mathematics

MDPI

Article

On Minimal Hypersurfaces of a Unit Sphere

Amira Ishan [1,†], Sharief Deshmukh [2,*,†], Ibrahim Al-Dayel [3,†] and Cihan Özgür [4,†]

1 Department of Mathematics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; a.ishan@tu.edu.sa
2 Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
3 Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, P.O. Box 65892, Riyadh 11566, Saudi Arabia; iaaldayel@imamu.edu.sa
4 Department of Mathematics, İzmir Democracy University, Karabağlar, 35140 İzmir, Turkey; cihan.ozgur@idu.edu.tr
* Correspondence: shariefd@ksu.edu.sa
† These authors contributed equally to this work.

Abstract: Minimal compact hypersurface in the unit sphere $\mathbb{S}^{n+1}$ having squared length of shape operator $\|A\|^2 < n$ are totally geodesic and with $\|A\|^2 = n$ are Clifford hypersurfaces. Therefore, classifying totally geodesic hypersurfaces and Clifford hypersurfaces has importance in geometry of compact minimal hypersurfaces in $\mathbb{S}^{n+1}$. One finds a naturally induced vector field w called the associated vector field and a smooth function ρ called support function on the hypersurface M of $\mathbb{S}^{n+1}$. It is shown that a necessary and sufficient condition for a minimal compact hypersurface M in $\mathbb{S}^5$ to be totally geodesic is that the support function ρ is a non-trivial solution of static perfect fluid equation. Additionally, this result holds for minimal compact hypersurfaces in $\mathbb{S}^{n+1}$, $(n > 2)$, provided the scalar curvature τ is a constant on integral curves of w. Yet other classification of totally geodesic hypersurfaces among minimal compact hypersurfaces in $\mathbb{S}^{n+1}$ is obtained using the associated vector field w an eigenvector of rough Laplace operator. Finally, a characterization of Clifford hypersurfaces is found using an upper bound on the integral of Ricci curvature in the direction of the vector field Aw.

Keywords: minimal hypersurfaces; totally geodesic hypersurfaces; sphere; clifford hypersurfaces

Citation: Ishan, A.; Deshmukh, S.; Al-Dayel, I.; Özgür, C. On Minimal Hypersurfaces of a Unit Sphere. *Mathematics* **2021**, *9*, 3161. https://doi.org/10.3390/math9243161

Academic Editor: Vladimir Balan

Received: 21 November 2021
Accepted: 6 December 2021
Published: 8 December 2021

1. Introduction

Minimal hypersurfaces in a unit sphere is a very important subject in differential geometry that has been investigated by many researchers (cf. [1–11]). An important property of these hypersurfaces is that, if the shape operator A of a minimal compact hypersurface M of $\mathbb{S}^{n+1}$ satisfies $\|A\|^2 < n$, then it is totally geodesic and if $\|A\|^2 = n$, then it is a Clifford hypersurface (cf. [1]). Note that most simple and natural hypersurface of $\mathbb{S}^{n+1}$ is the totally geodesic sphere $\mathbb{S}^n$. Moreover, important minimal hypersurfaces of of $\mathbb{S}^{n+1}$ are Clifford hypersurfaces. Characterizing totally geodesic hypersurfaces and Clifford hypersurfaces among minimal compact hypersurfaces of $\mathbb{S}^{n+1}$ is an important question in geometry of minimal hypersurfaces of $\mathbb{S}^{n+1}$.

The Ricci operator Q of a Riemannian manifold (M, g), is defined using Ricci tensor Ric, namely $Ric(X_1, X_2) = g(QX_1, X_2)$, $X_1, X_2 \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields. Moreover, the Laplace operator acting on vector fields, Δ is defined by

$$\Delta U = \sum_i \left(\nabla_{e_i} \nabla_{e_i} U - \nabla_{\nabla_{e_i} e_i} U \right), \quad U \in \mathfrak{X}(M), \tag{1}$$

where ∇ is the covariant derivative operator and $\{e_1, \ldots, e_n\}$ is a local frame on M, $n = dim M$. It is well known that this operator Δ is used for characterizing spheres and Euclidean spaces (cf. [12]). Additionally, on a Riemannian manifold (M, g), the static perfect fluid equation is (cf. [13–15])

$$fRic - Hess(f) = \frac{1}{n}(f\tau - \Delta f)g, \tag{2}$$

where τ is scalar curvature, $Hess(f)$ is the Hessian of the function f, Δ is the Laplacian that acts on smooth functions of M and $n = dimM$. This differential equation is known for its importance in general relativity and differential geometry. It is interesting to note that this differential equation plays an important role in characterizing totally geodesic hypersurfaces in $\mathbb{S}^{n+1}$ as observed in this paper.

Note that the unit sphere $\mathbb{S}^{n+1}$ as an embedded surface in the Euclidean space $\mathbb{E}^{n+2}$ having unit normal $\overline{N}$ has shape operator $-I$. For the vector field $Z = \frac{\partial}{\partial u^1}$ on $\mathbb{E}^{n+2}$, where $u^1, ..., u^{n+2}$ are coordinates on $\mathbb{E}^{n+2}$, we denote by v the projection of Z on the unit sphere $\mathbb{S}^{n+1}$. Then, it follows that

$$Z = v + \overline{\rho}\overline{N},$$

where $\overline{\rho} = \langle Z, \overline{N} \rangle$, $\langle , \rangle$ is the metric on $\mathbb{E}^{n+2}$. For a vector field U on the unit sphere $\mathbb{S}^{n+1}$, using fundamental equations for the hypersurface $\mathbb{S}^{n+1}$, we have

$$\overline{\nabla}_U v = -\overline{\rho}U, \quad grad\overline{\rho} = v, \tag{3}$$

where $\overline{\nabla}$ is the induced connection on $\mathbb{S}^{n+1}$ corresponding to the induced metric g and $grad\overline{\rho}$ is the gradient of $\overline{\rho}$ on $\mathbb{S}^{n+1}$. Thus, v is a concircular vector field on $\mathbb{S}^{n+1}$. Now, consider the totally geodesic sphere $\mathbb{S}^n$ as hypersurface of $\mathbb{S}^{n+1}$ with N the unit normal. We denote the metric on $\mathbb{S}^{n+1}$ and the induced metric on the hypersurface $\mathbb{S}^n$ by g and the induced connection on $\mathbb{S}^n$ by ∇. Additionally, we denote by ρ, the restriction of $\overline{\rho}$ to $\mathbb{S}^n$. Let w be the projection of the vector v to $\mathbb{S}^n$ and $f = g(v, N)$. Thus,

$$v = w + fN. \tag{4}$$

We call w and ρ, the associated vector field of $\mathbb{S}^n$ and the support function of $\mathbb{S}^n$, respectively. As $\mathbb{S}^n$ is totally geodesic, for a vector field U on $\mathbb{S}^n$, on using Equation (3), we find $U(f) = Ug(v, N) = g(-\overline{\rho}U, N) = 0$, that is, f is a constant c. Thus, using Equations (3) and (4), we get

$$grad\rho = (grad\overline{\rho})^T = w, \quad (grad\overline{\rho})^{\perp} = cN, \tag{5}$$

where $(grad\overline{\rho})^T$, $(grad\overline{\rho})^{\perp}$ are tangential and normal components of $grad\overline{\rho}$ to $\mathbb{S}^n$. Using Equation (3) and the fact that shape operator of $\mathbb{S}^n$ is zero, we have

$$\nabla_U w = -\rho U, \quad U \in \mathfrak{X}(\mathbb{S}^n). \tag{6}$$

Thus, using Equations (5) and (6), we observe that the function ρ on the hypersurface $\mathbb{S}^n$ satisfies $\Delta\rho = -n\rho$, $Hess(\rho) = -\rho g$. Using the expressions $Ric = (n-1)g$ and $\tau = n(n-1)$ for the sphere $\mathbb{S}^n$, we see that the support function ρ is solution of the static perfect fluid Equation (2) on the totally geodesic sphere $\mathbb{S}^n$.

Additionally, observe that using Equations (1), (5) and (6), we conclude

$$\Delta w = -w,$$

that is, the associated vector field w of $\mathbb{S}^n$ is the eigenvector of the Laplace operator Δ corresponding to eigenvalue 1 (it is customary to call a constant λ eigenvalue of Δ corresponding to eigenvector ξ if $\Delta\xi = -\lambda\xi$).

These raise two questions: (i) Given a minimal compact hypersurface M of $\mathbb{S}^{n+1}$ that has support function ρ a non-trivial solution of static perfect fluid equation necessarily totally geodesic? (ii) Given a compact hypersurface M of $\mathbb{S}^{n+1}$ with associated vector field

w an eigenvector of the Laplace operator corresponding to eigenvalue 1, is this hypersurface necessarily totally geodesic? In this paper, we answer these questions (cf. results in Section 3). We also find a characterization of a Clifford hypersurface of $\mathbb{S}^{n+1}$ (cf. the result in Section 4).

2. Preliminaries

Let N be the unit normal and A be the shape operator of an orientable minimal hypersurface M of $\mathbb{S}^{n+1}$. We denote by g the canonical metric on $\mathbb{S}^{n+1}$ and also for that is induced on M. We denote the Riemannian connections on $\mathbb{S}^{n+1}$ and the hypersurface M by $\overline{\nabla}$ and ∇, respectively. Then the fundamental equations for M are (cf. [16])

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + g(AX_1, X_2)N, \quad \overline{\nabla}_{X_1} N = -AX_1, \quad X_1, X_2 \in \mathfrak{X}(M). \tag{7}$$

The curvature tensor field R, the Ricci tensor field Ric and the scalar curvature τ of minimal hypersurface M are

$$R(X_1, X_2)X_3 = g(X_2, X_3)X_1 - g(X_1, X_3)X_2 + g(AX_2, X_3)AX_1 - g(AX_1, X_3)AX_2, \tag{8}$$

$X_1, X_2, X_3 \in \mathfrak{X}(M)$,

$$Ric(X_1, X_2) = (n-1)g(X_1, X_2) - g(AX_1, AX_2), \quad X_1, X_2 \in \mathfrak{X}(M), \tag{9}$$

and

$$\tau = n(n-1) - \|A\|^2. \tag{10}$$

The Codazzi equation of hypersurface gives

$$(\nabla A)(X_1, X_2) = (\nabla A)(X_2, X_1), \quad X_1, X_2 \in \mathfrak{X}(M), \tag{11}$$

where $(\nabla A)(X_1, X_2) = \nabla_{X_1} AX_2 - A\left(\nabla_{X_1} X_2\right)$. Taking a local frame $\{e_1, ..., e_n\}$ while using $Tr.A = 0$ and Equation (11) we get

$$\sum_i (\nabla A)(e_i, e_i) = 0. \tag{12}$$

Let v be the concircular vector field on $\mathbb{S}^{n+1}$ considered in the introduction, which satisfies Equation (3), where $\overline{\rho}$ is the function defined on $\mathbb{S}^{n+1}$ by $\overline{\rho} = \langle Z, \overline{N} \rangle$. We denote the restriction of $\overline{\rho}$ to M by ρ and the tangential projection of v on M by w that gives

$$v = w + fN, \quad f = g(v, N). \tag{13}$$

We call w the associated vector field on M and call the functions ρ, f the support function and the associated function, respectively, of M. It follows that $grad\rho = [grad\overline{\rho}]^T$ the tangential component of $grad\overline{\rho}$ and the normal component $[grad\overline{\rho}]^{\perp} = g(grad\overline{\rho}, N)N = g(v, N)N = fN$, that is, on using Equations (3) and (13), we have

$$grad\rho = w. \tag{14}$$

On differentiating Equation (13) and using Equations (3) and (7), we get on equating tangential and normal components

$$\nabla_{X_1} w = -\rho X_1 + fAX_1, \quad gradf = -Aw, \quad X_1 \in \mathfrak{X}(M). \tag{15}$$

Taking divergence in Equations (14) and (15) and using Equations (12), we get

$$\Delta\rho = -n\rho, \quad \Delta f = -f\|A\|^2. \tag{16}$$

The Hessian operator A_h of a smooth function h on a Riemannian manifold (M, g) is defined by

$$A_h(X_1) = \nabla_{X_1} gradh, \quad X_1 \in \mathfrak{X}(M), \tag{17}$$

and it is a symmetric operator. Furthermore, the Hessian $Hess(h)$ of h and A_h are related by

$$Hess(h)(X_1, X_2) = g(A_h(X_1), X_2), \quad X_1, X_2 \in \mathfrak{X}(M).$$

The Laplace operator Δ is defined by $\Delta h = div(gradh)$, which is also related to the operator A_h by

$$\Delta h = trA_h. \tag{18}$$

Well known Bochner's formula states

$$\int_M Ric(gradh, gradh) = \int_M \left((\Delta h)^2 - \|A_h\|^2\right). \tag{19}$$

Recall that for positive integers α, β, $\alpha + \beta = n$, a Clifford hypersurface is defined by

$$M = \mathbb{S}^\alpha\left(\sqrt{\frac{\alpha}{n}}\right) \times \mathbb{S}^\beta\left(\sqrt{\frac{\beta}{n}}\right) = \left\{(x, y) \in \mathbb{E}^{\alpha+1} \times \mathbb{E}^{\beta+1} : \|x\|^2 = \frac{\alpha}{n}, \|y\|^2 = \frac{\beta}{n}\right\},$$

and it is a minimal hypersurface of $\mathbb{S}^{n+1}$ with $\|A\|^2 = n$. We denote by N the unit normal vector to the Clifford hypersurface M in $\mathbb{S}^{n+1}$ by $\overline{N}$ the unit normal vector of $\mathbb{S}^{n+1}$ in the Euclidean space $\mathbb{E}^{n+2}$. Then, we have

$$N = \left(\sqrt{\frac{\beta}{n}}\zeta_1, -\sqrt{\frac{\alpha}{n}}\zeta_2\right), \quad \overline{N} = \left(\sqrt{\frac{\alpha}{n}}\zeta_1, \sqrt{\frac{\beta}{n}}\zeta_2\right),$$

where ζ_1 is unit normal to $\mathbb{S}^\alpha\left(\sqrt{\frac{\alpha}{n}}\right)$ in $\mathbb{E}^{\alpha+1}$ and ζ_2 is the unit normal to the hypersurface $\mathbb{S}^\beta\left(\sqrt{\frac{\beta}{n}}\right)$ in $\mathbb{E}^{\beta+1}$. It follows that the functions ρ and f satisfy

$$\rho = \sqrt{\frac{\alpha}{\beta}} f. \tag{20}$$

3. Characterizations of Totally Geodesic Hypersurfaces

Here, we find characterizations of totally geodesic hypersurfaces among minimal compact hypersurfaces of $\mathbb{S}^{n+1}$. Let M be the minimal hypersurface of $\mathbb{S}^{n+1}$ with support function ρ a non-trivial solution of Equation (2). Then, Equations (2) and (9) imply

$$\rho Q(X_1) - A_\rho(X_1) = \frac{\rho\tau}{n} X_1 - \frac{\Delta\rho}{n} X_1, \quad X_1 \in \mathfrak{X}(M). \tag{21}$$

Now, Equations (14) and (15) imply

$$A_\rho(X_1) = -\rho X_1 + fAX_1. \tag{22}$$

Using Equations (16) and (22) in Equation (21), we have

$$fAX_1 = \rho\left(Q(X_1) - \frac{\tau}{n} X_1\right), \quad X_1 \in \mathfrak{X}(M).$$

Differentiating this equation, we get

$$X_1(f)AX_2 + f(\nabla A)(X_1, X_2) = X_1(\rho)\left(Q(X_2) - \frac{\tau}{n} X_2\right) + \rho\left((\nabla Q)(X_1, X_2) - \frac{1}{n} X_1(\tau) X_2\right).$$

Choosing a local frame $\{e_1, ..., e_n\}$ and replacing X_1 and X_2 in above equation by e_i and taking sum, while using Equation (12), we conclude

$$A(gradf) = Q(grad\rho) - \frac{\tau}{n}grad\rho + \frac{1}{2}\rho grad\tau - \frac{\rho}{n}grad\tau, \tag{23}$$

where we have used the well known formula

$$\sum_i (\nabla Q)(e_i, e_i) = \frac{1}{2}grad\tau.$$

Now, using Equations (14) and (15) and $Q(X_1) = (n-1)X_1 - A^2X_1$ (see Equation (9)) in Equation (23), we have

$$(n-1)w - \frac{\tau}{n}w + \frac{n-2}{2n}\rho grad\tau = 0,$$

that is,

$$\left(n - 1 - \frac{\tau}{n}\right)w + \frac{n-2}{2n}\rho grad\tau = 0.$$

Using Equation (10) in above equation, we have

$$\frac{1}{n}\|A\|^2 w + \frac{n-2}{2n}\rho grad\tau = 0.$$

Taking divergence in above equation, while using $divw = -n\rho$ (outcome of Equations (14)) and (15), we get

$$\frac{1}{n}w\left(\|A\|^2\right) - \rho\|A\|^2 + \frac{n-2}{2n}w(\tau) + \frac{n-2}{2n}\rho\Delta\tau = 0,$$

that is, for $n > 2$, we have

$$\frac{1}{2n}w\left(2\|A\|^2 + (n-2)\tau\right) + \frac{n-2}{2n}\rho\left(\Delta\tau - \frac{2n}{n-2}\|A\|^2\right) = 0.$$

Using Equation (10), we get

$$\frac{n-4}{2n}w(\tau) + \frac{n-2}{2n}\rho\left(\Delta\tau - \frac{2n}{n-2}\|A\|^2\right) = 0. \tag{24}$$

Theorem 1. *A compact and connected minimal hypersurface M of the unit sphere $\mathbb{S}^5$ has support function ρ non-trivial solution of the static perfect fluid equation, if and only if, M is totally geodesic.*

Proof. For a compact and connected minimal hypersurface M of $\mathbb{S}^5$ with support function ρ a non-trivial solution of the static perfect fluid equation, using Equation (24), we have

$$\frac{1}{4}\rho\left(\Delta\tau - 4\|A\|^2\right) = 0.$$

Since, ρ is non-trivial solution, $\rho \neq 0$, above equation on connected M implies

$$\Delta\tau = 4\|A\|^2.$$

Consequently, we get that M is totally geodesic.

Conversely, we have observed in the introduction that on totally geodesic sphere $\mathbb{S}^4$ in the unit sphere $\mathbb{S}^5$, the support function ρ is a solution of the static perfect equation. We claim that ρ is non-trivial solution. If ρ is a constant, then Equation (6) implies $divw = -4\rho$, which gives $\rho = 0$ and Equation (5) implies $w = 0$. As f is constant c (see introduction) for totally geodesic hypersurface of $\mathbb{S}^5$, Equation (5) gives $grad\overline{\rho} = cN$. The sphere $\mathbb{S}^5$ being compact, there is a point $p \in \mathbb{S}^5$, with $(grad\overline{\rho})(p) = 0$, that is, $cN_p = 0$. With N being the unit vector field, we must have $c = 0$. Thus, we get the constant vector field $Z = 0$, contrary to our assumption that Z is a unit vector. This proves that, ρ is a non-trivial solution. □

As consequence of Equation (24), same as in Theorem 1, we have:

Theorem 2. *A compact and connected minimal hypersurface M of the unit sphere $\mathbb{S}^{n+1}$, $(n > 2)$, with scalar curvature τ constant along the integral curves of the associated vector field w, has support function ρ non-trivial solution of the static perfect fluid equation, if and only if, M is totally geodesic.*

Next, we use the associated vector w of M as an eigenvector of the Laplacian to get yet other characterization of the totally geodesic hypersurface of $\mathbb{S}^{n+1}$.

Theorem 3. *A compact and connected minimal hypersurface M of the unit sphere $\mathbb{S}^{n+1}$ is totally geodesic, if and only if, the associated vector field w of M satisfies $\Delta w = -w$.*

Proof. Suppose the associated vector field w of M satisfies $\Delta w = -w$. Using Equations (1) and (15), we get

$$\Delta w = -w - A^2 w,$$

which implies $A^2 w = 0$. Consequently, we get $Q(w) = (n-1)w$, that is,

$$Ric(w, w) = (n-1)\|w\|^2. \tag{25}$$

Additionally, using Equation (15), we have

$$|\pounds_w g|^2 = 4\left(n\rho^2 + f^2\|A\|^2\right), \tag{26}$$

$$\|\nabla w\|^2 = n\rho^2 + f^2\|A\|^2 \tag{27}$$

and $divw = -n\rho$, where $\pounds_w g$ is the Lie derivative of g. Using integral formula (cf. [17])

$$\int_M \left(Ric(w, w) + \frac{1}{2}|\pounds_w g|^2 - \|\nabla w\|^2 - (divw)^2 \right) = 0$$

and Equations (25)–(27), we get

$$\int_M \left((n-1)\|w\|^2 + f^2\|A\|^2 - n(n-1)\rho^2 \right) = 0. \tag{28}$$

Now, using Equation (14) and $divw = -n\rho$, we have $div(\rho w) = \|w\|^2 - n\rho^2$ and inserting it in Equation (28), we conclude

$$\int_M f^2\|A\|^2 = 0.$$

Thus, we have $f^2\|A\|^2 = 0$. If $f \neq 0$, we find M is totally geodesic. Furthermore, for $f = 0$, we see that Equations (14) and (15) imply

$$\nabla_{X_1} grad\rho = -\rho X_1, \quad X_1 \in \mathfrak{X}(M). \tag{29}$$

We proceed to show that ρ can not be a constant. If ρ is a constant, then integration of $divw = -n\rho$ implies $\rho = 0$ and this in turn by virtue of Equation (14) implies $w = 0$. Thus, we get the constant vector field $Z = 0$, that is a contradiction to the fact Z is a unit vector. Thus, ρ is a non-constant function satisfying Equation (29). This shows that M is isometric to $\mathbb{S}^n$ (cf. [18,19]) and therefore, M is totally geodesic. □

4. A Characterization of Clifford Hypersurfaces

In this section, we use an upper bound on the integral of Ricci curvature in the direction of the vector Aw to obtain a characterization of Clifford hypersurfaces.

Theorem 4. *A compact and connected non-totally geodesic minimal hypersurface M of the unit sphere $\mathbb{S}^{n+1}$ is a Clifford hypersurface, if and only if,*

$$\int_M Ric(Aw, Aw) \leq \int_M \left(nf^2 \left(2\|A\|^2 - n \right) - \left\| A_f \right\|^2 \right).$$

Proof. Let M be a compact and connected minimal hypersurface M of $\mathbb{S}^{n+1}$ with

$$\int_M Ric(Aw, Aw) \leq \int_M \left(nf^2 \left(2\|A\|^2 - n \right) - \left\| A_f \right\|^2 \right). \tag{30}$$

We have on using Equation (16)

$$(\Delta f + nf)^2 = (\Delta f)^2 + n^2 f^2 - 2nf^2 \|A\|^2.$$

Using Equations (15) and (19) in the integral of the above equation, we conclude

$$\int_M (\Delta f + nf)^2 = \int_M \left(Ric(Aw, Aw) + \left\| A_f \right\|^2 + n^2 f^2 - 2nf^2 \|A\|^2 \right),$$

that is,

$$\int_M (\Delta f + nf)^2 = \int_M \left(Ric(Aw, Aw) - \left(nf^2 \left(2\|A\|^2 - n \right) - \left\| A_f \right\|^2 \right) \right).$$

The above equation together with inequality (30) gives $\Delta f + nf = 0$. Thus, using Equation (16), we conclude

$$f \left(\|A\|^2 - n \right) = 0. \tag{31}$$

If $f = 0$, then by Equations (14) and (15), we get

$$\nabla_{X_1} grad\rho = -\rho X_1, \quad X_1 \in \mathfrak{X}(M)$$

and similar the proof of Theorem 3, we conclude M is totally geodesic (see Equation (29)). Since M is non-totally geodesic, we get $f \neq 0$. Thus, on connected M, Equation (31) implies $\|A\|^2 = n$. Hence, M is a Clifford hypersurface (cf. [1]).

Conversely, if M is a Clifford hypersurface in $\mathbb{S}^{n+1}$, then, $\|A\|^2 = n$ and using Equations (14), (15) and (20), we have

$$f = \sqrt{\frac{\beta}{\alpha}}\rho, \quad Aw = -\sqrt{\frac{\beta}{\alpha}}w, \tag{32}$$

that is,

$$Ric(Aw, Aw) = \frac{\beta}{\alpha} Ric(w, w) = \frac{\beta}{\alpha} \left((n-1)\|w\|^2 - \|Aw\|^2 \right). \tag{33}$$

Using Equation (16), we have

$$\rho\Delta\rho = -n\rho^2, \quad f\Delta f = -f^2\|A\|^2 = -nf^2$$

and integrating above equations by parts while using Equations (14) and (15), we conclude

$$\int_M \|w\|^2 = n \int_M \rho^2, \quad \int_M \|Aw\|^2 = n \int_M f^2. \tag{34}$$

Integrating the Equation (33) and using Equations (32) and (34), we get

$$\int_M Ric(Aw, Aw) = \frac{n\beta}{\alpha}\left(n - 1 - \frac{\beta}{\alpha}\right)\int_M \rho^2. \tag{35}$$

Additionally, using Equations (22) and (32), we have

$$A_f(X_1) = \sqrt{\frac{\beta}{\alpha}} A_\rho(X_1) = \sqrt{\frac{\beta}{\alpha}}(-\rho X_1 + fAX_1) = \sqrt{\frac{\beta}{\alpha}}\rho\left(-X_1 + \sqrt{\frac{\beta}{\alpha}} AX_1\right),$$

that is,

$$\left\|A_f\right\|^2 = \frac{\beta}{\alpha}\rho^2\left(n + n\frac{\beta}{\alpha}\right) = \frac{n\beta}{\alpha}\rho^2\left(1 + \frac{\beta}{\alpha}\right).$$

Thus, using Equation (32) and the above equation, we have

$$\int_M \left(nf^2\left(2\|A\|^2 - n\right) - \left\|A_f\right\|^2\right) = \int_M \left(\frac{n^2\beta}{\alpha}\rho^2 - \frac{n\beta}{\alpha}\rho^2\left(1 + \frac{\beta}{\alpha}\right)\right),$$

that is,

$$\int_M \left(nf^2\left(2\|A\|^2 - n\right) - \left\|A_f\right\|^2\right) = \frac{n\beta}{\alpha}\left(n - 1 - \frac{\beta}{\alpha}\right)\int_M \rho^2.$$

Combining the above equation with Equation (35), we get

$$\int_M Ric(Aw, Aw) = \int_M \left(nf^2\left(2\|A\|^2 - n\right) - \left\|A_f\right\|^2\right).$$

Hence, the required condition holds. □

Remark 1. *We have seen that the Theorem 1 holds for* 4*-dimensional minimal hypersurface of* $\mathbb{S}^5$*, where it is shown that for a compact minimal hypersurface of* $\mathbb{S}^5$ *to be totally geodesic, it is necessary and sufficient that the support function* ρ *is non-trivial solution of the differential Equation (2). In order that this result to hold for hypersurfaces of* $\mathbb{S}^5$ *for* $n > 2$*, in Theorem 2, we had to impose an additional restriction on the scalar curvature* τ *to satisfy* $w(\tau) = 0$*. Therefore, the question whether the result in Theorem 1 can be proved to dimension* $n \neq 4$ *is open.*

Author Contributions: A.I., S.D., I.A.-D. and C.Ö. contributed equally to this research. The research was carried out by A.I., S.D., I.A.-D. and C.Ö. and the manuscript was subsequently prepared together. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work is supported by Taif University Researchers Supporting Project number (TURSP-2020/223), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Chern, S.S.; do Carmo, M.; Kobayashi, S. Minimal submanifolds of a sphere with second fundamental form of constant length. In *Functional Analysis and Related Fields*; Springer: New York, NY, USA, 1970; pp. 59–75.
2. Deshmukh, S. First nonzero eigenvalue of a minimal hypersurface in the unit sphere. *Ann. Mat. Pura Appl.* **2012**, *191*, 529–537. [CrossRef]
3. Deshmukh, S.; Al-Dayel, I. A Note on Minimal Hypersurfaces of an Odd Dimensional Sphere. *Mathematics* **2020**, *8*, 294. [CrossRef]
4. Lawson, H.B., Jr. Local rigidity theorems for minimal hypersurfaces. *Ann. Math.* **1969**, *89*, 187–197. [CrossRef]
5. Deshmukh, S. Clifford hypersurfaces in a unit sphere. *Acta Math. Acad. Paedagog. Nyházi.* **2010**, *26*, 91–98.

6. Li, H.Z. A characterization of Clifford minimal hypersurfaces in S^4. *Proc. Am. Math. Soc.* **1995**, *123*, 3183–3187. [CrossRef]
7. Min, S-H.; Seo, K. A characterization of Clifford hypersurfaces among embedded constant mean curvature hypersurfaces in a unit sphere. *Math. Res. Lett.* **2017**, *24*, 503–534. [CrossRef]
8. Perdomo, O. Another proof for the rigidity of Clifford minimal hypersurfaces of S^n. *Mat. Ensen. Univ.* **2005**, *13*, 13–18.
9. Simons, J. Minimal varieties in riemannian manifolds. *Ann. Math.* **1968**, *88*, 62–105. [CrossRef]
10. Wang, Q. Rigidity of Clifford minimal hypersurfaces. *Monatsh. Math.* **2003**, *140*, 163–167. [CrossRef]
11. Deshmukh, S. Minimal hypersurfaces in a nearly Kaehler 6-sphere. *J. Geom. Phys.* **2010**, *60*, 623–625. [CrossRef]
12. García-Río, E.; Kupeli, D.N.; Ünal, B. On a differential equation characterizing Euclidean spheres. *J. Differ. Equations* **2003**, *194*, 287–299. [CrossRef]
13. Martin, D.; Visser, M. Algorithmic construction of static perfect fluid spheres. *Phys. Rev. D* **2004**, *69*, 104028. [CrossRef]
14. Coutinho, F.; Diógenes, R.; Leandro, B.; Ribeiro, E., Jr. Static perfect fluid space-time on compact manifolds. *Class. Quantum Gravity* **2020**, *37*, 015003. [CrossRef]
15. Oliynyk, T.A. Future stability of the FLRW fluid solutions in the presence of a positive cosmological constant. *Comm. Math. Phys.* **2016**, *346*, 293–312. [CrossRef]
16. Chen, B.-Y. *Total Mean Curvature and Submanifolds of Finite Type*; Series in Pure Mathematics; World Scientific Publishing Co.: Singapore, 1984.
17. Yano, K. *Integral Formulas in Riemannian Geometry*; Pure and Applied Mathematics; No. 1 Marcel Dekker, Inc.: New York, NY, USA, 1970.
18. Obata, M. Conformal transformations of Riemannian manifolds. *J. Differ. Geom.* **1970**, *4*, 311–333. [CrossRef]
19. Obata, M. The conjectures on conformal transformations of Riemannian manifolds. *J. Differ. Geom.* **1971**, *6*, 247–258. [CrossRef]