(t_1-s)^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}F_v(s)ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}F_v(s)ds \\
&\quad + \frac{t_2^{\delta-\alpha_{n-2}-1}-t_1^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}\int_0^1(1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\
&\quad + \frac{(t_2^{\delta-\alpha_{n-2}-1}-t_1^{\delta-\alpha_{n-2}-1})\Gamma(\delta-\alpha_{n-1})}{\triangle}\int_0^1\varphi[K(\cdot,s)]F_v(s)ds \\
&\le \int_0^{t_1}\frac{(t_2-s)^{\delta-\alpha_{n-2}-1}-(t_1-s)^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}F_v(s)ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}F_v(s)ds \\
&\quad + (t_2^{\delta-\alpha_{n-2}-1}-t_1^{\delta-\alpha_{n-2}-1})\left(\frac{1}{\Gamma(\delta-\alpha_{n-2})}+M\varphi_0\right)\kappa_2\|F_v\|_{L^p_{\delta-\delta_n}}.
\end{aligned} \tag{39}$$

We estimate the first two integral expressions of (39):

$$\begin{aligned}
&\int_0^{t_1}\frac{(t_2-s)^{\delta-\alpha_{n-2}-1}-(t_1-s)^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}F_v(s)ds \\
&\le \frac{\|F_v\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\alpha_{n-2})}\left(\int_0^{t_1}(t_2-t_1)^{(\delta-\alpha_{n-2}-1)q}s^{(\delta-\delta_n-1)q}ds\right)^{\frac{1}{q}} \\
&\le \frac{\|F_v\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\alpha_{n-2})}\frac{(t_2-t_1)^{\delta-\alpha_{n-2}-1}}{((\delta-\delta_n-1)q+1)^{\frac{1}{q}}}
\end{aligned}$$

and:

$$\begin{aligned}
\int_{t_1}^{t_2}\frac{(t_2-s)^{\delta-\alpha_{n-2}-1}}{\Gamma(\delta-\alpha_{n-2})}F_v(s)ds &\le \frac{\|F_v\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\alpha_{n-2})}\left(\int_{t_1}^{t_2}(t_2-s)^{(\delta-\alpha_{n-2}-1)q}(s-t_1)^{(\delta-\delta_n-1)q}\right)^{\frac{1}{q}} \\
&= \frac{\|F_v\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\alpha_{n-2})}\kappa_0(t_2-t_1)^{\delta-\alpha_{n-2}-1-\frac{1}{p}+\delta-\delta_n},
\end{aligned}$$

where κ_0 is defined in (10). Gathering together all these facts and combining with (39), we have:

$$|(Tv)(t_2)-(Tv)(t_1)| \to 0,\ t_2 \to t_1. \tag{40}$$

Analogously, due to (30), we obtain:

$$\begin{aligned}
&\left|D_{0+}^{\delta_{n-1}-\alpha_{n-2}}Tv(t_2)-D_{0+}^{\delta_{n-1}-\alpha_{n-2}}Tv(t_1)\right| \\
\leq &\int_0^{t_1}\frac{(t_2-s)^{\delta-\delta_{n-1}-1}-(t_1-s)^{\delta-\delta_{n-1}-1}}{\Gamma(\delta-\delta_{n-1})}F_v(s)ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\delta-\delta_{n-1}-1}}{\Gamma(\delta-\delta_{n-1})}F_v(s)ds \\
&+\frac{t_2^{\delta-\delta_{n-1}-1}-t_1^{\delta-\delta_{n-1}-1}}{\Gamma(\delta-\delta_{n-1})}\int_0^1(1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\
&+\frac{(t_2^{\delta-\delta_{n-1}-1}-t_1^{\delta-\delta_{n-1}-1})\Gamma(\delta-\alpha_{n-2})\Gamma(\delta-\alpha_{n-1})}{\Gamma(\delta-\delta_{n-1})\triangle}\int_0^1\varphi[K(t,s)]F_v(s)ds \\
\leq &\frac{\|F_v\|_{L^p_{\delta-\delta_n}}}{\Gamma(\delta-\delta_{n-1})}\left(\frac{(t_2-t_1)^{\delta-\delta_{n-1}-1}}{((\delta-\delta_n-1)q+1)^{\frac{1}{q}}}+\kappa_3(t_2-t_1)^{\delta-\delta_{n-1}-1-\frac{1}{p}+\delta-\delta_n}\right) \\
&+(t_2^{\delta-\delta_{n-1}-1}-t_1^{\delta-\delta_{n-1}-1})\left(\frac{1}{\Gamma(\delta-\delta_{n-1})}+\frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_{n-1})}M\varphi_0\right)\kappa_2\|F_v\|_{L^p_{\delta-\delta_n}},
\end{aligned}$$

where $\kappa_3=[\mathcal{B}((\delta-\delta_{n-1}-1)q+1,(\delta-\delta_n-1)q+1)]^{\frac{1}{q}}$. Hence:

$$|D_{0+}^{\delta_{n-1}-\alpha_{n-2}}Tv(t_2)-D_{0+}^{\delta_{n-1}-\alpha_{n-2}}Tv(t_1)|\to 0,\ t_2\to t_1. \tag{41}$$

From (32), imitating the method in Lemma 5, we can also deduce that:

$$|t_2^{1+\delta_n-\delta}D_{0+}^{\delta_n-\alpha_{n-2}}Tv(t_2)-t_1^{1+\delta_n-\delta}D_{0+}^{\delta_n-\alpha_{n-2}}Tv(t_1)|\to 0,\ t_2\to t_1. \tag{42}$$

Synthesizing the above conclusions (40)–(42), we know that $T(B)$ is equicontinuous.

In the end, we will prove that $T:P\to P$ is continuous. Suppose that $\{v_k\}\subset P$ is a convergent sequence and let $\lim\limits_{k\to\infty}\|v_k-v\|_E=0$. From Lemma 10, it follows that $\lim\limits_{k\to\infty}\|F_{v_k}-F_v\|_{L^p_{\delta-\delta_n}}=0$. We deduce from (34) that:

$$\|Tv_k-Tv\|_0\leq M(1+\varphi_0)\kappa_2\|F_{v_k}-F_v\|_{L^p_{\delta-\delta_n}}.$$

Likewise, by means of (37) and (38), we also obtain:

$$\begin{aligned}
\|Tv_k-Tv\|_1&=\max_{0\leq t\leq 1}|D_{0+}^{\delta_{n-1}-\alpha_{n-2}}Tv_k(t)-D_{0+}^{\delta_{n-1}-n+2}Tv(t)| \\
&\leq\left[M_1+\frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_{n-1})}M\varphi_0\right]\kappa_2\|F_{v_k}-F_v\|_{L^p_{\delta-\delta_n}}
\end{aligned}$$

and:

$$\begin{aligned}
\|Tv_k-Tv\|_2&=\max_{0\leq t\leq 1}|t^{1+\delta_n-\delta}D_{0+}^{\delta_{n-1}-\alpha_{n-2}}Tv_k(t)-t^{1+\delta_n-\delta}D_{0+}^{\delta_{n-1}-n+2}Tv(t)| \\
&\leq\left[\frac{\kappa_1+\kappa_2}{\Gamma(\delta-\delta_n)}+\frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_n)}M\varphi_0\kappa_2\right]\|F_{v_k}-F_v\|_{L^p_{\delta-\delta_n}}.
\end{aligned}$$

As a consequence, $\lim\limits_{k\to\infty}\|Tv_k-Tv\|=0$ and T is continuous.

From the above steps, we obtain a completely continuous operator $T:P\to P$. □

Corollary 5. *Suppose that* $(C_1),(C_2)$, *Hypotheses 1 and 3 hold. Then,* $T:P\to P$ *is completely continuous.*

The proof is similar to that of Lemma 11, so it is omitted.

For convenience, we introduce some donations which will be used in next theorem:

$$\overline{M} = \max\left\{ M(1+\varphi_0)\kappa_2, \left[M_1 + \frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_{n-1})} M\varphi_0 \right]\kappa_2, \frac{\kappa_1+\kappa_2}{\Gamma(\delta-\delta_n)} + \frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_n)} M\varphi_0\kappa_2 \right\},$$

$$B_i = \frac{\|b_i\|_{L^p_{\delta-\delta_n}}}{(\Gamma(\alpha_{n-2}-\delta_i+1))^{\rho_i}}, \ i=0,1,...,n-2, B_{n-1} = \|b_{n-1}\|_{L^p_{\delta-\delta_n}}, \tag{43}$$

$$B_n = \left(\int_0^1 (t^{(1+\delta_n-\delta)(1-\rho_n)} b_n(t))^p dt \right)^{\frac{1}{p}}.$$

From Hypothesis 2 or Hypothesis 3, we immediately infer that $0 \le B_i < \infty$ $(i = 0,1,...,n)$.

The main results of this section are the following existence results.

Theorem 3. *Let us assume that $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$, Hypotheses 1, 2 and 4 hold. Then, the boundary value problem (1) has at least a positive solution.*

Proof of Theorem 3. From Lemma 7, $v \in P$ is a solution of (2) if and only if v is a fixed point of T. By Lemma 3, the function $u(t) = I_{0+}^{\alpha_{n-2}} v(t)$ is a solution of problem (1). Choose:

$$r = \min\{r_0, m(1+\varphi_0) \int_0^1 s(1-s)^{\delta-\alpha_{n-2}-1} a(s)ds\}, \tag{44}$$

let $\Omega_1 = \{v \in E : \|v\|_E \le r\}$. For $v \in P \cap \partial\Omega_1, t \in [0,1]$, combining with Lemma 8 and Corollary 2, from Hypothesis 4, we obtain:

$$\begin{aligned} Tv(t) &= \int_0^1 K(t,s)F_v(s)ds + \frac{t^{\delta-\alpha_{n-2}-1}\Gamma(\delta-\alpha_{n-1})}{\triangle} \int_0^1 \varphi[K(\cdot,s)]F_v(s)ds \\ &\ge t^{\delta-\alpha_{n-2}-1}(m + \frac{\Gamma(\delta-\alpha_{n-1})}{\triangle} m\varphi[t^{\delta-\alpha_{n-2}-1}]) \int_0^1 s(1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds \\ &= t^{\delta-\alpha_{n-2}-1} m(1+\varphi_0) \int_0^1 s(1-s)^{\delta-\alpha_{n-1}-1} a(s)ds, \end{aligned}$$

so:

$$\|Tv\|_0 \ge m(1+\varphi_0) \int_0^1 s(1-s)^{\delta-\alpha_{n-1}-1} a(s)ds \ge r.$$

Hence, $\|Tv\|_E \ge \|v\|_E, \forall v \in P \cap \partial\Omega_1$.

For all $v \in P$, according to (28), by Minkowski inequality, we obtain:

$$\begin{aligned} \|F_v\|_{L^p_{\delta-\delta_n}} &\le \|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n-2} B_i\|v\|_0^{\rho_i} + B_{n-1}\|v\|_1^{\rho_{n-1}} + B_n\|v\|_2^{\rho_n} \\ &\le \|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i(\|v\|_E)^{\rho_i}. \end{aligned} \tag{45}$$

We choose:

$$R_1 > \max\left\{1, r, \left(\overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i)\right)^{\frac{1}{1-\overline{\rho}}}\right\},$$

where $\overline{\rho} = \max\limits_{0\le i\le n}\{\rho_i\}$. Let $\Omega_2 = \{v \in E : \|v\|_E \le R_1\}$, for all $v \in P \cap \partial\Omega_2$, then we derive from (36)–(38) and the definition of $\overline{M}$ in (43) that:

$$\|Tv\|_E \le \overline{M}\|F_v\|_{L^p_{\delta-\delta_n}} \le \overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_iR_1^{\rho_i}) \le \overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_iR_1^{\overline{\rho}}) \le R_1.$$

So $\|Tv\|_E \le \|v\|_E, \forall v \in P \cap \partial\Omega_2$.

Then, Theorem 1 ensures the existence of a fixed point $v \in P \cap (\overline{\Omega_2} \backslash \Omega_1)$ such that $v = Tv$, and thus the problem (2) possesses at least one positive solution. Then, by Lemma 3, we conclude that the boundary value problem (1) has at least a positive solution. □

Theorem 4. *Suppose that* $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$, *Hypotheses 1, 3 and 4 hold. In addition, let us assume that* $0 < \overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i) < 1$ *holds. Then, the boundary value problem (1) has at least a positive solution.*

Proof of Theorem 4. From Corollary 5, we know $T : P \to P$ is continuous. Let $\Omega_1 = \{v \in E : \|v\|_E \leq r\}, r$ be defined in (44). From Theorem 3, we know that $\|Tv\|_E \geq \|v\|_E, \forall v \in P \cap \partial\Omega_1$.

If $r < 1$, we choose $1 < r_2 \leq \left(\dfrac{1}{\overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i)} \right)^{\frac{1}{\overline{\rho}-1}}$, where $\overline{\rho} = \max_{0 \leq i \leq n} \{\rho_i\}$. If $r \geq 1$, we choose $\overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i) \leq r_2 < 1$. Let $\Omega_2 = \{v \in E : \|v\|_E \leq r_2\}$, for all $v \in P \cap \partial\Omega_2$, then we obtain:

$$\|Tv\|_E \leq \overline{M}\|F_v\|_{L^p_{\delta-\delta_n}} \leq \overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i r_2^{\rho_i}) \leq r_2.$$

Thus, $\|Tv\|_E \leq \|v\|_E, \forall v \in P \cap \partial\Omega_2$. Again, from Theorem 1, we know that the problem (2) has at least one positive solution $v \in P \cap (\overline{\Omega_1} \backslash \Omega_2)$ or $v \in P \cap (\overline{\Omega_2} \backslash \Omega_1)$, then the boundary value problem (1) has at least a positive solution. □

Theorem 5. *Suppose that* $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$, *Hypotheses 1, 2 and 5 hold. Then, the boundary value problem (1) has at least a positive solution u satisfying* $u(t) \geq \frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta)} \chi_1 t^{\delta-1}$, $t \in [0, 1]$.

Proof of Theorem 5. According to Lemma 11, we know that $T : P \to P$ is completely continuous. Let $\Omega = \{v \in P : \chi_1 t^{\delta-\alpha_{n-2}-1} \leq v(t) \leq \chi_2 t^{\delta-\alpha_{n-2}-1}, t \in [0, 1], \|v\|_E \leq \chi_2\}$. For any $v \in \Omega$, under Hypothesis 2, we derive:

$$\begin{aligned}
&\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1} F_v(s) ds \\
\leq &\int_0^1 (1-s)^{\delta-\alpha_{n-1}-1} \Big(b(s) + \sum_{i=0}^{n-2} b_i(s) |I_{0+}^{\alpha_{n-2}-\delta_i} v(s)|^{\rho_i} \\
&+ b_{n-1}(s) |D_{0+}^{\delta_{n-1}-\alpha_{n-2}} v(s)|^{\rho_{n-1}} + b_n(s) |D_{0+}^{\delta_n-\alpha_{n-2}} v(s)|^{\rho_n} \Big) \\
\leq &\kappa_2 \|b\|_{L^p_{\delta-\delta_n}} + \kappa_2 \sum_{i=0}^{n-2} B_i \|v\|_0^{\rho_i} + \kappa_2 B_{n-1} \|v\|_1^{\rho_{n-1}} + \kappa_2 B_n \|v\|_2^{\rho_n} \\
\leq &\kappa_2 (\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i \|v\|_E^{\rho_i}).
\end{aligned} \tag{46}$$

Hence, combining with (46), from Lemma 8, we have:

$$\begin{aligned}
Tv(t) \leq &t^{\delta-\alpha_{n-2}-1} [M(1+\varphi_0)] \int_0^1 (1-s)^{\delta-\alpha_{n-1}-1} F_v(s) ds \\
\leq &t^{\delta-\alpha_{n-2}-1} [M(1+\varphi_0)] \kappa_2 (\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i \|v\|_E^{\rho_i}) \\
\leq &t^{\delta-\alpha_{n-2}-1} \overline{M} (\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i \chi_2^{\rho_i}) \leq \chi_2 t^{\delta-\alpha_{n-2}-1},
\end{aligned}$$

so $\|Tv\|_0 \le \chi_2$. In addition., by Hypothesis 5, we obtain:

$$\begin{aligned}Tv(t) \ge& t^{\delta-\alpha_{n-2}-1}[m(1+\varphi_0)]\int_0^1 s(1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds\\ \ge& t^{\delta-\alpha_{n-2}-1}[m(1+\varphi_0)]\frac{\chi_1}{m(1+\varphi_0)} = \chi_1 t^{\delta-\alpha_{n-2}-1}.\end{aligned}$$

In addition, by (30) and (46), in view of Lemma 9 and Corollary 3, we have:

$$\begin{aligned}\|Tv\|_1 =& \max_{0\le t\le 1} D_{0+}^{\delta_{n-1}-\alpha_{n-2}}Tv(t)\\ \le& \max_{0\le t\le 1} t^{\delta-\delta_{n-1}-1}\left[M_1+\frac{\Gamma(\delta-\alpha_{n-2})}{\Gamma(\delta-\delta_{n-1})}M\varphi_0\right]\int_0^1(1-s)^{\delta-\alpha_{n-1}-1}F_v(s)ds\\ \le& \overline{M}(\|b\|_{L^p_{\delta-\delta_n}}+\sum_{i=0}^n B_i\chi_2^{\rho_i}) \le \chi_2.\end{aligned}$$

Similarly, we can deduce $\|Tv\|_2 \le \chi_2$. Therefore, $\|Tv\|_E \le \chi_2$, then $T(\Omega) \subset \Omega$. Theorem 2 ensures the existence of (at least) one fixed point $v \in \Omega$ such that $v = Tv$, and v is a positive solution. Thus, the problem (2) possesses at least a positive solution. Therefore, the boundary value problem (1) has at least a positive solution. □

Theorem 6. *Let us assume that $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$, Hypotheses 1, 3 and 5 hold. Then, the boundary value problem (1) has at least a positive solution provided that* $0 < \overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum\limits_{i=0}^n B_i) < 1$.

The proof is similar to that of Theorem 5, so it is omitted.

Theorem 7. *Assume that $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$, Hypothesis 1, Hypothesis 2 (or Hypothesis 3), Hypothesis 6 hold, then the following condition is also satisfied:*

$$\overline{M}\left(\sum_{i=0}^{n-2}\frac{\|a_i\|_{L^p}}{\Gamma(\alpha_{n-2}-\delta_i+1)}+\|a_{n-1}\|_{L^p}+\|a_n\|_{L^p}\right) < 1. \tag{47}$$

Then, the boundary value problem (1) has a unique nonegative solution.

Proof of Theorem 7. To obtain the conclusion, we just need to prove that T is a contraction. For any $v_1, v_2 \in P$, by Hypothesis 6, likewise, we obtain:

$$\begin{aligned}& t^{1+\delta_n-\delta}|F_{v_1}(t)-F_{v_2}(t)|\\ \le& \sum_{i=0}^{n-2} a_i(t)|I_{0+}^{\alpha_{n-2}-\delta_i}v_1(t)-I_{0+}^{\alpha_{n-2}-\delta_i}v_2(t)| + a_{n-1}(t)|D_{0+}^{\delta_{n-1}-\alpha_{n-2}}v_1(t)-D_{0+}^{\delta_{n-1}-\alpha_{n-2}}v_2(t)|\\ &+a_n(t)t^{1+\delta_n-\delta}|D_{0+}^{\delta_n-\alpha_{n-2}}v_1(t)-D_{0+}^{\delta_n-\alpha_{n-2}}v_2(t)|\\ \le& \sum_{i=0}^{n-2}\frac{a_i(t)}{\Gamma(\alpha_{n-2}-\delta_i+1)}\|v_1-v_2\|_0 + a_{n-1}(t)\|v_1-v_2\|_1 + a_n(t)\|v_1-v_2\|_2,\end{aligned}$$

then:

$$\|F_{v_1}-F_{v_2}\|_{L^p_{\delta-\delta_n}} \le \left(\sum_{i=0}^{n-2}\frac{\|a_i\|_{L^p}}{\Gamma(\alpha_{n-2}-\delta_i+1)}+\|a_{n-1}\|_{L^p}+\|a_n\|_{L^p}\right)\|v_1-v_2\|_E.$$

Therefore:

$$\begin{aligned}\|Tv_1-Tv_2\|_E \le& \overline{M}\|F_{v_1}-F_{v_2}\|_{L^p_{\delta-\delta_n}}\\ \le& \overline{M}\left(\sum_{i=0}^{n-2}\frac{\|a_i\|_{L^p}}{\Gamma(\alpha_{n-2}-\delta_i+1)}+\|a_{n-1}\|_{L^p}+\|a_n\|_{L^p}\right)\|v_1-v_2\|_E.\end{aligned}$$

From (47), we naturally infer that T is a contraction. By the Banach contraction mapping principle, we deduce that T has a unique fixed point which is obviously a solution of the problem (1). This ends the proof. $\square$

5. Examples

Example 1.

$$\begin{cases} D_{0+}^{\delta}u(t) + f(t, u(t), D_{0+}^{\delta_1}u(t), D_{0+}^{\delta_2}u(t), \dots, D_{0+}^{\delta_5}u(t)) = 0, \\ u(0) = D_{0+}^{\alpha_1}u(0) = D_{0+}^{\alpha_2}u(0) = D_{0+}^{\alpha_3}u(0) = 0, \\ D_{0+}^{\alpha_4}u(1) = \int_0^1 tD_{0+}^{\alpha_3}u(t)dt, \end{cases} \tag{48}$$

Taking $\delta = \frac{47}{10}, \delta_1 = \frac{1}{2}, \delta_2 = \frac{3}{2}, \delta_3 = \frac{5}{2}, \delta_4 = \frac{31}{10}, \delta_5 = \frac{83}{20}, n = 5, \alpha_1 = \frac{3}{2}, \alpha_2 = \frac{12}{5}, \alpha_3 = \frac{29}{10}, \alpha_4 = \frac{41}{10}$. *Let* $\varphi[g] = \int_0^1 (1-t)^{0.2}g(t)dt$, *then:*

$$\triangle = \Gamma(\delta - \alpha_3) - \Gamma(\delta - \alpha_4)\varphi[t^{\delta-\alpha_3-1}] = \Gamma(1.8) - \Gamma(0.6)\varphi[t^{0.8}] \approx 0.2946 > 0.$$

Choosing $p = 2, q = 2$, *it is obvious that* $\frac{1}{p} < \min\{\delta - \delta_4 - 1, \delta - \delta_5\}$. *Choosing* $a(t) = t^{-\frac{2}{5}}, a_0(t) = t^{-\frac{1}{5}}, a_1(t) = t^{-\frac{3}{10}}, a_2(t) = t^{-\frac{3}{20}}, a_3(t) = t^{-\frac{1}{4}}, a_4(t) = t^{-\frac{1}{3}}, a_5(t) = t^{-\frac{1}{10}}, \rho_i = \frac{1}{i+2}(0 \le i \le 4), \rho_5 = \frac{2}{3}$, *for any* $t \in (0,1], (x_0, \dots, x_4, x_5) \in (\mathbb{R}^+)^5 \times \mathbb{R}$:

$$f(t, x_0, .., x_5) = a(t) + \sum_0^4 \frac{a_i(t)}{\sqrt{i+1+x_i}} x_i^{\rho_i} + \frac{a_5(t)}{\sqrt{6+|x_5|}}|x_5|^{\frac{2}{3}}.$$

It is easy to show that $a, a_i \in L_{\delta-\delta_n}^p$ $(i = 0, 1, \dots, 4), t^{(1+\delta_n-\delta)(1-\rho_5)}a_5(t) \in L^p$ *are non-negative. Let* $b(t) = a(t), b_i(t) = \frac{a_i(t)}{\sqrt{i+1}}, i = 0, 1, \dots, 5$, *then we deduce that for any* $t \in (0,1], (x_0, \dots, x_4, x_5) \in (\mathbb{R}^+)^5 \times \mathbb{R}$:

$$|f(t, x_0, \dots, x_5)| \le b(t) + \sum_{i=0}^5 b_i(t)|x_i|^{\rho_i}.$$

This implies that Hypotheseis 1 and 2 hold. In addition:

$$f(t, x_0, \dots, x_5) \ge b(t), t \in (0,1], (x_0, \dots, x_4, x_5) \in (\mathbb{R}^+)^5 \times \mathbb{R}.$$

In conclusion, all the conditions $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$, *Hypotheses 1, 2 and 4 hold. From Theorem 3, we know that the boundary value problem (48) has at least one positive solution.*

Example 2. *We consider BVP (1) with* $\delta = \frac{23}{5}, \delta_5 = \frac{41}{10}, \delta_4 = \frac{31}{10}, \delta_3 = \frac{13}{5}, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = \frac{13}{5}, \alpha_4 = \frac{18}{5}$. *Let* $p = 3, p_1(t) = \frac{1}{50}t^{-\frac{1}{3}}, p_2(t) = \frac{1}{40}t^{\frac{1}{4}}, p_3(t) = \frac{1}{100}t^{-\frac{1}{6}}, g(t) = \frac{1}{100}(t^{\frac{1}{2}} + t^{\frac{1}{4}})$ *and:*

$$f(t, x, y, z) = g(t) + p_1(t)x + p_2(t)\sin(y^2) + p_3(t)|z|, t \in (0,1], u, v \ge 0, w \in \mathbb{R},$$

It is obvious that $0 < \frac{1}{p} < \frac{1}{2} = \delta - \delta_5$, $g, p_i (i = 1, 2, 3)$ *is a nonegative function and* $g, p_1, p_2 \in L_{\delta-\delta_5}^p, p_3 \in L^p$. *Then, the problem can be transformed into the following two-point boundary value problem:*

$$\begin{cases} v''(t) + p_1(t)v(t) + p_2(t)\sin(D_{0+}^{\frac{1}{2}}v(t))^2 + p_3(t)|D_{0+}^{\frac{3}{2}}v(t)| + g(t) = 0, t \in (0,1), \\ v(0) = 0, v'(1) = 0, \end{cases} \tag{49}$$

Through basic calculations and inferences, we conclude that the assumptions of Theorem 4 are satisfied. Hence, BVP (49) has at least one positive solution.

Remark 2. *Here we show an illustrate example (see [9] or [10]). Consider now the rigid plate of mass m immersed in a Newtonian fluid of infinite extent and connected by a massless spring of stiffness K to a fixed point. The system is depicted in Figure 1. We assume that the small motions of the spring do not disturb the fluid, and that the area A of the plate is sufficiently large as to produce the velocity field in the fluid adjacent to the plate. Given that the initial velocity of the fluid is zero, by Hook's law and Newton's second law, we find the differential equation describing the displacement X of the plate to be:*

$$mX''(t) + 2A\sqrt{\mu\rho}D_{0+}^{\frac{1}{2}}X'(t) + KX + F = 0, \tag{50}$$

where μ is the viscosity, ρ is the fluid density, and F is constant external force. Combining the boundary condition $X(0) = 0$, integrating by parts, we obtain:

$$\begin{aligned} D_{0+}^{\frac{3}{2}}X(t) =& \frac{1}{\Gamma(\frac{1}{2})}\frac{d^2}{dt^2}\int_0^t (t-s)^{-\frac{1}{2}}X(s)ds \\ =& \frac{1}{\Gamma(\frac{1}{2})}\frac{d^2}{dt^2}(2\int_0^t (t-s)^{\frac{1}{2}}X'(s)ds) \\ =& \frac{1}{\Gamma(\frac{1}{2})}\frac{d}{dt}(\int_0^t (t-s)^{-\frac{1}{2}}X'(s)ds) = D_{0+}^{\frac{1}{2}}X'(t). \end{aligned}$$

Let $m = 1, p_1(t) = K, p_2(t) = 0, p_3(t) = 2A\sqrt{\mu\rho}, g(t) = F$. Then, Equation (50) is transformed into:

$$X''(t) + p_1(t)X(t) + p_3(t)D_{0+}^{\frac{3}{2}}X(t) + g(t) = 0. \tag{51}$$

Obviously, it is a special case of equation in (49).

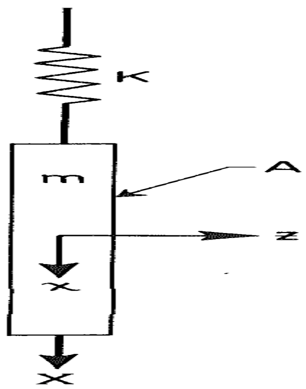

Figure 1. The immersed plate.

Example 3. *We consider BVP (1) with $\delta = \frac{7}{2}, \delta_4 = \frac{16}{5}, \delta_3 = 2, \delta_2 = \frac{3}{2}, \delta_1 = \frac{1}{2}, \alpha_3 = \frac{29}{10}, \alpha_2 = \frac{9}{5}, \alpha_1 = 1, p = 5, q = \frac{5}{4}, \varphi[u] = \int_0^1 t^{0.3}u(t)dt$. We know by calculation that the conditions $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$ are satisfied. Choosing:*

$$f(t, x_0, ..., x_4) = \sum_{i=0}^{3} a_i(t)\frac{i+2+(x_i)^{\rho_i}}{i+2+(sinx_i)^{\rho_i}} + a_4(t)|x_4|^{\rho_4}, \tag{52}$$

where $a_0(t) = \frac{1}{10}t^{-0.7}, a_1(t) = \frac{1}{5}t^{-0.72}, a_2(t) = \frac{1}{3}t^{-0.74}, a_3(t) = t^{-0.6}, a_4(t) = \frac{1}{5}t^{-0.608}, \rho_0 = 0.1, \rho_1 = 0.2, \rho_2 = 0.1, \rho_3 = 0.08, \rho_4 = 0.16$. Since $|sint| \le 1$, we know:

$$f(t, x_0, ..., x_4) \le \sum_{i=0}^{3}\frac{i+2}{i+1}a_i(t) + \sum_{i=0}^{3}\frac{a_i(t)}{i+1}(x_i)^{\rho_i} + a_4(t)|x_4|^{\rho_4}. \tag{53}$$

Let $b(t) = \sum_{i=0}^{3} \frac{i+2}{i+1} a_i(t), b_i(t) = \frac{a_i(t)}{i+1} (i = 0, 1, 2, 3), b_4(t) = a_4(t)$. *Equations (52) and (53) means that Hypotheses 1 and 2 hold. Moreover, we choose* $\chi_1 = 10, \chi_2 = 232$. *It follows from* $\sin x \leq x, \forall x \geq 0$ *that for any* $(x_0, x_1, ..., x_n) \in \Xi$*:*

$$\inf_{(x_0, x_1, ..., x_n) \in \Xi} f(s, x_0, ..., x_n) \geq \sum_{0}^{3} a_i(t) = \eta(t). \tag{54}$$

We can also verify that Hypothesis 5 is satisfied. From Theorem 5, we know that the boundary value problem has at least one positive solution.

Example 4. *We consider BVP (1) with* $\delta = 3.7, \delta_4 = 3.4, \delta_3 = 2.2, \delta_2 = 1.7, \delta_1 = 0.7, \alpha_3 = 3.1, \alpha_2 = 2, \alpha_1 = 1.2, p = 5, q = \frac{5}{4}, \varphi[u] = \int_0^1 t^{0.3} u(t) dt$. *The conditions* $(\boldsymbol{C}_1), (\boldsymbol{C}_2)$ *are satisfied by calculation. Define* $f(t, x_0, ..., x_4)$ *as (52), where* $a_0(t) = \frac{1}{200} t^{1.1}, a_1(t) = \frac{1}{150} t^{3.1}, a_2(t) = \frac{3}{400} t^{5.1}, a_3(t) = \frac{1}{125} t^{7.1}, a_4(t) = \frac{1}{500} t^{9.8}, \rho_0 = 2, \rho_1 = 3, \rho_2 = 1, \rho_3 = 4, \rho_4 = 1$. *Let* $b(t) = \sum_{i=0}^{3} \frac{i+2}{i+1} a_i(t), b_i(t) = \frac{a_i(t)}{i+1} (i = 0, 1, 2, 3), b_4(t) = a_4(t)$. *By similar deduction, we note that Hypotheses 1 and 3 hold. Moreover, we choose* $\chi_1 = 0.01, \chi_2 = 1$. *It follows from the same inference and calculation that Hypothesis 5 is satisfied and* $0 < \overline{M}(\|b\|_{L^p_{\delta-\delta_n}} + \sum_{i=0}^{n} B_i) < 1$. *From Theorem 6, we know that the boundary value problem has at least one positive solution.*

6. Conclusions

In the above research work, we studied the properties of integral operators and then successfully obtained the existence results of a nonlinear higher-order fractional differential equation with multi-term lower-order derivatives by means of the Guo–Krasnoselskii fixed point theorem and Schauder fixed point theorem. Furthermore, a uniqueness result was obtained by the Banach contraction mapping principle. The existence results were verified by considering an example where needed.

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