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Soft Semi ω-Open Sets

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Abstract: In this paper, we introduce the class of soft semi ω-open sets of a soft topological space (X, τ, A), using soft ω-open sets. We show that the class of soft semi ω-open sets contains both the soft topology τ_ω and the class of soft semi-open sets. Additionally, we define soft semi ω-closed sets as the class of soft complements of soft semi ω-open sets. We present here a study of the properties of soft semi ω-open sets, especially in (X, τ, A) and (X, τ_ω, A). In particular, we prove that the class of soft semi ω-open sets is closed under arbitrary soft union but not closed under finite soft intersections; we also study the correspondence between the soft topology of soft semi ω-open sets of a soft topological space and their generated topological spaces and vice versa. In addition to these, we introduce the soft semi ω-interior and soft semi ω-closure operators via soft semi ω-open and soft semi ω-closed sets. We prove several equations regarding these two new soft operators. In particular, we prove that these operators can be calculated using other usual soft operators in both of (X, τ, A) and (X, τ_ω, A), and some equations focus on soft anti-locally countable soft topological spaces.

Keywords: soft ω-open; soft semi-open; soft semi interior; soft semi interior; soft generated soft topological space; soft induced topological spaces

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1. Introduction and Preliminaries

In this work, we follow the notions and terminologies that appeared in [1–3]. Throughout this work, topological space and soft topological space will be denoted by TS and STS, respectively. In 1999, Molodtsov [4] introduced the concept of "soft sets", which can be seen as a new mathematical tool for dealing with uncertainties. The structure of STSs was introduced in [5]. Then, mathematicians modified several concepts of classical TSs to include STSs in [1–3,6–24], and others.

The generalizations of soft open sets play an effective role in the structure of soft topology by using them to redefine and investigate some soft topological concepts, such as soft continuity, soft compactness, soft separation axioms, etc. Soft ω-open sets in STSs were defined as an important generalization of soft open sets in [2]. Then, via ω-open sets, several research papers have appeared in [3,6–9]. Chen [25] introduced the concept of soft semi-open sets. Then, many research papers regarding soft semi-open sets appeared. The author in [3], studied ω_s-open sets as a class of soft sets, which lies strictly between soft open sets and soft semi-open sets. In this paper, we introduce the class of soft semi ω-open sets of a soft topological space (X, τ, A) using soft ω-open sets. We show that the class of soft semi ω-open sets contains both the soft topology τ_ω and the class of soft semi-open sets. Additionally, we define soft semi ω-closed sets as the class of soft complements of soft semi ω-open sets. We present here a study of the properties of soft semi ω-open sets, especially in (X, τ, A) and (X, τ_ω, A). In particular, we prove that the class of soft semi ω-open sets is closed under arbitrary soft union but not closed under finite soft intersections; also, we study the correspondence between the soft topology of soft semi ω-open sets of a soft topological space and their generated topological spaces and vice versa. In addition to these, we introduce the soft semi ω-interior and soft semi ω-closure operators via soft semi ω-open and soft semi ω-closed sets. We prove several equations regarding these two new soft operators. In particular, we prove that these operators can be calculated using other usual soft operators in both of (X, τ, A) and (X, τ_ω, A); also, some equations focus on soft anti-locally countable soft topological spaces.

The authors proved in [26,27] that soft sets are a class of special information systems. This is a strong motivation to study the structures of soft sets for information systems. Thus, this paper not only constitutes the theoretical basis for further applications of soft topology but also leads to the development of information systems.

The following definitions, results, and notations will be used in the sequel.

Definition 1. *Let X be a universal set and A be a set of parameters. A map $F : A \longrightarrow \mathcal{P}(X)$ is said to be a soft set of X relative to A. The collection of all soft sets of X relative to A will be denoted by $SS(X, A)$.*

In this paper, the *null soft set* and the *absolute soft* set of X relative to A will be denoted by 0_A and 1_A, respectively.

Definition 2. *Let Z be a universal set and B be a set of parameters. Then $H \in SS(Z, B)$ defined by:*

(a) Ref [1] $H(b) = \begin{cases} X & \text{if } b = a \\ \emptyset & \text{if } b \neq a \end{cases}$ will be denoted by a_X.

(b) Ref [28] $H(b) = X$ for all $b \in B$ will be denoted by C_X.

(c) Ref [29] $H(b) = \begin{cases} \{y\} & \text{if } b = a \\ \emptyset & \text{if } b \neq a \end{cases}$ will be denoted by a_y and will be called a soft point.

The set of all soft points in $SS(Z, B)$ will be denoted $SP(Z, B)$.

Definition 3 ([29])**.** *Let $H \in SS(Y, B)$ and $a_y \in SP(Y, B)$. Then a_y is said to belong to H (notation: $a_y \tilde{\in} H$ if $a_y \tilde{\subseteq} H$ or equivalently: $a_y \tilde{\in} H$ if and only if $y \in H(a)$.*

Definition 4. *Let $\tau \subseteq SS(X, A)$. Then τ is called a soft topology on X relative to A if*

(1) $0_A, 1_A \in \tau$,

(2) τ is closed under finite soft intersection,

(3) τ is closed under arbitrary soft union.

If τ is a soft topology on X relative to A, then the triplet (X, τ, A) will be called a *STS on X relative to A*. If (X, τ, A) is a STS and $F \in SS(X, A)$, then F is a *soft open set in* (X, τ, A) if $F \in \tau$ and F is a *soft closed set in* (X, τ, A) if $1_A - F$ is a soft open set in (X, τ, A). The family of all soft closed sets in the STS (X, τ, A) will be denoted by τ^c.

Definition 5 ([2])**.** *Let (X, τ, A) be a STS and let $H \in SS(X, A)$. Then H is said to be a soft ω-open set in (X, τ, A) if for every $a_x \tilde{\in} H$, there exist $K \in \tau$ and a countable soft set M such that $a_x \tilde{\in} K - M \tilde{\subseteq} H$. The collection of all soft ω-open sets in (X, τ, A) will be denoted by τ_ω.*

For a STS (X, τ, A), it is proved in [2] that τ_ω forms a soft topology on X relative to A that is finer than τ.

Theorem 1 ([5])**.** *If (X, τ, A) is a STS and $a \in A$, then the collection $\{H(a) : H \in \tau\}$ forms a topology on X. This topology will be denoted by τ_a .*

Theorem 2 ([30])**.** *If $(X, \Im)$ is a TS, then the collection*

$$\{H \in SS(X, A) : H(a) \in \Im \text{ for all } a \in A\}$$

forms a soft topology on X relative to A. This soft topology will be denoted by $\tau(\Im)$.

Theorem 3 ([1])**.** *Let X be an initial universe and let A be a set of parameters. Let $\{\Im_a : a \in A\}$ be an indexed family of topologies on X and let*

$$\tau = \{F \in SS(X, A) : F(a) \in \Im_a \text{ for all } a \in A\}.$$

Then τ defines a soft topology on X relative to A. This soft topology will be denoted by $\underset{a \in A}{\oplus} \Im_a$.

Let (X, τ, A) be a STS, $(X, \Im)$ be a TS, $M \in SS(X, A)$, and $S \subseteq X$. In this paper, the soft closure of M in (X, τ, A), the soft interior of M in (X, τ, A), the closure of S in $(X, \Im)$, and the interior of S in $(X, \Im)$, will be denoted by $Cl_\tau(M)$, $Int_\tau(M)$, $Cl_\Im(S)$, and $Int_\Im(S)$, respectively.

2. Soft Semi ω-Open Sets

In this section, we introduce the concepts of soft semi ω-open sets and soft semi ω-open sets and explore their essential properties. We will see that that class of semi ω-open sets forms a supra STS. To illustrate the relationships related to them, we give some examples.

Definition 6. *A soft set F in a STS (X, τ, A) is said to be a soft semi ω-open set in (X, τ, A) if there exists $K \in \tau_\omega$ such that $K \widetilde{\subseteq} F \widetilde{\subseteq} Cl_\tau(K)$. The collection of all soft semi ω-open sets in (X, τ, A) will be denoted by $S\omega O(X, \tau, A)$.*

Theorem 4. *Let (X, τ, A) be a STS and let $F \in SS(X, A)$. Then $F \in S\omega O(X, \tau, A)$ if and only if $F \widetilde{\subseteq} Cl_\tau(Int_{\tau_\omega}(F))$.*

Proof. *Necessity.* Suppose that $F \in S\omega O(X, \tau, A)$.Then there exists $H \in \tau_\omega$ such that $H \widetilde{\subseteq} F \widetilde{\subseteq} Cl_\tau(H)$. Since $H \in \tau_\omega$, then $Int_{\tau_\omega}(H) = H$. Since $H \widetilde{\subseteq} F$, then $H = Int_{\tau_\omega}(H) \widetilde{\subseteq} Int_{\tau_\omega}(F)$, and hence $F \widetilde{\subseteq} Cl_\tau(H) \widetilde{\subseteq} Cl_\tau(Int_{\tau_\omega}(F))$.

Sufficiency. Suppose that $F \widetilde{\subseteq} Cl_\tau(Int_{\tau_\omega}(F))$. Put $H = Int_{\tau_\omega}(F)$. Then $H \in \tau_\omega$ and $H \widetilde{\subseteq} F \widetilde{\subseteq} Cl_\tau(Int_{\tau_\omega}(F)) = Cl_\tau(H)$. Hence, $F \in S\omega O(X, \tau, A)$. □

Theorem 5. *For any STS (X, τ, A), $\tau_\omega \subseteq S\omega O(X, \tau, A)$.*

Proof. Let $F \in \tau_\omega$. Choose $H = F$. Then $H \in \tau_\omega$ and $H \widetilde{\subseteq} F \widetilde{\subseteq} Cl_\tau(H)$. Hence, $F \in S\omega O(X, \tau, A)$. □

The following example shows that the inclusion in Theorem 5 cannot be replaced by equality, in general:

Example 1. *Let $X = \mathbb{R}$, $A = \mathbb{Z}$, and $\tau = \{C_V : V \subseteq \mathbb{R}$ and $1 \notin V\} \cup \{C_V : V \subseteq \mathbb{R}$, $1 \in V$ and $\mathbb{R} - V$ is finite$\}$. Then $C_\mathbb{Q} \notin \tau_\omega$. On the other hand, since $C_{\mathbb{Q}-\{1\}} \in \tau_\omega$ and $C_{\mathbb{Q}-\{1\}} \widetilde{\subseteq} C_\mathbb{Q} \widetilde{\subseteq} Cl_\tau(C_{\mathbb{Q}-\{1\}})$, then $C_\mathbb{Q} \in S\omega O(X, \tau, A)$.*

Theorem 6. *For any STS (X, τ, A), $\omega_s(X, \tau, A) \subseteq SO(X, \tau, A) \subseteq S\omega O(X, \tau, A)$.*

Proof. By Theorem 4 of [3], we have $\omega_s(X, \tau, A) \subseteq SO(X, \tau, A)$. To see that $SO(X, \tau, A) \subseteq S\omega O(X, \tau, A)$, let $F \in SO(X, \tau, A)$, then there exists $H \in \tau \subseteq \tau_\omega$ such that $H \widetilde{\subseteq} F \widetilde{\subseteq} Cl_\tau(H)$. Hence, $F \in S\omega O(X, \tau, A)$. □

The author of [3] provided an example to demonstrate in general, that $\omega_s(X, \tau, A) \neq SO(X, \tau, A)$. The following example shows that the inclusion of $SO(X, \tau, A) \subseteq S\omega O(X, \tau, A)$ in Theorem 6 cannot be replaced by equality, in general.

Example 2. *Let $X = \mathbb{R}$, $A = \mathbb{Z}$, $\Im$ be the usual topology on $\mathbb{R}$, and $\tau = \{C_V : V \in \Im\}$. Let $F = C_{\mathbb{R}-\mathbb{Q}}$. Since $F \in \tau_\omega$, then by Theorem 5, $F \in S\omega O(X, \tau, A)$. On the other hand, since $Int_\tau(F) = 0_A$, then $F \notin SO(X, \tau, A)$.*

Lemma 1. *If (X, τ, A) is a soft anti-locally countable STS, then $Int_{\tau_\omega}(M) = Int_\tau(M)$ for each $M \in \tau_\omega^c$.*

Proof. Suppose that (X, τ, A) is soft anti-locally countable and let $M \in \tau_\omega^c$. Then $1_A - M \in \tau_\omega$ and by Theorem 14 of [2], $Cl_{\tau_\omega}(1_A - M) = Cl_\tau(1_A - M)$. So, $Int_{\tau_\omega}(M) = 1_A - Cl_{\tau_\omega}(1_A - M) = 1_A - Cl_\tau(1_A - M) = Int_\tau(M)$. □

Theorem 7. *If (X, τ, A) is a soft anti-locally countable STS, then $S\omega O(X, \tau, A) \cap \tau_\omega^c \subseteq SO(X, \tau, A)$.*

Proof. Let $F \in S\omega O(X, \tau, A) \cap \tau_\omega^c$. Since $F \in S\omega O(X, \tau, A)$, then by Theorem 4, $F \widetilde{\subseteq} Cl_\tau(Int_{\tau_\omega}(F))$. Since (X, τ, A) is a soft anti-locally countable and $F \in \tau_\omega^c$, then by Lemma 1, $Int_{\tau_\omega}(F) = Int_\tau(F)$. Hence, $F \widetilde{\subseteq} Cl_\tau(Int_{\tau_\omega}(F))$. Therefore, by Theorem 3.1 of [25], $F \in SO(X, \tau, A)$. □

The following example shows in that Theorem 7 the assumption of (X, τ, A) to be soft anti-locally countable is essential:

Example 3. *Let $X = \{1, 2, 3\}$, $A = \mathbb{Z}$, and $\tau = \left\{0_A, 1_A, C_{\{1\}}, C_{\{2,3\}}\right\}$. Let $F = C_{\{1,2\}}$. Since $F \in \tau_\omega$, then by Theorem 5, $F \in S\omega O(X, \tau, A)$. On the other hand, since $Cl_\tau(Int_\tau(F)) = Cl_\tau\left(C_{\{1\}}\right) = C_{\{1\}}$, then $F \notin SO(X, \tau, A)$.*

Theorem 8. *If (X, τ, A) is a soft locally countable STS, then $S\omega O(X, \tau, A) = SS(X, A)$.*

Proof. Since (X, τ, A) is soft locally countable, then by Corollary 5 of [2], $\tau_\omega = SS(X, A)$. Hence, by Theorem 5 we obtain the result. □

Theorem 9. *For any STS (X, τ, A), $\omega_s(X, \tau_\omega, A) = SO(X, \tau_\omega, A) = S\omega O(X, \tau, A)$.*

Proof. Let (X, τ, A) be a STS. By Theorem 2.7 of [3] and Theorem 6, we only need to show that $S\omega O(X, \tau_\omega, A) \subseteq SO(X, \tau_\omega, A)$. Let $F \in S\omega O(X, \tau_\omega, A)$, then there exists $H \in (\tau_\omega)_\omega$ such that $H \widetilde{\subseteq} F \widetilde{\subseteq} Cl_{(\tau_\omega)_\omega}(H)$. Since by Theorem 5 of [2], $(\tau_\omega)_\omega = \tau_\omega$, then $Cl_{(\tau_\omega)_\omega}(H) = Cl_{\tau_\omega}(H)$. Hence, $F \in SO(X, \tau_\omega, A)$. This ends the proof that $S\omega O(X, \tau_\omega, A) \subseteq SO(X, \tau_\omega, A)$. □

Theorem 10. *For any STS (X, τ, A), $S\omega O(X, \tau_\omega, A) \subseteq S\omega O(X, \tau, A)$.*

Proof. Let $F \in S\omega O(X, \tau_\omega, A)$, then by Theorem 9, $F \in SO(X, (\tau_\omega)_\omega, A) = SO(X, \tau_\omega, A)$. So, there exists $H \in \tau_\omega$ such that $H \widetilde{\subseteq} F \widetilde{\subseteq} Cl_{\tau_\omega}(H) \widetilde{\subseteq} Cl_\tau(H)$. Hence, $F \in S\omega O(X, \tau, A)$. □

The following example shows that the inclusion in Theorem 10 cannot be replaced by equality in general:

Example 4. *We consider the STS (X, τ, A) given in Example 1. We take $F = C_{\mathbb{Q}}$. Then $F \in S\omega O(X, \tau, A)$. On other hand, since $Cl_{\tau_\omega}(Int_{\tau_\omega}(F)) = C_{\mathbb{Q}-\{1\}}$, then $F \notin S\omega O(X, \tau_\omega, A)$.*

Theorem 11. *Let (X, τ, A) be a STS. If $\{S_i : i \in \Gamma\} \subseteq S\omega O(X, \tau, A)$, then $\widetilde{\bigcup_{i\in\Gamma}} S_i \in S\omega O(X, \tau, A)$.*

Proof. For every $i \in \Gamma$, choose $H_i \in \tau_\omega$ such that $H_i \widetilde{\subseteq} S_i \widetilde{\subseteq} Cl_{\tau_\omega}(H_i)$. Then $\widetilde{\bigcup_{i\in\Gamma}} H_i \in \tau_\omega$ and $\widetilde{\bigcup_{i\in\Gamma}} H_i \widetilde{\subseteq} \widetilde{\bigcup_{i\in\Gamma}} S_i \widetilde{\subseteq} \widetilde{\bigcup_{i\in\Gamma}} Cl_{\tau_\omega}(S_i) \widetilde{\subseteq} \widetilde{\bigcup_{i\in\Gamma}} Cl_{\tau_\omega}(H_i) \widetilde{\subseteq} Cl_{\tau_\omega}\left(\widetilde{\bigcup_{i\in\Gamma}} H_i\right)$. Hence, $\widetilde{\bigcup_{i\in\Gamma}} S_i \in S\omega O(X, \tau, A)$. □

The soft intersection of two soft semi ω-open sets is not in general soft semi ω-open as it is shown in the next example.

Example 5. *We consider the STS (X, τ, A) given in Example 1. We take $F = C_{\mathbb{Q}}$ and $G = C_{(\mathbb{R}-\mathbb{Q})\cup\{1\}}$. Then $F, G \in S\omega O(X, \tau, A)$. On the other hand, since $F\tilde{\cap}G = C_{\{1\}}$ and $Int_{\tau_\omega}(C_{\{1\}}) = 0_A$, then $F\tilde{\cap}G \notin S\omega O(X, \tau, A)$.*

Theorem 12. *Let (X, τ, A) be a STS. If $F \in \tau$ and $G \in S\omega O(X, \tau, A)$, then $F\tilde{\cap}G \in S\omega O(X, \tau, A)$.*

Proof. Let $F \in \tau$ and $G \in S\omega O(X, \tau, A)$. Since $G \in S\omega O(X, \tau, A)$, then we find $M \in \tau_\omega$ such that $M\tilde{\subseteq}G\tilde{\subseteq}Cl_\tau(M)$. Therefore, $F\tilde{\cap}M \in \tau_\omega$ and $F\tilde{\cap}M \tilde{\subseteq}F\tilde{\cap}G\tilde{\subseteq} F\tilde{\cap}Cl_\tau(G)\tilde{\subseteq}Cl_\tau(F\tilde{\cap}M)$. Hence, $F\tilde{\cap}G \in S\omega O(X, \tau, A)$. □

Theorem 13. *Let (X, τ, A) be a STS. If $F \in S\omega O(X, \tau, A)$ and $F\tilde{\subseteq}G\tilde{\subseteq}Cl_\tau(F)$, then $G \in S\omega O(X, \tau, A)$.*

Proof. Suppose that $F \in S\omega O(X, \tau, A)$ and $F\tilde{\subseteq}G\tilde{\subseteq}Cl_\tau(F)$. Since $F \in S\omega O(X, \tau, A)$, then there exists $H \in \tau_\omega$ such that $H\tilde{\subseteq}F \tilde{\subseteq}Cl_\tau(H)$. Since $F\tilde{\subseteq}Cl_\tau(H)$, then $Cl_\tau(F)\tilde{\subseteq}Cl_\tau(H)$. Thus, we have $H \in \tau_\omega$ and $H \tilde{\subseteq}F \tilde{\subseteq}G\tilde{\subseteq}Cl_\tau(F)\tilde{\subseteq}Cl_\tau(H)$. Hence, $G \in S\omega O(X, \tau, A)$. □

Theorem 14. *Let (X, τ, A) be a STS, Y be a nonempty subset of X, and $K \in SS(Y, A) \subseteq SS(X, A)$. If $K \in S\omega O(X, \tau, A)$, then $K \in S\omega O(Y, \tau_Y, A)$.*

Proof. Since $K \in S\omega O(X, \tau, A)$, then there exists $M \in \tau_\omega$ such that $M\tilde{\subseteq}K\tilde{\subseteq}Cl_\tau(M)$. So, we have $M = M\tilde{\cap}C_Y\tilde{\subseteq}K = K\tilde{\cap}C_Y\tilde{\subseteq}Cl_\tau(M)\tilde{\cap}C_Y = Cl_{\tau_Y}(M)$. Since $M\tilde{\subseteq}K \in SS(Y, A)$, then $M \in SS(Y, A)$. Since $M \in \tau_\omega$, then $M = M\tilde{\cap}C_Y \in (\tau_\omega)_Y$. So by Theorem 15 of [2], $M \in (\tau_Y)_\omega$. Therefore, $K \in S\omega O(Y, \tau_Y, A)$. □

The converse of Theorem 14 is not true in general, as we show in the next example:

Example 6. *We consider the STS (X, τ, A) given in Example 2. We take $Y = \mathbb{Q}$ and $K = C_{\{0\}}$. Then $K \in (\tau_Y)_\omega$ and by Theorem 5, $K \in S\omega O(Y, \tau_Y, A)$. On the other hand, since $Int_{\tau_\omega}(K) = 0_A$, then $K \notin S\omega O(X, \tau, A)$.*

Theorem 15. *Let (X, τ, A) be a STS, Y be a nonempty subset of X, and $K \in SS(Y, A)$. If $C_Y \in \tau_\omega$ and $K \in S\omega O(Y, \tau_Y, A)$, then $K \in S\omega O(X, \tau, A)$.*

Proof. Since $K \in S\omega O(Y, \tau_Y, A)$, then there exists $M \in (\tau_Y)_\omega$ such that $M\tilde{\subseteq}K\tilde{\subseteq}Cl_{\tau_Y}(M)$. Since $M \in (\tau_Y)_\omega$, then by Theorem 15 of [2], $M \in (\tau_\omega)_Y$. So, there exists $H \in \tau_\omega$ such that $M = H\tilde{\cap}C_Y$. Since $C_Y \in \tau_\omega$, then $M \in \tau_\omega$. As a result, we have $M\tilde{\subseteq}K\tilde{\subseteq}Cl_{\tau_Y}(M)\tilde{\subseteq}Cl_\tau(M)$ with $M \in \tau_\omega$, and thus $K \in S\omega O(X, \tau, A)$. □

The next example demonstrates that the assumption "$C_Y \in \tau_\omega$" in Theorem 15 cannot be weakened to "$C_Y \in S\omega O(X, \tau, A)$".

Example 7. *We consider the STS (X, τ, A) given in Example 1. We take $Y = \mathbb{Q}$ and $K = C_{\{0,1\}}$. Then $C_Y \in S\omega O(X, \tau, A) - \tau_\omega$. Additionally, $K \in S\omega O(Y, \tau_Y, A)$. On the other hand, since $Cl_\tau(Int_{\tau_\omega}(K)) = Cl_\tau\left(C_{\{0\}}\right) = C_{\{0\}}$, then $K \notin S\omega O(X, \tau, A)$.*

Theorem 16. *For any STS (X, τ, A), we have*
(a) $\tau = \{Int_\tau(F) : F \in S\omega O(X, \tau, A)\}$.
(b) $\tau_\omega = \{Int_{\tau_\omega}(F) : F \in S\omega O(X, \tau, A)\}$.

Proof. (a) Let $M \in \tau$, then $M = Int_\tau(M)$. On the other hand, by Theorem 5, we have $M \in S\omega O(X, \tau, A)$. Hence, $\tau \subseteq \{Int_\tau(F) : F \in S\omega O(X, \tau, A)\}$. Conversely, since $Int_\tau(F) \in \tau$ for every $F \in S\omega O(X, \tau, A)$, then $\{Int_\tau(F) : F \in S\omega O(X, \tau, A)\} \subseteq \tau$.

(b) Let $M \in \tau_\omega$, then $M = Int_{\tau_\omega}(M)$. On the other hand, by Theorem 5, we have $M \in S\omega O(X,\tau,A)$. Hence, $\tau_\omega \subseteq \{Int_{\tau_\omega}(F) : F \in S\omega O(X,\tau,A)\}$. Conversely, since $Int_{\tau_\omega}(F) \in \tau_\omega$ for every $F \in S\omega O(X,\tau,A)$, then $\{Int_{\tau_\omega}(F) : F \in S\omega O(X,\tau,A)\} \subseteq \tau_\omega$. □

Theorem 17. *Let X be a nonempty set and A be a set of parameters. Let τ and σ be two soft topologies on X relative to A. If $S\omega O(X,\tau,A) \subseteq S\omega O(X,\sigma,A)$, then $\tau \subseteq \sigma$ and $\tau_\omega \subseteq \sigma_\omega$.*

Proof. Suppose that $S\omega O(X,\tau,A) \subseteq S\omega O(X,\sigma,A)$, then by Theorem 16 *(a)*,

$$\begin{aligned} \tau &= \{Int_\tau(F) : F \in S\omega O(X,\tau,A)\} \\ &\subseteq \{Int_\tau(F) : F \in S\omega O(X,\sigma,A)\} \\ &= \sigma. \end{aligned}$$

and by Theorem 16 *(b)*,

$$\begin{aligned} \tau_\omega &= \{Int_{\tau_\omega}(F) : F \in S\omega O(X,\tau,A)\} \\ &\subseteq \{Int_{\tau_\omega}(F) : F \in S\omega O(X,\sigma,A)\} \\ &= \sigma_\omega. \end{aligned}$$

□

Corollary 1. *Let X be a nonempty set and A be a set of parameters. Let τ and σ be two soft topologies on X relative to A. If $S\omega O(X,\tau,A) = S\omega O(X,\sigma,A)$, then $\tau = \sigma$ and $\tau_\omega = \sigma_\omega$.*

The converse of Theorem 17 is not true in general, as shown by the next example:

Example 8. *Let $X = \mathbb{R}$, $\Im$ and $\aleph$ be the usual topology on $\mathbb{R}$ and Sorgenfrey line, respectively, $A = \mathbb{Z}$, $\tau = \{F \in SS(X,A) : F(a) \in \Im \text{ for alla } a \in A\}$, and $\sigma = \{F \in SS(X,A) : F(a) \in \sigma \text{ for all } a \in A\}$. Then $\tau \subseteq \sigma$. On the other hand, it is not difficult to check that $C_{(0,1]} \in S\omega O(X,\tau,A) - S\omega O(X,\sigma,A)$.*

Now we raise the following two natural questions.

Question 1. *Let (X,τ,A) and let $F \in S\omega O(X,\tau,A)$. Is it true that $F(a) \in S\omega O(X,\tau_a)$ for all $a \in A$?*

Question 2. *Let (X,τ,A) and let $F \in SS(X,A)$ such that $F(a) \in S\omega O(X,\tau_a)$ for all $a \in A$. Is it true that $F \in S\omega O(X,\tau,A)$?*

We leave Question 1 as an open question. However, the following example shows a negative answer to Question 2.

Example 9. *Let $X = \mathbb{R}$, $A = \{a,b\}$, and $M, N, F \in SS(X,A)$ defined as*

$$\begin{aligned} M(a) &= \mathbb{Q}^c, M(b) = \{1\}, \\ N(a) &= \{2\}, N(b) = \mathbb{Q}^c, \\ F(a) &= \{2\}, F(b) = \{1\}. \end{aligned}$$

Let $\tau = \{0_A, 1_A, M, N, M \tilde{\cup} N\}$. Then $F(a) = N(a) \in \tau_a \subseteq (\tau_a)_\omega \subseteq S\omega O(X,\tau_a)$ and $F(b) = M(b) \in \tau_b \subseteq S\omega O(X,\tau_b)$. On the other hand, it is not difficult to check that $Int_{\tau_\omega}(F) = 0_A$ and thus $F \notin S\omega O(X,\tau,A)$.

If we add the condition "τ is a generated soft topology," then Questions 1 and 2 will have positive answers.

Theorem 18. *Let $\{(X, \Im_a) : a \in A\}$ be an indexed family of TSs and let $\tau = \underset{a \in A}{\oplus} \Im_a$. Let $F \in SS(X, A)$. Then $F \in S\omega O(X, \tau, A)$ if and only if $F(a) \in S\omega O(X, \Im_a)$ for every $a \in A$.*

Proof. *Necessity*. Suppose that $F \in S\omega O(X, \tau, A)$ and let $a \in A$. Then we find $H \in \tau_\omega$ such that $H \tilde{\subseteq} F \tilde{\subseteq} Cl_\tau(H)$. So, $H(a) \subseteq F(a) \subseteq (Cl_\tau(H))(a)$. Since $H \in \tau_\omega$, then $H(a) \in (\tau_\omega)_a$ and thus by Theorem 7 of [2], $H(a) \in (\tau_a)_\omega = (\Im_a)_\omega$. Also, by Lemma 4.9 of [3], $(Cl_\tau(H))(a) = Cl_{\tau_a}(H(a))$. Hence, $F(a) \in S\omega O(X, \Im_a)$.

Sufficiency. Suppose that $F(a) \in S\omega O(X, \Im_a)$ for every $a \in A$. Then for every $a \in A$, there exists $V_a \in (\Im_a)_\omega = (\tau_a)_\omega$ such that $V_a \subseteq F(a) \subseteq Cl_{\tau_a}(V_a)$. Let $H \in SS(X, A)$ with $H(a) = V_a \in \Im_a$ for every $a \in A$. Then $H \in \left(\underset{a \in A}{\oplus} (\tau_a)_\omega\right) = \left(\underset{a \in A}{\oplus} \tau_a\right)_\omega = \tau_\omega$ and by Lemma 4.9 of [3], $(Cl_\tau(H))(a) = Cl_{\tau_a}(H(a)) = Cl_{\tau_a}(V_a)$ for all $a \in A$. Since for every $a \in A$, $H(a) = V_a \subseteq F(a) \subseteq Cl_{\tau_a}(V_a) = (Cl_\tau(H))(a)$, then $H \tilde{\subseteq} F \tilde{\subseteq} Cl_\tau(H)$. Therefore, $F \in S\omega O(X, \tau, A)$. □

Corollary 2. *Let $(X, \Im)$ be a TS and let A be a set of parameters. Let $F \in SS(X, A)$. Then $F \in S\omega O(X, \tau(\Im), A)$ if and only if $F(a) \in S\omega O(X, \Im_a)$ for every $a \in A$.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a \in A}{\oplus} \Im_a$. So by Theorem 18, we obtain the result. □

If the STS (X, τ, A) is an extended STS, then we can easily apply Theorem 3 of [31] to get positive answers to Questions 1 and 2.

Theorem 19. *If $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is a soft continuous function such that f_{pu}: $(X, \tau_\omega, A) \longrightarrow (Y, \sigma_\omega, B)$ is soft soft open, then we have $f_{pu}(F) \in S\omega O(Y, \sigma, B)$ for every $F \in S\omega O(X, \tau, A)$.*

Proof. Let $F \in S\omega O(X, \tau, A)$. Then we find $H \in \tau_\omega$ such that $H \tilde{\subseteq} F \tilde{\subseteq} Cl_\tau(H)$, and thus $f_{pu}(H) \tilde{\subseteq} f_{pu}(F) \tilde{\subseteq} f_{pu}(Cl_\tau(H))$. Since $f_{pu} : (X, \tau_\omega, A) \longrightarrow (Y, \sigma_\omega, B)$ is soft open, then $f_{pu}(H) \in \sigma_\omega$. Since $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is soft continuous, then $f_{pu}(Cl_\tau(H)) \tilde{\subseteq} Cl_\sigma(f_{pu}(H))$. Therefore, $f_{pu}(F) \in S\omega O(Y, \sigma, B)$. □

Definition 7. *Let (X, τ, A) be a STS and let $G \in SS(X, A)$. Then G is said to be soft semi ω-closed set in (X, τ, A) if $1_A - G \in S\omega O(X, \tau, A)$. The family of all semi ω-closed sets in (X, τ, A) will be denoted by $S\omega C(X, \tau, A)$.*

Theorem 20. *Let (X, τ, A) be a STS and let $G \in SS(X, A)$. Then $G \in S\omega C(X, \tau, A)$ if and only if $Int_\tau(Cl_{\tau_\omega}(G)) \tilde{\subseteq} G$.*

Proof. $G \in S\omega C(X, \tau, A)$ if and only if $1_A - G \in S\omega O(X, \tau, A)$ if and only if $1_A - G \tilde{\subseteq} Cl_\tau(Int_{\tau_\omega}(1_A - G))$ if and only if $1_A - Cl_\tau(Int_{\tau_\omega}(1_A - G)) \tilde{\subseteq} G$ if and only if $Int_\tau(1_A - Int_{\tau_\omega}(1_A - G)) \tilde{\subseteq} G$ if and only if $Int_\tau(Cl_{\tau_\omega}(G)) \tilde{\subseteq} G$. □

Theorem 21. *Let (X, τ, A) be a STS. If $\{T_i : i \in \Gamma\} \subseteq S\omega C(X, \tau, A)$, then $\underset{i \in \Gamma}{\tilde{\bigcap}} T_i \in S\omega C(X, \tau, A)$.*

Proof. For every $i \in \Gamma$, $1_A - T_i \in S\omega O(X, \tau, A)$. So by Theorem 11, $\underset{i \in \Gamma}{\tilde{\bigcup}} (1_A - T_i) = 1_A - \left(\underset{i \in \Gamma}{\tilde{\bigcap}} T_i\right) \in S\omega O(X, \tau, A)$. Hence, $\underset{i \in \Gamma}{\tilde{\bigcap}} T_i \in S\omega C(X, \tau, A)$ □

Theorem 22. *For any STS (X, τ, A), $\tau_\omega^c \subseteq S\omega C(X, \tau, A)$.*

Proof. Let $T \in \tau_{\omega}^{c}$, then $1_A - T \in \tau_{\omega}$. So by Theorem 5, $1_A - T \in S\omega O(X, \tau, A)$. Hence, $T \in S\omega C(X, \tau, A)$. $\square$

Theorem 23. *Let (X, τ, A) be a STS. If $T \in \tau^c$ and $G \in S\omega C(X, \tau, A)$, then $T \widetilde{\cup} G \in S\omega C (X, \tau, A)$.*

Proof. Let $T \in \tau^c$ and $G \in S\omega C(X, \tau, A)$. Then $1_A - T \in \tau$ and $1_A - G \in S\omega O(X, \tau, A)$. So by Theorem 12, $(1_A - T) \widetilde{\cap} (1_A - G) = 1_A - \left(T \widetilde{\cup} G\right) \in S\omega O(X, \tau, A)$. Hence, $T \widetilde{\cup} G \in S\omega C(X, \tau, A)$. $\square$

Theorem 24. *For any STS (X, τ, A), $SC(X, \tau, A) \subseteq S\omega C(X, \tau, A)$.*

Proof. Let $T \in SC(X, \tau, A)$, then $1_A - T \in SO(X, \tau, A)$. So by Theorem 6, $1_A - T \in S\omega O(X, \tau, A)$. Hence, $T \in S\omega C(X, \tau, A)$. $\square$

3. Soft Semi ω-Closure and Soft Semi ω-Interior

In this section, we introduce soft semi ω-interior and soft semi ω-closure as two new soft operators. We prove several equations regarding these operators. In particular, we prove that these operators can be calculated using other usual soft operators in both of (X, τ, A) and (X, τ_{ω}, A).

Definition 8. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. The soft semi ω-closure of M in (X, τ, A), denoted $S\omega\text{-}Cl_{\tau}(M)$, is defined by*

$$S\omega\text{-}Cl_{\tau}(M) = \widetilde{\cap}\{T : T \in S\omega C(X, \tau, A) \text{ and } M \widetilde{\subseteq} T\}.$$

Theorem 25. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. Then*
(a) $S\omega\text{-}Cl_{\tau}(M)$ is the smallest soft semi ω-closed in (X, τ, A) containing M.
(b) $M = S\omega\text{-}Cl_{\tau}(M)$ if and only if M is soft semi ω-closed in (X, τ, A).

Proof. (a) Follows from Theorem 21.
(b) Obvious. $\square$

Theorem 26. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. Then*

$$S\omega\text{-}Cl_{\tau}(M) = M \widetilde{\cup} Int_{\tau}(Cl_{\tau_{\omega}}(M)).$$

Proof. Since

$$Int_{\tau}(Cl_{\tau_{\omega}}(M \widetilde{\cup} Int_{\tau}(Cl_{\tau_{\omega}}(M)))) \widetilde{\subseteq} Int_{\tau}(Cl_{\tau_{\omega}}(M \widetilde{\cup} (Cl_{\tau_{\omega}}(M)))$$

$$= Int_{\tau}(Cl_{\tau_{\omega}}(Cl_{\tau_{\omega}}(M))) = Int_{\tau}(Cl_{\tau_{\omega}}(M)) \widetilde{\subseteq} M \widetilde{\cup} Int_{\tau}(Cl_{\tau_{\omega}}(M)),$$

then by Theorem 20, $M \widetilde{\cup} Int_{\tau}(Cl_{\tau_{\omega}}(M)) \in S\omega C(X, \tau, A)$. Since $M \widetilde{\subseteq} M \widetilde{\cup} Int_{\tau}(Cl_{\tau_{\omega}}(M))$, then by Theorem 25 (a), $S\omega\text{-}Cl_{\tau}(M) \widetilde{\subseteq} M \widetilde{\cup} Int_{\tau}(Cl_{\tau_{\omega}}(M))$. On the other hand, since by Theorem 25 (a), $S\omega\text{-}Cl_{\tau}(M) \in S\omega C(X, \tau, A)$, then by Theorem 20, $Int_{\tau}(Cl_{\tau_{\omega}}(S\omega\text{-}Cl_{\tau}(M))) \widetilde{\subseteq} S\omega\text{-}Cl_{\tau}(M)$. Thus,

$$Int_{\tau}(Cl_{\tau_{\omega}}(M)) \widetilde{\subseteq} Int_{\tau}(Cl_{\tau_{\omega}}(S\omega\text{-}Cl_{\tau}(M))) \widetilde{\subseteq} S\omega\text{-}Cl_{\tau}(M)$$

and consequently $M \widetilde{\cup} Int_{\tau}(Cl_{\tau_{\omega}}(M)) \widetilde{\subseteq} S\omega\text{-}Cl_{\tau}(M)$. Therefore, $S\omega\text{-}Cl_{\tau}(M) = M \widetilde{\cup} Int_{\tau} (Cl_{\tau_{\omega}}(M))$. $\square$

Theorem 27. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. Then*

$$S\omega\text{-}Cl_{\tau}(M) \widetilde{\subseteq} Cl_{\tau_{\omega}}(M) \widetilde{\cap} S\text{-}Cl_{\tau}(M).$$

Proof. Follows from the definitions and Theorems 22 and 24. □

The equality in Theorem 27 does not hold in general, as we show in the next example.

Example 10. *Let $X = \mathbb{R}$, $A = [0,1]$, and $\Im$ be the usual topology on $\mathbb{R}$. Consider $(X, \tau(\Im), A)$ and let $M = C_{\mathbb{Q}\cup(0,1)}$. Then $S\text{-}Cl_\tau(M) = 1_A$ and $Cl_{\tau_\omega}(M) = C_{\mathbb{Q}\cup[0,1]}$, and so $Cl_{\tau_\omega}(M)\widetilde{\cap}S\text{-}Cl_\tau(M) = C_{\mathbb{Q}\cup[0,1]}$. On the other hand, by Theorem 26, $S\omega\text{-}Cl_\tau(M) = M\widetilde{\cup}Int_\tau(Cl_{\tau_\omega}(M)) = C_{\mathbb{Q}\cup(0,1)}\widetilde{\cup}\ Int_\tau(C_{\mathbb{Q}\cup[0,1]}) = C_{\mathbb{Q}\cup(0,1)}\widetilde{\cup}C_{(0,1)} = M$.*

Definition 9. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. The soft semi ω-interior of M in (X, τ, A), denoted $S\omega\text{-}Int_\tau(M)$, and defined by*

$$S\omega\text{-}Int_\tau(M) = \widetilde{\cup}\{G : G \in S\omega O(X, \tau, A) \text{ and } G \widetilde{\subseteq} M\}.$$

Theorem 28. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. Then*
(a) $S\omega\text{-}Int_\tau(M)$ is the largest soft semi ω-open in (X, τ, A) contained in M.
(b) $M \in S\omega O(X, \tau, A)$ if and only if $M = S\omega\text{-}Int_\tau(M)$.

Proof. (a) Follows from Definition 9 and Theorem 11.
(b) Follows immediately by (a). □

Theorem 29. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. Then*

$$S\omega\text{-}Int_\tau(M) = M\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(M)).$$

Proof. Since

$$M\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(M))\widetilde{\subseteq}Cl_\tau(Int_{\tau_\omega}(M\widetilde{\cap}(Int_{\tau_\omega}(M))))\widetilde{\subseteq}Cl_\tau(Int_{\tau_\omega}(M\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(M)))),$$

then by Theorem 4, $M\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(M)) \in S\omega O(X, \tau, A)$. Since $M\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(M))\widetilde{\subseteq}M$, then by Theorem 28 (a), $M\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(M))\widetilde{\subseteq}S\omega\text{-}Int_\tau(M)$. On the other hand, since by Theorem 25 (a), $S\omega\text{-}Int_\tau(M) \in S\omega O(X, \tau, A)$, then by Theorem 4, $S\omega\text{-}Int_\tau(M)\widetilde{\subseteq}Cl_\tau(Int_{\tau_\omega}(S\omega\text{-}Int_\tau(M))\widetilde{\subseteq}Cl_\tau(Int_{\tau_\omega}(M))$. Hence, $S\omega\text{-}Int_\tau(M)\widetilde{\subseteq}M\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(M))$. Therefore, $S\omega\text{-}Int_\tau(M) = M\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(M))$. □

Theorem 30. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. Then*
(a) $S\omega\text{-}Int_\tau(1_A - M) = 1_A - S\omega\text{-}Cl_\tau(M)$.
(b) $S\omega\text{-}Cl_\tau(1_A - M) = 1_A - S\omega\text{-}Int_\tau(M)$.

Proof. (a) By Theorem 29, $S\omega\text{-}Int_\tau(1_A - M) = (1_A - M)\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(1_A - M))$. In addition, by Theorem 26.

$$\begin{aligned}
1_A - S\omega\text{-}Cl_\tau(M) &= 1_A - (M\widetilde{\cup}Int_\tau(Cl_{\tau_\omega}(M))) \\
&= (1_A - M)\widetilde{\cap}(1_A - Int_\tau(Cl_{\tau_\omega}(M))) \\
&= (1_A - M)\widetilde{\cap}(Cl_\tau(1_A - Cl_{\tau_\omega}(M)))) \\
&= (1_A - M)\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(1_A - M))
\end{aligned}$$

Thus, $S\omega\text{-}Int_\tau(1_A - M) = 1_A - S\omega\text{-}Cl_\tau(M)$.

(b) By (a), $S\omega\text{-}Int_\tau(M) = S\omega\text{-}Int_\tau(1_A - (1_A - M)) = 1_A - S\omega\text{-}Cl_\tau(1_A - M)$. So, $1_A - S\omega\text{-}Int_\tau(M) = S\omega\text{-}Cl_\tau(1_A - M)$. □

Theorem 31. *Let (X, τ, A) be a STS and $M \in SS(X, A)$. Then the following conditions are equivalent:*
(a) M is soft dense in (X, τ_ω, A).
(b) $S\omega\text{-}Cl_\tau(M) = 1_A$.

(c) If $N \in S\omega C(X,\tau,A)$ *and* $M\widetilde{\subseteq}N$*, then* $N = 1_A$*.*
(d) For every $G \in S\omega O(X,\tau,A) - \{0_A\}$*,* $G\widetilde{\cap}M \neq 0_A$*.*
(e) $S\omega\text{-}Int_\tau(1_A - M) = 0_A$*.*

Proof. (a) $\Longrightarrow$ (b): By (a), $Cl_{\tau_\omega}(M) = 1_A$. So by Theorem 26,

$$\begin{aligned} S\omega\text{-}Cl_\tau(M) &= M\widetilde{\cup}Int_\tau(Cl_{\tau_\omega}(M)) \\ &= M\widetilde{\cup}Int_\tau(1_A) \\ &= M\widetilde{\cup}1_A \\ &= 1_A. \end{aligned}$$

(b) $\Longrightarrow$ (c): Let $N \in S\omega C(X,\tau,A)$ with $M\widetilde{\subseteq}N$. Then by (b), $1_A = S\omega\text{-}Cl_\tau(M)\widetilde{\subseteq}S\omega\text{-}Cl_\tau(N) = N$. Thus, $N = 1_A$.

(c) $\Longrightarrow$ (d): Suppose to the contrary that there exists $G \in S\omega O(X,\tau,A) - \{0_A\}$ such that $G\widetilde{\cap}M = 0_A$. Then $M\widetilde{\subseteq}1_A - G$ and $1_A - G \in S\omega C(X,\tau,A)$. Thus, by (c), $1_A - G = 1_A$ and hence $G = 0_A$, a contradiction.

(d) $\Longrightarrow$ (e): Suppose to the contrary that $S\omega\text{-}Int_\tau(1_A - M) \neq 0_A$. Then we have $S\omega\text{-}Int_\tau(1_A - M) \in S\omega O(X,\tau,A) - \{0_A\}$ and by (d), $S\omega\text{-}Int_\tau(1_A - M)\widetilde{\cap}M \neq 0_A$. However, $S\omega\text{-}Int_\tau(1_A - M)\widetilde{\cap}M\widetilde{\subseteq}(1_A - M)\widetilde{\cap}M = 0_A$, a contradiction.

(e) $\Longrightarrow$ (a): By (e) and Theorem 30 (a), we have $0_A = 1_A - S\omega\text{-}Cl_\tau(M)$ and so $1_A = S\omega\text{-}Cl_\tau(M)\widetilde{\subseteq}Cl_{\tau_\omega}(M)$. Thus, $Cl_{\tau_\omega}(M) = 1_A$ and hence, M is soft dense in (X,τ_ω,A). □

Theorem 32. *Let* (X,τ,A) *be soft anti-locally countable and* $M \in SS(X,A)$*. Then*

$$S\omega\text{-}Int_\tau(S\omega\text{-}Cl_\tau(M)) = S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau(Int_\tau(Cl_{\tau_\omega}(M))).$$

Proof. Since (X,τ,A) be a soft anti-locally countable and $Cl_{\tau_\omega}(M) \in \tau_\omega^c$, then by Lemma 1, $Int_{\tau_\omega}(Cl_{\tau_\omega}(M)) = Int_\tau(Cl_{\tau_\omega}(M))$. So, by Theorems 27 and 29, we have

$$\begin{aligned} S\omega\text{-}Int_\tau(S\omega\text{-}Cl_\tau(M)) &= S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(S\omega\text{-}Cl_\tau(M))) \\ &\widetilde{\subseteq} S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(Cl_{\tau_\omega}(M))) \\ &= S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau(Int_\tau(Cl_{\tau_\omega}(M))). \end{aligned}$$

On the other hand, by Theorems 26 and 29, we have

$$\begin{aligned} S\omega\text{-}Int_\tau(S\omega\text{-}Cl_\tau(M)) &= S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau(Int_{\tau_\omega}(S\omega\text{-}Cl_\tau(M))) \\ &= S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau\big(Int_{\tau_\omega}\big(M\widetilde{\cup}Int_\tau(Cl_{\tau_\omega}(M))\big)\big) \\ &\widetilde{\supseteq} S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau\big(Int_{\tau_\omega}(M)\widetilde{\cup}Int_\tau(Cl_{\tau_\omega}(M))\big) \\ &\widetilde{\supseteq} S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau\big(Int_\tau(M)\widetilde{\cup}Int_\tau(Cl_{\tau_\omega}(M))\big) \\ &= S\omega\text{-}Cl_\tau(M)\widetilde{\cap}Cl_\tau(Int_\tau(Cl_{\tau_\omega}(M))). \end{aligned}$$

As we show in the next example, it is necessary for (X,τ,A) to be soft anti-locally countable in Theorem 32. □

Example 11. *Let* (X,τ,A) *be as in Example 3. Take* $M = C_{\{1,2\}}$*. Then by Theorems 26 and 29,*

$$\begin{aligned} S\omega\text{-}Cl_\tau(M) &= M\widetilde{\cup}Int_\tau(Cl_{\tau_\omega}(M)) \\ &= M\widetilde{\cup}Int_\tau(M) \\ &= M, \end{aligned}$$

$$\begin{aligned} S\omega\text{-}Int_\tau(S\omega\text{-}Cl_\tau(M)) &= S\omega\text{-}Int_\tau(M) \\ &= M\tilde{\cap}Cl_\tau(Int_{\tau_\omega}(M)) \\ &= M\tilde{\cap}Cl_\tau(M) \\ &= M, \end{aligned}$$

and

$$\begin{aligned} S\omega\text{-}Cl_\tau(M)\tilde{\cap}Cl_\tau(Int_\tau(Cl_{\tau_\omega}(M))) &= M\tilde{\cap}Cl_\tau(Int_\tau((M)) \\ &= M\tilde{\cap}Cl_\tau\left(C_{\{1\}}\right) \\ &= M\tilde{\cap}C_{\{1\}} \\ &= C_{\{1\}} \neq M. \end{aligned}$$

Theorem 33. *Let (X, τ, A) be a soft anti-locally countable STS and $M \in SS(X, A)$. Then*

$$S\omega\text{-}Cl_\tau(S\omega\text{-}Int_\tau(M)) = S\omega\text{-}Int_\tau(M)\tilde{\cup}Int_\tau(Cl_\tau(Int_{\tau_\omega}(M))).$$

Proof. Since (X, τ, A) is soft anti-locally countable and $Int_{\tau_\omega}(M) \in \tau_\omega$, then by Theorem 14 of [2], $Cl_{\tau_\omega}(Int_{\tau_\omega}(M)) = Cl_\tau(Int_{\tau_\omega}(M))$. So, by Theorem 26, we have

$$\begin{aligned} S\omega\text{-}Cl_\tau(S\omega\text{-}Int_\tau(M)) &= S\omega\text{-}Int_\tau(M)\tilde{\cup}Int_\tau(Cl_{\tau_\omega}(S\omega\text{-}Int_\tau(M))) \\ &\tilde{\supseteq} S\omega\text{-}Int_\tau(M)\tilde{\cup}Int_\tau(Cl_{\tau_\omega}(Int_{\tau_\omega}(M)) \\ &= S\omega\text{-}Int_\tau(M)\tilde{\cup}Int_\tau(Cl_\tau(Int_{\tau_\omega}(M))). \end{aligned}$$

On the other hand, by Theorems 26 and 29, we have

$$\begin{aligned} S\omega\text{-}Cl_\tau(S\omega\text{-}Int_\tau(M)) &= S\omega\text{-}Int_\tau(M)\tilde{\cup}Int_\tau(Cl_{\tau_\omega}(S\omega\text{-}Int_\tau(M))) \\ &= S\omega\text{-}Int_\tau(M)\tilde{\cup}Int_\tau(Cl_{\tau_\omega}(M\tilde{\cap}Cl_\tau(Int_{\tau_\omega}(M))) \\ &\tilde{\subseteq} S\omega\text{-}Int_\tau(M)\tilde{\cup}\big(Int_\tau\big((Cl_{\tau_\omega}(M))\tilde{\cap}Cl_\tau(Int_{\tau_\omega}(M))\big)\big) \\ &= S\omega\text{-}Int_\tau(M)\tilde{\cup}\big(Int_\tau(Cl_{\tau_\omega}(M))\tilde{\cap}Int_\tau(Cl_\tau(Int_{\tau_\omega}(M)))\big) \\ &\tilde{\subseteq} S\omega\text{-}Int_\tau(M)\tilde{\cup}Int_\tau(Cl_\tau(Int_{\tau_\omega}(M))). \end{aligned}$$

□

4. Conclusions

As a weaker form of soft ω-open sets and soft semi-open sets, the concept of soft semi ω-open sets is introduced and studied. It is proved that the class of soft semi ω-open sets is closed under an arbitrary soft union but not closed under finite soft intersections. The correspondence between the soft topology of soft semi ω-open sets of a soft topological space and their generated topological spaces and vice versa is studied. In addition to these, soft semi ω-interior and soft semi ω-closure as two new soft operators are introduced. Several characterizations, relationships, and examples regarding our new concepts are given. The following topics could be considered in future studies: (1) to define soft semi ω-continuous functions; (2) to define soft semi ω-open functions; and (3) to define new separation axioms via soft semi ω-open sets.

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