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# Asymptotic Properties of Solutions to Delay Differential Equations Describing Plankton—Fish Interaction

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**Abstract:** We consider a system of differential equations with two delays describing plankton–fish interaction. We analyze the case when the equilibrium point of this system corresponding to the presence of only phytoplankton and the absence of zooplankton and fish is asymptotically stable. In this case, the asymptotic behavior of solutions to the system is studied. We establish estimates of solutions characterizing the stabilization rate at infinity to the considered equilibrium point. The results are obtained using Lyapunov–Krasovskii functionals.

**Keywords:** predator–prey model; plankton–fish interaction; delay differential equations; equilibrium point; asymptotic stability; estimates for solutions; Lyapunov–Krasovskii functionals

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## 1. Introduction

At present, there exist a large number of works devoted to the study of biological models described by delay differential equations (see, for example, the monographs [1–4] and bibliography therein). Among studied models, the models of population dynamics are widespread, in particular, models of predator–prey type, based on the classical predator–prey model that was introduced independently by A.J. Lotka [5] and V. Volterra [6]. An overview of some results for predator–prey models with delay is contained, for example, in [7,8].

In particular, predator–prey models are used when describing plankton–fish interaction (see, for example, [9–14]). Taking into account a model proposed in [14], in the present paper, we study the model of the following form:

$$\begin{cases} \frac{d}{dt}x(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) - c_1x(t)y(t), \\ \frac{d}{dt}y(t) = -d_1y(t) + e_1c_1x(t-\tau_1)y(t-\tau_1) - c_2y(t)z(t), \\ \frac{d}{dt}z(t) = -d_2z(t) + e_2c_2y(t-\tau_2)z(t-\tau_2), \end{cases} \tag{1}$$

which can be also considered a model of plankton–fish interaction (note that, in [14], it was considered the case that $\tau_2 = 0$, but nonlinear terms had a more general form). In this system, $x(t)$ is the amount of phytoplankton, $y(t)$ is the amount of zooplankton, and $z(t)$ is the number of fish. It is assumed that phytoplankton is the favorite food of zooplankton, which serves as the favorite food of fish. The delay parameter $\tau_1 \geq 0$ is responsible for time required for the appearance of new zooplankton, and the delay parameter $\tau_2 \geq 0$ is responsible for time of fish maturation. The coefficients of the system have the following meaning: $r > 0$ is intrinsic growth rate of phytoplankton, $K > 0$ is environmental carrying capacity of phytoplankton, $d_1 > 0$ is mortality rate of zooplankton, $d_2 > 0$ is mortality rate of fish, $c_1 \geq 0$ is predation rate of zooplankton, $c_2 \geq 0$ is predation rate of fish, $e_1 = b_1e^{-c_1\tau_1}$, $b_1 \geq 0$ is birth rate of zooplankton, $e_2 = b_2e^{-c_2\tau_2}$, $b_2 \geq 0$ is birth rate of fish.

We consider system (1) for $t > 0$, assuming that the initial conditions are given on the segment $\theta \in [-\tau_{\max}, 0]$, $\tau_{\max} = \max\{\tau_1, \tau_2\}$:

$$\begin{cases} x(\theta) = \varphi(\theta) \geq 0, & \theta \in [-\tau_1, 0], \quad x(+0) = \varphi(0) > 0, \\ y(\theta) = \psi(\theta) \geq 0, & \theta \in [-\tau_{\max}, 0], \quad y(+0) = \psi(0), \\ z(\theta) = \eta(\theta) \geq 0, & \theta \in [-\tau_2, 0], \quad z(+0) = \eta(0), \end{cases} \tag{2}$$

where $\varphi(\theta)$, $\psi(\theta)$, $\eta(\theta)$ are continuous functions. It is well known that a solution to the initial value problems (1) and (2) exists and is unique. Moreover, by analogy with [14], it is not difficult to show that the solution is defined on the entire right half-axis $\{t > 0\}$, has non-negative components, and each component of the solution is a bounded function. In other words, there exist constants $M_1,\ M_2,\ M_3 > 0$ such that, for all $t > 0$, the inequalities are valid

$$0 \leq x(t) \leq M_1, \quad 0 \leq y(t) \leq M_2, \quad 0 \leq z(t) \leq M_3,$$

i.e., the amount of plankton and fish cannot increase indefinitely.

One of the stationary solutions to system (1) is the equilibrium point $(x(t), y(t), z(t))^{\mathrm{T}} = (K, 0, 0)^{\mathrm{T}}$, corresponding to the presence of only phytoplankton in the system and the absence of zooplankton and fish. We say that equilibrium point $(K, 0, 0)^{\mathrm{T}}$ is asymptotically stable, if

1. $$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \max_{\theta \in [-\tau_1, 0]} |\varphi(\theta) - K| + \max_{\theta \in [-\tau_{\max}, 0]} |\psi(\theta)| + \max_{\theta \in [-\tau_2, 0]} |\eta(\theta)| < \delta$$
$$\Longrightarrow \quad |x(t) - K| + |y(t)| + |z(t)| < \varepsilon \quad \forall t > 0;$$
2. $$\exists \rho > 0 : \quad \max_{\theta \in [-\tau_1, 0]} |\varphi(\theta) - K| + \max_{\theta \in [-\tau_{\max}, 0]} |\psi(\theta)| + \max_{\theta \in [-\tau_2, 0]} |\eta(\theta)| < \rho$$
$$\Longrightarrow \quad |x(t) - K| + |y(t)| + |z(t)| \to 0 \quad \text{as} \quad t \to +\infty.$$

In [14], it was noted that, for $\tau_2 = 0$, a sufficient condition for the asymptotic stability of this equilibrium point is the condition

$$e_1 c_1 K < d_1, \tag{3}$$

meaning a sufficiently high mortality of zooplankton. By analogy with [14], it is not difficult to establish that, for $\tau_2 > 0$, condition (3) is also a sufficient condition for the asymptotic stability of the considered equilibrium point. Moreover, it can be shown that all solutions to system (1) with initial conditions of the form (2) are stabilized at infinity to this equilibrium point.

The aim of the present paper is, under condition (3), to establish estimates for all components of the solution to the initial value problems (1) and (2) characterizing the stabilization rate at infinity to equilibrium point $(K, 0, 0)^{\mathrm{T}}$. To achieve our goal, we will use Lyapunov–Krasovskii functionals. Note that such functionals are actively used to obtain estimates of solutions to various classes of systems with delay (see, for example, [15–18], where systems of differential equations of delayed type were considered, and [19–29], where systems of differential equations of neutral type were considered). For specific biological models, the results based on the use of Lyapunov–Krasovskii functionals are contained, for example, in [30–35].

## 2. Main Results

As already noted above, throughout the paper, we assume that condition (3) is fulfilled, which guarantees the asymptotic stability of equilibrium point $(K, 0, 0)^{\mathrm{T}}$ of system (1). We obtain estimates for all components of the solution to the initial value problems (1) and (2).

First, we formulate an auxiliary statement.

**Lemma 1.** *Let $(x(t), y(t), z(t))^{\mathrm{T}}$ be the solution to the initial value problems (1) and (2). Then, for the first component of the solution $x(t)$, the estimate holds*

$$0 < x(t) \leq K + (\varphi(0) - K)e^{-rt}, \quad t > 0. \tag{4}$$

**Proof.** Due to the initial condition (2), we have $x(+0) = \varphi(0) > 0$. Hence, from the first equation of system (1), it follows that $x(t) > 0$ for all $t > 0$. Further, since $y(t) \geq 0$ for all $t > 0$, from this equation, we obtain the inequality

$$\frac{d}{dt}x(t) \leq rx(t)\left(1 - \frac{x(t)}{K}\right).$$

Changing the variables

$$\overline{x}(t) = x(t) - K,$$

we establish the following estimate:

$$\frac{d}{dt}\overline{x}(t) \leq -\frac{r}{K}\overline{x}(t)(\overline{x}(t) + K) \leq -r\overline{x}(t).$$

From here, it is not difficult to obtain the inequality

$$\overline{x}(t) \leq \overline{x}(0)e^{-rt}.$$

Taking into account that $\overline{x}(t) = x(t) - K$, we establish (4).
Lemma is proved. □

Now, we establish estimates for the second component of the solution $y(t)$.

**Lemma 2.** *Let $(x(t), y(t), z(t))^{\mathrm{T}}$ be the solution to the initial value problems (1) and (2). Then, for the second component of the solution $y(t)$, the estimate holds*

$$0 \leq y(t) \leq \alpha, \quad \alpha = \psi(0) + e_1 c_1 \int_{-\tau_1}^{0} \varphi(s)\psi(s)ds, \quad t \in [0, \tau_1]. \tag{5}$$

**Proof.** Let $t \in [0, \tau_1]$. Taking into account the initial conditions (2), and considering $y(t) \geq 0$ and $z(t) \geq 0$, from the second equation of system (1), it is not difficult to obtain the estimate

$$\frac{d}{dt}y(t) \leq -d_1 y(t) + e_1 c_1 \varphi(t - \tau_1)\psi(t - \tau_1).$$

This estimate can be rewritten in an equivalent form:

$$\frac{d}{dt}\left(e^{d_1 t} y(t)\right) \leq e_1 c_1 e^{d_1 t}\varphi(t - \tau_1)\psi(t - \tau_1),$$

from which follows the inequality

$$y(t) \leq \psi(0)e^{-d_1 t} + e_1 c_1 \int_{0}^{t} e^{-d_1(t-s)}\varphi(s - \tau_1)\psi(s - \tau_1)ds$$

$$\leq \psi(0) + e_1 c_1 \int_{0}^{\tau_1} \varphi(s - \tau_1)\psi(s - \tau_1)ds,$$

which coincides with (5).
Lemma is proved. □

Let us proceed to obtaining estimates for the second component of the solution $y(t)$ for $t > \tau_1$. To do this, consider the Lyapunov–Krasovskii functional of the following form

$$V(t, y) = y^2(t) + \int_{t-\tau_1}^{t} e^{-k_1(t-s)} m_1(s) y^2(s)ds, \tag{6}$$

where $k_1 > 0$ is such a number that the inequality is satisfied

$$e_1 c_1 K e^{k_1 \tau_1/2} < d_1$$

(by virtue of condition (3), such $k_1 > 0$ exists),

$$m_1(t) = e_1 c_1 e^{k_1 \tau_1/2}\Big(K + (\varphi(0) - K)e^{-rt}\Big). \tag{7}$$

The following theorem is valid.

**Theorem 1.** *Let condition (3) be satisfied, and let $(x(t), y(t), z(t))^{\mathrm{T}}$ be the solution to the initial value problems (1) and (2). Then, for the second component of the solution $y(t)$, the estimate holds*

$$0 \le y(t) \le \sqrt{V(\tau_1, \alpha)} e^{J_1/2} e^{-\varepsilon(t-\tau_1)/2}, \quad t > \tau_1, \tag{8}$$

*where $\alpha$ is defined in (5),*

$$J_1 = \begin{cases} 0, & \text{if} \quad 0 < \varphi(0) \le K, \\ e_1 c_1 e^{k_1\tau_1/2}(\varphi(0) - K)\dfrac{(1 + e^{-r\tau_1})}{r}, & \text{if} \quad \varphi(0) > K, \end{cases} \tag{9}$$

$$\varepsilon = \min\Big\{2\Big(d_1 - e_1 c_1 K e^{k_1\tau_1/2}\Big), k_1\Big\} > 0. \tag{10}$$

**Proof.** Let $t > \tau_1$. Using inequality (4), and considering $y(t) \ge 0$ and $z(t) \ge 0$, from the second equation of system (1), we obtain the estimate

$$\frac{d}{dt} y(t) \le -d_1 y(t) + e_1 c_1 \Big(K + (\varphi(0) - K)e^{-r(t-\tau_1)}\Big) y(t - \tau_1).$$

Now we consider Lyapunov–Krasovskii functional (6). Differentiating it along the solution to the initial value problems (1) and (2), we establish validity of the following inequality

$$\frac{d}{dt} V(t, y) = 2y(t)\frac{d}{dt} y(t) + m_1(t) y^2(t) - e^{-k_1\tau_1} m_1(t - \tau_1) y^2(t - \tau_1)$$

$$-k_1 \int_{t-\tau_1}^{t} e^{-k_1(t-s)} m_1(s) y^2(s) ds$$

$$\le -(2d_1 - m_1(t)) y^2(t) + 2e_1 c_1 \Big(K + (\varphi(0) - K)e^{-r(t-\tau_1)}\Big) y(t) y(t - \tau_1)$$

$$-e^{-k_1\tau_1} m_1(t - \tau_1) y^2(t - \tau_1) - k_1 \int_{t-\tau_1}^{t} e^{-k_1(t-s)} m_1(s) y^2(s) ds.$$

Hence, using the inequality

$$2e_1 c_1 \Big(K + (\varphi(0) - K)e^{-r(t-\tau_1)}\Big) y(t) y(t - \tau_1) - e^{-k_1\tau_1} m_1(t - \tau_1) y^2(t - \tau_1)$$

$$\le \frac{e_1^2 c_1^2 \Big(K + (\varphi(0) - K)e^{-r(t-\tau_1)}\Big)^2}{e^{-k_1\tau_1} m_1(t - \tau_1)} y^2(t)$$

and considering definition (7) of function $m_1(t)$, we obtain the estimate

$$\frac{d}{dt} V(t, y) \le -2\Big(d_1 - e_1 c_1 K e^{k_1\tau_1/2}\Big) y^2(t) + e_1 c_1 e^{k_1\tau_1/2}(\varphi(0) - K)(1 + e^{r\tau_1}) e^{-rt} y^2(t)$$

$$-k_1 \int_{t-\tau_1}^{t} e^{-k_1(t-s)} m_1(s) y^2(s) ds.$$

By virtue of definition (10) of $\varepsilon$, from here, the inequality follows:

$$\frac{d}{dt} V(t,y) \le -\varepsilon V(t,y) + e_1 c_1 e^{k_1 \tau_1 / 2} (\varphi(0) - K)(1 + e^{-r\tau_1}) e^{-r(t-\tau_1)} y^2(t). \tag{11}$$

First, let $0 < \varphi(0) \le K$; then,

$$\frac{d}{dt} V(t,y) \le -\varepsilon V(t,y);$$

therefore,

$$V(t,y) \le V(\tau_1, y) e^{-\varepsilon(t-\tau_1)}.$$

From definition (6) and inequality (5), the estimates follow: $y^2(t) \le V(t,y)$, $V(\tau_1, y) \le V(\tau_1, \alpha)$, from which we establish (8).

Now, let $\varphi(0) > K$. Then, from inequality (11), we obtain the estimate

$$\frac{d}{dt} V(t,y) \le \left( -\varepsilon + J_1 r e^{-r(t-\tau_1)} \right) V(t,y),$$

where $J_1$ is defined in (9). Hence, the inequality follows

$$V(t,y) \le V(\tau_1, y) \exp \left( \int_{\tau_1}^{t} \left( -\varepsilon + J_1 r e^{-r(s-\tau_1)} \right) ds \right) \le V(\tau_1, y) e^{J_1} e^{-\varepsilon(t-\tau_1)}.$$

This estimate directly implies (8).

Theorem is proved. □

Now, we obtain estimates for the third component of the solution $z(t)$. First, we consider the case $t \in [0, \tau_2]$.

**Lemma 3.** *Let $(x(t), y(t), z(t))^{\mathrm{T}}$ be the solution to the initial value problems (1) and (2). Then, for the third component of the solution $z(t)$, the estimate holds*

$$0 \le z(t) \le \beta, \quad \beta = \eta(0) + e_2 c_2 \int_{-\tau_2}^{0} \psi(s) \eta(s) ds, \quad t \in [0, \tau_2]. \tag{12}$$

**Proof.** Let $t \in [0, \tau_2]$. Taking into account the initial conditions (2), from the third equation of system (1), it is not difficult to obtain the representation

$$z(t) = \eta(0) e^{-d_2 t} + e_2 c_2 \int_{0}^{t} e^{-d_2(t-s)} \psi(s - \tau_2) \eta(s - \tau_2) ds;$$

hence, estimate (12) follows.

Lemma is proved. □

Now, suppose that $t \in [\tau_2, \tau_1 + \tau_2]$. Consider the Lyapunov–Krasovskii functional

$$U(t,z) = z^2(t) + e_2 c_2 \alpha \int_{t-\tau_2}^{t} z^2(s) ds, \tag{13}$$

where $\alpha$ is defined in (5).

The following statement takes place.

**Lemma 4.** *Let $(x(t), y(t), z(t))^{\mathrm{T}}$ be the solution to the initial value problems (1) and (2). Then, for the third component of the solution $z(t)$, the estimate holds*

$$0 \leq z(t) \leq \gamma, \quad \gamma = \sqrt{U(\tau_2, \beta)} e^{\omega \tau_1 / 2}, \quad t \in [\tau_2, \tau_1 + \tau_2], \tag{14}$$

*where $\beta$ is defined in (12),*

$$\omega = \max\{2(e_2 c_2 \alpha - d_2), 0\}.$$

**Proof.** Let $t \in [\tau_2, \tau_1 + \tau_2]$. Using inequality (5), from the third equation of system (1), we obtain the estimate

$$\frac{d}{dt} z(t) \leq -d_2 z(t) + e_2 c_2 \alpha z(t - \tau_2).$$

Now, we consider the Lyapunov–Krasovskii functional (13). Differentiating it along the solution to the initial value problems (1) and (2), it is not difficult to establish the following inequality:

$$\frac{d}{dt} U(t, z) = 2z(t) \frac{d}{dt} z(t) + e_2 c_2 \alpha z^2(t) - e_2 c_2 \alpha z^2(t - \tau_1)$$

$$\leq (e_2 c_2 \alpha - 2d_2) z^2(t) + 2 e_2 c_2 \alpha z(t) z(t - \tau_2) - e_2 c_2 \alpha z^2(t - \tau_1)$$

$$\leq 2(e_2 c_2 \alpha - d_2) z^2(t) \leq \omega U(t, z).$$

From this inequality, we obtain the estimate

$$U(t, z) \leq U(\tau_2, z) e^{\omega (t - \tau_2)} \leq U(\tau_2, z) e^{\omega \tau_1}.$$

From definition (13) and inequality (12), the estimates follow: $z^2(t) \leq U(t, z)$, $U(\tau_2, z) \leq U(\tau_2, \beta)$, from which we establish (14).

Lemma is proved. □

To obtain estimates for the third component of the solution $z(t)$ for $t > \tau_1 + \tau_2$, we consider the Lyapunov–Krasovskii functional of the following form:

$$W(t, z) = z^2(t) + \int_{t - \tau_2}^{t} e^{-k_2 (t - s)} m_2(s) z^2(s) ds, \tag{15}$$

where $k_2 > 0$ is arbitrary,

$$m_2(t) = e_2 c_2 e^{k_2 \tau_2 / 2} \sqrt{V(\tau_1, \alpha)} e^{J_1 / 2} e^{-\varepsilon (t - \tau_1) / 2}, \tag{16}$$

$V(t, y)$ is defined in (6), $\alpha$ is defined in (5), $J_1$ is defined in (9), and $\varepsilon$ is defined in (10).

The following theorem is valid.

**Theorem 2.** *Let condition (3) be satisfied, and let $(x(t), y(t), z(t))^{\mathrm{T}}$ be the solution to the initial value problems (1) and (2). Then, for the third component of the solution $z(t)$, the estimate holds*

$$0 \leq z(t) \leq \sqrt{W(\tau_1 + \tau_2, \gamma)} e^{J_2 / 2} e^{-\sigma (t - \tau_1 - \tau_2) / 2}, \quad t > \tau_1 + \tau_2, \tag{17}$$

*where $\gamma$ is defined in (14),*

$$J_2 = e_2 c_2 e^{k_2 \tau_2 / 2} \sqrt{V(\tau_1, \alpha)} e^{J_1 / 2} \frac{(1 + e^{-\varepsilon \tau_2 / 2})}{(\varepsilon / 2)}, \tag{18}$$

$$\sigma = \min\{2d_2, k_2\}. \tag{19}$$

**Proof.** Let $t > \tau_1 + \tau_2$. Using inequality (8), from the third equation of system (1), we obtain the estimate

$$\frac{d}{dt}z(t) \leq -d_2 z(t) + e_2 c_2 \sqrt{V(\tau_1, \alpha)} e^{J_1/2} e^{-\varepsilon(t-\tau_1-\tau_2)/2} z(t-\tau_2).$$

Now, we consider the Lyapunov–Krasovskii functional (15). Differentiating it along the solution to the initial value problems (1) and (2), we establish the validity of the following inequality:

$$\frac{d}{dt}W(t,z) = 2z(t)\frac{d}{dt}z(t) + m_2(t)z^2(t) - e^{-k_2\tau_2} m_2(t-\tau_2) z^2(t-\tau_2)$$

$$-k_2 \int\limits_{t-\tau_2}^{t} e^{-k_2(t-s)} m_2(s) z^2(s) ds$$

$$\leq -(2d_2 - m_2(t))z^2(t) + 2e_2 c_2 \sqrt{V(\tau_1, \alpha)} e^{J_1/2} e^{-\varepsilon(t-\tau_1-\tau_2)/2} z(t) z(t-\tau_2)$$

$$-e^{-k_2\tau_2} m_2(t-\tau_2) z^2(t-\tau_2) - k_2 \int\limits_{t-\tau_2}^{t} e^{-k_2(t-s)} m_2(s) z^2(s) ds.$$

Hence, using the inequality

$$2e_2 c_2 \sqrt{V(\tau_1, \alpha)} e^{J_1/2} e^{-\varepsilon(t-\tau_1-\tau_2)/2} z(t) z(t-\tau_2) - e^{-k_2\tau_2} m_2(t-\tau_2) z^2(t-\tau_2)$$

$$\leq \frac{\left(e_2 c_2 \sqrt{V(\tau_1, \alpha)} e^{J_1/2} e^{-\varepsilon(t-\tau_1-\tau_2)/2}\right)^2}{e^{-k_2\tau_2} m_2(t-\tau_2)} z^2(t)$$

and considering definition (16) of function $m_2(t)$, we obtain the estimate

$$\frac{d}{dt}W(t,z) \leq -2d_2 z^2(t) + e_2 c_2 e^{k_2\tau_2/2} \sqrt{V(\tau_1, \alpha)} e^{J_1/2} (1 + e^{-\varepsilon\tau_2/2}) e^{-\varepsilon(t-\tau_1-\tau_2)/2} z^2(t)$$

$$-k_2 \int\limits_{t-\tau_2}^{t} e^{-k_2(t-s)} m_2(s) z^2(s) ds.$$

By virtue of definition (19) of $\sigma$ and definition (18) of $J_2$, the inequality follows:

$$\frac{d}{dt}W(t,z) \leq \left(-\sigma + J_2 \frac{\varepsilon}{2} e^{-\varepsilon(t-\tau_1-\tau_2)/2}\right) W(t,z).$$

Therefore,

$$W(t,z) \leq W(\tau_1+\tau_2, z) \exp\left( \int\limits_{\tau_1+\tau_2}^{t} \left(-\sigma + J_2 \frac{\varepsilon}{2} e^{-\varepsilon(s-\tau_1-\tau_2)/2}\right) ds \right)$$

$$\leq W(\tau_1+\tau_2, z) e^{J_2} e^{-\sigma(t-\tau_1-\tau_2)}.$$

Since $z^2(t) \leq W(t,z)$ and $W(\tau_1+\tau_2, z) \leq W(\tau_1+\tau_2, \gamma)$, where $\gamma$ is defined in (14), this estimate directly implies (17).

Theorem is proved. □

Finally, we obtain estimates for the first component of the solution $x(t)$.

The following theorem is valid.

**Theorem 3.** *Let condition (3) be satisfied, and let $(x(t), y(t), z(t))^{\mathrm{T}}$ be the solution to the initial value problems (1) and (2). Then, for the first component of the solution $x(t)$, the estimate holds*

$$K - x(t) \leq \frac{Ke^{c_1 Y}}{\min\{K, \varphi(0)\}}\left(X_1 e^{-rt} + X_2\,(t-\tau_1)\,e^{-\min\{r,(\varepsilon/2)\}(t-\tau_1)}\right), \quad t > \tau_1, \tag{20}$$

*where*

$$Y = \alpha\tau_1 + \frac{2}{\varepsilon}\sqrt{V(\tau_1,\alpha)e^{J_1/2}}, \tag{21}$$

$$X_1 = \max\{K-\varphi(0), 0\} + \varphi(0)\,c_1\alpha\frac{(e^{r\tau_1}-1)}{r}, \tag{22}$$

$$X_2 = \varphi(0)\,c_1\sqrt{V(\tau_1,\alpha)e^{J_1/2}}, \tag{23}$$

*$\alpha$ is defined in (5), $V(t,y)$ is defined in (6), $J_1$ is defined in (9), and $\varepsilon$ is defined in (10).*

**Proof.** Let $t > \tau_1$. From the first equation of system (1), it is not difficult to obtain the representation

$$K - x(t) = K\left[K - \varphi(0) + \varphi(0)\,c_1\int_0^t y(s)e^{rs}\exp\left(-c_1\int_0^s y(\xi)d\xi\right)ds\right]$$
$$\times\left[K + \varphi(0)\,r\int_0^t e^{rs}\exp\left(-c_1\int_0^s y(\xi)d\xi\right)ds\right]^{-1}.$$

Hence, the inequality follows:

$$K - x(t) \leq K\left[\max\{K-\varphi(0),0\} + \varphi(0)\,c_1\int_0^t y(s)e^{rs}ds\right]$$
$$\times\left[K + \varphi(0)\,r\int_0^t e^{rs}\exp\left(-c_1\int_0^s y(\xi)d\xi\right)ds\right]^{-1}.$$

By virtue of the estimate

$$K + \varphi(0)\,r\int_0^t e^{rs}\exp\left(-c_1\int_0^s y(\xi)d\xi\right)ds$$
$$\geq \left[K + \varphi(0)\,r\int_0^t e^{rs}ds\right]\exp\left(-c_1\int_0^t y(\xi)d\xi\right)$$
$$\geq \min\{K,\varphi(0)\}e^{rt}\exp\left(-c_1\int_0^t y(\xi)d\xi\right),$$

we obtain the inequality

$$K - x(t) \leq \frac{K}{\min\{K,\varphi(0)\}}\exp\left(c_1\int_0^t y(\xi)d\xi\right)$$
$$\times\left[\max\{K-\varphi(0),0\}e^{-rt} + \varphi(0)\,c_1\int_0^t y(s)e^{-r(t-s)}ds\right].$$

Now, we estimate the integrals. By virtue of the inequalities (5) and (8), we have

$$\int_0^t y(\xi)d\xi \le \int_0^{\tau_1} \alpha d\xi + \int_{\tau_1}^{\infty} \sqrt{V(\tau_1,\alpha)}e^{J_1/2}e^{-\varepsilon(\xi-\tau_1)/2}d\xi = \alpha\tau_1 + \frac{2}{\varepsilon}\sqrt{V(\tau_1,\alpha)}e^{J_1/2} = Y;$$

hence,

$$K - x(t) \le \frac{Ke^{c_1 Y}}{\min\{K,\varphi(0)\}}\left[\max\{K-\varphi(0),0\}e^{-rt} + \varphi(0)\,c_1\int_0^t y(s)e^{-r(t-s)}ds\right]. \tag{24}$$

Next,

$$\int_0^t y(s)e^{-r(t-s)}ds \le \int_0^{\tau_1} \alpha e^{-r(t-s)}ds + \int_{\tau_1}^{t} \sqrt{V(\tau_1,\alpha)}e^{J_1/2}e^{-\varepsilon(s-\tau_1)/2}e^{-r(t-s)}ds$$

$$\le \alpha\frac{(e^{r\tau_1}-1)}{r}e^{-rt} + \sqrt{V(\tau_1,\alpha)}e^{J_1/2}(t-\tau_1)e^{-\min\{r,(\varepsilon/2)\}(t-\tau_1)}.$$

By virtue of this inequality, taking into account notations (22) and (23), from estimate (24), we obtain inequality (20).

Theorem is proved. □

From Theorem 3 and Lemma 1, the statement follows.

**Corollary 1.** *Let condition (3) be satisfied, and let $(x(t),y(t),z(t))^{\mathrm{T}}$ be the solution to the initial value problems (1) and (2). Then, for $t > \tau_1$, for the first component of the solution $x(t)$, the estimate holds*

$$K - \frac{Ke^{c_1 Y}}{\min\{K,\varphi(0)\}}\left(X_1e^{-rt} + X_2\,(t-\tau_1)\,e^{-\min\{r,(\varepsilon/2)\}(t-\tau_1)}\right) \le x(t) \le K + (\varphi(0)-K)e^{-rt},$$

*where $Y$, $X_1$, and $X_2$ are defined in (21)–(23), and $\varepsilon$ is defined in (10).*

## 3. Conclusions

In the present paper, we have considered a system of differential equations with two delays describing the interaction between fish, zooplankton, and phytoplankton. Provided that zooplankton mortality was sufficiently high, we have established estimates of solutions that characterize the decay rates of the amount of zooplankton and fish, as well as the change rate to a positive constant value of the amount of phytoplankton. The obtained estimates are constructive, and all the values responsible for the stabilization rates are indicated explicitly. To obtain the results, special Lyapunov–Krasovskii functionals were constructed.

Note that the results of this paper have been obtained under the assumption that the equilibrium point corresponding to the presence of only phytoplankton in the system is asymptotically stable. Under certain conditions on the coefficients of the system, there also exist other equilibrium points corresponding to the presence of all populations in the system. Finding the conditions of asymptotic stability of these equilibrium points, as well as obtaining estimates of the stabilization rate of solutions and estimates for attraction sets, is of interest both from a mathematical and biological points of view. The construction of special Lyapunov–Krasovskii functionals to study the asymptotic behavior of solutions in the case of asymptotic stability of equilibrium points corresponding to the presence of all populations in the system is the goal of further research.

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