Article

On the Search for a Measure to Compare Interval-Valued Fuzzy Sets

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Abstract: Multiple definitions have been put forward in the literature to measure the differences between two interval-valued fuzzy sets. However, in most cases, the outcome is just a real value, although an interval could be more appropriate in this environment. This is the starting point of this contribution. Thus, we revisit the axioms that a measure of the difference between two interval-valued fuzzy sets should satisfy, paying special attention to the condition of monotonicity in the sense that the closer the intervals are, the smaller the measure of difference between them is. Its formalisation leads to very different concepts: distances, divergences and dissimilarities. We have proven that distances and divergences lead to contradictory properties for this kind of sets. Therefore, we conclude that dissimilarities are the only appropriate measures to measure the difference between two interval-valued fuzzy sets when the outcome is an interval.

Keywords: interval-valued fuzzy set; interval order; difference; distance; divergence; dissimilarity

1. Introduction

It us usually understood that knowledge of comparisons of objects, opinions, etc. are incomplete. A widely accepted theory (and methodology) to cope with imprecision is fuzzy sets theory, where elements are not necessarily in a set or out of it, but rather intermediate degrees of membership are allowed. In this context, the classical ways to contrast sets do not apply, and several measures for comparing fuzzy sets have been introduced and can be found in the literature. An in-depth study was carried out by Bouchon-Meunier et al. in 1996 [1]. After this, many other measures have been proposed. Some of them are constructive definitions, i.e., specific formulae (see, among many others, Refs. [2–5]) and others are based on axiomatic definitions (see, for example, Refs. [6–8]).

The presence of imprecision in real-life situations has been a challenge even from a theoretical point of view. In order to cope with this handicap, different extensions of fuzzy sets have been proposed. Interval-valued fuzzy sets (IVFSs) are one of the most successful and challenging extensions. This generalization was introduced independently and almost simultaneously by Zadeh [9], Grattan-Guiness [10], Jahn [11], and Sambuc [12]. Interval-valued fuzzy sets are a useful tool. They are used to model situations where the "classical" fuzzy sets are not appropriate. This occurs in the case when an objective procedure to determine crisp membership degrees is not available. IVFSs show high potential in practical applications. They were used in medical diagnosis in thyrodian pathology (see Sambuc [12]), in approximate reasoning (see, for instance, the contributions of Bustince [13] and Gozalczany [14]) and Cornelis et al. [15] and Turksen [16] applied this theory in logic.

Due to its potential utility, different notions and tools connected to this extension must be studied. In particular, our interest is focused on the measures of comparison of two interval-valued fuzzy sets, which have been studied in the last years. Some of them are based on comparing the degree of similarity between them (see, e.g., [17–20]). However,

it is also possible to consider a dual approach, based on measuring the difference (see, e.g., [21]). Another previous study related to this topic can be obtained from the related concept of intuitionistic fuzzy sets. It was introduced by Atanassov [22].

Closely connected to IVFSs is the theory of intuitionistic fuzzy sets, introduced by Atanassov [22] about a decade after IVFSs were defined. Despite they are semantically different, it is widely known that intuitionistic and interval-valued fuzzy sets are equipollent (see, for instance, [23,24]); that is, there is a bijective function that maps one onto the other. Measures to compare intuitionistic fuzzy sets have already been introduced (see, for example, [25,26]). These proposals could provide us with an initial idea on the way to compare two interval-valued fuzzy sets. However, they cannot be directly used, as was shown in [27–29].

The previously introduced measures provide a unique real value as the result of the comparison. However, this is not a desirable result. If we are dealing with interval-valued fuzzy sets from an epistemic point of view, even the absolute similarity between incomplete descriptions does not guarantee the absolute similarity of the described elements. In order to cope with this situation, it could be more appropriate to formalize the idea of similarity using a range of values. However, this perspective is not the usual one. To the best of our knowledge, the literature where we can find this approach is rather limited [30–33]. These papers can be considered as the starting point of our research [34]. Thus, our main purpose is to study the different approaches considered in the literature that measure the degree of difference between two interval-valued fuzzy sets by means of an interval, in order to preserve the uncertainty that we have about the description of the involved sets. In this paper, we will consider the different approaches, compare them and conclude which ones are the best axioms in order to characterise a measure of the difference between two interval-valued fuzzy sets.

More precisely, we will focus on distances, divergences, and dissimilarities and study how sound these definitions are. We provide examples that show that distances lead to counterintuitive situations and that the axioms involved in the definition of an interval-valued divergence are conflicting. Therefore, we consider dissimilarities as the only reasonable way to compare IVFSs among the three considered. As a consequence, we finally compare the different proposals given in the literature for this concept.

The contribution is organised as follows. In Section 2, basic concepts and results are introduced, and the notation used in the subsequent sections is fixed. Section 3 is devoted to studying the possible definitions of a measure of how different two interval-valued fuzzy sets can be. Section 4 closes the contribution with some conclusions. We also put forward some questions that remain open in this section.

2. Basic Concepts

In this section, we recall some basic notions and properties that are important to understand the following section of this contribution. We begin with the classical theory of fuzzy sets.

Let X denote the universe of discourse. A fuzzy set in X is a mapping $A : X \to [0,1]$ where $A(x)$ stands for the degree to which element x belongs to the subset A of X. We will denote $FS(X)$ the family of all the fuzzy sets defined on the universe X.

An interval-valued fuzzy subset (IVFS for short) of X is a mapping $A : X \to L([0,1])$ such that $A(x) = [\underline{A}(x), \overline{A}(x)]$, where $L([0,1])$ denotes the family of closed intervals included in the unit interval $[0,1]$. It is therefore easy to check that an interval-valued fuzzy set A is characterized by two mappings, $\underline{A}$ and $\overline{A}$, from X into $[0,1]$ such that $\underline{A}(x) \le \overline{A}(x), \forall x \in X$. These functions provide the lower and upper bounds, respectively, of the associated intervals. Observe that if $\underline{A}(x) = \overline{A}(x), \forall x \in X$, then A is a classical fuzzy set. The abbreviation $IVFS(X)$ stands for set of all the interval-valued fuzzy sets in X.

For IVFSs, we can consider the epistemic or the ontic interpretation. In our study, the former is chosen. Thus, we assume that there is one actual, real-valued membership degree of an element inside the membership interval of possible membership degrees.

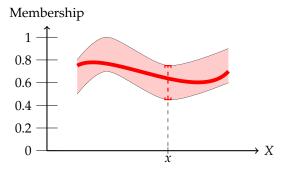**Example 1.** *Consider the IVFS drawn in Figure 1. The IVFS assigns to element x the interval* [0.45, 0.75].

Membership

Figure 1. Idea of IVFS.

This means that the real membership degree for x may be 0.65, *but we are not sure about it and we can only say that it is between* 0.45 *and* 0.75.

As we explained in detail at the Introduction, we define a measure to compare two IVFSs such that the value of this comparison is again an interval. In order to do this, some operations and previous concepts have to be fixed.

2.1. Inclusion

The inclusion for IVFSs is directly connected to an order relation between intervals. In [35], we can find a summary of the main interval orders.

Definition 1. *([35]) Let* $a = [\underline{a}, \overline{a}]$ *and* $b = [\underline{b}, \overline{b}]$ *be two intervals in* $L([0, 1])$. *Then a is smaller than or equal to b for the following orders between intervals if:*

- *Interval dominance [36]:* $a \preceq_{ID} b$ *if* $\overline{a} \leq \underline{b}$.
- *Lattice order [37]:* $a \preceq_{Lo} b$ *if* $\underline{a} \leq \underline{b}$ *and* $\overline{a} \leq \overline{b}$, *which is induced by the usual partial order in* $\mathbb{R}^2$.
- *Lexicographical order of type 1 [38]:* $a \preceq_{Lex1} b$ *if* $\underline{a} < \underline{b}$ *or* $(\underline{a} = \underline{b}$ *and* $\overline{a} \leq \overline{b})$.
- *Lexicographical order of type 2 [38]:* $a \preceq_{Lex2} b$ *if* $\overline{a} < \overline{b}$ *or* $(\overline{a} = \overline{b}$ *and* $\underline{a} \leq \underline{b})$.
- *The Xu and Yager order [39]:* $a \preceq_{XY} b$ *if* $\underline{a} + \overline{a} < \underline{b} + \overline{b}$ *or* $(\underline{a} + \overline{a} = \underline{b} + \overline{b}$ *and* $\overline{a} - \underline{a} \leq \overline{b} - \underline{b})$.
- *Maximin order [40,41]:* $a \preceq_{Mm} b$ *if* $\underline{a} \leq \underline{b}$.
- *Maximax order [42]:* $a \preceq_{MM} b$ *if* $\overline{a} \leq \overline{b}$.
- *Hurwicz order [43]:* $a \preceq_{H(\alpha)} b$ *if* $\alpha \cdot \underline{a} + (1 - \alpha) \cdot \overline{a} \leq \alpha \cdot \underline{b} + (1 - \alpha) \cdot \overline{b}$ *with* $\alpha \in [0, 1]$.
- *Weak order [44]:* $a \preceq_{wo} b$ *if* $\underline{a} \leq \overline{b}$.

Given an order $\preceq_o$, the equality between intervals can be defined as follows: $a =_o b$ if and only if $a \preceq_o b$ and $b \preceq_o a$.

Most of the previously recalled orders are connected. First of all, it is well known that if one interval *a* is lower than or equal to another interval *b* w.r.t. interval dominance, *a* is also lower than or equal to *b* w.r.t. the lattice order. Interval dominance is also a stronger relation than the lexicographical order of type 1, which implies the maximax order which, in turn, implies the weak order. Figure 2 summarizes these and other similar connections.
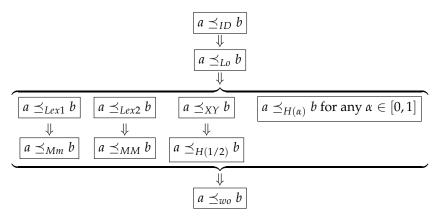
Figure 2. Relationships among the different relations.

Observe that *ID* is the strongest relation in the sense that if two intervals are connected by it, then they are connected by any of the other relations previously recalled.

Apart from that, it is important to notice that, although all of them are called orders, they are not really orders, in the mathematical sense, in all the cases, as we can see in Table 1. Thus, only the lattice order, the lexicographical orders and the Xu-Yager order are really orders and the first one is not a total order.

Table 1. Some properties of the considered relations on $L([0,1])$.

	Reflexive	Antisymmetric	Transitive	Total	Preorder	Order
ID	✗	✓	✓	✗	✗	✗
Lo	✓	✓	✓	✗	✓	✓
Lex_1	✓	✓	✓	✓	✓	✓
Lex_2	✓	✓	✓	✓	✓	✓
XY	✓	✓	✓	✓	✓	✓
Mm	✓	✗	✓	✓	✓	✗
MM	✓	✗	✓	✓	✓	✗
$H(\alpha)$	✓	✗	✓	✓	✓	✗
wo	✓	✗	✗	✓	✗	✗

Regarding total orders in $L([0,1])$, we consider the so-called admissible orders, whose definition we now recall.

Definition 2. *([38]) An admissible order on $L([0,1])$ is a total order $\preceq_{to}$ that refines the lattice order; that is, for every $a,b \in L([0,1])$, if $a \preceq_{Lo} b$ then $a \preceq_{to} b$.*

An interesting feature of admissible orders is that they can be built using aggregation functions, as stated in the following result. Recall that an aggregation function is an increasing function $\mathcal{A} : [0,1]^n \to [0,1]$ with $\mathcal{A}(0,\ldots,0) = 0$ and $\mathcal{A}(1,\ldots,1) = 1$ (see [45]).

Observe that there is an easy bijection between the sets $L([0,1])$ and $K([0,1]) = \{(u,v) \in [0,1]^2 \mid u \leq v\}$. It assigns to each interval $[\underline{a},\overline{a}]$ the point in $\mathbb{R}^2$ whose coordinates are the extreme values of the interval, i.e., $(\underline{a},\overline{a})$ (see [38]). Therefore, aggregation functions can be used to summarize the information provided by an interval. This idea is beneath the following method provided by Bustince et al. to build admissible orders.

Proposition 1. *([38]) Let $\mathcal{A}$ and $\mathcal{B} : [0,1]^2 \to [0,1]$ be continuous aggregation functions, verifying that for all $(u,v),(w,z) \in K([0,1])$, the equalities $\mathcal{A}(u,v) = \mathcal{A}(w,z)$ and $\mathcal{B}(u,v) = \mathcal{B}(w,z)$ can only hold if $(u,v) = (w,z)$. Define the relation $\preceq_{\mathcal{A},\mathcal{B}}$ on $L([0,1])$ by:*

$$a \preceq_{\mathcal{A},\mathcal{B}} b \text{ if } \mathcal{A}(\underline{a},\overline{a}) < \mathcal{A}(\underline{b},\overline{b}) \text{ or } (\mathcal{A}(\underline{a},\overline{a}) = \mathcal{A}(\underline{b},\overline{b}) \text{ and } \mathcal{B}(\underline{a},\overline{a}) \leq \mathcal{B}(\underline{b},\overline{b})).$$

Then $\preceq_{\mathcal{A},\mathcal{B}}$ is an admissible order on $L([0,1])$.

The weighted mean provides a particular way to obtain admissible orders on $L([0,1])$. The definition is as follows (see [46]):

$$K_\alpha(u,v) = (1-\alpha) \cdot u + \alpha \cdot vs. \text{ with } \alpha \in [0,1].$$

This operator can be interpreted as the α-quantile of a probability distribution uniformly distributed on the interval $[u,v]$. Applying Proposition 1 to the aggregation operators K_α and K_β with $\alpha \neq \beta$, the admissible order $\preceq_{K_\alpha,K_\beta}$ is obtained. For the sake of simplicity, it is denoted $\preceq_{\alpha,\beta}$.

Particular cases of admissible orders obtained by the weighted mean are the lexicographical orders of type one and two and the Xu and Yager order. Note that $\preceq_{Lex1} \equiv \preceq_{0,1}$, $\preceq_{Lex2} \equiv \preceq_{1,0}$ and $\preceq_{XY} \equiv \preceq_{1/2,\beta}$ for β any value in $(1/2,1]$ (see [38]).

Any order $\preceq_o$ defined over $L([0,1])$ induces an order over $IVFS(X)$ that is the content relation derived from this order ($\subseteq_o$). The following result formalized what said above and is straightforward to prove it.
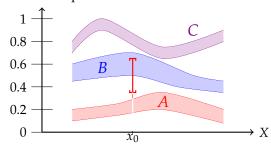
Proposition 2. *Let $\preceq_o$ be an interval order in $L([0,1])$ and A and B in $IVFS(X)$. Then $\subseteq_o$ defined as*

$$A \subseteq_o B \text{ iff } A(x) \preceq_o B(x), \forall x \in X.$$

is a partial order in $IVFS(X)$.

Example 2. *If we consider the IVFSs A, B, and C represented in Figure 3, it is clear that $A, B \subseteq_{ID} C$ and therefore they are included in C with respect to any of the orders recalled in Definition 1. We also have that $A \subseteq_{Lo} B$ but $A \not\subseteq_{ID} B$. Thus, A is included in B for any considered order except for the interval dominance. Finally, we can say that neither B nor C are included in A for any order.*

Figure 3. Membership degrees for A, B and C.

On the other hand, $\subseteq_o$ is not a total order in general. Consider for instance the lattice order $\subseteq_{Lo}$ and the IVFSs given in Figure 3, A and B are incomparable. In fact, we can obtain incomparable IVFSs even in the case we are considering a total order.

2.2. Embedding

Another important partial order on $IVFS(X)$ could be defined as follows.

Definition 3. *Let $\subseteq$ be the usual inclusion between intervals and A and B in $IVFS(X)$. It is said that A is embedded in B, and it is denoted as $A \sqsubseteq B$ if and only if $A(x) \subseteq B(x), \forall x \in X$.*

The following example shows the idea behind this definition.

Example 3. *If we consider the IVFSs A and B represented in Figure 4, we have that A is embedded in B, since $A(x) \subseteq B(x), \forall x \in X$.*
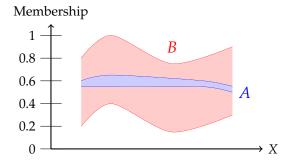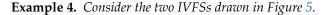
Membership

Figure 4. *A is embedded in B.*

Nor is it a total order, as the following example shows.

Example 4. *Consider the two IVFSs drawn in Figure 5.*
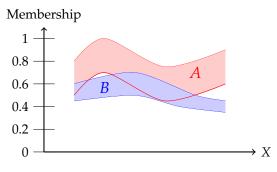
Membership

Figure 5. Not embedded IVFSs.

It is clear that A is not embedded in B and B is not embedded in A.

2.3. Intersection

There are different proposals to formalize the notion of intersection in the literature. We will base our definition on the idea that the intersection of two sets is the greatest set contained in both departing sets. Since this definition is based on contents, we will obtain a different definition of intersection for each order we consider in $IVFS(X)$ as explained in [35,47].

Definition 4. *Let A, B be two interval-valued fuzzy sets in X and let $\preceq_o$ be an order relation between intervals in $L([0, 1])$. We define the o-intersection of A and B, and we denote it by $A \cap_o B$ as the greatest interval-valued fuzzy set such that $A \cap_o B \subseteq_o A$ and $A \cap_o B \subseteq_o B$.*

For any two interval orders $\preceq_{o_1}$ and $\preceq_{o_2}$ in $L([0, 1])$ such that $a \preceq_{o_1} b$ implies that $a \preceq_{o_2} b, \forall a, b \in L([0, 1])$, we have that $A \cap_{o_1} B \subseteq_{o_2} A \cap_{o_2} B$ for any $A, B \in IVFS(X)$.

Considering the connection among the orders in Definition 1, we next discuss the definition of intersection obtained for each of them. If possible, we describe general behaviours.

Proposition 3. *([35]) Let A, B be two sets in $IVFS(X)$. Then, for any $x \in X$, we have that:*

- *Interval dominance: $A \cap_{ID} B(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\overline{A(x)}, \overline{B(x)}\}]$.*

- *Lattice order: $A \cap_{Lo} B(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\overline{A(x)}, \overline{B(x)}\}]$.*

- *Lexicographical order of type 1: $A \cap_{Lex1} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{Lex1} B(x) \\ B(x) & \text{if } B(x) \preceq_{Lex1} A(x) \end{cases}$*

- *Lexicographical order of type 2: $A \cap_{Lex2} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{Lex2} B(x) \\ B(x) & \text{if } B(x) \preceq_{Lex2} A(x) \end{cases}$*

- *Xu and Yager order: $A \cap_{XY} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{XY} B(x) \\ B(x) & \text{if } B(x) \preceq_{XY} A(x) \end{cases}$*

- *Maximim order:* $A \cap_{Mm} B(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, v]$ *for v any number in the interval* $[\min\{\underline{A(x)}, \underline{B(x)}\}, 1]$.
- *Maximax order:* $A \cap_{MM} B(x) = [u, \min\{\overline{A(x)}, \overline{B(x)}\}]$ *for u any number in the interval* $[0, \min\{\overline{A(x)}, \overline{B(x)}\}]$.
- *Hurwicz order:* $A \cap_{H(\alpha)} B(x) = \left[u, \frac{k - \alpha \cdot u}{1 - \alpha}\right]$ *for* $k = \min\{\alpha \cdot \underline{A(x)} + (1 - \alpha) \cdot \overline{A(x)}, \alpha \cdot \underline{B(x)} + (1 - \alpha) \cdot \overline{B(x)}\}$ *and u any value in the interval* $\left[\max\{0, \frac{k - (1 - \alpha)}{\alpha}\}, k\right]$.
- *Weak order:* $A \cap_{wo} B(x) = [u, v]$ *for u any value in the interval* $[0, \min\{\underline{A(x)}, \underline{B(x)}\}]$ *and v any value in the interval* $[\min\{\overline{A(x)}, \overline{B(x)}\}, 1]$.

Lexicographical orders and the Xu and Yager order are particular cases of admissible orders, and the associated intersections are obtained as a consequence of the following result.

Corollary 1. *Let A and $B \in FS(X)$ and denote A' and B' as the previous fuzzy sets written in terms of IVFSs: $A'(x) = [A(x), A(x)]$ and $B'(x) = [B(x), B(x)]$ for every $x \in X$. Let $\preceq_0$ be the interval dominance, the lattice order, the lexicographical order of types 1 and 2, or the Xu and Yager order. Then $A' \cap_0 B' = (A \cap B)'$, where $\cap$ denotes the classical intersection of fuzzy sets based on the minimum.*

Proof. Fix $x \in X$. Denote $A(x) = a$ and $B(x) = b$, then $A'(x) = [a, a]$ and $B'(x) = [b, b]$.

On the one hand it holds that $(A \cap B)(x) = \min(A(x), B(x))$. Therefore, $(A \cap B)'(x) = [\min(a, b), \min(a, b)]$.

On the other hand, $\underline{A'(x)} = \overline{A'(x)} = a$ and $\underline{B'(x)} = \overline{B'(x)} = b$ and it follows from Proposition 3 that $A' \cap_0 B'(x) = [\min(a, b), \min(a, b)]$. $\square$

Proposition 4. *([35]) Let $\mathcal{A}$ and $\mathcal{B} : [0, 1]^2 \to [0, 1]$ be two continuous aggregation functions such that $\forall (u, v), (u', v') \in K([0, 1])$, $\mathcal{A}(u, v) = \mathcal{A}(u', v')$ and $\mathcal{B}(u, v) = \mathcal{B}(u', v')$ hold simultaneously if and only if $(u, v) = (u', v')$. Let $\preceq_{\mathcal{A}, \mathcal{B}}$ be the admissible order on $L([0, 1])$ induced by these aggregation functions. For all $A, B \in IVFS(X)$, the $\mathcal{A}, \mathcal{B}$-intersection of A and B is the interval-valued fuzzy set defined by:*

$$A \cap_{\mathcal{A}, \mathcal{B}} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{\mathcal{A}, \mathcal{B}} B(x) \\ B(x) & \text{if } B(x) \preceq_{\mathcal{A}, \mathcal{B}} A(x) \end{cases}$$

Taking into account Proposition 3, we can see that in some cases the intersection is not uniquely defined for the four last relations. Moreover, for the first one, we have that the intersection of two IVFSs is just a fuzzy set. This is summarized in Table 2.

Table 2. Uniqueness of the intersection of IVFSs.

Interval Order	Is the Intersection Unique?	Is the Intersection an IVFS?
Interval dominance	✓	✗
Lattice order	✓	✓
Lex. order type 1	✓	✓
Lex. order type 2	✓	✓
Xu and Yager order	✓	✓
Maximim order	✗	
Maximax order	✗	
Hurwicz order	✗	
Weak order	✗	

The following examples can help to clarify the previous remarks.

Example 5. *Let us consider the case $X = \{x\}$ and the interval-valued fuzzy sets A and B defined by $A(x) = [0.4, 0.8]$ and $B(x) = [0.2, 0.9]$. Then, the intersection for the four last orders is given in Table 3 and shown in Figure 6.*

Table 3. No uniqueness of the intersection of IVFSs for some orders.

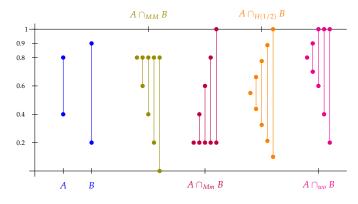| $A \cap_{MM} B(x)$	$A \cap_{Mm} B(x)$	$A \cap_{H(1/2)} B(x)$	$A \cap_{wo} B(x)$
$[u, 0.8]$	$[0.2, v]$	$[u, 1.1 - u]$	$[u, v]$
$u \in [0, 0.8]$	$v \in [0.2, 1]$	$u \in [0.1, 0.55]$	$u \in [0, 0.8]$ $v \in [0.8, 1]$

Figure 6. Intersection for different orders.

If we consider the orders that lead to a unique set as intersection, we obtain an interval uniquely defined, as we can see in Table 4. A graphical representation is shown in Figure 7.

Table 4. Uniqueness of the intersection of IVFSs for some orders.

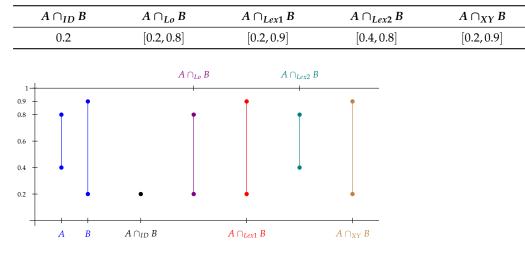| $A \cap_{ID} B$	$A \cap_{Lo} B$	$A \cap_{Lex1} B$	$A \cap_{Lex2} B$	$A \cap_{XY} B$
0.2	$[0.2, 0.8]$	$[0.2, 0.9]$	$[0.4, 0.8]$	$[0.2, 0.9]$

Figure 7. Intersection w.r.t. *ID*, *Lo*, *Lex*1, *Lex*2 and *XY*.

It is clear that the intersection is just a fuzzy set for the case of the interval dominance.

In this case the lexicographical order of type 1 and the Xu and Yager order provide the same intersection, but, of course, this does not hold in general. For example, if we consider C an IVFS such that $C(x) = [0.4, 0.5]$, we have that $B \preceq_{Lex1} C$ and $C \preceq_{XY} B$ and therefore $B \cap_{Lex1} C = B \neq B \cap_{XY} C = C$.

This example also emphasises that the intersection depends on the considered order, which is logical from the considered definition.

2.4. Union

In this subsection, we reproduce for the union the discussion included in the previous one concerning intersection.

We assume that the union of two sets is the smallest set that contains both sets. Then we have a different definition of union for every order we consider in $L([0,1])$.

Definition 5. *Let $A, B \in IVFS(X)$ and let $\preceq_o$ be an order in $L([0,1])$. The o-union of A and B, denoted $A \cup_o B$, is the smallest interval-valued fuzzy set such that $A \subseteq_o A \cup_o B$ and $B \subseteq_o A \cup_o B$.*

Thus, for the orders where the intersection is unique, we have that:

Proposition 5. *([35]) Let A, B be two sets in $IVFS(X)$. Then, for any $x \in X$, we have that:*

- *Interval dominance: $A \cup_{ID} B(x) = [\max\{\overline{A(x)}, \overline{B(x)}\}, \max\{\overline{A(x)}, \overline{B(x)}\}]$.*
- *Lattice order: $A \cup_{Lo} B(x) = [\max\{\underline{A(x)}, \underline{B(x)}\}, \max\{\overline{A(x)}, \overline{B(x)}\}]$.*
- *Lexicographical order of type 1: $A \cup_{Lex1} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{Lex1} B(x) \\ A(x) & \text{if } B(x) \preceq_{Lex1} A(x) \end{cases}$*
- *Lexicographical order of type 2: $A \cup_{Lex2} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{Lex2} B(x) \\ A(x) & \text{if } B(x) \preceq_{Lex2} A(x) \end{cases}$*
- *Xu and Yager order: $A \cup_{XY} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{XY} B(x) \\ A(x) & \text{if } B(x) \preceq_{XY} A(x) \end{cases}$*

We can prove again that the considered definition preserves the classical definition of union for the particular case of fuzzy sets.

Corollary 2. *Let $A, B \in FS(X)$ and denote A' and B' the previous fuzzy sets written in terms of IVFSs: $A'(x) = [A(x), A(x)]$ and $B'(x) = [B(x), B(x)]$ for every $x \in X$. Let $\preceq_0$ be the interval dominance, the lattice order, the lexicographical order of types 1 and 2 or the Xu and Yager order. Then $A' \cup_0 B' = (A \cup B)'$, where $\cup$ denotes the classical union of fuzzy sets based on the maximum.*

Proof. Fix $x \in X$. Denote $A(x) = a$ and $B(x) = b$, then $A'(x) = [a,a]$ and $B'(x) = [b,b]$.

On the one hand, it holds that $(A \cup B)(x) = \max(A(x), B(x))$. Therefore, $(A \cup B)'(x) = [\max(a,b), \max(a,b)]$.

On the other hand, $\underline{A'(x)} = \overline{A'(x)} = a$ and $\underline{B'(x)} = \overline{B'(x)} = b$, and it follows from Proposition 5 that $A' \cup_0 \overline{B'(x)} = [\max(a,b), \max(a,b)]$. □
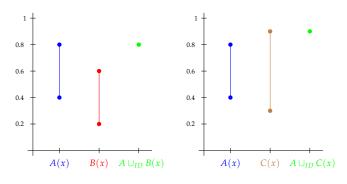
The lexicographical orders and the Xu and Yager order are particular cases of admissible order, and the union can also be obtained as a consequence of the following general result.
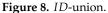
Proposition 6. *([35]) Let $\mathcal{A}, \mathcal{B} : [0,1]^2 \to [0,1]$ be two continuous aggregation functions, such that $\forall (u,v), (u',v') \in K([0,1])$, $\mathcal{A}(u,v) = \mathcal{A}(u',v')$ and $\mathcal{B}(u,v) = \mathcal{B}(u',v')$ hold simultaneously if and only if $(u,v) = (u',v')$. Let $\preceq_{\mathcal{A},\mathcal{B}}$ be the admissible order on $L([0,1])$ induced by them. For any $A, B \in IVFS(X)$, the $\mathcal{A}, \mathcal{B}$-union of A and B is the IVFS defined by:*

$$A \cup_{\mathcal{A},\mathcal{B}} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{\mathcal{A},\mathcal{B}} B(x) \\ A(x) & \text{if } B(x) \preceq_{\mathcal{A},\mathcal{B}} A(x) \end{cases}$$

Example 6. *Let the universe $X = \{x\}$ and let $A, B, C \in IVFS(X)$ such that $A(x) = [0.4, 0.8]$, $B(x) = [0.2, 0.6]$ and $C(x) = [0.3, 0.9]$.*

- *The ID-union of A and B is the IVFS $A \cup_{ID} B(x) = [0.8, 0.8]$ and the ID-union of A and C is the IVFS $A \cup_{ID} C(x) = [0.9, 0.9]$. Figure 8 provides a graphical representation.*

Figure 8. *ID*-union.

It is clear that $A \cap_{ID} B \neq A$ and $A \cap_{ID} B \neq B$.

- The *Lo*-union of A and B is the IVFS $A \cup_{Lo} B(x) = [0.2, 0.6]$ and the *Lo*-union of A and C is the IVFS $A \cup_{Lo} C(x) = [0.3, 0.8]$.

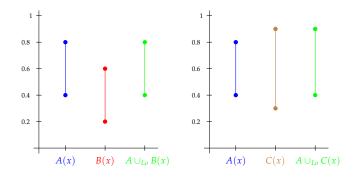 As we can see in Figure *9*, $A \cup_{Lo} B = B$, but $A \cup_{Lo} C \neq A$ and $A \cup_{ID} C \neq C$.

Figure 9. *Lo*-union.

- The *Lex*1-union of A and B is the IVFS $A \cup_{Lex1} B(x) = [0.4, 0.8]$ and the *Lex*1-union of A and C is the IVFS $A \cup_{Lex1} C(x) = [0.4, 0.8]$. Thus, in this case, $A \cup_{Lex1} B = A \cup_{Lex1} C = A$.
- The *Lex*2-union of A and B is the IVFS $A \cup_{Lex2} B(x) = [0.4, 0.8]$. and the *Lex*2-union of A and C is the IVFS $A \cup_{Lex2} C(x) = [0.3, 0.9]$. Thus, in this case, $A \cup_{Lex2} B = A$ and $A \cup_{Lex2} C = C$.
- The *XY*-union of A and B is the IVFS $A \cup_{XY} B(x) = [0.4, 0.8]$, and the *XY*-union of A and C is the IVFS $A \cup_{XY} C(x) = [0.3, 0.9]$. Thus, again $A \cup_{XY} B = A$ and $A \cup_{XY} C = C$ and the union obtained for *Lex*2 and for *XY* are the same.

  However, this is not true in general, since *Lex*2 compares the right endpoint of intervals and *XY* the sum of both endpoints. For instance, if we consider $D(x) = [0.2, 0.9]$, the *XY*-union of A and D is the IVFS $A \cup_{XY} D(x) = [0.4, 0.8]$, but their *Lex*2-union is $A \cup_{Lex2} D(x) = [0.2, 0.9]$, as we can see in Figure *10*.
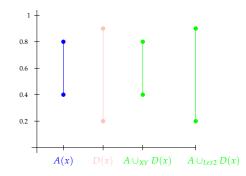

Figure 10. *Lex*2-union is different from *XY*-union.

Once we have introduced the basic concepts about different operations between IVFSs, we can start to think about the necessary requirements of a measure to be an appropriate way to quantify the difference between two IVFSs.

3. How to Compare Two Interval-Valued Fuzzy Sets?

As we described in detail at the Introduction, most of the measures of comparison between IVFSs found in the literature provide a unique number as final outcome. Such a simplification necessarily means a loss of information. In order to keep the idea underlying IVFSs, the result of the comparison should not be an isolated value. This section contains a discussion on the definition of measure of comparison between IVFSs. We consider the axioms that should be included in the definition.

There are some natural requirements that underlie the idea of difference between two interval-valued fuzzy sets:

REQ1   Non-negativity;
REQ2   Symmetry;
REQ3   It becomes zero when the two sets are "equal";
REQ4   It takes into account the uncertainty associated to the width of the intervals;
REQ5   It decreases when the sets are closer.

Requirements REQ1, REQ2, and REQ3 are the usual ones for comparing any set, in particular fuzzy sets. Requirement REQ4 gives expression to the idea that the width of the interval is important. Requirement REQ5 describes the idea of proximity, and, as will later be shown, it will be the characteristic axiom.

Let us study them in detail one by one.

3.1. Non-Negativity

Initially, the degree of difference between two IVFSs A and B is a closed interval in $\mathbb{R}$, that is, $D(A, B) \in L(\mathbb{R})$.

It seems natural to require that $D(A, B)$ is "non-negative". This is required as follows:

$$\underline{D(A, B)} \geq 0$$

and therefore the codomain of D is not $L(\mathbb{R})$ in general, but $L([0, \infty))$.

We can relate this requirement to the different orders among intervals as follows:

Proposition 7. *Let D be a map from $IVFS(X) \times IVFS(X)$ into $L(\mathbb{R})$ and consider the orders recalled in subsection. For the statements*

i) $\underline{D(A, B)} \geq 0$
ii) $[0, 0] \preceq_{ID} D(A, B)$
iii) $[0, 0] \preceq_{Lo} D(A, B)$
iv) $[0, 0] \preceq_{Lex1} D(A, B)$
v) $[0, 0] \preceq_{Lex2} D(A, B)$
vi) $[0, 0] \preceq_{XY} D(A, B)$
vii) $[0, 0] \preceq_{Mm} D(A, B)$
viii) $[0, 0] \preceq_{MM} D(A, B)$
ix) $[0, 0] \preceq_{H(\alpha)} D(A, B)$
x) $[0, 0] \preceq_{wo} D(A, B)$
xi) $[0, 0] \preceq_{\mathcal{AB}} D(A, B)$
xii) $[0, 0] \preceq_{to} D(A, B)$

we have that

$$i) \Leftrightarrow ii) \Leftrightarrow iii) \Leftrightarrow iv) \Leftrightarrow vii)$$

and $i)$ implies $v)$, $vi)$, $viii)$, $ix)$, $x)$, $xi)$, and $xii)$, but the converse is not true in general.

Proof. Since $0 \leq \underline{D(A,B)}$ implies that $[0,0] \preceq_{ID} D(A,B)$, by the relationship among the orders, we have the implication from $i)$ to any other statement. For the last two cases we have used that $[0,0] \preceq_{Lo} D(A,B)$ implies $[0,0] \preceq_{to} D(A,B)$ and therefore, in particular, $[0,0] \preceq_{AB} D(A,B)$.

On the other hand, if $[0,0] \preceq_{Mm} D(A,B)$, then $0 \leq \overline{D(A,B)}$. Again taking into account the relationship among the orders, we have $vii) \Rightarrow i)$ and therefore also $ii), iii)$ and $iv)$ implies $i)$.

However, we have that $[0,0] \preceq_{Lex2} [-0.1,0.2]$, $[0,0] \preceq_{XY} [-0.1,0.2]$, $[0,0] \preceq_{MM} [-0.1,0.2]$, $[0,0] \preceq_{H(\alpha)} [-0.1,b]$ for any $b \geq \frac{0.1\alpha}{1-\alpha}$ and $[0,0] \preceq_{wo} [-0.1,0.2]$, but $0 \not\leq -0.1$, so the converse implication is not fulfilled for these orders.

Since the lexicographical order of type 2 is an example of an $\mathcal{AB}$-admissible order, the converse implication is not fulfilled for this particular case of admissible order and, in general, for admissible orders. $\square$

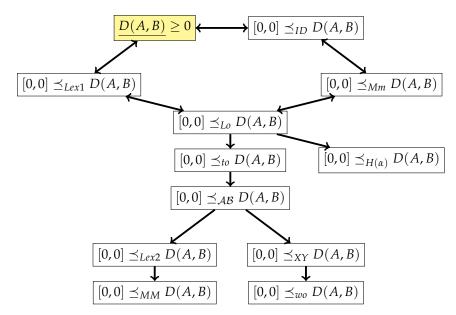This proposition is represented in Figure 11.

Figure 11. Non-negativity for different orders.

Thus, the first axiom for a measure of difference could be described as follows, depending on the order considered:

A1 $[0,0] \preceq_o D(A,B)$

On the other hand, if we suppose that the measure is upper bounded, then we can normalize it and work in the same spaces where the IVFSs are defined, that is,

$$D(A,B) \in L(\mathbb{R}) \overset{\text{Axiom 1}}{\Rightarrow} D(A,B) \in L([0,\infty)) \overset{\text{Upper bound}}{\Rightarrow} D(A,B) \in L([0,1]))$$

We will therefore assume that every measure of the difference, D, will have $L([0,1])$ as codomain:

$$
\begin{array}{rccc}
D: & IVFS(X) \times IVFS(X) & \longrightarrow & L([0,1]) \\
& (A,B) & \rightsquigarrow & [\underline{D(A,B)}, \overline{D(A,B)}]
\end{array}
$$

3.2. Symmetry

Taking into account the previous comments, the logical way to formalize symmetry is:

A2 $D(A,B) =_o D(B,A)$

Thus, this axiom depends on the considered order and so the measure of difference. If the relation is antisymmetric, it is clear that this requirement means that both intervals are exactly the same. Thus, this happens for any real order (reflexive, antisymmetri,c and transitive), but this is not true for any relation considered in Definition 1.

Proposition 8. *Let D be a map from* $IVFS(X) \times IVFS(X)$ *into* $L([0,1])$. *For the statements*

i) $\underline{D(A,B)} = \underline{D(B,A)}$ *and* $\overline{D(A,B)} = \overline{D(B,A)}$
ii) $D(A,B) =_{ID} D(B,A)$
iii) $D(A,B) =_{Lo} D(B,A)$
iv) $D(A,B) =_{Lex1} D(B,A)$
v) $D(A,B) =_{Lex2} D(B,A)$
vi) $D(A,B) =_{XY} D(B,A)$
vii) $D(A,B) =_{Mm} D(B,A)$
viii) $D(A,B) =_{MM} D(B,A)$
ix) $D(A,B) =_{H(\alpha)} D(B,A)$
x) $D(A,B) =_{wo} D(B,A)$
xi) $D(A,B) =_{\mathcal{AB}} D(B,A)$

we have that

$$i) \Leftrightarrow iii) \Leftrightarrow iv) \Leftrightarrow v) \Leftrightarrow vi) \Leftrightarrow xi)$$

and they imply vii), viii), ix), and x), but the converse is not true in general. Moreover, ii) implies i), but the converse is not true.

Proof. The equivalences are clear by antisymmetry and reflexivity of the involved orders (see Table 1).

For the maximax order, we have that $[0.2, 0.6] =_{MM} [0.3, 0.6]$. Thus, $viii) \not\Rightarrow i)$. Since $viii) \Rightarrow x)$, we also have proven that $x) \not\Rightarrow i)$.

We also have that $[0.2, 0.5] =_{Mm} [0.2, 0.6]$ and so $vii) \not\Rightarrow i)$.

Furthermore, $[0.2, 0.6] =_{H(1/2)} [0.3, 0.5]$ and then $ix) \not\Rightarrow i)$.

Finally, $ii) \Rightarrow i)$ follows from the antisymmetry of interval dominance. Furthermore, $i) \not\Rightarrow ii)$ follows from the fact that interval dominance is not reflexive. □
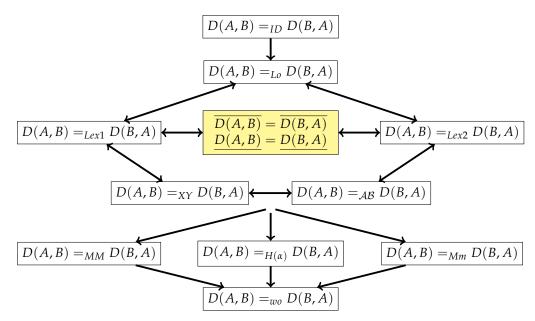
This proposition is represented in Figure 12.

Figure 12. Symmetry for different orders.

The fact that the equality given by the interval dominance is stronger that Condition i) in the previous proposition should not be undervalued. If we study in depth Condition ii), we find the following lemma.

Lemma 1. *Let D be a map from $IVFS(X) \times IVFS(X)$ into $L([0,1])$. Then D is symmetric with respect to the interval dominance if and only if its image is a number in $[0,1]$; that is, if and only if $D(A, B)$ is a unique value (not an interval) for any $A, B \in IVFS(X)$.*

Proof. Take A, B any two $IVFS(X)$ and call $D(A, B) = [\underline{d}, \overline{d}]$ and $D(B, A) = [\underline{d'}, \overline{d'}]$. In order for $D(A, B) =_{ID} D(B, A)$, it should hold both $D(A, B) \leq_{ID} D(B, A)$ and $D(B, A) \leq_{ID} D(A, B)$.

Now $D(A, B) \leq_{ID} D(B, A)$ holds if and only if $\overline{d} \leq \underline{d'}$ and $D(B, A) \leq_{ID} D(A, B)$ holds if and only if $\overline{d'} \leq \underline{d}$.

So $\overline{d} \leq \underline{d'} \leq \overline{d'} \leq \underline{d}$. Then, necessarily, $\overline{d} = \underline{d'} = \overline{d'} = \underline{d}$ and $D(A, B)$ becomes a number for any pair of IVFSs, A and B, considered. □

Thus, if interval dominance is the interval order chosen the measure of difference between any two IVFSs has to be a unique value. However, this is counterintuitive as we have explained above: the measure that quantifies how different two IVFSs are should be an interval. This is again an argument to consider orders in $L([0,1])$ and not any relation in Definition 1.

3.3. Zero Difference

Another condition that is assumed to be logical when measuring differences is that the difference should be zero only when the two sets compared are the same. The original idea would be that

$$D(A, B) =_0 [0,0] \qquad \text{if and only if} \qquad A = B \text{, for } A, B \in IVFS(X) \text{,}$$

where the equality between IVFSs is the classical equality between sets: $A(x) = B(x)$ for all $x \in X$. However, according to the epistemic interpretation, two elements with the same interval membership need not necessarily have the same (unknown) actual real-valued membership degree, as we can see with the following example.

Example 7. *If we consider the IVFSs A and B represented in Figure 13, where the known membership degree is represented as well as the (unknown) real membership function, we have that*

$$A(x) = B(x), \forall x \in X \text{ but } A \neq B$$

Figure 13. Comparing the real value of the sets.
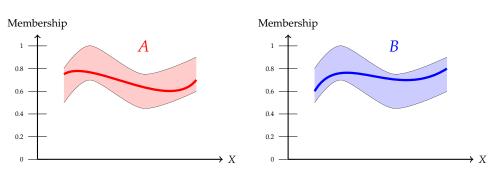
Thus, under the epistemic viewpoint, two IVFSs are only considered to be truly equal if they necessarily take the same value, i.e., if they are the same fuzzy set. So the difference between two IVFSs has to be zero if and only if both are fuzzy sets and they are equal. The axiom can be written as follows:

A3 $\qquad D(A, B) =_o [0,0]$ iff $A, B \in FS(X)$ and $A = B$.

The equality above depends on the order considered between IVFSs. Next, we study for which of the orders considered in Definition 1 the previous equality actually means $D(A, B) = [0, 0]$.

Proposition 9. *Let D be a map from $IVFS(X) \times IVFS(X)$ into $L([0, 1])$, and consider the orders recalled in Section 2.1. For the statements*

i) $\underline{D(A, B)} = \overline{D(A, B)} = 0$
ii) $D(A, B) =_{ID} [0, 0]$
iii) $D(A, B) =_{Lo} [0, 0]$
iv) $D(A, B) =_{Lex1} [0, 0]$
v) $D(A, B) =_{Lex2} [0, 0]$
vi) $D(A, B) =_{XY} [0, 0]$
vii) $D(A, B) =_{Mm} [0, 0]$
viii) $D(A, B) =_{MM} [0, 0]$
ix) $D(A, B) =_{H(\alpha)} [0, 0]$
x) $D(A, B) =_{wo} [0, 0]$
xi) $D(A, B) =_{\mathcal{AB}} [0, 0]$

we have that

$$i) \Leftrightarrow ii) \Leftrightarrow iii) \Leftrightarrow iv) \Leftrightarrow v) \Leftrightarrow vi) \Leftrightarrow viii) \Leftrightarrow ix) \Leftrightarrow xi)$$

and they imply vii) , and x) but the converse is not true in general.

Proof. By simplicity, we denote $D(A, B)$ by $a = [\underline{a}, \overline{a}]$.

By antisymmetry, it is clear that

$$i) \Leftrightarrow ii) \Leftrightarrow iii) \Leftrightarrow iv) \Leftrightarrow v) \Leftrightarrow vi) \Leftrightarrow xi)$$

From Proposition 8 we know that $\underline{a} = \overline{a} = 0$ implies that $[\underline{a}, \overline{a}] =_{Mm} [0, 0]$, $[\underline{a}, \overline{a}] =_{MM} [0, 0]$, $[\underline{a}, \overline{a}] =_{H(\alpha)} [0, 0]$, and $[\underline{a}, \overline{a}] =_{wo} [0, 0]$. Conversely, it is trivial to prove that $[\underline{a}, \overline{a}] =_{MM} [0, 0]$ is only fulfilled if both numbers are zero. Moreover, if $[\underline{a}, \overline{a}] =_{H(\alpha)} [0, 0]$, we obtain that $\alpha \underline{a} + (1 - \alpha)\overline{a} = 0$, and this is equivalent to saying that $\underline{a} = \overline{a} = 0$. So the equivalence is also obtained for the maximax and the Hurwicz orders.

However, $[0, 0] =_{Mm} [0, 0.2]$ and $[0, 0] =_{wo} [0, 0.2]$, and therefore the reciprocal is not fulfilled for these orders. □

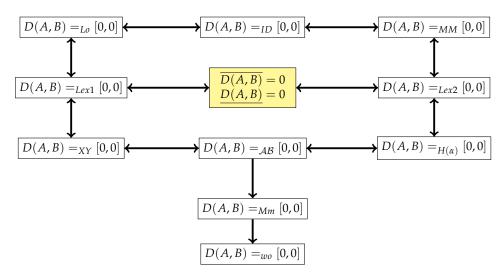The above proposition is summarized in Figure 14.

Figure 14. Zero difference for different orders.

3.4. The Importance of the Widths of the Intervals

The previous axioms are just direct translations from the ones considered in the context of fuzzy sets, and they will be the same even in the case the measure of difference is just a number. However, now we have to take into account the widths of the intervals. The next requirement is considered in order to deal properly with this uncertainty.

First of all, we will consider the following example for understanding the idea we are trying to formalise.

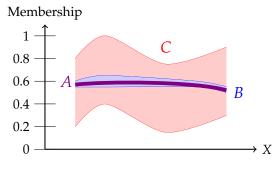**Example 8.** *Let A, B, and C be the IVFSs represented in Figure 15.*

Figure 15. Related IVFSs with different widths.

It is clear that B is embedded in C, which is denoted by $B \sqsubseteq C$, since

$$B(x) \subseteq C(x), \forall x \in X$$

As a consequence, for any third IVFS, A, the uncertainty when comparing A and C must be greater than the uncertainty when comparing A and B. Thus, for instance, in Figure 15 we are almost sure that A and B are very similar and the difference should be something similar to $D(A,B) \approx [0, 0.1]$, but when we compare A to C, we find that they could be equal but they could also be very different. A reasonable value could be $D(A,C) \approx [0, 0.7]$.

Thus, bigger uncertainty of the IVFS *C* with respect to *B* should mean bigger uncertainty in the measure of difference between *C* and a third IVFS *A* than between *B* and *A*. This implies that $D(A,C)$ is a more imprecise interval than $D(A,B)$, which is equivalent to saying that

$$D(A,B) \subseteq D(A,C)$$

In general, this requirement can be formalized as follows:

A4　If $B \sqsubseteq C$, then $D(A,B) \subseteq D(A,C)$

Here no order between intervals is involved, just the classical content between intervals, and therefore, no study about the behaviour of the different interval orders is required.

Corollary 3. *Let $D : IVFS(X) \times IVFS(X) \to L([0,1])$ satisfying Axioms A2, A3, and A4. If A and B are two IVFSs satisfying that $A(x) \cap B(x) \neq \varnothing$ for every $x \in X$, then it holds that $D(A,B) = 0$.*

Proof. Assume that $\alpha_x \in A(x) \cap B(x)$. Now take C the IVFS $C(x) = [\alpha_x, \alpha_x]$ for every $x \in X$; that is, C is a fuzzy set. According to Axiom A3, $D(C,C) = [0,0]$. On the other hand, $C \sqsubseteq A$ and $C \sqsubseteq B$; therefore, applying Axiom A4 twice and the symmetry (Axiom A2), we have that $0 = D(C,C) \geq D(A,C) \geq D(A,B)$.　□

Moreover, this axiom ensures that the imprecision about the difference between any two interval-valued fuzzy sets is greater than or equal to the one between the two furthest apart (fuzzy) sets in A and B, as we can see from the following corollary.

Corollary 4. *Let $D : IVFS(X) \times IVFS(X) \to \underline{L}([0,1])$ satisfying Axioms A2 and A4. Let A and B be any two IVFSs defined as $A(x) = [\underline{A(x)}, \overline{A(x)}]$ and $B(x) = [\underline{B(x)}, \overline{B(x)}]$ for any $x \in X$. If we consider the fuzzy set $\overline{A}$ and $\underline{B}$ defined as $\overline{A}(x) = \overline{A(x)}$ and $\underline{B}(x) = \underline{B(x)}$ for any $x \in X$, we have that $D(\overline{A}, \underline{B}) \subseteq D(A, B)$.*

Proof. It is clear that $\overline{A} \sqsubseteq A$ and $\underline{B} \sqsubseteq B$. Then, $D(\overline{A}, \underline{B}) \subseteq D(\overline{A}, B) = D(B, \overline{A}) \subseteq D(B, A) = D(A, B)$, by applying twice Axioms A2 and A4. $\square$

3.5. Proximity

It is clear that every definition of the measure of the comparison between two IVFSs should satisfy the four properties REQ1–REQ4 and, from the previous subsection, they could be immediately rewritten as:

A1. $[0,0] \preceq_o D(A, B)$.
A2. $D(A, B) =_o D(B, A)$.
A3. $D(A, B) =_o [0,0]$ iff $A, B \in FS(X)$ and $A = B$.
A4. If $B \sqsubseteq C$, then $D(A, B) \subseteq D(A, C)$.

However, they should also fulfil the fifth natural requirement:

> REQ5 For closer IVFs, the difference measure has to be smaller.

The four previous conditions are commonly accepted in the literature in the sense that most authors formalise them in the same way. However, for this fifth condition, multiple (quite different) alternatives have been proposed, leading to different definitions such us the notion of distance, divergence, or dissimilarity. We next revisit the three definitions and discuss about their convenience to model differences among IVFSs quantified by means of intervals.

3.5.1. Distances

In the definition of distance, Requirement REQ5 is formalized by means of the well-known triangular inequality:

DIST.A5 Triangular inequality: $D(A, B) \preceq_o D(A, C) + D(C, B)$.

which is here adapted to the case of IVFSs and orders in $L([0,1])$.

However, this could be a little difficult to justify if we consider that the interval which represents the membership function is just an imprecise information, as we can see with the following example.

Example 9. *Let us consider a referential X and $A(x) = [0, 0.2]$, $B(x) = [0.8, 1]$ and $C = [0.1, 0.9]$ for any $x \in X$.*

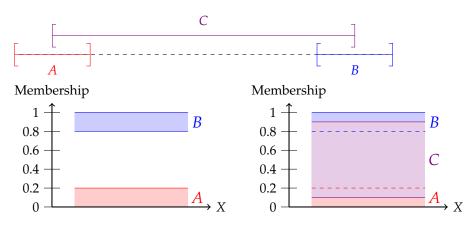*These sets are graphically represented in Figure 16.*

Figure 16. Distance between different IVFSs.

Then $D(A, B)$ seems to be greater than 0, since they have never the same membership function. However, $A(x)$ and $C(x)$ could have the same value, and, from Corollary 3, we have that $D(A, C) = [0, \alpha]$. The same happens for B and C, since $B(x) \cap C(x) \neq \varnothing, \forall x \in X$. Thus, $D(C, B) = [0, \beta]$ and $D(A, C) + D(C, B) = [0, \gamma]$, and it is not greater than or equal to $D(A, B)$ for the lattice order or the lexicographical order type 1.

Remark 1. *About distances we have yet another problem apart from the previous counterintuitive example we considered for the triangular inequality. This is that if we deal with fuzzy sets, the distance is a number. If we deal with IVFSs, the distance should be an interval. In both cases, we can define the sum. However, what happens if we consider lattice-valued fuzzy sets, for instance, if the membership function assumes values that are colours? In that case, the definition of the sum is not so immediate. However, if we just consider an order, as we do for dissimilarities and divergences, we can deal with this concept in a more general environment.*

3.5.2. Divergences

Trying to avoid the previous problems, the fifth axiom should not be based on a triangular inequality, since we are not trying to measure a distance, but the difference between two sets, which is not exactly the same in general. Now we are not studying if they are "close" or "far", but if they are similar in the sense of the description of the set.

In this sense, the fifth requirement expresses the idea that the more similar the sets are, the lower the measure of difference between them. For fuzzy sets this condition can be formalised as follows:

$$D(A \cap C, B \cap C) \leq D(A, B) \text{ and } D(A \cup C, B \cup C) \leq D(A, B)$$

and the result is the notion of divergence between fuzzy sets. Montes et al. [8] showed that this is a good option to compare fuzzy sets. Then, it is natural to think of translating this property into the context of IVFSs and that it could perform well in this context too. Consider that the value of the divergence is now an interval and taking into account any order in $L([0, 1])$, Axiom 5 could be as follows:

DIV.A5 $D(A \cap_o C, B \cap_o C) \preceq_o D(A, B)$ and $D(A \cup_o C, B \cup_o C) \preceq_o D(A, B)$.

It is again based on the interval order chosen, but if we consider a total order, we are requiring these conditions for any $A, B, C \in IVFS(X)$. For the particular case of the lattice order, we can obtain some nice properties that follow from this condition.

Proposition 10. *Let $D : IVFS(X) \times IVFS(X) \to L([0, 1])$ satisfy Axiom DIV.A5. Then, $\forall A, B \in IVFS(X)$*

1. $D(A \cap_{Lo} B, B) \preceq_{Lo} D(A, A \cup_{Lo} B)$.
2. $D(A \cap_{Lo} B, B) \preceq_{Lo} D(A, B)$.
3. $D(A \cap_{Lo} B, B) \preceq_{Lo} D(A \cap_{Lo} B, A \cup_{Lo} B)$.

4. $D(B, A \cup_{Lo} B) \preceq_{Lo} D(A \cap_{Lo} B, A \cup_{Lo} B)$.

Proof. 1.   Call $E = A$, $F = A \cup_{Lo} B$ and $G = B$ and apply the first part of Axiom DIV.A5:

$$D(E \cap_{Lo} G, F \cap_{Lo} G) \preceq_{Lo} D(E, F),$$

since $F \cap_{Lo} G = (A \cup_{Lo} B) \cap_{Lo} B = B$, the inequality follows.

2.   It Follows from the first part of Axiom DIV.A5 taking $C = B$.
3.   Call $E = A \cap_{Lo} B$, $F = A \cup_{Lo} B$ and $G = B$. Applying the first condition in Axiom DIV.A5:

$$D(E \cap_{Lo} G, F \cap_{Lo} G) \preceq_{Lo} D(E, F),$$

since $F \cap_{Lo} G = (A \cup_{Lo} B) \cap_{Lo} B = B$, the inequality follows.

4.   It follows from applying the second condition in Axiom DIV.A5 to the sets $E = A \cap_{Lo} B$ and $F = A \cup_{Lo} B$ and $G = B$.   □

In general, for any order $\preceq_o$, we obtain the following definition of divergence in $IVFS(X)$.

Definition 6. *A mapping $D : IVFS(X) \times IVFS(X) \to L([0,1])$ satisfying*

A1. $[0,0] \preceq_o D(A, B)$
A2. $D(A, B) =_o D(B, A)$
A3. $D(A, B) =_o [0,0]$ *iff* $A, B \in FS(X)$ *and* $A = B$
A4. *If* $B \sqsubseteq C$, *then* $D(A, B) \subseteq D(A, C)$
DIV.A5 $D(A \cap_o C, B \cap_o C) \preceq_o D(A, B)$ *and* $D(A \cup_o C, B \cup_o C) \preceq_o D(A, B)$.

is a divergence between IVFSs.

For the particular case of the lattice order or the interval dominance, we can obtain divergences between fuzzy sets from divergences between IVFSs as follows.

Proposition 11. *Let $\preceq_0$ be the lattice order or the interval dominance and let $\mathcal{A}$ be an aggregation function.*
Let D be a divergence measure in $IVFS(X)$. Then the map $D|_{FS(X)} : FS(X) \times FS(X) \to FS(X)$, defined as

$$D|_{FS(X)}(A, B) = \mathcal{A}(\underline{D(A', B')}, \overline{D(A', B')})$$

with $A'(x) = [A(x), A(x)]$ and $B'(x) = [B(x), B(x)]$ for any $x \in X$ is a divergence measure in $FS(X)$.

Proof. We have to check that $D|_{FS(X)}$ satisfies the three conditions of the definition of divergence between fuzzy sets.

1.   Symmetry of $D|_{FS(X)}$ follows from symmetry of D.
2.   Since $D(A', A') = [0,0]$ for every $A \in FS(X)$, also $\mathcal{A}(\underline{D(A', A')}, \overline{D(A', A')}) = \mathcal{A}(0,0) = 0$.
3.   Let us first prove that $D|_{FS(X)}(A \cap C, B \cap C) \leq D|_{FS(X)}(A, B)$ for any $C \in FS(X)$.
      By definition, $D|_{FS(X)}(A, B) = \mathcal{A}(\underline{D(A', B')}, \overline{D(A', B')})$ and

$$D|_{FS(X)}(A \cap C, B \cap C) = \mathcal{A}(\underline{D((A \cap C)', (B \cap C)')}, \overline{D((A \cap C)', (B \cap C)')}).$$

According to Corollary 1, $A' \cap_0 C' = (A \cap C)' = [\min(a, c), \min(a, c)]$ and $B' \cap_0 C' = (B \cap C)' = [\min(b, c), \min(b, c)]$. Then $D((A \cap C)', (B \cap C)') = D(A' \cap_0 C', B' \cap_0 C') \preceq_0 D(A', B')$, where the inequality follows from Axiom DIV.A5. For the interval dominance order this implies that $\underline{D(A' \cap_0 C', B' \cap_0 C')} \leq \underline{D(A', B')}$ and $\overline{D(A' \cap_0 C', B' \cap_0 C')} \leq \overline{D(A', B')}$ and the proof follows from the monotonicity of $\mathcal{A}$.

The proof for the union is totally analogous.

Therefore, $D|_{FS(X)}$ is a divergence between fuzzy sets. $\square$

The previous results seems to strengthen the idea of divergence. However, for most of the interval orders recalled in Section 2.1, Axiom DIV.A5 is incompatible with the other axioms.

For Axiom A3, we obtain the following lemma, which could be considered as a stronger version of this axiom.

Lemma 2. *For every mapping $D : IVFS(X) \times IVFS(X) \to L([0,1])$ satisfying Axioms A3 and DIV.A5 for one of the following interval orders: lattice order, lexicographic order of type 1 or type 2 or Xu and Yager, it holds that*

$$D(A, A) = [0, 0]$$

for every $A \in IVFS(X)$.

Proof. Let A be any element in $IVFS(X)$ and let B be the element in $IVFS(X)$ defined as:

$$B(x) = \left[\sup_{x \in X} \overline{A(x)}, \sup_{x \in X} \overline{A(x)}\right]$$

for any $x \in X$.

According to Proposition 3, using the lattice order, any of the lexicographic orders or Xu and Yager order, we get that $A \cap_o B = A$.

Then, for a measure of difference that satisfies Axiom DIV.A5, it holds that

$$D(A, A) = D(A \cap_o B, A \cap_o B) \preceq_0 D(B, B).$$

According to Axiom A3, $D(B, B) = [0, 0]$ for any B being a fuzzy set. So we get $D(A, A) \preceq_o [0, 0]$. However, for the interval orders considered above (lattice order, lexicographic orders and Xu and Yager order), the only possibility then is $D(A, A) = [0, 0]$. $\square$

Even if we consider IVFSs from an ontic point of view and we relax Axiom A3, that is, even if we admit $D(A, A) =_o [0, 0]$ for any A as a reasonable property, Axiom A4 forces the difference between any set and its subsets to be zero:

Corollary 5. *For any $D : IVFS(X) \times IVFS(X) \to L([0,1])$ satisfying Axioms A2, A3, A4, and DIV.A5 for one of the following interval orders: lattice order, lexicographic order of type 1 or type 2 or Xu and Yager, it holds that:*

$$D(A, B) = [0, 0] \text{ for any IVFSs such that } A \sqsubseteq B.$$

Proof. To prove this, it suffices to apply Axioms A2 and A4: $D(A, B) \subseteq D(B, B)$. However, as proven in Lemma 2, $D(B, B) = [0, 0]$ and therefore $D(A, B) = [0, 0]$. $\square$

Furthermore, this implies that the difference between any two IVFSs is zero, as we will see now.

Corollary 6. *For any $D : IVFS(X) \times IVFS(X) \to L([0,1])$ satisfying Axioms A2, A3, A4, and DIV.A5 for one of the interval orders lattice order, lexicographic order of type 1 or type 2, or Xu and Yager, it holds that:*

$$D(A, B) = [0, 0] \text{ for any } A, B \in IVFS(X).$$

Proof. Take A, B, and any IVFSs. Since $A \sqsubseteq O_1$, where $O_1(x) = [0, 1], \forall x \in X$, by Axioms A2 and A4, it holds that $D(A, B) \subseteq D(O_1, B)$. However, also $B \sqsubseteq O_1$, then

$D(O_1, B) = [0, 0]$ as proven above. Therefore, $D(A, B) \subseteq [0, 0]$. Equivalently, $D(A, B) = [0, 0]$. $\square$

Thus, if Axiom DIV.A5 is kept and we consider the most common interval orders, including lattice order, a contradiction between Axiom DIV.A5 and the other axioms in the definition of divergence arrives. Even if we admit a weaker version of Axiom 3, the combination of this relaxed version of Axiom 3, Axiom 2, Axiom 4, and Axiom DIV.A5 makes the constant function that assigns to every pair of IVFSs the value $[0, 0]$, the only possible measure of difference between IVFSs.

Therefore, the combination of Axioms 2, 3, 4, and DIV.A5 forces the use of interval dominance to compare the intervals. However, interval dominance is not an order, and due to the lack of reflexivity, it also leads to the constant function if we combine it with Axiom 2, as proven in Lemma 1.

By all theses studies, we can conclude that the use of divergences is not appropriate for the case of IVFSs.

3.5.3. Dissimilarities

We have seen that the notion of distance, in particular the triangular inequality, is not appropriate to capture the idea of difference between two IVFSs. However, we find intuitive a property of the type "the closer the sets, the smaller the difference". We have seen that the attempt to formalize "closer" by intersections and unions of IVFSs, that is, by generalizing divergencies to IVFSs leads to incompatibilities among axioms. Then, an alternative way to express the closeness of IVFSs must be considered. Dissimilarities use interval orders to capture the proximity notion: given an interval order and three IVFSs A, B, and C, A is supposed to be closer to B than to C if $A \subseteq_o B \subseteq_o C$, and since A is closer to B than to C and, on the contrary, C is closer to B than to A, the corresponding dissimilarities should be ordered in accordance with this idea of proximity.

A5   If $A \subseteq_o B \subseteq_o C$, then $D(A, B) \preceq_o D(A, C)$ and $D(B, C) \preceq_o D(A, C)$.

With this condition, the definition of dissimilarity between IVFSs would look as follows:

Definition 7. *Let $\preceq_o$ be any of the orders recalled in Section 2.1. A mapping $D : IVFS(X) \times IVFS(X) \to L([0, 1])$ satisfying*

A1. $[0, 0] \preceq_o D(A, B)$
A2. $D(A, B) =_o D(B, A)$
A3. $D(A, B) =_o [0, 0]$ iff $A, B \in FS(X)$ and $A = B$
A4. If $B \sqsubseteq C$, then $D(A, B) \subseteq D(A, C)$
A5. If $A \subseteq_o B \subseteq_o C$, then $D(A, B) \preceq_o D(A, C)$ and $D(B, C) \preceq_o D(A, C)$

is a dissimilarity between IVFSs.

Example 10. •   *The map*

$$D_0(A, B) = \begin{cases} [0, 0] & \text{if } A, B \in FS(X), A = B, \\ [0, 1] & \text{otherwise.} \end{cases}$$

is a dissimilarity w.r.t. the lattice order since for any $A, B, C \in IVFS(X)$ we have that:

A1. $[0, 0] \preceq_{Lo} D_0(A, B)$.
   By definition $0 \leq \underline{D_0(A, B)}$ and $0 \leq \overline{D_0(A, B)}$ for every $A, B \in IVFS(x)$.
A2. $D_0(A, B) =_{Lo} D_0(B, A)$.
   Symmetry also follows immediately from the definition.
A3. $D_0(A, B) =_{Lo} [0, 0]$ iff $A, B \in FS(X)$ and $A = B$.
   As proven in Proposition 8, $D_0(A, B) =_{Lo} [0, 0]$ if and only if $D_0(A, B) = [0, 0]$. (Remember that this is not always the case. For instance, if we set the Mm-order, the equality $D_0(A, B) =_{Mm} [0, 0]$ holds for any $A, B \in IVFS(X)$).

A4. If $B \sqsubseteq C$, then $D_0(A, B) \subseteq D_0(A, C)$.

If $D_0(A, B)$ takes the value $[0, 0]$, it is trivial.

If $D_0(A, B)$ takes the value $[0, 1]$, this means that either A is not a fuzzy set, or B is not a fuzzy set or both of them are fuzzy sets but they are not equal. In the first case, $D_0(A, C)$ is also $[0, 1]$. In the second case, since $B \sqsubseteq C$, C is not a fuzzy set and therefore $D_0(A, C)$ also coincides with $[0, 1]$. In the third case, since B is a fuzzy set different from A and $B \sqsubseteq C$, we have that either

– C is the same fuzzy set as B and then A and C are two different fuzzy sets and $D_0(A, C)$ is $[0, 1]$.
– Or C is a proper IVFS containing B. Since C is not a fuzzy set, then $D_0(A, C)$ is $[0, 1]$.

A5. If $A \subseteq_{Lo} B \subseteq_{Lo} C$, then $D_0(A, B) \preceq_{Lo} D_0(A, C)$ and $D_0(B, C) \preceq_{Lo} D_0(A, C)$.

If $D_0(A, C)$ is $[0, 1]$ the proof is trivial.

If $D_0(A, C)$ is $[0, 0]$, then A and C are the same fuzzy set. From $A \subseteq_{Lo} B \subseteq_{Lo} C$ we have that then B is the same fuzzy set and the proof is concluded.

As a direct consequence of Propositions *7*, *8* and *9* D_0 also fulfils Axioms *A1*, *A2*, *A3* and *A4* for any $\mathcal{AB}$-order (recall that Axiom *A4* does not depend on the order considered). Moreover, Axiom *A5* is also fulfilled for any $\mathcal{AB}$-order, by taking into account that $[0, 0] \preceq_{\mathcal{AB}} [0, 1]$ by the monotonicity of the aggregation functions, and then we could provide a proof similar to the previous one.

Thus, D_0 is an Lo-dissimilarity and an $\mathcal{AB}$-dissimilarity, and it is called the trivial dissimilarity.

- For X a finite set, the dissimilarity induced by a numerical distance:

$$D_1(A, B) = \frac{1}{|X|} \sum_{x \in X} \left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right]$$

is a dissimilarity with respect to the lattice order.

Axiom A1: Follows from the fact that $|a - b| \geq 0$ for any two values a and $b \in \mathbb{R}$.

Axiom A2: Follows from the symmetry of the absolute value of the difference: $|a - b| = |b - a|$ for any two values a and $b \in \mathbb{R}$.

Axiom A3: $D_1(A, B) = [0, 0]$ if and only if

$$\left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right] = [0, 0]$$

for all $x \in X$. For each $x \in X$, this happens if and only if $|a - b| = 0$ for all $a \in A(x), b \in B(x)$; therefore, if and only if $A(x) = B(x)$ and equal to just one value. If this happens for all $x \in X$, then it is equivalent to A and B being the same fuzzy set.

Axiom A4: Assume $B \sqsubseteq C$. We have to prove that $D_1(A, B) \subseteq D_1(A, C)$. It is sufficient to prove that for every $x \in X$ it holds that

$$\left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right] \leq \left[\inf_{a \in A(x), c \in C(x)} |a - c|, \sup_{a \in A(x), c \in C(x)} |a - c| \right].$$

Equivalently, we will prove that

(I) $\inf\limits_{a \in A(x), c \in C(x)} |a - c| \leq \inf\limits_{a \in A(x), b \in B(x)} |a - b|$.

(II) $\sup\limits_{a \in A(x), b \in B(x)} |a - b| \leq \sup\limits_{a \in A(x), c \in C(x)} |a - c|$.

Call $A(x) = [\underline{a}, \overline{a}]$, $B(x) = [\underline{b}, \overline{b}]$ and $C(x) = [\underline{c}, \overline{c}]$. Since $B \sqsubseteq C$, it holds that $[\underline{b}, \overline{b}] \subseteq [\underline{c}, \overline{c}]$. Equivalently, $\underline{c} \leq \underline{b} \leq \overline{b} \leq \overline{c}$.

(I)   To prove that $\inf\limits_{a \in A(x), c \in C(x)} |a - c| \leq \inf\limits_{a \in A(x), b \in B(x)} |a - b|$, we distinguish three cases:

* $\underline{a} < \underline{c}$ *(then $[\underline{a}, \overline{a}] \cap [\underline{c}, \overline{c}] = \varnothing$).*
  In this case

$$\inf_{a \in A(x), c \in C(x)} |a - c| = |\overline{a} - \underline{c}| \leq |\overline{a} - \underline{b}| = \inf_{a \in A(x), b \in B(x)} |a - b|.$$

* $[\underline{a}, \overline{a}] \cap [\underline{c}, \overline{c}] \neq \varnothing$
  In this case

$$\inf_{a \in A(x), c \in C(x)} |a - c| = 0 \leq \inf_{a \in A(x), b \in B(x)} |a - b|.$$

* $\overline{c} < \underline{a}$ *(then $[\underline{a}, \overline{a}] \cap [\underline{c}, \overline{c}] = \varnothing$).*
  In this case,

$$\inf_{a \in A(x), c \in C(x)} |a - c| = |\overline{c} - \underline{a}| \leq |\overline{b} - \underline{a}| = \inf_{a \in A(x), b \in B(x)} |a - b|.$$

In any case, (I) follows.

In order to prove (II), let us note the following: for any closed intervals $D = [\underline{d}, \overline{d}]$ and $E = [\underline{e}, \overline{e}]$ in $\mathbb{R}$ it holds that

$$\sup_{d \in D, e \in E} |d - e| = \max\{|\underline{d} - \overline{e}|, |\overline{d} - \underline{e}|\}.$$

The equality $|\underline{d} - \overline{e}| = |\overline{d} - \underline{e}|$ can only hold if $D = E$. If this is not the case, $\max\{|\overline{d} - \underline{e}|, |\overline{e} - \underline{d}|\} = |\underline{d} - \overline{e}| > |\overline{d} - \underline{e}|$ implies $\underline{d} < \overline{e}$ (otherwise $\overline{d} \geq \underline{d} \geq \overline{e} \geq \underline{e}$ and $|\underline{d} - \overline{e}| \leq |\overline{d} - \underline{e}|$. A contradiction).

(II) *The proof of $\displaystyle\sup_{a \in A(x), b \in B(x)} |a - b| \leq \sup_{a \in A(x), c \in C(x)} |a - c|$ follows from the previous remark.*

* *If $[\underline{a}, \overline{a}] = [\underline{b}, \overline{b}]$, then $[\underline{a}, \overline{a}] \subseteq [\underline{c}, \overline{c}]$ and*

$$\sup_{a \in A(x), b \in B(x)} |a - b| = |\overline{b} - \underline{b}| \leq \max\{|\overline{b} - \underline{c}|, |\overline{c} - \underline{b}|\} \leq \max\{|\overline{a} - \underline{c}|, |\overline{c} - \underline{a}|\} = \sup_{a \in A(x), c \in C(x)} |a - c|.$$

Otherwise,

* *If $\displaystyle\sup_{a \in A(x), b \in B(x)} |a - b| = |\underline{a} - \overline{b}|$ then $\underline{a} < \overline{b} \leq \overline{c}$ so that*

$$\sup_{a \in A(x), b \in B(x)} |a - b| = |\underline{a} - \overline{b}| \leq |\underline{a} - \overline{c}| \leq \sup_{a \in A(x), c \in C(x)} |a - c|.$$

* *Analogously, if $\displaystyle\sup_{a \in A(x), b \in B(x)} |a - b| = |\overline{a} - \underline{b}|$ then $\underline{c} \leq \underline{b} < \overline{a}$ so that*

$$\sup_{a \in A(x), b \in B(x)} |a - b| = |\overline{a} - \underline{b}| \leq |\overline{a} - underline{c}| \leq \sup_{a \in A(x), c \in C(x)} |a - c|.$$

Axiom A5: Assume $A \subseteq_{Lo} B \subseteq_{Lo} C$. Observe that

$$D_1(A, B) = \left[\frac{1}{|X|} \sum_{x \in X} \inf_{a \in A(x), b \in B(x)} |a - b|, \frac{1}{|X|} \sum_{x \in X} \sup_{a \in A(x), b \in B(x)} |a - b| \right].$$

Then, in order to prove that $D_1(A, B) \preceq_{Lo} D_1(A, C)$, it suffices to prove that for every $x \in X$,

$$\inf_{a \in A(x), b \in B(x)} |a - b| \leq \inf_{a \in A(x), c \in C(x)} |a - c| \quad \text{and} \quad \sup_{a \in A(x), b \in B(x)} |a - b| \leq \sup_{a \in A(x), c \in C(x)} |a - c|.$$

Fix an element $x \in X$ and call $A(x) = [\underline{a}, \overline{a}]$, $B(x) = [\underline{b}, \overline{b}]$ and $C(x) = [\underline{c}, \overline{c}]$. Since $A \subseteq_{Lo} B \subseteq_{Lo} C$, $\underline{a} \leq \overline{b} \leq \overline{c}$, so that

$$\sup_{a \in A(x), b \in B(x)} |a - b| = \overline{b} - \underline{a} \leq \overline{c} - \underline{a} = \sup_{a \in A(x), c \in C(x)} |a - c|.$$

We now prove $\inf\limits_{a \in A(x), b \in B(x)} |a - b| \leq \inf\limits_{a \in A(x), c \in C(x)} |a - c|$.

- if $\overline{a} \leq \underline{b}$, then $\inf\limits_{a \in A(x), b \in B(x)} |a - b| = \underline{b} - \overline{a} \leq \underline{c} - \overline{a} = \inf\limits_{a \in A(x), c \in C(x)} |a - c|$.

- If $\overline{a} > \underline{b}$, then $\inf\limits_{a \in A(x), b \in B(x)} |a - b| = 0 \leq \inf\limits_{a \in A(x), c \in C(x)} |a - c|$.

Therefore, in any case,

$$D_1(A, B) = \frac{1}{|X|} \sum_{x \in X} \left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right] \preceq_{Lo}$$

$$\frac{1}{|X|} \sum_{x \in X} \left[\inf_{a \in A(x), c \in C(x)} |a - c|, \sup_{a \in A(x), c \in C(x)} |a - c| \right] = D_1(A, C).$$

- For X, a non-finite set, the previous function may not be a dissimilarity.
  Take $X = [0, 1]$ and

$$D_1(A, B) = \frac{1}{|X|} \int_{x \in X} \left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right]$$

$$= \int_0^1 \left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right]$$

Take $A(x) = [0, 0]$ for $x \in X$ and $B(x) = [0, 1]$ for $x = \frac{1}{n}$ and $B(x) = 0$ elsewhere. Then, $\sup\limits_{a \in A(x), b \in B(x)} |a - b| = 1$ for $x \in \{\frac{1}{n} | n \in \mathbb{N}\}$ and $\sup\limits_{a \in A(x), b \in B(x)} |a - b| = 0$, elsewhere.

Since $\sup\limits_{a \in A(x), b \in B(x)} |a - b| = 0$ almost everywhere, $\int_{x \in X} \sup\limits_{a \in A(x), b \in B(x)} |a - b| = 0$ and $D_1(A, B) = [0, 0]$ despite they are not the same fuzzy set. We have then proven that D_1 does not satisfy Axiom A3.

- Let $\mathcal{A}$ and $\mathcal{B}$ be two continuous aggregation functions. The dissimilarity induced by a numerical distance:

$$D_1(A, B) = \frac{1}{|X|} \sum_{x \in X} \left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right]$$

is NOT necessarily a dissimilarity with respect to the admissible order $\preceq_{\mathcal{A}, \mathcal{B}}$.

Take as an example the aggregation functions $\mathcal{A} = \min$ and $\mathcal{B} = \max$. Consider the universe $X = \{x\}$ and the IVFSs $A(x) = [0.2, 0.8]$, $B(x) = [0.3, 0.6]$ and $C(x) = [0.45, 0.55]$. Then clearly $A \subseteq_{\mathcal{A}, \mathcal{B}} B \subseteq_{\mathcal{A}, \mathcal{B}} C$ but

$$D_1(A, B) = \left[\inf_{a \in A(x), b \in B(x)} |a - b|, \sup_{a \in A(x), b \in B(x)} |a - b| \right] = [0, 0.5] \npreceq_{\mathcal{A}, \mathcal{B}}$$

$$[0, 0.35] = \left[\inf_{a \in A(x), c \in C(x)} |a - c|, \sup_{a \in A(x), c \in C(x)} |a - c| \right] = D_1(A, C).$$

Axiom A5 is a generalization of the condition found in Torres-Manzanera et al. [32]:

TOR.A5 If $A \subseteq_{Lo} B \subseteq_{Lo} C$, then $D(A, B) \preceq_{Lo} D(A, C)$ and $D(B, C) \preceq_{Lo} D(A, C)$.

This is Axiom A5 for the particular case of the lattice order.

Takáč et al. [31] provided a similar condition but only for intervals with the same width.

TAK.A4　If $A \subseteq_{Lo} B \subseteq_{Lo} C$ and $w(A(x)) = w(B(x)) = w(C(x))$ for all $x \in X$, then $D(A, B) \preceq_{Lo} D(A, C)$ and $D(B, C) \preceq_{Lo} D(A, C)$,

where for any $[a, b] \subseteq [0, 1]$, $w([a, b])$ is the width of the interval, that is, $w([a, b]) = b - a$.

Despite its similarity to our Axiom A5, we have called it here TAK.A4, since it is the forth axiom in the definition of dissimilarity considered in [31]. These authors do not include any condition similar to Axiom A4 in their definition. For the sake of completeness, we next recall the definition given by Takáč et al.:

Definition 8. *[31] Let $\preceq_{Lo}$ be the lattice order. A mapping $D : IVFS(X) \times IVFS(X) \to L([0, 1])$ is a dissimilarity measure in $IVFS(X)$ if it satisfies:*

TAK.A1　$D(A, B) = D(B, A)$;

TAK.A2　$D(A, B) =_{Lo} [0, 0]$ *if and only if $A = B$ and $A, B \in FS(X)$;*

TAK.A3　$D(A, B) =_{Lo} [1, 1]$ *if and only if $A(x), B(x) = 0, 0], [1, 1]$ for all $x \in X$;*

TAK.A4　*If $A \subseteq_{Lo} A' \subseteq_{Lo} B' \subseteq_{Lo} B$ and $w(A(x)) = w(A'(x)) = w(B'(x)) = w(B(x))$ for all $x \in X$, then $D(A, B) \preceq_{Lo} D(A', B')$.*

This definition is clearly less restrictive than Definition 7. Condition TAK.A4 is less restrictive than A5. It neither implies Axiom A4 as we prove next.

Proposition 12. *Consider the lattice order.*

- *Axiom A5 implies Condition TAK.A4.*
- *Condition TAK.A4 does not imply Axiom A4, even in the case Conditions TAK.A1, TAK.A2, and TAK.A3 are fulfilled.*
- *Condition TAK.A4 does not imply Axiom A5, even in the case Conditions TAK.A1, TAK.A2, and TAK.A3 are fulfilled.*

Proof. • Condition TAK.A4 is a particular case of Axiom A5, so the implication is immediate.

- Let us now see that Axiom TAK.A4 does not imply Axiom A4. Take $X = \{x\}$. Then the function

$$D(A, B) = \begin{cases} [1, 1] & \text{if } \{A(x), B(x)\} = \{[1, 1], [0, 0]\} \\ [0, w(A)] & \text{if } A(x) = B(x) \\ [0.2, 1] & \text{otherwise.} \end{cases}$$

is a dissimilarity measure in the sense of Takáč et al. In fact, conditions TAK.A1, TAK.A2 and TAK.A3 are satisfied by the definition of D. Condition TAK.A4 also holds for the lattice order: we will prove that if $A \subseteq_{Lo} B \subseteq_{Lo} C$ and $w(A(x)) = w(B(x)) = w(C(x))$, then $D(A, B) \preceq_{Lo} D(A, C)$ (the case $D(B, C) \preceq_{Lo} D(A, C)$ being analogous). If $D(A, C) = [1, 1]$, then the condition holds trivially. Now assume $D(A, C) \neq [1, 1]$; then $A \neq [0, 0]$ or $C \neq [1, 1]$. If $A = C$, then also $A = B = C$, and the inequality also holds trivially.

If $A \neq C$, then $D(A, C) = [0.2, 1]$. If $A = B$, then $D(A, B) = [0, w(A)] \preceq_{Lo} [0.2, 1]$ whatever $w(A)$ is. Furthermore, if $A \neq B$, then $D(A, B) = [0.2, 1] = D(A, C)$ and the inequality also holds.

However, this function does not satisfy A4. Consider $B(x) = [0.3, 0.4]$ and $C(x) = [0.2, 0.5]$; we have that $B \sqsubseteq C$ and

$$D(B, B) = [0, 0.1] \not\sqsubseteq D(B, C) = [0.2, 1].$$

- Take $X = \{x\}$ and the function $D : IVFS(X) \times IVFS(X) \to L([0,1])$ defined as:

$$D(A,B) = \begin{cases} [0,0] & \text{if } A = B \in FS(X) \\ [1,1] & \text{if } \{A(x), B(x)\} = \{[0,0], [1,1]\} \\ [0.2, 0.2] & \text{if } w(A(x)) = w(B(x)), A \neq B \\ [0.4, 0.4] & \text{if } w(A(x)) \neq w(B(x)) \end{cases}$$

It is straightforward to check that D satisfies Definition 8 for a dissimilarity. However, it does not satisfy Axiom A5: consider $A = [0.2, 0.3]$, $B = [0.4, 0.6]$ and $C = [0.7, 0.8]$. Then $D(A, B) = [0.4, 0.4] \npreceq_{Lo} [0.2, 0.2] = D(A, C)$.
  $\square$

Condition TAK.A4 is weaker than Condition TOR.A5, even if we also impose the other four axioms we have discussed above, i.e.,

$$A1 + A2 + A3 + A4 + TAK.A4 \nRightarrow A5$$

If we take the lattice order as the interval order, even if we combine the previous axioms with Condition TAK.A3, Axiom TOR.A5 is not guaranteed:

$$\left. \begin{array}{l} A1 \\ A2 \\ A3 \\ A4 \\ TAK.A3 \\ TAK.A4 \end{array} \right\} \nRightarrow A5$$

as the following example shows.

Example 11. *Take $X = \{x\}$ and $D : IVFS(X) \times IVFS(X) \to L([0,1])$ defined as*

$$D(A,B) = \begin{cases} [0,0] & \text{if } A = B \in FS(X), \\ [1,1] & \text{if } \{A,B\} = \{[0,0],[1,1]\}, \\ [0,1] & \text{if } A = [0,\alpha] \text{ and } B = [\beta,1] \text{ or } B = [0,\alpha] \text{ and } A = [\beta,1] \\ & \text{but } \{A,B\} \neq \{[0,0],[1,1]\}, \\ [0,0.3] & \text{if } 0 \notin A \text{ or } 1 \notin B \text{ (and the opposite: } 0 \notin B \text{ or } 1 \notin A) \\ & \text{and } \max(w(A(x)), w(B(x))) \leq 0.1 \text{ and if} \\ & \{A,B\} \in FS(x), \text{ then } A \neq B \\ [0,0.4] & \text{if } 0 \notin A \text{ or } 1 \notin B \text{ (and the opposite: } 0 \notin B \text{ or } 1 \notin A) \\ & \text{and } \max(w(A(x)), w(B(x))) > 0.1 \end{cases}$$
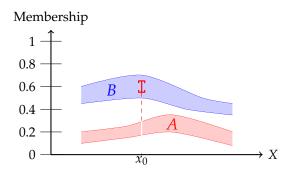
It is easy to check that D satisfies conditions A1, A2, A3, A4, and TAK.A3 and TAK.A4. However, it does not satisfy Axiom A5 for the lattice order. It suffices to take $A = [0.1, 0.2]$, $B = [0.3, 0.6]$ and $C = [0.7, 0.8]$. It holds that $A \subseteq_{Lo} B \subseteq_{Lo} C$ but $D(A, B) = [0, 0.4] \npreceq_{Lo} [0, 0.3] = D(A, C)$.

Dissimilarities are a frequent tool to compare two sets. However, they are based on a partial order. Thus, in our case, one of the main properties only applies for some of the elements in $IVFS(X)$. This is an important drawback. This also happens for fuzzy sets, where the same problem arises. For IVFSs, we have considered that

$$A \subseteq_o B \Leftrightarrow A(x) \preceq_o B(x), \forall x \in X$$

It is clearly not unique since it depends on the interval order $\preceq_o$ considered to compare IVFSs. However, in all the cases, even for total orders between intervals, we cannot obtain a total order for the family of IVFSs even if the order does not hold just for one point as the following example shows:

Example 12. *Let us take $A(x) \preceq_o B(x), \forall x \in X - \{x_0\}$ and $B(x_0) \prec_o A(x_0)$; that is, A contained in B for all the elements of the universe except for one. An example of A and B in this situation is represented in Figure 17.*

Figure 17. *A* and *B* are not comparable due to a single element x_0.

Even if the cardinality of X is infinite, one point is enough to state that A and B are incomparable and then Axiom A5 is not applicable.

We have that $A(x) \preceq_{Lo} B(x), \forall x \in X - \{x_0\}$ but $A(x_o)$ and $B(x_o)$ are incomparable w.r.t. $\preceq_{Lo}$ for instance.

The previous example shows that although Axiom A5 is without any doubt a desirable property, it is may be too weak in the sense that it only applies to a few number of IVFSs. The departing condition, $A \subseteq_o B \subseteq_o C$ (partial order) is maybe too restrictive and should be relaxed in order to apply conditions $D(A, B) \preceq_o D(A, C)$ and $D(B, C) \preceq_o D(A, C)$ to more triplets A, B, C.

Yet, although Axiom A5 has its own drawbacks, it does not lead to counterintuitive situations. We have provided examples that show that this is the case for the fifth axioms associated with distances and divergences, but we have not found any example that leads to a contradiction with the notion of dissimilarity given in Definition 7. Since the characteristic axiom, Axiom A5, is based on a partial order, it is probably not a "definitive Axiom 5", but to the best of our knowledge, it is the best way to formalize the idea of "the closer, the less different" and therefore, the best way to compare two IVFSs would be a measure that satisfies Axioms A1 to A5; that is, the measure provided in Definition 7.

4. Concluding Remarks

In this contribution, we have recalled the basic conditions that a function should satisfy in order to formalise the differences between IVFSs. We have seen that interval orders appear naturally in the formalisation of these axioms. Furthermore, since the definitions depend on the interval order, they do not have an associated definitive expression but a different one for each interval order considered.

We have also seen that the fifth logical requirement is the most problematic one to be formalised, and we have discussed the suitability of the most popular proposals: distances, divergences, and dissimilarities. We have shown that distances and divergences lead to unnatural situations, and therefore, they are not appropriate to formalise the differences between IVFSs. However, Axiom A5, the one considered in the definition of dissimilarity, does not lead to counterintuitive situations and therefore is the most appropriate among the three definitions studied in detail. Thus, our main conclusion is that dissimilarities are the only appropriate way to compare two IVFSs by means of an interval. Graphically, this is represented in Figure 18.
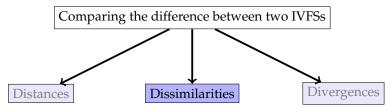
Figure 18. Suitable way to compare IVFSs.

Thus, our final recommendation after this study on different possible approaches to compare IVFSs is to consider dissimilarity measures assuming values in $L([0,1])$ where a partial order (lattice order) or a total order (lexicographical orders, Xu-Yager order or, in general, admissible orders) should be considered.

Apart from that, we have compared our proposal for the definition of dissimilarity for $IVFS(X)$ assuming values in $L([0,1])$ with the two other approaches that we have found in the literature.

The drawback of Axiom A5 is the departing point for a future work: it is necessary to find an axiom by collecting the ideas discussed in Example 12, that is, not only for the very restricted content relation in $IVFS(X)$. In a more applied future work, we would like to study the behaviour of this definition in the comparison of two colour images.

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References

1. Bouchon-Meunier, B.; Rifqi, M.; Bothorel, S. Towards general measures of comparison of objects. *Fuzzy Sets Syst.* **1996**, *84*, 143–153. [CrossRef]
2. Anthony, M.; Hammer, P.L. A boolean measure of similarity. *Discret. Appl. Math.* **2006**, *154*, 2242–2246. [CrossRef]
3. Valverde, L.; Ovchinnikov, S. Representations of T-similarity relations. *Fuzzy Sets Syst.* **2008**, *159*, 211–220. [CrossRef]
4. Wilbik, A.; Keller, J.M. A Fuzzy Measure Similarity Between Sets of Linguistic Summaries. *IEEE Trans. Fuzzy Syst.* **2013**, *21*, 183–189. [CrossRef]
5. Zhang, C.; Fu, H. Similarity measures on three kinds of fuzzy sets. *Pattern Recognit. Lett.* **2006**, *27*, 1307–1317. [CrossRef]
6. Couso, I.; Montes, S. An axiomatic definition of fuzzy divergence measures. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2008**, *16*, 1–17. [CrossRef]
7. Kobza, V.; Janis, V.; Montes, S. Generalized local divergence measures. *J. Intell. Fuzzy Syst.* **2017**, *33*, 337–350. [CrossRef]
8. Montes, S.; Couso, I.; Gil, P.; Bertoluzza, C. Divergence measure between fuzzy sets. *Int. J. Approx. Reason.* **2002**, *30*, 91–105. [CrossRef]
9. Zadeh, L. The concept of a linguistic variable and its application to approximate reasoning-I. *Inf. Sci.* **1975**, *8*, 199–249. [CrossRef]
10. Grattan-Guinness, I. Fuzzy Membership Mapped onto Intervals and Many-Valued Quantities. *Math. Log. Q.* **1976**, *22*, 149–160. [CrossRef]
11. Jahn, K. Intervall-wertige Mengen. *Math. Nach.* **1975**, *68*, 115–132. [CrossRef]
12. Sambuc, R. Fonctions and Floues: Application a l'aide au Diagnostic en Pathologie Thyroidienne. Ph.D. Thesis, Faculté de Médecine de Marseille, Marseille, France, 1975.
13. Bustince, H.; Burillo, P. Mathematical analysis of interval-valued fuzzy relations: Application to approximate reasoning. *Fuzzy Sets Syst.* **2000**, *113*, 205–219. [CrossRef]

14. Gorzałczany, M. A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets Syst.* **1987**, *21*, 1–17. [CrossRef]
15. Cornelis, C.; Deschrijver, G.; Kerre, E. Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: Construction, classification, application. *Int. J. Approx. Reason.* **2004**, *35*, 55–95. [CrossRef]
16. Turksen, I.; Zhong, Z. An approximate analogical reasoning schema based on similarity measures and interval-valued fuzzy sets. *Fuzzy Sets Syst.* **1990**, *34*, 323–346. [CrossRef]
17. Arefi, M.; Taheri, S. Weighted similarity measure on interval-valued fuzzy sets and its application to pattern recognition. *Iran. J. Fuzzy Syst.* **2014**, *11*, 67–79.
18. Deng, G.; Song, L.; Jiang, Y.; Fu, J. Monotonic Similarity Measures of Interval-Valued Fuzzy Sets and Their Applications. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2017**, *25*, 515–544. [CrossRef]
19. Pękala, B.; Dyczkowski, K.; Grzegorzewski, P.; Bentkowska, U. Inclusion and similarity measures for interval-valued fuzzy sets based on aggregation and uncertainty assessment. *Inf. Sci.* **2021**, *547*, 1182–1200. [CrossRef]
20. Wu, C.; Luo, P.; Li, Y.; Ren, X. A New Similarity Measure of Interval-Valued Intuitionistic Fuzzy Sets Considering Its Hesitancy Degree and Applications in Expert Systems. *Math. Probl. Eng.* **2014**, *2014*, 359214. [CrossRef]
21. Zhang, H.; Zhang, W.; Mei, C. Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure. *Knowl.-Based Syst.* **2009**, *22*, 449–454. [CrossRef]
22. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
23. Atanassov, K.; Gargov, G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [CrossRef]
24. Deschrijver, G.; Kerre, E. On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets Syst.* **2003**, *133*, 227–235. [CrossRef]
25. Montes, I.; Pal, N.R.; Janis, V.; Montes, S. Divergence measures for intuitionistic fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2015**, *23*, 444–456. [CrossRef]
26. Montes, I.; Janiš, V.; Pal, N.R.; Montes, S. Local divergences for Atanassov intuitionistic fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2016**, *24*, 360–373. [CrossRef]
27. Chen, T.Y. A note on distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. *Fuzzy Sets Syst.* **2007**, *158*, 2523–2525. [CrossRef]
28. Szmidt, E.; Kacprzyk, J. Intuitionistic fuzzy sets—Two and three term representations in the context of a Hausdorff distance. *Acta Univ. Matthiae Belii Ser. Math.* **2011**, *19*, 53–62.
29. Szmidt, E.; Kacprzyk, J. A Perspective on Differences between Atanassov's Intuitionistic Fuzzy Sets and Interval-Valued Fuzzy Sets. In *Fuzzy Sets, Rough Sets, Multisets and Clustering*; Studies in Computational Intelligence; Torra, V., Dahlbom, A., Narukawa, Y., Eds.; Springer: Berlin/Heidelberg, Germany, 2017; Volume 671, pp. 221–237. [CrossRef]
30. Bustince, H.; Marco-Detchart, C.; Fernandez, J.; Wagner, C.; Garibaldi, J.; Takáč, Z. Similarity between interval-valued fuzzy sets taking into account the width of the intervals and admissible orders. *Fuzzy Sets Syst.* **2020**, *390*, 23–47. [CrossRef]
31. Takáč, Z.; Bustince, H.; Pintor, J.; Marco-Detchart, C.; Couso, I. Width-Based Interval-Valued Distances and Fuzzy Entropies. *IEEE Access* **2019**, *7*, 14044–14057. [CrossRef]
32. Torres-Manzanera, E.; Kral, P.; Janis, V.; Montes, S. Uncertainty-Aware Dissimilarity Measures for Interval-Valued Fuzzy Sets. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2020**, *28*, 757–768. [CrossRef]
33. Żywica, P.; Stachowiak, A. Uncertainty-aware similarity measures-properties and construction method. In Proceedings of the 11th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2019), Prague, Czech Republic, 9–13 September 2019; pp. 512–519. [CrossRef]
34. Díaz, S.; Díaz, I.; Montes, S. An Interval-Valued Divergence for Interval-Valued Fuzzy Sets. In *Information Processing and Management of Uncertainty in Knowledge-Based Systems*; Lesot, M.J., Vieira, S., Reformat, M.Z., Carvalho, J.P., Wilbik, A., Bouchon-Meunier, B., Yager, R.R., Eds.; Springer International Publishing: Cham, Switzerland, 2020; pp. 241–249.
35. Huidobro, P.; Alonso, P.; Janiš, V.; Montes, S. Convexity and level sets for interval-valued fuzzy sets. *Fuzzy Optim. Decis. Mak.* **2021**, in press.
36. Fishburn, P. *Interval Orderings*; Wiley: New York, NY, USA, 1987.
37. Goguen, J.A. L-fuzzy sets. *J. Math. Anal. Appl.* **1967**, *18*, 145–174. [CrossRef]
38. Bustince, H.; Fernandez, J.; Kolesárová, A.; Mesiar, R. Generation of linear orders for intervals by means of aggregation functions. *Fuzzy Sets Syst.* **2013**, *220*, 69–77. [CrossRef]
39. Xu, Z.; Yager, R.R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gen. Syst.* **2006**, *35*, 417–433. [CrossRef]
40. Sniedovich, M. Wald's maximin model: A treasure in disguise! *J. Risk Financ.* **2008**, *9*, 287–291. [CrossRef]
41. Wald, A. Statistical decision functions which minimize the maximum risk. *Ann. Math.* **1945**, *46*, 265–280. [CrossRef]
42. Satia, J.; Lave, R. Markovian decision processes with uncertain transition probabilities. *Oper. Res.* **1973**, *21*, 728–740. [CrossRef]
43. Hurwicz, L. A class of criteria for decision-making under ignorance. *Cowles Com. Discuss. Pap. Stat.* **1951**, *370*, 1–16.
44. Bogart, K.P.; Bonin, J.; Mitas, J. Interval orders based on weak orders. *Discret. Appl. Math.* **1995**, *60*, 93–98. [CrossRef]
45. Mesiar, R.; Komorníková, M. Aggregation Functions on Bounded Posets. In *35 Years of Fuzzy Set Theory. Studies in Fuzziness and Soft Computing*; Cornelis, C., Deschrijver, G., Nachtegael, M., Schockaert, S., Shi, Y., Eds.; Springer: Berlin/Heidelberg, Germany, 2010; Volume 261, pp. 3–17. [CrossRef]

46.　Bustince, H.; Calvo, T.; De Baets, B.; Fodor, J.; Mesiar, R.; Montero, J.; Paternain, D.; Pradera, A. A class of aggregation functions encompassing two-dimensional OWA operators. *Inf. Sci.* **2010**, *180*, 1977–1989. [CrossRef]

47.　Huidobro, P.; Alonso, P.; Janiš, V.; Montes, S. Orders Preserving Convexity Under Intersections for Interval-Valued Fuzzy Sets. In Proceedings of the International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, Lisbon, Portugal, 15–19 June 2020; Springer: Cham, Switzerland, 2020; pp. 493–505.