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Determination of Bounds for the Jensen Gap and Its Applications

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Abstract: The Jensen inequality has been reported as one of the most consequential inequalities that has a lot of applications in diverse fields of science. For this reason, the Jensen inequality has become one of the most discussed developmental inequalities in the current literature on mathematical inequalities. The main intention of this article is to find some novel bounds for the Jensen difference while using some classes of twice differentiable convex functions. We obtain the proposed bounds by utilizing the power mean and Höilder inequalities, the notion of convexity and the prominent Jensen inequality for concave function. We deduce several inequalities for power and quasi-arithmetic means as a consequence of main results. Furthermore, we also establish different improvements for Hölder inequality with the help of obtained results. Moreover, we present some applications of the main results in information theory.

Keywords: convex function; Jensen's inequality; means; Hölder inequality; information theory

MSC: 26A51; 26D15; 68P30

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1. Introduction

The notion of convexity has played a very fundamental role in the last century with a dynamic impact on the several areas of science including Engineering [1], Statistics [2], Economics [3], Optimization [4] and Information Theory [5], etc. The most diligent and dynamic area for the notion of convexity is the field of mathematics [6]. Moreover, the class of convex functions has a vital history and has been an intense topic for the researchers during the last century [7,8]. Due to its extensive importance and gravity, different extensions, generalizations, expansions and variations of this class have been introduced in diverse directions while utilizing some techniques and behavior of convex functions [9]. In the classical approach, a real valued function $\Psi : I \to \mathbb{R}$ will be convex on the interval I, if

$$\Psi(\alpha x + (1-\alpha)y) \leq \alpha\Psi(x) + (1-\alpha)\Psi(y) \tag{1}$$

holds, for all $x, y \in I$ and $\alpha \in [0,1]$. If the inequality (1) holds in the reverse direction, then Ψ will be concave.

The class of convex functions has some interesting properties and due to such properties and its behavior dealing with problems a lot of inequalities have been established for this class of functions. Some of the well-known inequalities for the class of convex functions are majorization [10], Hermite–Hadamrd [6] and Favard [11] inequalities, etc. Among these inequalities, one of the most dynamic and favorable inequalities is the Jensen inequality [5]. Jensen's inequality and convex functions have a very deep pertinency in the view that they generalize the definition of the convex function. Furthermore, this inequality also generalized the renowned triangular inequality. In addition to this, Jensen's inequality is also of great importance in the sense that many classical inequalities can easily be deduced from this inequality. In an elegant manner, Jensen's inequality can be stated in the following way:

Theorem 1. *Assume that Ψ is a real valued convex function defined on interval I and $x_i \in I$, $p_i > 0$ for $i = 1, 2, \cdots, n$ with $P_n := \sum_{i=1}^{n} p_i$. Then*

$$\Psi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \Psi(x_i). \tag{2}$$

Inequality (2) will become true in the opposite sense, if the function Ψ is concave.

In the continuous case, Jensen's inequality can be verbalized in the following fashion:

Theorem 2. *Assume that Ψ is a real valued convex function defined on interval $[a, b]$ and ρ, $\varphi : I \to [a, b]$ are integrable functions such that $\rho > 0$, then*

$$\Psi\left(\frac{1}{\int_a^b \rho(x)dx}\int_a^b \rho(x)\varphi(x)dx\right) \leq \frac{1}{\int_a^b \rho(x)dx}\int_a^b \rho(x)\Psi(\varphi(x))dx. \tag{3}$$

For the concave function Ψ, the inequality (3) will hold in the opposite direction.

Jensen's inequality has a variety of applications in almost all areas of science. Particularly, Jensen's inequality has recorded very good performance in mathematics, statistics and information theory. Due to vast applications of Jensen's inequality, many researchers have worked on this inequality and several important and useful refinements, improvements and generalizations of this inequality have been established in several directions. Ivelić and Pečarić [12] substantiated a generalization for the converse of Jensen's inequality by exploiting a convex function defined on convex hulls and also gave generalizations of the Hermite–Hadamard inequality as a consequence of main results. They also presented some more results, which are actually the generalizations of some existing results. Zabandan and Kilicman [13] utilized a convex function defined on rectangles and established an extension of Jensen's inequality under uniform circumstances and also obtained some more important inequalities. Nakasuji and Takahasi [14] used a convex function from topological abelian semi group to topological ordered abelian semigroup and obtained a finite form of Jensen's inequality. Moreover, they also gave an application of their main result in the form a refinement of mean inequality. Dragomir et al. [15] presented a refinement of Jensen's inequality and its generalization for linear functionals. They also provided some applications of their work in information theory. In 2019, Bibi et al. [16] utilized k-convex functions and obtained several inequalities of the Jensen type and its converses for the diamond integrals. Khan et al. [17] proposed a new method of finding estimates for the Jensen differences by choosing differentiable functions and discussed some improvements of Hermite–Hadamard and Hölder inequalities. They also deliberated inequalities for different means and granted applications of their main results in information theory. In 2021, Deng et al. [5] proved some refinements of Jensen's inequality with the help of majorization results while using the notion of convexity. Furthermore, they provided some refinements for classical inequalities and also presented applications of main results. Bakula and Pečarić [18] used a convex function on rectangles and obtained several inequalities of Jensen's type which are basically the generalizations of some results already subsistent in the literature. For some important literature concerns regarding Jensen's inequality, see [19–21].

2. Main Results

In the present section, we attempt to procure some vital bounds for the Jensen difference. For the obtaining of proposed bounds, we shall use the notion of convexity, Jensen's inequality for concave functions, Hölder and power mean inequalities. Now, we commence this section with the following theorem, in which a bound for the Jensen difference is obtained with the help of Hölder inequality and definition of convex function.

Theorem 3. *Let $\Psi : [a,b] \to \mathbb{R}$ be a twice differentiable function such that $|\Psi''|^q$ be convex for $q > 1$. Additionally, suppose that $x_i \in [a,b]$, $p_i \in \mathbb{R}$, for each $i = 1,2,\cdots,n$ with $P_n := \sum_{i=1}^{n} p_i \neq 0$ and $\overline{x} := \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in [a,b]$. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\overline{x})\right| \leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\frac{(q+1)\left|\Psi''(\overline{x})\right|^q + \left|\Psi''(x_i)\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \quad (4)$$

Proof. Without misfortune of sweeping statement, assume that $\overline{x} \neq x_i$ for all $i = 1,2,\cdots,n$. By using integration by parts, we have

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^{n} p_i(\overline{x}-x_i)^2\int_0^1 t\Psi''(t\overline{x}+(1-t)x_i)dt \\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i(\overline{x}-x_i)^2\left(\frac{t}{\overline{x}-x_i}\Psi'(t\overline{x}+(1-t)x_i)\Big|_0^1 - \frac{1}{\overline{x}-x_i}\int_0^1 \Psi'(t\overline{x}+(1-t)x_i)dt\right) \\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i(\overline{x}-x_i)^2\left(\frac{\Psi'(\overline{x})}{\overline{x}-x_i} - \frac{1}{(\overline{x}-x_i)^2}\Psi(t\overline{x}+(1-t)x_i)\Big|_0^1\right) \\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i(\overline{x}-x_i)^2\left(\frac{\Psi'(\overline{x})}{\overline{x}-x_i} - \frac{1}{(\overline{x}-x_i)^2}\Big(\Psi(\overline{x})-\Psi(x_i)\Big)\right) \\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i(\overline{x}-x_i)\Psi'(\overline{x}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\Big(\Psi(\overline{x})-\Psi(x_i)\Big) \\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\overline{x}),
\end{aligned}$$

which implies that,

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\overline{x}) = \frac{1}{P_n}\sum_{i=1}^{n} p_i(\overline{x}-x_i)^2\int_0^1 t\Psi''(t\overline{x}+(1-t)x_i)dt. \quad (5)$$

Now, taking absolute of (5) and then applying triangular inequality, we get

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\overline{x})\right| \leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\int_0^1 \left|t\Psi''(t\overline{x}+(1-t)x_i)\right|dt. \quad (6)$$

Inequality (6) can also be written as

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\overline{x})\right| \leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\int_0^1 t\left|\Psi''(t\overline{x}+(1-t)x_i)\right|dt. \quad (7)$$

Instantly, applying Hölder inequality on the right-hand side of (6), we obtain

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i)-\Psi(\overline{x})\right| \\
&\qquad \leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\int_0^1 \left|t\Psi''(t\overline{x}+(1-t)x_i)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad = \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\int_0^1 t^q\left|\Psi''(t\overline{x}+(1-t)x_i)\right|^q dt\right)^{\frac{1}{q}}. \qquad (8)
\end{aligned}$$

Thus, the function $|\Psi''|^q$ is convex on $[a,b]$. Therefore, utilizing the definition of convex function on the right side of (8), we obtain

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i)-\Psi(\overline{x})\right| \\
&\quad \leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\int_0^1 t^q\left(t\left|\Psi''(\overline{x})\right|^q+(1-t)\left|\Psi''(x_i)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad = \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\left|\Psi''(\overline{x})\right|^q\int_0^1 t^{q+1}dt+\left|\Psi''(x_i)\right|^q\int_0^1 (t^q-t^{q+1})dt\right)^{\frac{1}{q}} \\
&\quad = \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\frac{\left|\Psi''(\overline{x})\right|^q}{q+2}+\frac{\left|\Psi''(x_i)\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}},
\end{aligned}$$

which is equivalent to (4). □

In the succeeding theorem, another bound for the Jensen difference is acquired by utilizing the definition of convex function and the famous Hölder inequality.

Theorem 4. *Assume that all the hypotheses of Theorem 3 are true. Additionally, if $p>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i)-\Psi(\overline{x})\right| \\
&\qquad \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\frac{\left|\Psi''(\overline{x})\right|^q+\left|\Psi''(x_i)\right|^q}{2}\right)^{\frac{1}{q}}. \qquad (9)
\end{aligned}$$

Proof. Applying Hölder inequality on the right-hand side of (6), we obtain

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i)-\Psi(\overline{x})\right| \\
&\qquad \leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\int_0^1 t^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\Psi''(t\overline{x}+(1-t)x_i)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad = \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\overline{x}-x_i)^2\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left|\Psi''(t\overline{x}+(1-t)x_i)\right|^q dt\right)^{\frac{1}{q}}. \qquad (10)
\end{aligned}$$

Since, the function $|\Psi''|^q$ is convex. Therefore, utilizing the definition of convex function on the right hand side of (10), we obtain

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\bar{x})\right| \\
&\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(|\Psi''(\bar{x})|^q\int_0^1 t dt + |\Psi''(x_i)|^q\int_0^1(1-t)dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\frac{|\Psi''(\bar{x})|^q}{2} + \frac{|\Psi''(x_i)|^q}{2}\right)^{\frac{1}{q}}, \qquad (11)
\end{aligned}$$

which is the required inequality. □

The following bound for the Jensen difference is achieved by exploiting the Hölder inequality the Jensen inequality for concave function.

Theorem 5. *Let $\Psi : [a,b] \to \mathbb{R}$ be a twice differentiable function such that $|\Psi''|^q$ be concave for $q > 1$. Furthermore, assume that $x_i \in [a,b]$, $p_i \in \mathbb{R}$, for each $i = 1,2,\cdots,n$ with $P_n := \sum_{i=1}^{n} p_i \neq 0$ and $\bar{x} := \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in [a,b]$. Then*

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\bar{x})\right| \\
&\qquad = \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\frac{(q+1)\bar{x}+x_i}{q+2}\right)\right|. \qquad (12)
\end{aligned}$$

Proof. Utilizing (8), we can write that:

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\bar{x})\right| \\
&\qquad \leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\int_0^1 t^q|\Psi''(t\bar{x}+(1-t)x_i)|^q dt\right)^{\frac{1}{q}} \\
&\qquad = \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\frac{\int_0^1 t^q|\Psi''(t\bar{x}+(1-t)x_i)|^q dt}{(q+1)\int_0^1 t^q dt}\right)^{\frac{1}{q}}. \qquad (13)
\end{aligned}$$

As, the function $|\Psi'|^q$ is concave. Therefore applying the integral Jensen inequality on the right side of (13), we obtain

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\bar{x})\right| \\
&\leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\frac{1}{q+1}\left|\Psi''\left(\frac{\bar{x}\int_0^1 t^{q+1}dt + x_i\int_0^1(t^q - t^{q+1})dt}{\int_0^1 t^q dt}\right)\right|^q\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\left|\Psi''\left(\frac{\frac{\bar{x}}{q+2} + \frac{x_i}{(q+1)(q+2)}}{\frac{1}{q+1}}\right)\right|^q\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\frac{(q+1)\bar{x}+x_i}{q+2}\right)\right|,
\end{aligned}$$

which is the inequality (12). □

We receive a bound for the Jensen difference, which is given in the next theorem.

Theorem 6. *Let all the conditions of Theorem 5 be satisfied. Moreover, if $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\bar{x})\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\frac{\bar{x}+x_i}{2}\right)\right|. \quad (14)$$

Proof. Since the function $|\Psi''|^q$ is concave. Therefore, applying the integral Jensen inequality on the right-hand side of (10), we get

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\bar{x})\right| \\
&\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\left|\Psi''\left(\int_0^1 \left(t\bar{x}+(1-t)x_i\right)\right)dt\right|^q\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\bar{x}\int_0^1 t\,dt + x_i\int_0^1 (1-t)dt\right)\right| \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\frac{\bar{x}}{2} + x_i(1-\frac{1}{2})dt\right)\right|,
\end{aligned}$$

which is equivalent to (14). □

In the next theorem, we secure a bound for the Jensen difference while utilizing the power mean inequality and definition of convex function.

Theorem 7. *Suppose that all the hypotheses of Theorem 3 are true, then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\bar{x})\right| \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\frac{2|\Psi''(\bar{x})|^q + |\Psi''(x_i)|^q}{6}\right)^{\frac{1}{q}}. \quad (15)$$

Proof. Applying power mean inequality on the right hand side of (7), we obtain

$$\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \Psi(\bar{x})\right| \\
&\leq \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|\Psi''(t\bar{x}+(1-t)x_i)|^q dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\int_0^1 t|\Psi''(t\bar{x}+(1-t)x_i)|^q dt\right)^{\frac{1}{q}}. \quad (16)
\end{aligned}$$

Since, the function $|\Psi''|^q$ is convex. Therefore, utilizing the definition of convex function on the right hand side of (16), we arrive at

$$
\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i)-\Psi(\bar{x})\right| \\
&\quad\leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\int_0^1 t\left|\Psi''(t\bar{x}+(1-t)x_i)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad= \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\int_0^1 t\left(t\left|\Psi''(\bar{x})\right|^q+(1-t)\left|\Psi''(x_i)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad= \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\left|\Psi''(\bar{x})\right|^q\int_0^1 t^2dt+\left|\Psi''(x_i)\right|^q\int_0^1 t(1-t)dt\right)^{\frac{1}{q}} \\
&\quad= \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\frac{\left|\Psi''(\bar{x})\right|^q}{3}+\left|\Psi''(x_i)\right|^q\left(\frac{1}{2}-\frac{1}{3}\right)\right)^{\frac{1}{q}}. \qquad (17)
\end{aligned}
$$

Now, simplifying (17), we obtain (15). □

Another bound for the Jensen difference is acquired with the support of power mean inequality and Jensen inequality for the concave function. This is formally stated in the below theorem.

Theorem 8. *Let all the assumptions of Theorem 5 be valid. Then*

$$
\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i)-\Psi(\bar{x})\right| \leq \frac{1}{2}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\frac{2\bar{x}+x_i}{3}\right)\right|. \qquad (18)
$$

Proof. From inequality (16), we can write that:

$$
\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i)-\Psi(\bar{x})\right| \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\frac{\int_0^1 t\left|\Psi''(t\bar{x}+(1-t)x_i)\right|^q dt}{2\int_0^1 t\,dt}\right)^{\frac{1}{q}}. \qquad (19)
$$

Since, the function $|\Psi''|^q$ is concave. Therefore, applying the integral Jensen inequality on the right-hand side of (19), we obtain

$$
\begin{aligned}
&\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i)-\Psi(\bar{x})\right| \\
&\quad\leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left(\left|\Psi''\left(\frac{\int_0^1 t(t\bar{x}+(1-t)x_i)dt}{\int_0^1 t\,dt}\right)\right|^q\right)^{\frac{1}{q}} \\
&\quad= \frac{1}{2}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\frac{\bar{x}\int_0^1 t^2dt+x_i\int_0^1 t(1-t)dt}{\frac{1}{2}}\right)\right| \\
&\quad= \frac{1}{2}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\frac{\frac{\bar{x}}{3}+\frac{x_i}{6}}{\frac{1}{2}}\right)\right| \\
&\quad= \frac{1}{2}\sum_{i=1}^{n}\left|\frac{p_i}{P_n}\right|(\bar{x}-x_i)^2\left|\Psi''\left(\frac{2\bar{x}+x_i}{3}\right)\right|. \qquad (20)
\end{aligned}
$$

Hence, (18) is proved. □

3. Applications for the Hölder Inequality

The Hölder inequality is one of the distinguished inequalities in the current literature of mathematical inequalities due its vast applications in the fields of pure and applied mathematics. Additionally, due to the structural importance of this inequality many researchers dedicated their work to this fundamental inequality and obtained several extensions, improvements and generalizations in various ways. This section of the paper is devoted to the improvements of the Hölder inequality. The intended improvements shall be obtained by keeping some particular functions in the main results.

We commence this section with the following proposition, in which an improvement of reverse of Hölder inequality is acquired with the help of Theorem 3.

Proposition 1. *Let $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ be two positive n-tuples and $q > 1$. If $\alpha > 1$ with $\alpha \notin (2, 2 + \frac{1}{q})$ and $\beta > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then*

$$\Big(\sum_{i=1}^{n} \zeta_i^{\alpha}\Big)^{\frac{1}{\alpha}} \Big(\sum_{i=1}^{n} \gamma_i^{\beta}\Big)^{\frac{1}{\beta}} - \sum_{i=1}^{n} \gamma_i \zeta_i$$
$$\leq \Bigg[\frac{\Big(\alpha(\alpha-1)\Big)}{\Big((q+1)(q+2)\Big)^{\frac{1}{q}}} \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}} \right)^2$$
$$\times \left((q+1) \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} \right)^{q(\alpha-2)} + \Big(\zeta_i \gamma_i^{-\frac{\beta}{\alpha}} \Big)^{q(\alpha-2)} \right)^{\frac{1}{q}} \Bigg]^{\frac{1}{\alpha}} \Big(\sum_{i=1}^{n} \gamma_i^{\beta} \Big)^{\frac{1}{\beta}}. \tag{21}$$

Proof. Let $\Psi(x) = x^{\alpha}$, $x \in (0, \infty)$. Then $\Psi''(x) = \alpha(\alpha-1)x^{\alpha-2}$ and $\big(|\Psi''(x)|^q\big)'' = q^2\big(\alpha(\alpha-1)\big)^q(\alpha-2)\big((\alpha-2)-1\big)x^{q(\alpha-2)-2}$. Clearly both Ψ'' and $\Big(|\Psi''|^q\Big)''$ are non-negative on $(0, \infty)$. This substantiates the convexity of Ψ and $|\Psi''|^q$. Hence, substitute $\Psi(x) = x^{\alpha}$, $|\Psi''(x)|^q = \big(\alpha(\alpha-1)\big)^q x^{q(\alpha-2)}$, $p_i = \gamma_i^{\beta}$ and $x_i = \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}$ in (4), we receive

$$\Big(\sum_{i=1}^{n} \zeta_i^{\alpha}\Big) \Big(\sum_{i=1}^{n} \gamma_i^{\beta}\Big)^{\alpha-1} - \Big(\sum_{i=1}^{n} \gamma_i \zeta_i\Big)^{\alpha}$$
$$\leq \frac{\alpha(\alpha-1)}{\Big((q+1)(q+2)\Big)^{\frac{1}{q}}} \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}} \right)^2$$
$$\times \left((q+1) \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} \right)^{q-2} + \Big(\zeta_i \gamma_i^{-\frac{\beta}{\alpha}} \Big)^{q-2} \right)^{\frac{1}{q}} \Big(\sum_{i=1}^{n} \gamma_i^{\beta} \Big)^{\alpha-1}. \tag{22}$$

Now, taking power $\frac{1}{\alpha}$ of (22), we acquire

$$\left(\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}-\left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^{\alpha}\right)^{\frac{1}{\alpha}}$$
$$\leq\left[\frac{\alpha(\alpha-1)}{\left((q+1)(q+2)\right)^{\frac{1}{q}}}\sum_{i=1}^{n}\gamma_i^{\beta}\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^{\beta}}-\zeta_i\gamma_i^{-\frac{\beta}{\alpha}}\right)^{2}\right.$$
$$\left.\times\left((q+1)\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^{\beta}}\right)^{q(\alpha-2)}+\left(\zeta_i\gamma_i^{-\frac{\beta}{\alpha}}\right)^{q(\alpha-2)}\right)^{\frac{1}{q}}\right]^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\frac{1}{\beta}}. \tag{23}$$

Since, the inequality

$$c^l-d^l\leq(c-d)^l \tag{24}$$

holds for all c, $d\in(0,\infty)$ and $l\in(0,1)$. Therefore, substituting $c=\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}$, $d=\left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^{\frac{1}{\alpha}}$ and $l=\frac{1}{\alpha}$ in (24), we obtain

$$\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)^{\alpha}\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\beta}-\sum_{i=1}^{n}\gamma_i\zeta_i$$
$$\leq\left(\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}-\left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^{\alpha}\right)^{\frac{1}{\alpha}}. \tag{25}$$

Now, comparing (23) and (25), we get (21). □

We obtain another improvement of the Hölder inequality by taking the particular convex function $\Psi(x)=x^{\frac{1}{\alpha}},\ \ x>0$ in (4).

Corollary 1. *Let $\boldsymbol{m}_1=(\zeta_1,\zeta_2,\cdots,\zeta_n)$, $\boldsymbol{m}_2=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ be two n-tuples such that $\zeta_i,\gamma_i\in(0,\infty)$ for each $i=1,2,\cdots,n$ and $q>1$. If $\alpha\in(0,1)$ with $\frac{1}{\alpha}\notin(2,2+\frac{1}{q})$ and $\beta=\frac{\alpha}{\alpha-1}$, then*

$$\sum_{i=1}^{n}\gamma_i\zeta_i-\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\frac{1}{\beta}}$$
$$\leq\frac{\frac{1}{\alpha}(\frac{1}{\alpha}-1)}{\left((q+1)(q+2)\right)^{\frac{1}{q}}}\sum_{i=1}^{n}\gamma_i^{\beta}\left(\frac{\sum_{i=1}^{n}\zeta_i^{\alpha}}{\sum_{i=1}^{n}\gamma_i^{\beta}}-\zeta_i^{\alpha}\gamma_i^{-\beta}\right)^{2}$$
$$\times\left((q+1)\left(\frac{\sum_{i=1}^{n}\zeta_i^{\alpha}}{\sum_{i=1}^{n}\gamma_i^{\beta}}\right)^{q(\frac{1}{\alpha}-2)}+\left(\zeta_i^{\alpha}\gamma_i^{-\beta}\right)^{q(\frac{1}{\alpha}-2)}\right)^{\frac{1}{q}}. \tag{26}$$

Proof. Consider the function $\Psi(x)=x^{\frac{1}{\alpha}}$, where $x>0$. Then certainly the functions Ψ and $|\Psi''|^q$ both are convex on $(0,\infty)$. Therefore, applying (4) while choosing $\Psi(x)=x^{\frac{1}{\alpha}}$, $|\Psi''(x)|^q=\left(\frac{1}{\alpha}(\frac{1}{\alpha}-1)\right)^q x^{q\left(\frac{1}{\alpha}-2\right)}$, $p_i=\gamma_i^{\beta}$ and $x_i=\zeta_i^{\alpha}\gamma_i^{-\beta}$, we get (26). □

We utilized inequality (9) for a particular convex function and obtain an improvement of the Hölder inequality, which is verbal in the below proposition.

Proposition 2. *Assume that* $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$*,* $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *are two n-tuples such that* $\zeta_i, \gamma_i \in (0, \infty)$ *for each* $i = 1, 2, \cdots, n$ *and* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $\alpha, \beta \in (1, \infty)$ *such that* $\alpha \notin (2, 2 + \frac{1}{q})$ *and* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$*, then*

$$\begin{aligned}
&\left(\sum_{i=1}^{n} \zeta_i^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}} - \sum_{i=1}^{n} \gamma_i \zeta_i \\
&\quad \leq \left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \Big(\alpha(\alpha-1)\Big) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^2 \right. \\
&\qquad \left. \times \left(\frac{\left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}}\right)^{q(\alpha-2)} + \left(\zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^{q(\alpha-2)}}{2}\right)^{\frac{1}{q}}\right]^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}}. \qquad (27)
\end{aligned}$$

Proof. Since the functions $\Psi(x) = x^{\alpha}$ and $|\Psi''(x)|^q = \Big(\alpha(\alpha-1)\Big)^q x^{q(\alpha-2)}$ both are convex on $(0, \infty)$ for the mentioned value of α and q. Therefore, utilizing (9) for $\Psi(x) = x^{\alpha}$, $|\Psi''(x)|^q = (\alpha(\alpha-1))^q x^{q(\alpha-2)}$, $p_i = \gamma_i^{\beta}$ and $x_i = \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}$, and then taking power $\frac{1}{\alpha}$, we acquire

$$\begin{aligned}
&\left(\left(\sum_{i=1}^{n} \zeta_i^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}} - \sum_{i=1}^{n} \gamma_i \zeta_i\right)^{\frac{1}{\alpha}} \\
&\quad \leq \left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \Big(\alpha(\alpha-1)\Big) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^2 \right. \\
&\qquad \left. \times \left(\frac{\left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}}\right)^{q(\alpha-2)} + \left(\zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^{q(\alpha-2)}}{2}\right)^{\frac{1}{q}}\right]^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}}. \qquad (28)
\end{aligned}$$

Now, comparing (28) and (25), we deduce (27). □

The following corollary is the consequence of Theorem 4, in which an improvement of the Hölder inequality is achieved.

Corollary 2. *Assume that,* $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$*,* $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *are two n-tuples such that* $\zeta_i, \gamma_i \in (0, \infty)$ *for each* $i = 1, 2, \cdots, n$ *and* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $\alpha \in (0, 1)$ *with* $\frac{1}{\alpha} \notin \left(2, 2 + \frac{1}{q}\right)$ *and* $\beta = \frac{\alpha}{\alpha - 1}$*, then*

$$\sum_{i=1}^{n} \gamma_i \zeta_i - \left(\sum_{i=1}^{n} \zeta_i^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}} \leq \left(\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)\right) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \zeta_i^{\alpha}}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i^{\alpha} \gamma_i^{-\beta}\right)^2 \times \left(\frac{\left(\frac{\sum_{i=1}^{n} \zeta_i^{\alpha}}{\sum_{i=1}^{n} \gamma_i^{\beta}}\right)^{q\left(\frac{1}{\alpha} - 2\right)} + \left(\zeta_i^{\alpha} \gamma_i^{-\beta}\right)^{q\left(\frac{1}{\alpha} - 2\right)}}{2}\right)^{\frac{1}{q}}. \tag{29}$$

Proof. The functions $\Psi(x) = x^{\frac{1}{\alpha}}$ and $|\Psi''(x)|^q = \left(\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)\right)^q x^{q\left(\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)\right)}$ both are convex on $(0, \infty)$. Therefore, inequality (29) can easily be obtained from inequality (9) by just putting $\Psi(x) = x^{\frac{1}{\alpha}}$, $|\Psi''(x)|^q = \left(\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)\right)^q x^{q\left(\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)\right)}$, $p_i = \gamma_i^{\beta}$ and $x_i = \zeta_i^{\alpha} \gamma_i^{-\beta}$. □

We consider the concave function $\Psi(x) = x^{\alpha}$, $\alpha \in \left(2, 2 + \frac{1}{q}\right)$, $q > 1$ in (12) and receive an improvement of the Hölder inequality, which is given in the coming proposition.

Proposition 3. *Assume that,* $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$*,* $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *are two n-tuples such that* $\zeta_i, \gamma_i \in (0, \infty)$ *for each* $i = 1, 2, \cdots, n$ *and* $q > 1$*. If* $\alpha \in (2, 2 + \frac{1}{q})$ *and* $\beta \in (1, \infty)$ *such that* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$*, then*

$$\sum_{i=1}^{n} \gamma_i \zeta_i - \left(\sum_{i=1}^{n} \zeta_i^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}} \leq \left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\alpha(1 - \alpha)\right) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^2 \times \left(\frac{(q+1)\left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}}\right) + \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}}{q+2}\right)^{(\alpha - 2)}\right]^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}}. \tag{30}$$

Proof. Let $\Psi(x) = x^{\alpha}$, $x \in (0, \infty)$. Then $\Psi''(x) = \alpha(\alpha - 1)x^{\alpha - 2}$ and $\left(|\Psi''(x)|^q\right)'' = q^2\left(\alpha(\alpha - 1)\right)^q (\alpha - 2)\left((\alpha - 2) - 1\right)x^{q(\alpha - 2) - 2}$. Clearly, $\Psi''(x) > 0$ and $\left(|\Psi''|^q(x)\right)'' < 0$ on $(0, \infty)$ for $\alpha \in \left(2, 2 + \frac{1}{q}\right)$. This substantiates the convexity of Ψ and concavity of $|\Psi''|^q$. Therefore, use $\Psi(x) = x^{\alpha}$, $|\Psi''(x)| = \alpha(\alpha - 1)x^{\alpha - 2}$, $p_i = \gamma_i^{\beta}$ and $x_i = \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}$ in (12), we receive

$$\begin{aligned}&\left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^{\alpha}-\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}\\&\quad\leq\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\Big(\alpha(1-\alpha)\Big)\sum_{i=1}^{n}\gamma_i^{\beta}\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^{\beta}}-\zeta_i\gamma_i^{-\frac{\beta}{\alpha}}\right)^2\\&\quad\times\left(\frac{(q+1)\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^{\beta}}\right)+\zeta_i\gamma_i^{-\frac{\beta}{\alpha}}}{q+2}\right)^{(\alpha-2)}\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}.\end{aligned}\tag{31}$$

Now, taking power $\frac{1}{\alpha}$ of (31), we acquire

$$\begin{aligned}&\left(\left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^{\alpha}-\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}\right)^{\frac{1}{\alpha}}\\&\quad\leq\Bigg[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\Big(\alpha(1-\alpha)\Big)\sum_{i=1}^{n}\gamma_i^{\beta}\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^{\beta}}-\zeta_i\gamma_i^{-\frac{\beta}{\alpha}}\right)^2\\&\quad\times\left(\frac{(q+1)\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^{\beta}}\right)+\zeta_i\gamma_i^{-\frac{\beta}{\alpha}}}{q+2}\right)^{(\alpha-2)}\Bigg]^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\frac{1}{\beta}}.\end{aligned}\tag{32}$$

Since, the inequality

$$c^l-d^l\leq(c-d)^l\tag{33}$$

holds for all $c,d\in(0,\infty)$ and $l\in(0,1)$. Therefore, substituting $c=\left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^{\alpha}$, $d=\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}$ and $l=\frac{1}{\alpha}$ in (33), we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\gamma_i\zeta_i-\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)^{\alpha}\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\beta}\\&\quad\leq\left(\left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^{\alpha}-\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}\right)^{\frac{1}{\alpha}}.\end{aligned}\tag{34}$$

Now, comparing (32) and (34), we obtain (30). □

The following is another improvement for the Hölder inequality.

Corollary 3. *Let $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ be two n-tuples such that $\zeta_i, \gamma_i \in (0, \infty)$ for each $i = 1, 2, \cdots, n$ and $q > 1$. If $\alpha \in (0, 1)$ such that $\frac{1}{\alpha} \in \left(2, 2 + \frac{1}{q}\right)$ and $\beta = \frac{\alpha}{\alpha - 1}$. Then*

$$\left(\sum_{i=1}^{n} \zeta_i^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}} - \sum_{i=1}^{n} \gamma_i \zeta_i$$
$$\leq \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum\limits_{i=1}^{n} \zeta_i^{\alpha}}{\sum\limits_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i^{\alpha} \gamma_i^{-\beta}\right)^2$$
$$\times \left(\frac{(q+1)\frac{\sum_{i=1}^{n} \zeta_i^{\alpha}}{\sum_{i=1}^{n} \gamma_i^{\beta}} + \zeta_i^{\alpha}\gamma_i^{-\beta}}{q+2}\right)^{(\frac{1}{\alpha} - 2)}. \tag{35}$$

Proof. Consider the function $\Psi(x) = x^{\frac{1}{\alpha}}$, where $x > 0$. Then certainly the function Ψ is convex and the function $|\Psi''|^q$ is concave on $(0, \infty)$ for $\frac{1}{\alpha} \in \left(2, 2 + \frac{1}{q}\right)$ and $q > 1$. Therefore, applying (12) while choosing $\Psi(x) = x^{\frac{1}{\alpha}}$, $|\Psi''(x)| = \frac{1}{\alpha}(\frac{1}{\alpha} - 1)x^{\frac{1}{\alpha} - 2}$, $p_i = \gamma_i^{\beta}$ and $x_i = \zeta_i^{\alpha}\gamma_i^{-\beta}$, we obtain (35). $\square$

Utilizing the inequality (14), we deduce an improvement of the Hölder inequality, which is stated in the below proposition.

Proposition 4. *Assume that $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ are two n-tuples such that $\zeta_i, \gamma_i \in (0, \infty)$ for each $i = 1, 2, \cdots, n$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $\beta \in (1, \infty)$ and $\alpha \in \left(2, 2 + \frac{1}{q}\right)$, such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then*

$$\sum_{i=1}^{n} \gamma_i \zeta_i - \left(\sum_{i=1}^{n} \zeta_i^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}}$$
$$\leq \left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\alpha(1-\alpha)\right) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum\limits_{i=1}^{n} \gamma_i \zeta_i}{\sum\limits_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^2\right.$$
$$\left. \times \left(\frac{\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} + \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}}{2}\right)^{(\alpha - 2)}\right]^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}}. \tag{36}$$

Proof. Since the function $\Psi(x) = x^\alpha$ is convex and the function $|\Psi''(x)|^q$ is concave on $(0,\infty)$ for the given value of α and q. Therefore, utilizing (14) for $\Psi(x) = x^\alpha$, $|\Psi''(x)| = \big(\alpha(\alpha-1)\big)x^{(\alpha-2)}$, $p_i = \gamma_i^\beta$ and $x_i = \zeta_i\gamma_i^{-\frac{\beta}{\alpha}}$ and then taking power $\frac{1}{\alpha}$, we acquire

$$\left(\sum_{i=1}^n \gamma_i\zeta_i - \left(\sum_{i=1}^n \zeta_i^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^n \gamma_i^\beta\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$$
$$\leq \left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\big(\alpha(1-\alpha)\big)\sum_{i=1}^n \gamma_i^\beta\left(\frac{\sum_{i=1}^n \gamma_i\zeta_i}{\sum_{i=1}^n \gamma_i^\beta} - \zeta_i\gamma_i^{-\frac{\beta}{\alpha}}\right)^2\right.$$
$$\left.\times\left(\frac{\frac{\sum_{i=1}^n \gamma_i\zeta_i}{\sum_{i=1}^n \gamma_i^\beta} + \zeta_i\gamma_i^{-\frac{\beta}{\alpha}}}{2}\right)^{(\alpha-2)}\right]^{\frac{1}{\alpha}}\left(\sum_{i=1}^n \gamma_i^\beta\right)^{\frac{1}{\beta}}. \tag{37}$$

To deduce inequality (36), compare (37) and (34). □

As a consequence of Theorem 6, we obtain the following improvement of the Hölder inequality.

Corollary 4. *Assume that* $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *are two n-tuples such that* $\zeta_i, \gamma_i \in (0,\infty)$ *for each* $i = 1, 2, \cdots, n$ *and ,* p, $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $\alpha \in (0,1)$ *such that* $\frac{1}{\alpha} \in \left(2, 2+\frac{1}{q}\right)$ *and* $\beta = \frac{\alpha}{\alpha-1}$ *satisfy* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, *then*

$$\left(\sum_{i=1}^n \zeta_i^\alpha\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^n \gamma_i^\beta\right)^{\frac{1}{\beta}} - \sum_{i=1}^n \gamma_i\zeta_i$$
$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)\right)\sum_{i=1}^n \gamma_i^\beta\left(\frac{\sum_{i=1}^n \zeta_i^\alpha}{\sum_{i=1}^n \gamma_i^\beta} - \zeta_i\gamma_i^{-\frac{\beta}{\alpha}}\right)^2$$
$$\times\left(\frac{\frac{\sum_{i=1}^n \zeta_i^\alpha}{\sum_{i=1}^n \gamma_i^\beta} + \zeta_i\gamma_i^{-\frac{\beta}{\alpha}}}{2}\right)^{\left(\frac{1}{\alpha}-2\right)}. \tag{38}$$

Proof. The function $\Psi(x) = x^{\frac{1}{\alpha}}$ is convex and the function $|\Psi''(x)|^q$ is concave on $(0,\infty)$ for the given value of α and q. Therefore, utilizing (14) for $\Psi(x) = x^{\frac{1}{\alpha}}$, $|\Psi''(x)| = \frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)x^{\frac{1}{\alpha}-2}$, $p_i = \gamma_i^\beta$ and $x_i = \zeta_i^\alpha\gamma_i^{-\beta}$, we acquire (38). □

In the following proposition, we acquire a relation with the help of Theorem 7 which gives an improvement of the Hölder inequality.

Proposition 5. *Assume that* $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *are two n-tuples such that* $\zeta_i, \gamma_i \in (0, \infty)$ *for each* $i = 1, 2, \cdots, n$ *and* $q > 1$. *If* $\alpha,\ \beta \in (1, \infty)$ *and* $\alpha \notin \left(2, 2 + \frac{1}{q}\right)$, *such that* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, *then*

$$\begin{aligned}
&\left(\sum_{i=1}^{n} \zeta_i^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}} - \sum_{i=1}^{n} \gamma_i \zeta_i \\
&\quad \leq \left[\left(\frac{1}{2}\right)^{\frac{1}{p}} \Big(\alpha(\alpha-1)\Big) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^2 \right. \\
&\quad\quad \left. \times \left(\frac{2\left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}}\right)^{q(\alpha-2)} + \left(\zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^{q(\alpha-2)}}{6}\right)^{\frac{1}{q}}\right]^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}}. \qquad (39)
\end{aligned}$$

Proof. Let us take the function $\Psi(x) = x^{\alpha}$, where $x > 0$. Then, clearly, both the functions $\Psi(x)$ and $|\Psi''(x)|^q$ are convex for the specified values of α and q. Therefore, utilizing (15) by choosing $\Psi(x) = x^{\alpha}$, $|\Psi''(x)|^q = \big(\alpha(\alpha-1)\big)^q x^{q(\alpha-2)}$, $p_i = \gamma_i^{\beta}$ and $x_i = \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}$ and then taking power $\frac{1}{\alpha}$, we get

$$\begin{aligned}
&\left(\left(\sum_{i=1}^{n} \zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\alpha-1} - \left(\sum_{i=1}^{n} \gamma_i \zeta_i\right)^{\alpha}\right)^{\frac{1}{\alpha}} \\
&\quad \leq \left[\left(\frac{1}{2}\right)^{\frac{1}{p}} \Big(\alpha(\alpha-1)\Big) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^2 \right. \\
&\quad\quad \left. \times \left(\frac{2\left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}}\right)^{q(\alpha-2)} + \left(\zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^{q(\alpha-2)}}{6}\right)^{\frac{1}{q}}\right]^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} \gamma_i^{\beta}\right)^{\frac{1}{\beta}}. \qquad (40)
\end{aligned}$$

Instantly, comparing (40) and (25), we obtain (39). □

The next corollary provides an improvement of the Hölder inequality, which has been deduced from Theorem 7.

Corollary 5. *Assume that $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ are two n-tuples such that $\zeta_i, \gamma_i \in (0, \infty)$ for each $i = 1, 2, \cdots, n$ and $q > 1$. If $\alpha \in (0, 1)$ with $\frac{1}{\alpha} \notin \left(2, 2 + \frac{1}{q}\right)$ and $\beta = \frac{\alpha}{\alpha - 1}$, then*

$$\begin{aligned}
&\sum_{i=1}^{n} \gamma_i \zeta_i - \Big(\sum_{i=1}^{n} \zeta_i^{\alpha}\Big)^{\frac{1}{\alpha}} \Big(\sum_{i=1}^{n} \gamma_i^{\beta}\Big)^{\frac{1}{\beta}} \\
&\leq \left(\frac{1}{2}\right)^{\frac{1}{p}} \Big(\frac{1}{\alpha}\big(\frac{1}{\alpha} - 1\big)\Big) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \zeta_i^{\alpha}}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i^{\alpha} \gamma_i^{-\beta}\right)^2 \\
&\times \left(\frac{2\left(\frac{\sum_{i=1}^{n} \zeta_i^{\alpha}}{\sum_{i=1}^{n} \gamma_i^{\beta}}\right)^{q(\frac{1}{\alpha} - 2)} + \left(\zeta_i^{\alpha} \gamma_i^{-\beta}\right)^{q(\frac{1}{\alpha} - 2)}}{6}\right)^{\frac{1}{q}}. \qquad (41)
\end{aligned}$$

Proof. Inequality (41) can easily be acquired by putting $\Psi(x) = x^{\frac{1}{\alpha}}$, $|\Psi''(x)|^q = (\frac{1}{\alpha}(\frac{1}{\alpha} - 1))^q x^{q\left(\frac{1}{\alpha} - 2\right)}$, $p_i = \gamma_i^{\beta}$ and $x_i = \zeta_i^{\alpha} \gamma_i^{-\beta}$ in (15). □

We use Theorem 8 and achieve the following improvement on the Hölder inequality.

Proposition 6. *Assume that $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, $\boldsymbol{m}_2 = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ are two n-tuples such that $\zeta_i, \gamma_i \in (0, \infty)$ for each $i = 1, 2, \cdots, n$ and $q > 1$. Also, let $\alpha \in \left(2, 2 + \frac{1}{q}\right)$ and $\beta \in (1, \infty)$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then*

$$\begin{aligned}
&\sum_{i=1}^{n} \gamma_i \zeta_i - \Big(\sum_{i=1}^{n} \zeta_i^{\alpha}\Big)^{\frac{1}{\alpha}} \Big(\sum_{i=1}^{n} \gamma_i^{\beta}\Big)^{\frac{1}{\beta}} \\
&\leq \left[\frac{1}{2}\Big(\alpha(1 - \alpha)\Big) \sum_{i=1}^{n} \gamma_i^{\beta} \left(\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} - \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}\right)^2 \right. \\
&\left. \times \left(\frac{2\frac{\sum_{i=1}^{n} \gamma_i \zeta_i}{\sum_{i=1}^{n} \gamma_i^{\beta}} + \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}}{3}\right)^{(\alpha - 2)}\right]^{\frac{1}{\alpha}} \Big(\sum_{i=1}^{n} \gamma_i^{\beta}\Big)^{\frac{1}{\beta}}. \qquad (42)
\end{aligned}$$

Proof. Let take the function $\Psi(x) = x^{\alpha}$, where $x > 0$. Then, clearly, $\Psi'' > 0$ and $\left(|\Psi''|^q\right)'' < 0$ on $(0, \infty)$ for the specified values of α and q. This confirms the convexity of Ψ and concavity of $|\Psi''|^q$. Therefore, utilizing (18) by choosing $\Psi(x) = x^{\alpha}$, $|\Psi''(x)| = \alpha(\alpha - 1)\, x^{\alpha - 2}$, $p_i = \gamma_i^{\beta}$ and $x_i = \zeta_i \gamma_i^{-\frac{\beta}{\alpha}}$ and then taking power $\frac{1}{\alpha}$, we get

$$\left(\left(\sum_{i=1}^{n}\gamma_i\zeta_i\right)^{\alpha}-\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\alpha-1}\right)^{\frac{1}{\alpha}}$$
$$\leq\left[\frac{1}{2}\Big(\alpha(1-\alpha)\Big)\sum_{i=1}^{n}\gamma_i^{\beta}\left(\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^{\beta}}-\zeta_i\gamma_i^{-\frac{\beta}{\alpha}}\right)^2\right.$$
$$\left.\times\left(\frac{2\frac{\sum_{i=1}^{n}\gamma_i\zeta_i}{\sum_{i=1}^{n}\gamma_i^{\beta}}+\zeta_i\gamma_i^{-\frac{\beta}{\alpha}}}{3}\right)^{\alpha-2}\right]^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\frac{1}{\beta}}. \tag{43}$$

Instantly, comparing (43) and (34), we obtain (42). □

Another improvement of the Hölder inequality is given in the next corollary.

Corollary 6. *Assume that $\boldsymbol{m}_1=(\zeta_1,\zeta_2,\cdots,\zeta_n)$, $\boldsymbol{m}_2=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ are two n-tuples such that $\zeta_i,\gamma_i\in(0,\infty)$ for each $i=1,2,\cdots,n$ and $q>1$. If $\alpha\in(0,1)$ such that $\frac{1}{\alpha}\in\left(2,2+\frac{1}{q}\right)$ and $\beta=\frac{\alpha}{\alpha-1}$, then*

$$\left(\sum_{i=1}^{n}\zeta_i^{\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\gamma_i^{\beta}\right)^{\frac{1}{\beta}}-\sum_{i=1}^{n}\gamma_i\zeta_i$$
$$\leq\frac{1}{2}\left(\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)\right)\sum_{i=1}^{n}\gamma_i^{\beta}\left(\frac{\sum_{i=1}^{n}\zeta_i^{\alpha}}{\sum_{i=1}^{n}\gamma_i^{\beta}}-\zeta_i^{\alpha}\gamma_i^{-\beta}\right)^2$$
$$\times\left(\frac{2\frac{\sum_{i=1}^{n}\zeta_i^{\alpha}}{\sum_{i=1}^{n}\gamma_i^{\beta}}+\zeta_i^{\alpha}\gamma_i^{-\beta}}{3}\right)^{\left(\frac{1}{\alpha}-2\right)}. \tag{44}$$

Proof. Inequality (44) can easily be acquired by putting $\Psi(x)=x^{\frac{1}{\alpha}}$, $|\Psi''(x)|=\frac{1}{\alpha}(\frac{1}{\alpha}-1)x^{\frac{1}{\alpha}-2}$, $p_i=\gamma_i^{\beta}$ and $x_i=\zeta_i^{\alpha}\gamma_i^{-\beta}$ in (18). □

4. Applications for Means

The significance of means has been fully accepted since 1930 and a number of researchers then gave full attention to the properties and applications of means [22,23]. A variety of article devoted to means in which they are studied very deeply in all directions. Recently, many mathematical inequalities have been published for different means and then these inequalities are generalized, extended and improved in several direction while utilizing some approaches and techniques [24,25]. This section of the paper is devoted to power and quasi-arithmetic means. Here, we will establish a number inequalities for the mentioned means with the help of our main results. The proposed inequalities will be acquired by putting some particular functions in the main results.

Now, we commence this section with the definition of power mean.

Definition 1. *Let* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ *and* $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ *be two n-tuples with* $P_n := \sum_{i=1}^{n} p_i$. *Then the power mean of order* $r \in \mathbb{R}$ *is defined by*

$$M_r(\boldsymbol{p};\boldsymbol{x}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0. \end{cases}$$

Instantly, we give some inequalities for the power mean with the support of Theorem 3.

Proposition 7. *Assume that* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ *and* $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ *are two n-tuples such that* $x_i, p_i > 0$ *for all* $i \in \{1, 2, \cdots, n\}$ *with* $P_n := \sum_{i=1}^{n} p_i$. *Additionally, suppose that* $q > 1$ *and* r, t *are non-zero real numbers such that* $t < r$, *then the following statements are true:*
(i) *If both r and t are positive, then*

$$M_r^t(\boldsymbol{p};\boldsymbol{x}) - M_t^t(\boldsymbol{p};\boldsymbol{x}) \leq \frac{t(r-t)}{r^2 P_n}\sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{(q+1)M_r^{q(t-2r)}(\boldsymbol{p};\boldsymbol{x}) + x_i^{q(t-2r)}}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \tag{45}$$

(ii) *If both r and t are negative with* $\frac{t}{r} \notin \left(2, 2 + \frac{1}{q}\right)$, *then*

$$M_t^t(\boldsymbol{p};\boldsymbol{x}) - M_r^t(\boldsymbol{p};\boldsymbol{x}) \leq \frac{t(t-r)}{r^2 P_n}\sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{(q+1)M_r^{q(t-2r)}(\boldsymbol{p};\boldsymbol{x}) + x_i^{q(t-2r)}}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \tag{46}$$

(iii) *If r is positive and t is negative, then* (46) *holds.*

Proof. First, we prove (i). For this, consider $\Psi(x) = x^{\frac{t}{r}}$, $(x > 0)$, then $\Psi''(x) = \frac{t}{r}\left(\frac{t}{r} - 1\right)x^{\frac{t}{r}-2}$ and $\left(|\Psi''(x)|^q\right)'' = \left(|\frac{t}{r}|(|\frac{t}{r} - 1|)\right)^q q(\frac{t}{r} - 2)\left(q(\frac{t}{r} - 2) - 1\right)x^{q\left(\frac{t}{r}-2\right)-2}$. Clearly for the mentioned values of r, t and q, $\Psi''(x) \leq 0$ and $\left(|\Psi''(x)|^q\right)'' \geq 0$, for all $x > 0$. This explains that $\Psi(x)$ is a concave function and $|\Psi''(x)|^q$ is a convex function. Therefore, utilizing (4) for $\Psi(x) = x^{\frac{t}{r}}$ and $x_i = x_i^r$, we get (45).

Now, we prove the case (ii). For the specified values of r, t and q both the functions $\Psi(x)$ and $|\Psi''(x)|^q$ are convex for all $x > 0$. Therefore, by copying the procedure of (i), one can obtain easily the inequality (46).

Instantly, we prove the last case. For the stated conditions on r, t and q both $\Psi(x)$ and $|\Psi''(x)|^q$ are convex functions. Therefore, inequality (46) can easily be achieved by just adopting the procedure of case (ii). □

As a consequence of Theorem 4, we establish some new inequalities for the difference of two power means, which are disposed in the following proposition.

Proposition 8. *Assume that* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ *and* $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ *are two positive n-tuples with* $P_n := \sum_{i=1}^{n} p_i$. *Also, let* p, $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* r, t *be non-zero real numbers*

with $t < r$. Then the following statements are true:
(i) If both r and t are positive, then

$$M_r^t(\boldsymbol{p};\boldsymbol{x}) - M_t^t(\boldsymbol{p};\boldsymbol{x}). \le \frac{t(r-t)}{r^2 P_n}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{M_r^{q(t-2r)}(\boldsymbol{p};\boldsymbol{x}) + x_i^{q(t-2r)}}{2}\right)^{\frac{1}{q}}. \tag{47}$$

(ii) If both r and t are negative with $\frac{t}{r} \notin \left(2, 2+\frac{1}{q}\right)$, then

$$M_t^t(\boldsymbol{p};\boldsymbol{x}) - M_r^t(\boldsymbol{p};\boldsymbol{x}) \le \frac{t(t-r)}{r^2 P_n}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{M_r^{q(t-2r)}(\boldsymbol{p};\boldsymbol{x}) + x_i^{q(t-2r)}}{2}\right)^{\frac{1}{q}}. \tag{48}$$

(iii) If r is positive and t is negative, then (48) holds.

Proof. (i) Let $\Psi(x) = x^{\frac{t}{r}}, \ x > 0$. Then for the given value of r, t and q, $\Psi'' \le 0$ and $\left(|\Psi''(x)|^q\right)'' \ge 0$. This confirm that the function $\Psi(x)$ is concave and the function $|\Psi''(x)|^q$ is convex . Therefore, utilizing (9) for $\Psi(x) = x^{\frac{t}{r}}$ and $x_i = x_i^r$, we obtain (47).

(ii) Recently, we have proven the second case. Both the functions $\Psi(x)$ and $|\Psi''(x)|^q$ are convex for the mentioned values of r, t and q. Therefore, following the procedure of (i), we obtain (48).

(iii) Now, we proceed for the third case. The functions $\Psi(x)$ and $|\Psi''(x)|^q$ are convex with the specified conditions. Therefore, adopting the method of (ii), we acquire (48). □

The following is the consequence of Theorem 5, in which we obtain an inequality for power means.

Proposition 9. *Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ be two n-tuples such that $x_i, p_i > 0$ for all $i \in \{1, 2, \cdots, n\}$ with $P_n := \sum_{i=1}^{n} p_i$. Also, let r and t be negative real numbers such that $t < r$. If $q > 1$ and $\frac{t}{r} \in \left(2, 2+\frac{1}{q}\right)$, then*

$$M_t^t(\boldsymbol{p};\boldsymbol{x}) - M_r^t(\boldsymbol{p};\boldsymbol{x}) \le \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \frac{t(t-r)}{r^2 P_n} \sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{(q+1)M_r^r(\boldsymbol{p};\boldsymbol{x}) + x_i^r}{(q+2)}\right)^{\frac{t}{r}-2}. \tag{49}$$

Proof. Consider the function $\Psi(x) = x^{\frac{t}{r}}$ defined on $(0, \infty)$, then certainly $\Psi(x)$ is convex and $|\Psi''(x)|^q$ is concave on $(0, \infty)$ for given values of r, t and q. Therefore, using (12) for $\Psi(x) = x^{\frac{t}{r}}$ and $x_i = x_i^r$, we get (49). □

The following proposition is the consequence of Theorem 6 in which we obtain a relation for the power means.

Proposition 10. *Assume that* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ *and* $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ *are two n-tuples such that* $x_i, p_i > 0$ *for all* $i \in \{1, 2, \cdots, n\}$ *with* $P_n := \sum_{i=1}^{n} p_i$*. Additionally, let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* r, t *be negative real numbers with* $t < r$*. If* $\frac{t}{r} \in \left(2, 2 + \frac{1}{q}\right)$*, then*

$$M_t^t(\boldsymbol{p};\boldsymbol{x}) - M_r^t(\boldsymbol{p};\boldsymbol{x}) \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \frac{t(t-r)}{r^2 P_n} \sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{M_r^r(\boldsymbol{p};\boldsymbol{x}) + x_i^r}{2}\right)^{\frac{t}{r}-2}. \tag{50}$$

Proof. Let $\Psi(x) = x^{\frac{t}{r}},\ x > 0$. Then obviously $\Psi''(x) \geq 0$ and $\left(|\Psi''(x)|^q\right)'' \leq 0$ on $(0, \infty)$ for the given values of r, t and q. This explains that the function Ψ is convex and function $|\Psi''(x)|^q$ is concave. Therefore, utilizing (14) for $\Psi(x) = x^{\frac{t}{r}}$ and $x_i = x_i^r$, we obtain (50). □

In the below proposition, some relations for the power means is achieved by utilizing Theorem 7.

Proposition 11. *Suppose that* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ *and* $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ *are positive n-tuples with* $P_n := \sum_{i=1}^{n} p_i$*. Additionally, assume that* $q > 1$ *and* r, t *are non-zero real numbers such that* $t < r$*. Then the following statements are valid:*
(i) *If both r and t are positive, then*

$$M_r^t(\boldsymbol{p};\boldsymbol{x}) - M_t^t(\boldsymbol{p};\boldsymbol{x}) \leq \frac{t(r-t)}{r^2 P_n} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{2M_r^{q(t-2r)}(\boldsymbol{p};\boldsymbol{x}) + x_i^{q(t-2r)}}{3}\right)^{\frac{1}{q}}. \tag{51}$$

(ii) *If both r and t are negative with* $\frac{t}{r} \neq \left(2, 2 + \frac{1}{q}\right)$*, then*

$$M_t^t(\boldsymbol{p};\boldsymbol{x}) - M_r^t(\boldsymbol{p};\boldsymbol{x}) \leq \frac{t(t-r)}{r^2 P_n} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{2M_r^{q(t-2r)}(\boldsymbol{p};\boldsymbol{x}) + x_i^{q(t-2r)}}{3}\right)^{\frac{1}{q}}. \tag{52}$$

(iii) *If r is positive and t is negative, then* (52) *holds.*

Proof. (i) The function $\Psi(x) = x^{\frac{t}{r}}$ is concave and the function $|\Psi''(x)|^q = \left(|\frac{t}{r}|(|\frac{t}{r} - 1|)\right)^q x^{q\left(\frac{t}{r}-2\right)}$ is convex on $(0, \infty)$ for the given values of r, t and q. Therefore, applying inequality (15) by choosing $\Psi(x) = x^{\frac{t}{r}}$ and $x_i = x_i^r$, we obtain (51).

(ii) For the mentioned values of r, t and q both the functions $\Psi(x) = x^{\frac{t}{r}}$ and $|\Psi''(x)|^q = \left(|\frac{t}{r}|(|\frac{t}{r} - 1|)\right)^q x^{q\left(\frac{t}{r}-2\right)}$ are convex. Therefore, inequality (52) can easily be deduced from (15) by picking $\Psi(x) = x^{\frac{t}{r}}$ and $x_i = x_i^r$.

(iii) For the given values of r, t and q both the functions $\Psi(x) = x^{\frac{t}{r}}$ and $|\Psi''(x)|^q = \left(|\frac{t}{r}|(|\frac{t}{r}-1|)\right)^q x^{q\left(\frac{t}{r}-2\right)}$ are convex. Therefore, following case (ii), we acquire the required inequality. □

The following is the consequence of Theorem 8 for power means.

Proposition 12. *Assume that $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ are two n-tuples such that $x_i, p_i > 0$ for all $i \in \{1, 2, \cdots, n\}$ with $P_n = \sum\limits_{i=1}^{n} p_i$. Additionally, let r and t are negative real numbers such that $t < r$. If $q > 1$ and $\frac{t}{r} \in \left(2, 2 + \frac{1}{q}\right)$, then*

$$M_t^t(\boldsymbol{p};\boldsymbol{x}) - M_r^t(\boldsymbol{p};\boldsymbol{x}) \leq \frac{t(t-r)}{2r^2 P_n} \sum_{i=1}^{n} p_i \left(M_r^r(\boldsymbol{p};\boldsymbol{x}) - x_i^r\right)^2 \times \left(\frac{2M_r^r(\boldsymbol{p};\boldsymbol{x}) + x_i^r}{3}\right)^{\frac{t}{r}-2}. \tag{53}$$

Proof. Consider the function $\Psi(x) = x^{\frac{t}{r}}$ defined on $(0, \infty)$, then certainly Ψ is convex and $|\Psi''(x)|^q$ is concave on $(0, \infty)$ for given values of r, t and q. Therefore, using (18) for $\Psi(x) = x^{\frac{t}{r}}$ and $x_i = x_i^r$, we get (53). □

In the rest of this section, we shall discuss some interesting consequences of our main results for quasi–arithmetic mean. These consequences shall provide different estimates for quasi-arithmetic mean.

Definition 2. *Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ be two positive n-tuples with $P_n := \sum\limits_{i=1}^{n} p_i$. If the function ϕ is both continuous and strictly monotonic, then quasi-arithmetic mean is defined by:*

$$M_\phi(\boldsymbol{p}, \boldsymbol{x}) = \phi^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \phi(x_i)\right).$$

Now, let us initiate with the following result in which a relation for the quasi mean is secured while utilizing Theorem 3.

Corollary 7. *Assume that $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ are positive n-tuples with $P_n := \sum\limits_{i=1}^{n} p_i$ and ϕ is a strictly monotonic continuous function. Additionally, suppose that $q > 1$ and $\Psi \circ \phi^{-1}$ is a twice differentiable function such that $|(\Psi \circ \phi)''|^q$ is convex, then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \Psi(x_i) - \Psi(M_\phi(\boldsymbol{p}, \boldsymbol{x}))\right| \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \left(\phi(M_\phi(\boldsymbol{p}, \boldsymbol{x})) - \phi(x_i)\right)^2 \times \left(\frac{(q+1)|(\Psi \circ \phi^{-1})''(\phi(M_\phi(\boldsymbol{p}, \boldsymbol{x})))|^q + |(\Psi \circ \phi^{-1})''(\phi(x_i))|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \tag{54}$$

Proof. Inequality (54) can easily be deduced by taking $\Psi = \Psi \circ \phi^{-1}$ and $x_i = \phi(x_i)$ in (4). □

A relation for the quasi mean is achieved with the support of Theorem 4, which is given in the following corollary.

Corollary 8. *Assume that $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ are positive n-tuples with $P_n := \sum_{i=1}^{n} p_i$ and ϕ is a strictly monotonic continuous function. Additionally, let $\Psi \circ \phi^{-1}$ be a function such that its double derivative exists. If $q, p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and the function $|(\Psi \circ \phi)''|^q$ is convex, then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \Psi(x_i) - \Psi(M_\phi(\boldsymbol{p},\boldsymbol{x}))\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \frac{1}{P_n}\sum_{i=1}^{n} p_i \Big(\phi(M_\phi(\boldsymbol{p},\boldsymbol{x})) - \phi(x_i)\Big)^2 \times \left(\frac{|(\Psi \circ \phi^{-1})''(\phi(M_\phi(\boldsymbol{p},\boldsymbol{x})))|^q + |(\Psi \circ \phi^{-1})''(\phi(x_i))|^q}{2}\right)^{\frac{1}{q}}. \tag{55}$$

Proof. To acquire (55), just put $\Psi = \Psi \circ \phi^{-1}$ and $x_i = \phi(x_i)$ in (9). □

As a consequence of Theorem 5, we give the following relation for quasi mean.

Corollary 9. *Assume that $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ are positive n-tuples with $P_n := \sum_{i=1}^{n} p_i$ and ϕ is a strictly monotonic continuous function. Additionally, let $\Psi \circ \phi^{-1}$ be a twice differentiable function such that $|(\Psi \circ \phi)''|^q$ is concave for $q > 1$. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \Psi(x_i) - \Psi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big)\right| \leq \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \frac{1}{P_n}\sum_{i=1}^{n} p_i \Big(\phi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big) - \phi(x_i)\Big)^2 \times \left|(\Psi \circ \phi^{-1})''\left(\frac{(q+1)\phi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big) + \phi(x_i)}{q+2}\right)\right|. \tag{56}$$

Proof. Utilize (12) for $\Psi = \Psi \circ \phi^{-1}$ and $x_i = \phi(x_i)$, we get (56). □

We present a consequence of Theorem 6 in next corollary in the form of bound for the quasi mean.

Corollary 10. *Assume that $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ are positive n-tuples with $P_n := \sum_{i=1}^{n} p_i$ and ϕ is a strictly monotonic continuous function. Furthermore, let $q, p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\Psi \circ \phi^{-1}$ is a twice differentiable function such that $|(\Psi \circ \phi)''|^q$ be concave. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \Psi(x_i) - \Psi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big)\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \frac{1}{P_n}\sum_{i=1}^{n} p_i \Big(\phi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big) - \phi(x_i)\Big)^2 \times \left|(\Psi \circ \phi^{-1})''\left(\frac{\phi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big) + \phi(x_i)}{2}\right)\right|. \tag{57}$$

Proof. Applying (14) while choosing $\Psi = \Psi \circ \phi^{-1}$ and $x_i = \phi(x_i)$, we obtain (57). □

We extract the following result from Theorem 7 for the quasi mean.

Corollary 11. *Assume ϕ is a strictly monotonic continuous function and $\Psi\circ\phi^{-1}$ is a twice differentiable function such that $|(\Psi\circ\phi)''|^q$ is convex for $q>1$. Additionally, suppose that $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$ and $\boldsymbol{p}=(p_1,p_2,\cdots,p_n)$ are positive n-tuples with $P_n:=\sum_{i=1}^{n}p_i$. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n}p_i\Psi(x_i)-\Psi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big)\right|\leq\frac{1}{P_n}\sum_{i=1}^{n}p_i\Big(\phi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big)-\phi(x_i)\Big)^2$$
$$\times\left(\frac{2\left|\left(\Psi\circ\phi^{-1}\right)''\Big(\phi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big)\Big)\right|^q+\left|\left(\Psi\circ\phi^{-1}\right)''(\phi(x_i))\right|^q}{6}\right)^{\frac{1}{q}}. \tag{58}$$

Proof. Inequality (58) can easily be obtained by putting $\Psi=\Psi\circ\phi^{-1}$ and $x_i=\phi(x_i)$ in (15). $\square$

In the below corollary, we obtain a bound for the quasi mean as a consequence of Theorem 8.

Corollary 12. *Assume that the function ϕ is a strictly monotonic and continuous and $\Psi\circ\phi^{-1}$ is a twice differentiable function such that $|(\Psi\circ\phi)''|^q$ is convex for $q>1$. Let also $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$ and $\boldsymbol{p}=(p_1,p_2,\cdots,p_n)$ be positive n-tuples with $P_n:=\sum_{i=1}^{n}p_i$. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n}p_i\Psi(x_i)-\Psi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big)\right|$$
$$\leq\frac{1}{P_n}\sum_{i=1}^{n}p_i\Big(\phi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big)-\phi(x_i)\Big)^2$$
$$\times\left|\left(\Psi\circ\phi^{-1}\right)''\left(\frac{2\phi\Big(M_\phi(\boldsymbol{p},\boldsymbol{x})\Big)+\phi(x_i)}{3}\right)\right|. \tag{59}$$

Proof. By taking $\Psi=\Psi\circ\phi^{-1}$ and $x_i=\phi(x_i)$ in (18), we obtain (59). $\square$

5. Applications in Information Theory

This section is dedicated to the applications of main results. The applications shall be discuss in information theory. These applications shall provide different bounds for Csiszár and Rényi divergences, Shannon entropy and Bhattacharyya coefficient.

Now, let us first recall the definition of Csiszár divergence.

Definition 3. *Let $\Psi:I\to\mathbb{R}$ be a function and $\boldsymbol{r}=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ and $\boldsymbol{z}=(\zeta_1,\zeta_2,\cdots,\zeta_n)$ be positive tuples such that $\frac{\zeta_i}{\gamma_i}\in[a,b]$ for all $i\in\{1,2,\cdots,n\}$. Then the Csiszár divergence is defined by*

$$\widetilde{D}_c(\boldsymbol{r},\boldsymbol{z})=\sum_{i=1}^{n}\gamma_i\Psi\left(\frac{\zeta_i}{\gamma_i}\right).$$

In the next theorem, we obtain a bound for the Csiszár divergence while using Theorem 3.

Theorem 9. *Assume that $\Psi : I \to \mathbb{R}$ is a twice differentiable function such that $|\Psi''(x)|^q$ is convex for $q > 1$. Also, let $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $\boldsymbol{z} = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be positive tuples with $\overline{\gamma} := \sum_{i=1}^{n} \gamma_i$ and $\overline{\zeta} := \sum_{i=1}^{n} \zeta_i$. If $\frac{\zeta_i}{\gamma_i}, \frac{\overline{\zeta}}{\overline{\gamma}} \in [a, b]$. then*

$$\left|\frac{1}{\overline{\gamma}}\widetilde{D}_c(\boldsymbol{r}, \boldsymbol{z}) - \Psi\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right| \leq \frac{1}{\overline{\gamma}}\sum_{i=1}^{n}\gamma_i\left(\frac{\overline{\zeta}}{\overline{\gamma}} - \frac{\zeta_i}{\gamma_i}\right)^2 \times \left(\frac{(q+1)\left|\Psi''\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right|^q + \left|\Psi''\left(\frac{\zeta_i}{\gamma_i}\right)\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \tag{60}$$

Proof. Inequality (60) can easily be acquired by putting $p_i = \frac{\gamma_i}{\overline{\gamma}}$ and $x_i = \frac{\zeta_i}{\gamma_i}$ in (4). □

The following result is an application of Theorem 4 for Csiszáar divergence.

Theorem 10. *Assume that, all the postulates of Theorem 9 are true. Moreover, if $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{1}{\overline{\gamma}}\widetilde{D}_c(\boldsymbol{r}, \boldsymbol{z}) - \Psi\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{1}{\overline{\gamma}}\sum_{i=1}^{n}\gamma_i\left(\frac{\overline{\zeta}}{\overline{\gamma}} - \frac{\zeta_i}{\gamma_i}\right)^2 \times \left(\frac{\left|\Psi''\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right|^q + \left|\Psi''\left(\frac{\zeta_i}{\gamma_i}\right)\right|^q}{2}\right)^{\frac{1}{q}}. \tag{61}$$

Proof. Utilizing (9) while choosing $p_i = \frac{\gamma_i}{\overline{\gamma}}$ and $x_i = \frac{\zeta_i}{\gamma_i}$, we obtain (61). □

Application of Theorem 5 is given in the coming results in which we get a bound for Csiszár divergence.

Theorem 11. *Let $\Psi : I \to \mathbb{R}$ be a twice differentiable function such that $|\Psi''(x)|^q$ be concave for $q > 1$. Assume also that $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $\boldsymbol{z} = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ are n-tuples with positive entries, $\overline{\gamma} := \sum_{i=1}^{n} \gamma_i$ and $\overline{\zeta} := \sum_{i=1}^{n} \zeta_i$. If $\frac{\zeta_i}{\gamma_i}, \frac{\overline{\zeta}}{\overline{\gamma}} \in [a, b]$, then*

$$\left|\frac{1}{\overline{\gamma}}\widetilde{D}_c(\boldsymbol{r}, \boldsymbol{z}) - \Psi\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right| \leq \left(\frac{1}{q+1}\right)^{\frac{1}{q}}\frac{1}{\overline{\gamma}}\sum_{i=1}^{n}\gamma_i\left(\frac{\overline{\zeta}}{\overline{\gamma}} - \frac{\zeta_i}{\gamma_i}\right)^2 \times \left|\Psi''\left(\frac{(q+1)\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right) + \frac{\zeta_i}{\gamma_i}}{q+2}\right)\right|. \tag{62}$$

Proof. Taking $p_i = \frac{\gamma_i}{\overline{\gamma}}$ and $x_i = \frac{\zeta_i}{\gamma_i}$ in (12), we obtain (62). □

We deduce a bound for the Csiszár divergence as an application of Theorem 6, which is stated in the below theorem.

Theorem 12. *Let all the assumptions of Theorem 11 be hold. Furthermore, if $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{1}{\overline{\gamma}}\widetilde{D}_c(\boldsymbol{r}, \boldsymbol{z}) - \Psi\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{1}{\overline{\gamma}}\sum_{i=1}^{n}\gamma_i\left(\frac{\overline{\zeta}}{\overline{\gamma}} - \frac{\zeta_i}{\gamma_i}\right)^2\left|\Psi''\left(\frac{\frac{\overline{\zeta}}{\overline{\gamma}} + \frac{\zeta_i}{\gamma_i}}{2}\right)\right|. \tag{63}$$

Proof. Use (14) for $p_i = \frac{\gamma_i}{\overline{\gamma}}$ and $x_i = \frac{\zeta_i}{\gamma_i}$, we deduce (63). □

We acquire the following application of Theorem 7.

Theorem 13. *Suppose that, all the conditions of Theorem 9 are valid. Then*

$$\left|\frac{1}{\overline{\gamma}}\widetilde{D}_c(\boldsymbol{r},\boldsymbol{z})-\Psi\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right| \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{1}{\overline{\gamma}}\sum_{i=1}^{n}\gamma_i\left(\frac{\overline{\zeta}}{\overline{\gamma}}-\frac{\zeta_i}{\gamma_i}\right)^2 \times\left(\frac{2\left|\Psi''\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right|^q+\left|\Psi''\left(\frac{\zeta_i}{\gamma_i}\right)\right|^q}{6}\right)^{\frac{1}{q}}. \tag{64}$$

Proof. Inequality (64) can easily be assumed from (15) by taking $p_i=\frac{\gamma_i}{\overline{\gamma}}$ and $x_i=\frac{\zeta_i}{\gamma_i}$. $\square$

The following is the application of Theorem 8.

Theorem 14. *Let all the assumptions of Theorem 11 be valid. Then*

$$\left|\frac{1}{\overline{\gamma}}\widetilde{D}_c(\boldsymbol{r},\boldsymbol{z})-\Psi\left(\frac{\overline{\zeta}}{\overline{\gamma}}\right)\right| \leq \frac{1}{2\overline{\gamma}}\sum_{i=1}^{n}\gamma_i\left(\frac{\overline{\zeta}}{\overline{\gamma}}-\frac{\zeta_i}{\gamma_i}\right)^2\left|\Psi''\left(\frac{2\frac{\overline{\zeta}}{\overline{\gamma}}+\frac{\zeta_i}{\gamma_i}}{3}\right)\right|. \tag{65}$$

Proof. Choosing $p_i=\frac{\gamma_i}{\overline{\gamma}}$ and $x_i=\frac{\zeta_i}{\gamma_i}$ in (18), we obtain (65). $\square$

Definition 4. *Let $\boldsymbol{r}=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ and $\boldsymbol{z}=(\zeta_1,\zeta_2,\cdots,\zeta_n)$ be positive probability distributions and $\alpha\in(0,\infty)$ such that $\alpha\neq 1$. Then Rényi divergence is defined as*

$$\widetilde{D}_{re}(\boldsymbol{r},\boldsymbol{z})=\frac{1}{\alpha-1}\log\left(\sum_{i=1}^{n}\gamma_i^{\alpha}\zeta_i^{1-\alpha}\right).$$

We give some more applications of our results for Rényi divergence, which are given in the following three corollaries.

Corollary 13. *Assume that, $\boldsymbol{r}=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ and $\boldsymbol{z}=(\zeta_1,\zeta_2,\cdots,\zeta_n)$ be positive probability distributions and α, $q>1$. Then*

$$\widetilde{D}_{re}(\boldsymbol{r},\boldsymbol{z})-\frac{1}{\alpha-1}\sum_{i=1}^{n}\gamma_i\log\left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1} \leq\frac{1}{\alpha-1}\sum_{i=1}^{n}\gamma_i\left(\sum_{i=1}^{n}\gamma_i^{\alpha}\zeta_i^{1-\alpha}-\left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1}\right)^2 \times\left(\frac{(q+1)\left(\sum_{i=1}^{n}\gamma_i^{\alpha}\zeta_i^{1-\alpha}\right)^{-2q}+\left(\frac{\gamma_i}{\zeta_i}\right)^{-2q(\alpha-1)}}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \tag{66}$$

Proof. Let $\Psi(x)=-\frac{1}{\alpha-1}\log x$, $x>0$. Then $\Psi''(x)=\frac{1}{(\alpha-1)x^2}$ and $\left(|\Psi''(x)|^q\right)''=\frac{2q(2q-1)}{(\alpha-1)^q}x^{-2q-2}$. Clearly both $\Psi''(x)$ and $\left(|\Psi''(x)|^q\right)''$ are positive on $(0,\infty)$ for $\alpha,q>1$. This confirms that the functions $\Psi(x)$ and $|\Psi''(x)|^q$ both are convex on $(0,\infty)$ for $\alpha,q>1$. Therefore, utilizing (4) for $\Psi(x)=-\frac{1}{\alpha-1}\log x$, $p_i=\gamma_i$ and $x_i=\left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1}$, we get (66). $\square$

Corollary 14. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, $\boldsymbol{z} = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be positive probability distributions. If $\alpha > 1$, then*

$$\begin{aligned}\widetilde{D}_{re}(\boldsymbol{r}, \boldsymbol{z}) - \frac{1}{\alpha - 1}\sum_{i=1}^{n}\gamma_i \log\left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1} &\\ \leq \frac{1}{\alpha - 1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{i=1}^{n}\gamma_i\left(\sum_{i=1}^{n}\gamma_i^{\alpha}\zeta_i^{1-\alpha} - \left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1}\right)^2 &\\ \times\left(\frac{\left(\sum\limits_{i=1}^{n}\gamma_i^{\alpha}\zeta_i^{1-\alpha}\right)^{-2q} + \left(\frac{\gamma_i}{\zeta_i}\right)^{-2q(\alpha-1)}}{2}\right)^{\frac{1}{q}}. &\qquad (67)\end{aligned}$$

Proof. Using (9) by choosing $\Psi(x) = -\frac{1}{\alpha-1}\log x$, $p_i = \gamma_i$ and $x_i = \left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1}$, we obtain (67). □

Corollary 15. *Let $q > 1$ and $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, $\boldsymbol{z} = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ be two positive probability distributions. Then*

$$\begin{aligned}\widetilde{D}_{re}(\boldsymbol{r}, \boldsymbol{z}) - \frac{1}{\alpha - 1}\sum_{i=1}^{n}\gamma_i \log\left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1} &\\ \leq \frac{1}{\alpha - 1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\sum_{i=1}^{n}\gamma_i\left(\sum_{i=1}^{n}\gamma_i^{\alpha}\zeta_i^{1-\alpha} - \left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1}\right)^2 &\\ \times\left(\frac{2\left(\sum\limits_{i=1}^{n}\gamma_i^{\alpha}\zeta_i^{1-\alpha}\right)^{-2q} + \left(\frac{\gamma_i}{\zeta_i}\right)^{-2q(\alpha-1)}}{3}\right)^{\frac{1}{q}}. &\qquad (68)\end{aligned}$$

Proof. Applying (15) while taking $\Psi(x) = -\frac{1}{\alpha-1}\log x$, $x_i = \left(\frac{\gamma_i}{\zeta_i}\right)^{\alpha-1}$ and $p_i = \gamma_i$, we receive (68). □

Definition 5. *For a positive probability distribution $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, the Shannon entropy is defined as*

$$E_s(\boldsymbol{r}) = -\sum_{i=1}^{n}\gamma_i \log \gamma_i.$$

Now, we are going to discuss some applications of our results for Shannon entropy, which are given in the next three corollaries.

Corollary 16. *Assume that $q > 1$ and $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ is a positive distribution such that $\sum\limits_{i=1}^{n}\gamma_i = 1$. Then*

$$\log n - E_s(\boldsymbol{r}) \leq \sum_{i=1}^{n}\gamma_i\left(n - \frac{1}{\gamma_i}\right)^2\left(\frac{(q+1)n^{-2q} - \gamma_i^{2q}}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \qquad (69)$$

Proof. Let $\Psi(x) = -\log x$, $x > 0$. Then $\Psi''(x) = x^{-2}$ and $|\Psi''(x)|^q = 2q(2q+1)x^{-2q-2}$, which implies that both $\Psi''(x)$ and $\left(|\Psi''(x)|^q\right)''$ are positive for all $x \in (0, \infty)$ and $q > 1$. This confirms the convexity of $\Psi''(x)$ and $\left(|\Psi''(x)|^q\right)''$ on $(0, \infty)$ for $q > 1$. Therefore, applying (60) by picking $\Psi(x) = -\log x$, and $\zeta_i = 1$ for all $i \in \{1, 2, \cdots, n\}$, we get (69). □

Corollary 17. *Let* $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *be a positive distribution such that* $\sum_{i=1}^{n} \gamma_i = 1$. *Let also* $q, p > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\log n - E_s(\boldsymbol{r}) \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n} \gamma_i \left(n - \frac{1}{\gamma_i}\right)^2 \left(\frac{n^{-2q} - \gamma_i^{2q}}{2}\right)^{\frac{1}{q}}. \tag{70}$$

Proof. Utilizing (61) for $\Psi(x) = -\log x$, and $\zeta_i = 1$ for all $i \in \{1, 2, \cdots, n\}$, we receive (70). □

Corollary 18. *suppose that all the hypotheses of Corollary 16 are true, then*

$$\log n - E_s(\boldsymbol{r}) \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \sum_{i=1}^{n} \gamma_i \left(n - \frac{1}{\gamma_i}\right)^2 \left(\frac{2n^{-2q} - \gamma_i^{2q}}{3}\right)^{\frac{1}{q}}. \tag{71}$$

Proof. Use $\Psi(x) = -\log x$, and $\zeta_i = 1$ for all $i \in \{1, 2, \cdots, n\}$ in (64) we receive (71). □

Definition 6. *For any positive probability distributions* $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *and* $\boldsymbol{z} = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, *the Bhattacharyya coefficient is defined as*

$$C_b(\boldsymbol{r}, \boldsymbol{z}) = \sum_{i=1}^{n} \sqrt{\gamma_i \zeta_i}.$$

We extract some bounds for Bhattacharyya coefficient from our results which are given in the following corollaries.

Corollary 19. *Let* $\boldsymbol{r} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ *and* $\boldsymbol{z} = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ *be positive probability distribution and* $q > 1$. *Then*

$$1 - C_b(\boldsymbol{r}, \boldsymbol{z}) \leq \frac{1}{4} \sum_{i=1}^{n} \gamma_i \left(1 - \frac{\zeta_i}{\gamma_i}\right)^2 \left(\frac{q + 1 + \left(\frac{\gamma_i}{\zeta_i}\right)^{\frac{3q}{2}}}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \tag{72}$$

Proof. Consider the function $\Psi(x) = -\sqrt{x}$, $x \in (0, \infty)$. Clearly the function Ψ is convex because $\Psi''(x) = \frac{1}{4} x^{-\frac{3}{2}} > 0$. Additionally, the function $|\Psi''(x)|^q$ is convex because $\left(|\Psi''(x)|^q\right)'' = \left(\frac{1}{4}\right)^q \frac{3q}{2} \left(\frac{3q}{2} + 1\right) x^{-\left(\frac{3q}{2} + 2\right)} > 0$. Therefore, utilizing (60) by choosing $\Psi(x) = -\sqrt{x}$, we acquire (72). □

Corollary 20. *Assume that the hypotheses of Corollary 19 hold. Moreover, if* $p > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$1 - C_b(\boldsymbol{r}, \boldsymbol{z}) \leq \frac{1}{4} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{i=1}^{n} \gamma_i \left(1 - \frac{\zeta_i}{\gamma_i}\right)^2 \left(\frac{1 + \left(\frac{\gamma_i}{\zeta_i}\right)^{\frac{3q}{2}}}{2}\right)^{\frac{1}{q}}. \tag{73}$$

Proof. To deduce (73), putting $\Psi(x) = -\sqrt{x}$ in (61). □

Corollary 21. *Suppose that, the conditions of Corollary 19 are true, then*

$$1 - C_b(\boldsymbol{r}, \boldsymbol{z}) \leq \frac{1}{4} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \sum_{i=1}^{n} \gamma_i \left(1 - \frac{\zeta_i}{\gamma_i}\right)^2 \left(\frac{2 + \left(\frac{\gamma_i}{\zeta_i}\right)^{\frac{3q}{2}}}{3}\right)^{\frac{1}{q}}. \tag{74}$$

Proof. Utilize (64) for $\Psi(x) = -\sqrt{x}$, we obtain (74). $\square$

6. Conclusions

The field of mathematical inequalities has performed a very consequential role in all areas of science, especially in mathematics. There were a lot of problems which were not possible to explain or solve with out mathematical inequalities. There are many well-known inequalities which have accomplished eminent performance in solving many problems in the fields of science. Among these inequalities, one of the weighty inequalities of great interest is the Jensen inequality. This inequality is of sublime importance in the sense that several inequalities can easily be deduced from it. In this article, we obtained some interesting bounds for the Jensen difference. We acquired the desired bounds by utilizing the definition of convex function, the integral Jensen inequality for concave function, the Hölder and power mean inequalities. By taking some particular functions in the main results, we deduced several improvements of the Hölder inequalities and also concluded different inequalities for quasi-arithmetic and power means. Finally, we presented some useful applications of our main results in information theory. These applications contain several bounds for Csiszár divergence, Rényi divergence, Shannon entropy and Bhattacharyya coefficient.

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