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A General Class of Differential Hemivariational Inequalities Systems in Reflexive Banach Spaces

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Abstract: We consider an abstract system consisting of the parabolic-type system of hemivariational inequalities (SHVI) along with the nonlinear system of evolution equations in the frame of the evolution triple of product spaces, which is called a system of differential hemivariational inequalities (SDHVI). A hybrid iterative system is proposed via the temporality semidiscrete technique on the basis of the Rothe rule and feedback iteration approach. Using the surjective theorem for pseudomonotonicity mappings and properties of the partial Clarke's generalized subgradient mappings, we establish the existence and priori estimations for solutions to the approximate problem. Whenever studying the parabolic-type SHVI, the surjective theorem for pseudomonotonicity mappings, instead of the KKM theorems exploited by other authors in recent literature for a SHVI, guarantees the successful continuation of our demonstration. This overcomes the drawback of the KKM-based approach. Finally, via the limitation process for solutions to the hybrid iterative system, we derive the solvability of the SDHVI with no convexity of functions $\mathbf{u} \mapsto f_l(t, \mathbf{x}, \mathbf{u}), l = 1, 2$ and no compact property of C_0-semigroups $e^{A_l(t)}, l = 1, 2$.

Keywords: systems of differential hemivariational inequalities; C_0-semigroup; Rothe rule; Pseudomonotonicity; Partial Clarke's generalized subdifferential

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1. Introduction

For convenience, let the EPs, VIs, EFs, HVIs, CGS, DVIs, DHVIs, DMVIs, DMHVIs, EE, PDEs and PCGDDs represent the equilibrium problems, variational inequalities, energy functionals, hemivariational inequalities, Clarke's generalized subdifferential, differential variational inequalities, differential hemivariational inequalities, differential mixed variational inequalities, differential mixed hemivariational inequalities, evolution equation, partial differential equations and partial Clarke's generalized directional derivatives, respectively.

To the best of our knowledge, the theory of VIs, which was first extended to treat the EPs, is intently relevent to the convexity of EFs, and is the basis of various arguments of monotonicity. In case the relevent EFs are of nonconvexity (i.e., superpotentials), the other type of inequalities emerges as the variational formula of a problem. They are referred to as HVIs and their derivation is based on the properties of the CGS formulated for locally Lipschitz functionals. In comparision with the VIs, the stationary HVIs do not coincide with minimization problems, they yield substationarity problems, whose research began at the originating work in [1]. Various problems are formulated via nonsmooth superpotentials, so it is very natural that, in the past three decades, a lot of authors paid attention to developing the theory of HVIs and applications, e.g., in contact mechanics [2,3], well-posedness [4,5], control problems [6,7], inclusion problems without convexity and smoothness [8–10], etc.

The concept of DHVIs was first put forth in [11]. Interest in the DHVIs started with the similar interest in the DVIs. The DVIs were first methodically considered in [12] in finite-dimensional spaces, since the DVIs are helpful to formulate models about both constraints and dynamics in the pattern of inequalities which emerge in a lot of applied topics, e.g., Coulomb friction issues for contacting bodies, electrical circuits with ideal diodes, dynamic traffic networks, economical dynamics, mechanical impact issues, etc. After the work [12], a large number of authors were interested in promoting the growth of theory of DVIs and applications. In particular, the existence of periodic solutions to a class of DVIs and global bifurcation issues were discussed in [13] in finite dimensional spaces via the topological approaches of the theory of set-valued mappings and some versions of the technique of guiding functions. Meanwhile, the stability theorem for a novel class of DVIs was derived in [14] via the monotonicity technique and the approach of Mosco's convergence. Moreover, the dynamic Nash EP of multiple players with shared constraints and dynamic decision processes was investigated in [15] via the idea of DVIs; see [14] for more details.

It is worth noting that the above works were implemented only in Euclidean spaces. However, a large number of applied issues in physical sciences, economical dynamics, operations research, engineering and so forth, are more exactly formulated via PDEs. According to this motivation, the existence of solutions for a class of DMVIs in Banach spaces was demonstrated in [16,17] by using fixed point results for condensing multivalued mappings, the Filippov implicit function lemma and the theory of semigroups. However, up to now, only one reference (i.e., [11]), investigated the DHVI in Banach spaces consisting of the EE and elliptic-type HVI instead of the parabolic-type. Moreover, it was assumed in [11] that the constraint set K is of boundedness, the function $u \mapsto f(t,x,u)$ maps convex subsets of K to convex sets and the C_0-semigroup $e^{A(t)}$ is of compactness. In this way, to overcome those drawbacks, the authors [18] filled a gap and provided new mathematical techniques and approaches for DHVIs. Furthermore, motivated by a class of DMVIs in [17], the authors [19] proposed a class of DMHVIs systems with PCGDDs and demonstrated the nonemptiness and compactness of their solution sets.

Let V, E, X and Y be reflexive, separable Banach spaces, H be a separable Hilbert space, $A : D(A) \subset E \to E$ be the infinitesimal generator of C_0-semigroup e^{At} in E and

$$\begin{aligned} &f : (0,T) \times E \times Y \to E, \quad \vartheta : H \to Y, \quad \mathcal{N} : V \to V^*, \\ &M : V \to X, \quad J : E \times X \to \mathbf{R}, \quad F : (0,T) \times E \to V^* \end{aligned}$$

be given mappings. In 2018, Migorski and Zeng [18] investigated the abstract problem constituted by the parabolic-type HVI along with the abstract EE, formulated below:

Find $u : (0,T) \to V$ and $x : (0,T) \to E$ s.t.

$$\begin{aligned} &x'(t) = Ax(t) + f(t,x(t),\vartheta u(t)) \ \text{ for a.e. } t \in (0,T), \\ &(u'(t),v)_H + \langle \mathcal{N}(u(t)),v\rangle + J^\circ(x(t),Mu(t);Mv) \geq \langle F(t,x(t)),v\rangle, \ \ \forall v \in V, \ \text{a.e. } t \in (0,T), \\ &x(0) = x_0 \text{ and } u(0) = u_0. \end{aligned}$$

It is worth pointing out that the authors [18], via the Rothe rule, first studied the parabolic-type HVI driven by the abstract EE. Up to now, there have been only a few papers devoted to the Rothe rule for HVIs, see [20]. It is worth mentioning that these were focused only on a single HVI via the Rothe rule.

Next, for convenience, let the AS, AP, SHVI, SEE, ETPS, SDHVI, HIS and PCGS represent an abstract system, an approximate problem, a system of hemivariational inequalities, a nonlinear system of evolution equations, an evolution triple of product spaces, a system of differential hemivariational inequalities, a hybrid iterative system and partial CGS, respectively. Inspired by recent works in [18,19], we introduce and consider the AS, which is constituted by the parabolic-type SHVI along with the SEE, in the frame of the ETPS, which is referred to as the SDHVI. The HIS is proposed via the temporality semidiscrete technique on basis of the backward Euler difference formula (i.e., the Rothe rule), and the feedback iteration approach. Using the surjective theorem for pseudomonotonicity mappings and properties of PCGS mappings, we demonstrate the existence of solutions to the AP and provide the priori estimation for solutions to the AP. At the end, via the limitation process for solutions to the HIS, we derive the solvability of the SDHVI with no convexity of functions $\mathbf{u} \mapsto f_l(t, \mathbf{x}, \mathbf{u}), l = 1, 2$ and no compact property of C_0-semigroups $e^{A_l(t)}, l = 1, 2$.

Until now, except for the DHVI considered in [18], many works about the DVIs were boosted only by elliptic-type VIs/HVIs. Here, we first consider the SDHVI driven by the parabolic-type SHVI. In addition, except for the DHVI considered in [18], in contrast to the previous works [11,16,17,19], in this article we assume no convexity condition on the functions $\mathbf{u} \mapsto f_l(t, \mathbf{x}, \mathbf{u}), l = 1, 2$ and no compactness condition on C_0-semigroups $e^{A_l(t)}, l = 1, 2$.

The article is assigned below. In Section 2, we recall some concepts and basic results about nonsmooth and nonlinear analysis, and present the formulation of the AS. In Section 3, we formulate a solution to the AS, and then give the formulation of the HIS. We obtain the solvability of the HIS via the surjective theorem for pseudomonotonicity mappings and derive the priori estimation of solutions to the HIS. Finally, via the limitation process for solutions to the HIS, we establish the existence of solutions to the AS.

It is also worthy of note that there are evident disadvantages of the method based on the KKM approach for studying the parabolic-type SHVI. Indeed, if the mappings in the method based on the KKM approach are not the KKM ones, then there are several possibilities which happen in the demonstration process, e.g., in particular, whenever studying the parabolic-type SHVI. This might result in an unsuccessful continuation of the demonstration. Practically, this is precisely the shortcoming of the KKM-based approach.

2. Preliminaries

We first recall some notations, concepts and basic results, and then give the formulation of the AS. We start with definitions and properties of semicontinuous set-valued mappings. Suppose that E and F both are topological spaces. The setvalued operator $\Gamma : E \to 2^F$ is referred to as being

(i) of upper semicontinuity (u.s.c.) at $x \in E$ iff, for each open $O \subset F$ with $\Gamma x \subset O$, $\exists U_x$ (i.e., a neighborhood of x) s.t.

$$\Gamma(U_x) := \bigcup_{y \in U_x} \Gamma y \subset O. \tag{1}$$

In case the above relation holds for all $x \in E$, Γ is said to be u.s.c.

(ii) of lower semicontinuity (l.s.c.) at $x \in E$ iff, for each open $O \subset F$ with $\Gamma x \cap O \neq \emptyset$, $\exists U_x$ (i.e., a neighborhood of x) s.t.

$$\Gamma y \cap O \neq \emptyset, \quad \forall y \in U_x.$$

In case the above relation holds for all $x \in E$, Γ is said to be l.s.c.

(iii) of continuity at $x \in E$ iff, Γ not only is u.s.c. at $x \in X$ and but also is l.s.c. at $x \in X$. In case this holds for all $x \in E$, Γ is said to be continuous.

Proposition 1 (see [3]). *The assertions below are of equivalence mutually:*

(i) $\Gamma : E \to 2^F$ *is u.s.c.;*

(ii) for each closed $C \subset F$, $\Gamma^-(C) := \{x \in E | \Gamma x \cap C \neq \emptyset\}$ *is of closedness in* E*;*

(iii) for each open $O \subset F$, $\Gamma^+(O) := \{x \in E | \Gamma x \subset O\}$ *is of openness in* E.

In what follows, we assume that X is a reflexive Banach space with its dual X^*. A single-valued mapping $A : X \to X^*$ is referred to as being pseudomonotone, if A is of boundedness and for each sequence $\{x_n\} \subseteq X$ converging weakly to $x \in X$ s.t. $\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle_{X^*\times X} \le 0$, one has

$$\langle Ax, x - y\rangle_{X^*\times X} \le \liminf_{n\to\infty} \langle Ax_n, x_n - y\rangle_{X^*\times X}, \quad \forall y \in X. \tag{2}$$

Recall that a mapping $A : X \to X^*$ is of pseudomonotonicity, iff $x_n \to x$ weakly in X and $\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle_{X^*\times X} \le 0$ entails

$$\lim_{n\to\infty} \langle Ax_n, x_n - x\rangle_{X^*\times X} = 0 \text{ with weak convergence of } \{Ax_n\} \text{ to } Ax.$$

In addition, in case $A \in \mathcal{L}(X, X^*)$ is of nonnegativity, A is of pseudomonotonicity.

Recall that a multivalued operator $T : X \to 2^{X^*}$ is said to be pseudomonotone if

(a) for every $v \in X$, the set $Tv \subset X^*$ is nonempty, closed and convex;

(b) T is u.s.c. from each finite dimensional subspace of X to X^* endowed with the weak topology;

(c) for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ s.t. $u_n \to u$ weakly in X, $u_n^* \in Tu_n$ for all $n \ge 1$ and $\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0$, one has that $\forall v \in X$, $\exists u^*(v) \in Tu$ s.t.

$$\langle u^*(v), u - v\rangle_{X^*\times X} \le \liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{X^*\times X}.$$

Also, recall the CGS of locally Lipschitz functional; see [3]. Suppose that X is a Banach space and $h : X \to \mathbf{R}$ is locally Lipschitz. Given $u, v \in X$ arbitrarily. The CGDD of h at point $u \in X$ in direction $v \in X$, written as $h^\circ(u; v)$, is formulated below

$$h^\circ(u; v) = \limsup_{w\to u, \lambda\downarrow 0,} \frac{h(w + \lambda v) - h(w)}{\lambda}. \tag{3}$$

The CGS of h at $u \in X$, written as $\partial h(u)$, is the set in X^*, formulated below

$$\partial h(u) = \{\xi \in X^* : h^\circ(u; v) \ge \langle \xi, v\rangle_{X^*\times X}, \forall v \in X\}. \tag{4}$$

The following lemma provides some basic properties for the CGDD and CGS; see [3].

Lemma 1. *Suppose that $h : X \to \mathbf{R}$ is locally Lipschitz. Given $u, v \in X$ arbitrarily. Then*

(i) *$v \mapsto h^\circ(u; v)$ is positively homogeneous, subadditive and finite, and hence convex;*

(ii) *$h^\circ(u; v)$ is u.s.c. on $X \times X$ as a functional of (u, v), and as a functional of v alone, is Lipschitz continuous;*

(iii) *$h^\circ(u; -v) = (-h)^\circ(u; v)$;*

(iv) *$\partial h(u)$ is nonempty, weak*-compact, bounded and convex in X^* for each $u \in X$;*

(v) *for all $v \in X$, one has $h^\circ(u; v) = \max\{\langle \xi, v\rangle_{X^*\times X} : \xi \in \partial h(u)\}$;*

(vi) *$\partial h(u)$ has the closed graph in $X \times (w^*\text{-}X^*)$ topology, with $(w^*\text{-}X^*)$ being the space X^* endowed with weak* topology, i.e., whenever $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ are sequences s.t. $u_n^* \in \partial h(u_n)$, $u_n \to u$ in X and $u_n^* \to u^*$ weak*ly in X^*, one has $u^* \in \partial h(u)$.*

Proposition 2 (see [21]). *Suppose that U and Y are reflexive Banach spaces and $S : U \to Y$ is the linear continuous mapping with compactness. One denotes by $S^* : Y^* \to U^*$ the adjoint mapping of S. Let $h : Y \to \mathbf{R}$ be locally Lipschitz s.t.*

$$\|\partial h(u)\|_{Y^*} \le c_h(1 + \|u\|_Y), \quad \forall u \in U$$

with $c_h > 0$. Then the setvalued mapping $G : U \to 2^{U^*}$ formulated below

$$G(u) = S^*\partial h(Su), \quad \forall u \in U$$

is of pseudomonotonicity.

The surjective theorem below can be found in [2,22].

Theorem 1. *Suppose that Y is a reflexive Banach space and $\Gamma : Y \to 2^{Y^*}$ is a coercive operator with pseudomonotonicity. Then Γ is of surjectivity, i.e., $\forall \phi \in Y^*$, $\exists y \in Y$ s.t. $\Gamma y \ni \phi$.*

We construct the spaces of functions, defined on $[0, T]$ with $0 < T < \infty$. Let π indicate a division of $(0, T)$ via a pool of subintervals $\sigma_l = (a_l, b_l)$ s.t. $[0, T] = \bigcup_{l=1}^{n} \overline{\sigma}_l$. Let $\mathcal{F}$ denote the family of all such divisions. For a Banach space X and $1 \leq q < \infty$, we construct the space

$$BV^q(0, T; X) = \{v : [0, T] \to X \mid \sup_{\pi \in \mathcal{F}} \{ \sum_{\sigma_l \in \pi} \|v(b_l) - v(a_l)\|_X^q \} < \infty\} \tag{5}$$

and formulate the seminorm of $v : [0, T] \to X$ below

$$\|v\|_{BV^q(0,T;X)}^q = \sup_{\pi \in \mathcal{F}} \{ \sum_{\sigma_l \in \pi} \|v(b_l) - v(a_l)\|_X^q \}. \tag{6}$$

Suppose that the Banach spaces X, Z are s.t. $X \subset Z$ with continuous embedding. For $1 \leq p \leq \infty$ and $1 \leq q < \infty$ we construct the Banach space below

$$M^{p,q}(0, T; X, Z) = L^p(0, T; X) \bigcap BV^q(0, T; Z), \tag{7}$$

equipped with norm $\|\cdot\|_{L^p(0,T;X)} + \|\cdot\|_{BV^q(0,T;Z)}$.

Proposition 3 (see [23])**.** *Suppose that $X_1 \subset X_2 \subset X_3$ are Banach spaces s.t. X_1 is reflexive, the embedding $X_1 \hookrightarrow X_2$ is of compactness, and the embedding $X_2 \hookrightarrow X_3$ is of continuity. In case the set B is of boundedness in $M^{p,q}(0, T; X_1, X_3)$ with $p, q \in [1, \infty)$, B is of relative compactness in $L^p(0, T; X_2)$.*

We recall the discrete form of Gronwall's inequality below.

Lemma 2 (see [24])**.** *Given $T \in (0, \infty)$. For integer $N \geq 1$, we define $\tau = \frac{T}{N}$. Suppose that $\{g_i\}_{i=1}^N$ and $\{e_i\}_{i=1}^N$ both are sequences of nonnegative reals s.t.*

$$e_i \leq \bar{c} g_i + \bar{c}\tau \sum_{i=1}^{n-1} e_i \quad \text{for } n = 1, \ldots, N$$

with constant $\bar{c} > 0$ independent of N (or τ). Then $\exists c > 0$, independent of N (or τ), s.t.

$$e_n \leq c(g_n + \sum_{i=1}^{n-1} g_i) \quad \text{for } n = 1, \ldots, N.$$

Put $l, k = 1, 2$ and $k \neq l$. Let V_l, E_l, X_l and Y_l be reflexive, separable Banach spaces, H_l be a separable Hilbert space, let $\mathrm{Z} = Z_1 \times Z_2$, $\forall Z_l \in \{V_l, E_l, X_l, Y_l\}$, and suppose that $A_l : D(A_l) \subset E_l \to E_l$ is the infinitesimal generator of C_0-semigroup $e^{A_l t}$ in E_l and

$$\begin{array}{l} f_l : (0, T) \times \mathrm{E} \times \mathrm{Y} \to E_l, \;\; \vartheta_l : H_l \to Y_l, \;\; \mathcal{N}_l : V_l \to V_l^*, \\ M_l : V_l \to X_l, \;\; J : \mathrm{E} \times \mathrm{X} \to \mathbf{R}, \;\; F_l : (0, T) \times E_k \to V_l^* \end{array}$$

are the mappings given. Inspired by [18,19], we formulate the AS below

Find $\mathbf{u} : (0, T) \to \mathrm{V}$ and $\mathbf{x} : (0, T) \to \mathrm{E}$ with $\mathbf{u} = (u_1, u_2)$ and $\mathbf{x} = (x_1, x_2)$, s.t.

$$\begin{cases} x_1'(t) = A_1 x_1(t) + f_1(t, \mathbf{x}(t), \vartheta\mathbf{u}(t)), \;\; \text{a.e. } t \in (0, T), \\ x_2'(t) = A_2 x_2(t) + f_2(t, \mathbf{x}(t), \vartheta\mathbf{u}(t)), \;\; \text{a.e. } t \in (0, T), \end{cases} \tag{8}$$

$$
\begin{cases}
(u_1'(t), v_1)_{H_1} + \langle \mathcal{N}_1(u_1(t)), v_1 \rangle_{V_1^* \times V_1} + J_1^\circ(\mathbf{x}(t), M_1 u_1(t), M_2 u_2(t); M_1 v_1) \\
\quad \geq \langle F_1(t, x_2(t)), v_1 \rangle_{V_1^* \times V_1}, \ \ \forall v_1 \in V_1, \text{ a.e. } t \in (0, T), \\
(u_2'(t), v_2)_{H_2} + \langle \mathcal{N}_2(u_2(t)), v_2 \rangle_{V_2^* \times V_2} + J_2^\circ(\mathbf{x}(t), M_1 u_1(t), M_2 u_2(t); M_2 v_2) \\
\quad \geq \langle F_2(t, x_1(t)), v_2 \rangle_{V_2^* \times V_2}, \ \ \forall v_2 \in V_2, \text{ a.e. } t \in (0, T),
\end{cases} \tag{9}
$$

and

$$
\mathrm{x}(0) = \mathrm{x}_0 \text{ and } \mathrm{u}(0) = \mathrm{u}_0, \tag{10}
$$

where $\mathrm{x}_0 = (x_1^0, x_2^0)$, $\mathrm{u}_0 = (u_1^0, u_2^0)$, $\vartheta\mathbf{u} = (\vartheta u_1, \vartheta u_2)$, $\langle \cdot, \cdot \rangle_{V_l^* \times V_l}$ is the duality pairing between V_l and V_l^* and $J_l^\circ(\mathbf{x}, z_1, z_2; v_l)$ is the partial Clarke's generalized directional derivative (PCGDD, for short) of the locally Lipschitz functional $J : \mathrm{E} \times X_1 \times X_2 \to \mathbf{R}$ w.r.t. the l-th argument at the point $z_l \in X_l$ in the direction $v_l \in X_l$ for the given $z_k \in X_k$.

In what follows, we consider an example of the AS, where locally Lipschitz J and functions $F_l, l = 1, 2$ are supposed to be independent of $\mathbf{x}$. Hence, the AS reverts to the parabolic-type SHVI below.

Find $\mathbf{u} : (0, T) \to \mathrm{V}$ s.t. $\mathrm{u}(0) = \mathrm{u}_0$ and

$$
\begin{cases}
(u_1'(t), v_1)_{H_1} + \langle \mathcal{N}_1(u_1(t)), v_1 \rangle_{V_1^* \times V_1} + J_1^\circ(M_1 u_1(t), M_2 u_2(t); M_1 v_1) \\
\quad \geq \langle F_1(t), v_1 \rangle_{V_1^* \times V_1}, \ \ \forall v_1 \in V_1, \text{ a.e. } t \in (0, T), \\
(u_2'(t), v_2)_{H_2} + \langle \mathcal{N}_2(u_2(t)), v_2 \rangle_{V_2^* \times V_2} + J_2^\circ(M_1 u_1(t), M_2 u_2(t); M_2 v_2) \\
\quad \geq \langle F_2(t), v_2 \rangle_{V_2^* \times V_2}, \ \ \forall v_2 \in V_2, \text{ a.e. } t \in (0, T).
\end{cases} \tag{11}
$$

It is easy to see that problem (11) is a generalization of the parabolic-type HVI below.

Find $u : (0, T) \to V$ s.t. $u(0) = u_0$ and

$$
(u'(t), v)_H + \langle \mathcal{N}(u(t)), v \rangle_{V^* \times V} + J^\circ(Mu(t); Mv) \geq \langle F(t), v \rangle_{V^* \times V}, \ \ \forall v \in V, \text{ a.e. } t \in (0, T). \tag{12}
$$

It is worth mentioning that this problem was considered only in [10,23,25].

3. Existence and Priori Estimation

In what follows, the process of demonstration involves the properties of PCGS, the surjective of setvalued pseudomonotonicity mappings, Rothe's rule, and convergent analysis.

We start this section with the normal symbos and functions; see [19,22]. For $l = 1, 2$, let the Banach space $(V_l, \|\cdot\|_{V_l})$ be a separable and reflexive one with the dual V_l^*, the Hilbert space H_l be a separable one, and the Banach space $(Y_l, \|\cdot\|_{Y_l})$ be the other separable and reflexive one. Later on, we suppose that the ones $V_l \subset H_l \subset V_l^*$ (or (V_l, H_l, V_l^*)) constitute the ETS [3] with dense continuity compactness embeddings. Let $\mathrm{V} = V_1 \times V_2$. Endowed with the norm defined by $\|\mathbf{u}\|_{\mathrm{V}} := \|u_1\|_{V_1} + \|u_2\|_{V_2}$ for all $\mathbf{u} = (u_1, u_2) \in \mathrm{V}$, V is a reflexive Banach space ([22]) with its dual V^* and the duality pairing between V and V^* is formulated below

$$
\langle \mathbf{u}^*, \mathbf{u} \rangle_{\mathrm{V}^* \times \mathrm{V}} = \langle u_1^*, u_1 \rangle_{V_1^* \times V_1} + \langle u_2^*, u_2 \rangle_{V_2^* \times V_2}, \ \ \forall \mathbf{u}^* = (u_1^*, u_2^*) \in \mathrm{V}^*, \ \mathbf{u} = (u_1, u_2) \in \mathrm{V}.
$$

Similarly, we can construct the product space $\mathrm{H} = H_1 \times H_2$. It is clear that the ones $\mathrm{V} \subset \mathrm{H} \subset \mathrm{V}^*$ (or $(\mathrm{V}, \mathrm{H}, \mathrm{V}^*)$) constitute the ETPS. For $l = 1, 2$, the embedding injection from V_l to H_l is denoted by $\iota_l : V_l \hookrightarrow H_l$. Moreover, for $l = 1, 2$, let $(X_l, \|\cdot\|_{X_l})$ and $(E_l, \|\cdot\|_{E_l})$ be reflexive and separable Banach space with their dual X_l^* and E_l^*, respectively. For $l = 1, 2$ and $0 < T < +\infty$, in the sequel, we use the standard Bochner-Lebesgue function spaces $\mathcal{V}_l = L^2(0, T; V_l)$, $\mathcal{H}_l = L^2(0, T; H_l)$, $\mathcal{X}_l = L^2(0, T; X_l)$, $\mathcal{V}_l^* = L^2(0, T; V_l^*)$ and $\mathcal{W}_l = \{v_l \in \mathcal{V}_l \mid v_l' \in \mathcal{V}_l^*\}$, here v_l' indicates the derivative of v_l to time. The symbol $\langle \cdot, \cdot \rangle_{\mathcal{V}_l^* \times \mathcal{V}_l}$ denotes the dual pairing between $\mathcal{V}_l$ and $\mathcal{V}_l^*$ and the space of linear continuous operators of V_l into X_l is written as $\mathcal{L}(V_l, X_l)$ for $l = 1, 2$. Of course, we can also construct the product spaces $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$, $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, $\mathcal{V}^* = \mathcal{V}_1^* \times \mathcal{V}_2^*$ and $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2$.

To prove the solvability of the AS, for $l, k = 1, 2$ and $k \neq l$ we always assume that the conditions below hold.

$\underline{H(A)}$: $A_l : D(A_l) \subset E_l \to E_l$ is an infinitesimal generator of C_0-semigroup $e^{A_l t}$.

$\underline{H(\mathcal{N})}$: $\mathcal{N}_l : V_l \to V_l^*$ is of pseudomonotonicity s.t.

(i) $\langle \mathcal{N}_l v_l, v_l \rangle_{V_l^* \times V_l} \geq a_{l,0} \|v_l\|_{V_l}^2 - a_{l,1} \|v_l\|_{H_l}^2, \ \forall v_l \in V_l$.

(ii) one of two hypotheses is valid below

a $\mathcal{N}_l$ satisfies the growth property

$$\|\mathcal{N}_l(v_l)\|_{V_l^*} \leq a_2 + a_3 \|v_l\|_{V_l}, \ \ \forall v_l \in V_l \text{ with } a_2 \geq 0, \ a_3 > 0.$$

b $\widetilde{\mathcal{N}_l}$ is bounded in $\mathcal{V}_l \cap L^\infty(0, T; H_l)$ and

$$\widetilde{\mathcal{N}_l}(u_l^n) \to \widetilde{\mathcal{N}_l}(u_l) \text{ weakly in } \mathcal{V}_l^*$$

for any sequence $\{u_l^n\}$ with $u_l^n \to u_l$ weakly in $\mathcal{V}_l$, where $\widetilde{\mathcal{N}_l} : \mathcal{V}_l \to \mathcal{V}_l^*$ is Nemytskii's mapping for $\mathcal{N}_l$ written as $(\widetilde{\mathcal{N}_l} u_l)(t) = \mathcal{N}_l(u_l(t))$ for $t \in [0, T]$.

$\underline{H(J)}$: $J : \mathrm{E} \times \mathrm{X} \to \mathbf{R}$ is s.t.

(i) for each fixed $\mathbf{x} \in \mathrm{E}$, the functional $\tilde{J}_{\mathbf{x}}(\cdot, \cdot) = J(\mathbf{x}, \cdot, \cdot) : X_1 \times X_2 \to \mathbf{R}$ is locally Lipschitz
with respect to first variable and second variable on $X_1 \times X_2$.

(ii) $\exists c_{l,J} > 0$ s.t.

$$\|\partial_l J(\mathbf{x}, \mathbf{u})\|_{X_l^*} \leq c_{l,J}(1 + \|u_l\|_{X_l}), \ \ \forall \mathbf{u} = (u_1, u_2) \in \mathrm{X}, \ \forall \mathbf{x} \in \mathrm{E}.$$

(iii) for each fixed $\mathbf{x} \in \mathrm{E}$, $\tilde{J}_{\mathbf{x}}(u_1, u_2) + \tilde{J}_{\mathbf{x}}(v_1, v_2) = \tilde{J}_{\mathbf{x}}(u_1, v_2) + \tilde{J}_{\mathbf{x}}(v_1, u_2)$ $\forall \mathbf{u}, \mathbf{v} \in \mathrm{X}$ with $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

(iv) $J(\mathbf{x}, u_1, u_2) + J(\mathbf{y}, v_1, u_2) = J(\mathbf{x}, v_1, u_2) + J(\mathbf{y}, u_1, u_2)$ and $J(\mathbf{x}, u_1, u_2) + J(\mathbf{y}, u_1, v_2) = J(\mathbf{x}, u_1, v_2) + J(\mathbf{y}, u_1, u_2)$ for all $\mathbf{x}, \mathbf{y} \in \mathrm{E}$ and $\mathbf{u}, \mathbf{v} \in \mathbf{X}$ with $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$\underline{H(M)}$: $M_l \in \mathcal{L}(V_l, X_l)$ and its Nemytskii operator $\mathcal{M}_l : M^{2,2}(0, T; V_l, V_l^*) \to \mathcal{X}_l$ formulated below $\mathcal{M}_l(u_l(t)) = M_l u_l(t)$ for $t \in [0, T]$ is of compactness.

$\underline{H(F)}$: $F_l : (0, T) \times E_k \to V_l^*$ is the mapping s.t.

(i) $t \mapsto F_l(t, x_k)$ is of measurability to $x_k \in E_k$.
(ii) $x_k \mapsto F_l(t, x_k)$ is of continuity to $t \in [0, T]$.
(iii) $\exists m_{F_l} > 0$ s.t. $\|F_l(t, x_k)\|_{V_l^*} \leq m_{F_l}$ for all $(t, x_k) \in (0, T) \times E_k$.

$\underline{H(0)}$: $\min_{l \in \{1,2\}} a_{l,0} > \max_{l \in \{1,2\}} c_{l,J} \|M_l\|^2$.

$\underline{H(\vartheta)}$: $\vartheta_l : H_l \to Y_l$ is of compactness.

$\underline{H(f)}$: $f_l : (0, T) \times \mathrm{E} \times \mathrm{Y} \to E_l$ is s.t.

(i) $t \mapsto f_l(t, \mathbf{x}, \mathbf{u})$ is of measurability to $(\mathbf{x}, \mathbf{u}) \in \mathrm{E} \times \mathrm{Y}$.
(ii) $(\mathbf{x}, \mathbf{u}) \mapsto f_l(t, \mathbf{x}, \mathbf{u})$ is of continuity to a.e. $t \in (0, T)$.
(iii) $\exists$(positive)$\varphi_l \in L^2(0, T)$ s.t.

$$\begin{cases} \|f_l(t, \mathbf{x}_1, \mathbf{u}) - f_l(t, \mathbf{x}_2, \mathbf{u})\|_{E_l} \leq \varphi_l(t) \|x_l^1 - x_l^2\|_{E_l}, \\ \|f_l(t, \mathbf{0}, \mathbf{u})\|_{E_l} \leq \varphi_l(t)(1 + \|u_l\|_{Y_l}), \end{cases}$$

for a.e. $t \in (0, T)$, all $\mathbf{x}_1 = (x_1^1, x_2^1)$, $\mathbf{x}_2 = (x_1^2, x_2^2) \in \mathrm{E}$ and $\mathbf{u} = (u_1, u_2) \in \mathrm{Y}$.

It is worth pointing out that Migorski and Zeng provided two examples of operator $\mathcal{N} : V \to V^*$ in Problem MZ, which satisfies the hypotheses $H(\mathcal{N})$; see [18], Remark [14]. Inspired by Wang et al. [10] (Lemma 3.6), we first present an important result.

Proposition 4. *Assume that hypotheses $H(J)$ (i), $H(J)$ (iii) and $H(J)$ (iv) hold. Then, for any sequences $\mathbf{x}^n \in \mathrm{E}$ converging strongly to $\mathbf{x} \in \mathrm{E}$, $\mathbf{u}^n = (u_1^n, u_2^n) \in \mathrm{X}$ converging strongly to*

$\mathbf{u} = (u_1, u_2) \in X$ *and* $v_l^n \in X_l$ *converging strongly to* $v_l \in X_l$, $\limsup_{n\to\infty} J_l^\circ(\mathbf{x}^n, \mathbf{u}^n; v_l^n) \leq J_l^\circ(\mathbf{x}, \mathbf{u}; v_l)$, *i.e.,*

$$\limsup_{n\to\infty} J_l^\circ(\mathbf{x}^n, u_1^n, u_2^n; v_l^n) \leq J_l^\circ(\mathbf{x}, u_1, u_2; v_l),$$

with $l = 1, 2$.

Proof. Let $\mathbf{x}^n \in E$ converge strongly to $\mathbf{x} \in E$, $\mathbf{u}^n = (u_1^n, u_2^n) \in X$ converge strongly to $\mathbf{u} = (u_1, u_2) \in X$ and, for $l = 1, 2$, $v_l^n \in X_l$ converge strongly to $v_l \in X_l$. Note that, the CGDD of $J(\mathbf{x}^n, \cdot, u_2^n)$ at u_1^n in the direction v_1^n is formulated as

$$J_1^\circ(\mathbf{x}^n, u_1^n, u_2^n; v_1^n) = \limsup_{w_1 \to u_1^n, t\downarrow 0} \frac{J(\mathbf{x}^n, w_1 + tv_1^n, u_2^n) - J(\mathbf{x}^n, w_1, u_2^n)}{t}.$$

For each integer $n \geq 1$, by the definition of the limsup, there exist $w_1^n \in X_1$ and $t^n > 0$ such that

$$\|w_1^n - u_1^n\|_{X_1} + t^n < \frac{1}{n}$$

and

$$\frac{J(\mathbf{x}^n, w_1^n + t^n v_1^n, u_2^n) - J(\mathbf{x}^n, w_1^n, u_2^n)}{t^n} > J_1^\circ(\mathbf{x}^n, u_1^n, u_2^n; v_1^n) - \frac{1}{n}.$$

In terms of hypotheses $H(J)$ (i) and $H(J)$ (iii), we have

$$\begin{aligned}
&\tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1^n,u_2^n)-J(\mathbf{x}^n,w_1^n,u_2^n)}{t^n} = \tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1,u_2)-J(\mathbf{x}^n,w_1^n,u_2)}{t^n} \\
&\quad + \tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1^n,u_2^n)-J(\mathbf{x}^n,w_1^n+t^nv_1,u_2)+J(\mathbf{x}^n,w_1^n,u_2)-J(\mathbf{x}^n,w_1^n,u_2^n)}{t^n} \\
&= \tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1,u_2)-J(\mathbf{x}^n,w_1^n,u_2)}{t^n} + \tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1^n,u_2)-J(\mathbf{x}^n,w_1^n+t^nv_1,u_2)}{t^n} \\
&\quad + \tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1^n,u_2^n)-J(\mathbf{x}^n,w_1^n+t^nv_1^n,u_2)+J(\mathbf{x}^n,w_1^n,u_2)-J(\mathbf{x}^n,w_1^n,u_2^n)}{t^n} \\
&\leq \tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1,u_2)-J(\mathbf{x}^n,w_1^n,u_2)}{t^n} + L_{u_1}\|v_1^n - v_1\|_{X_1},
\end{aligned} \tag{13}$$

where L_{u_1} is the local Lipschitz constant of functional $J(\mathbf{x}^n, \cdot, u_2)$ at u_1. It follows from the above inequalities and $H(J)$ (iv) that

$$\begin{aligned}
J_1^\circ(\mathbf{x}^n, u_1^n, u_2^n; v_1^n) - \tfrac{1}{n} &< \tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1,u_2)-J(\mathbf{x}^n,w_1^n,u_2)}{t^n} + L_{u_1}\|v_1^n - v_1\|_{X_1} \\
&= \tfrac{J(\mathbf{x},w_1^n+t^nv_1,u_2)-J(\mathbf{x},w_1^n,u_2)}{t^n} + L_{u_1}\|v_1^n - v_1\|_{X_1} \\
&\quad + \tfrac{J(\mathbf{x}^n,w_1^n+t^nv_1,u_2)-J(\mathbf{x},w_1^n+t^nv_1,u_2)+J(\mathbf{x},w_1^n,u_2)-J(\mathbf{x}^n,w_1^n,u_2)}{t^n} \\
&\leq \tfrac{J(\mathbf{x},w_1^n+t^nv_1,u_2)-J(\mathbf{x},w_1^n,u_2)}{t^n} + L_{u_1}\|v_1^n - v_1\|_{X_1}.
\end{aligned}$$

Taking the limsup as $n \to \infty$ at both sides of the last inequality yields

$$\limsup_{n\to\infty} J_1^\circ(\mathbf{x}^n, u_1^n, u_2^n; v_1^n) \leq J_1^\circ(\mathbf{x}, u_1, u_2; v_1).$$

Similarly, we can prove that

$$\limsup_{n\to\infty} J_2^\circ(\mathbf{x}^n, u_1^n, u_2^n; v_2^n) \leq J_2^\circ(\mathbf{x}, u_1, u_2; v_2).$$

This completes the proof. □

First of all, we claim that condition $H(J)$ ensures the u.s.c. of the PCGS $\partial_l J$ for $l = 1, 2$.

Lemma 3. *Assume that* $H(J)$ *holds. Then for* $l = 1, 2$, *the PCGS mapping*

$$E \times X \ni (\mathbf{y}, \mathbf{x}) \mapsto \partial_l J(\mathbf{y}, \mathbf{x}) \subset X_l^*$$

is u.s.c. from $\mathrm{E} \times \mathrm{X}$ *equipped with the norm topology to the subsets of* X_l^* *equipped with the weak topology.*

Proof. According to Proposition 1, it is sufficient to show that for each weakly closed $D_l \subset X_l^*$, the weak inverse image $(\partial_l J)^{-1}(D_l)$ is of norm closedness, with

$$(\partial_l J)^{-1}(D_l) = \{(\mathbf{y},\mathbf{x}) \in \mathrm{E} \times \mathrm{X} \mid \partial_l J(\mathbf{y},\mathbf{x}) \cap D_l \neq \emptyset\}.$$

Suppose that $\{(\mathbf{y}_n,\mathbf{x}_n)\} \subset (\partial_l J)^{-1}(D_l)$ is s.t. $(\mathbf{y}_n,\mathbf{x}_n) \to (\mathbf{y},\mathbf{x})$ in $\mathrm{E} \times \mathrm{X}$ as $n \to \infty$, and $\{\xi_l^n\} \subset X_l^*$ is s.t. $\xi_l^n \in \partial_l J(\mathbf{y}_n,\mathbf{x}_n) \cap D_l$, $\forall n \geq 1$. Hypothesis $H(J)$ (ii) implies that the sequence $\{\xi_l^n\}$ is bounded in X_l^*. Hence, by the reflexivity of X_l^*, without loss of generality, we may assume that $\xi_l^n \to \xi_l$ weakly in X_l^*. The weak closedness of D_l guarantees that $\xi_l \in D_l$. On the other hand, from Lemma 1 (v) we know that $\xi_l^n \in \partial_l J(\mathbf{y}_n,\mathbf{x}_n)$ entails

$$\langle \xi_l^n, z_l \rangle_{X_l^* \times X_l} \leq J_l^\circ(\mathbf{y}_n,\mathbf{x}_n; z_l), \quad \forall z_l \in X_l.$$

Utilizing Proposition 4 and passing to the limsup as $n \to \infty$, we deduce that

$$\langle \xi_l, z_l \rangle_{X_l^* \times X_l} = \limsup_{n\to\infty} \langle \xi_l^n, z_l \rangle_{X_l^* \times X_l} \leq \limsup_{n\to\infty} J_l^\circ(\mathbf{y}_n,\mathbf{x}_n; z_l) \leq J_l^\circ(\mathbf{y},\mathbf{x}; z_l)$$

for all $z_l \in X_l$. Thus $\xi_l \in \partial_l J(\mathbf{y},\mathbf{x})$, and hence, one gets $\xi_l \in \partial_l J(\mathbf{y},\mathbf{x}) \cap D_l$, that is, $(\mathbf{y},\mathbf{x}) \in (\partial_l J)^{-1}(D_l)$. □

In the rest of this paper, the range of variable t is always assumed to be the a.e. $t \in (0,T)$. For the convenience, we naturally omit the description of the a.e. $t \in (0,T)$. It is clear that the AS is equivalent to the problem below.

RAS. Find $\mathbf{x} : (0,T) \to \mathrm{E}$ and $\mathbf{u} : (0,T) \to \mathrm{V}$ with $\mathbf{x} = (x_1,x_2)$ and $\mathbf{u} = (u_1,u_2)$, s.t.

$$\begin{cases} x_1'(t) = A_1 x_1(t) + f_1(t,\mathbf{x}(t),\vartheta\mathbf{u}(t)), \\ x_2'(t) = A_2 x_2(t) + f_2(t,\mathbf{x}(t),\vartheta\mathbf{u}(t)), \end{cases}$$

$$\begin{cases} u_1'(t) + \mathcal{N}_1(u_1(t)) + M_1^*\partial_1 J(\mathbf{x}(t), M\mathbf{u}(t)) \ni F_1(t,x_2(t)), \\ u_2'(t) + \mathcal{N}_2(u_2(t)) + M_2^*\partial_2 J(\mathbf{x}(t), M\mathbf{u}(t)) \ni F_2(t,x_1(t)), \end{cases} \tag{14}$$

and

$$\mathbf{x}(0) = \mathbf{x}_0 \text{ and } \mathbf{u}(0) = \mathbf{u}_0,$$

where $\mathbf{x}_0 = (x_1^0, x_2^0)$, $\mathbf{u}_0 = (u_1^0, u_2^0)$, $M\mathbf{u} := (M_1 u_1, M_2 u_2)$ and $\vartheta\mathbf{u} := (\vartheta_1 u_1, \vartheta_2 u_2)$.

According to the previous works [16–19], we give the following definition of a mild solution to the RAS.

Definition 1. *The* $(\mathbf{x},\mathbf{u},\xi)$ *with* $\mathbf{x} = (x_1,x_2) \in C(0,T;\mathrm{E})$, $\mathbf{u} = (u_1,u_2) \in \mathcal{W}$ *and* $\xi = (\xi_1,\xi_2) \in \mathcal{X}^*$ *is referred to as a mild solution to the RAS, iff*

$$\begin{cases} x_1(t) = e^{A_1 t} x_1^0 + \int_0^t e^{A_1(t-s)} f_1(s,\mathbf{x}(s),\vartheta\mathbf{u}(s))ds, \\ x_2(t) = e^{A_2 t} x_2^0 + \int_0^t e^{A_2(t-s)} f_2(s,\mathbf{x}(s),\vartheta\mathbf{u}(s))ds, \end{cases}$$

$$\begin{cases} u_1'(t) + \mathcal{N}_1(u_1(t)) + M_1^*\xi_1(t) = F_1(t,x_2(t)), \\ u_2'(t) + \mathcal{N}_2(u_2(t)) + M_2^*\xi_2(t) = F_2(t,x_1(t)), \end{cases} \tag{15}$$

and

$$\mathbf{x}(0) = \mathbf{x}_0 \text{ and } \mathbf{u}(0) = \mathbf{u}_0,$$

where $\mathbf{x}_0 = (x_1^0, x_2^0)$, $\mathbf{u}_0 = (u_1^0, u_2^0)$, $M\mathbf{u} := (M_1 u_1, M_2 u_2)$, $\vartheta\mathbf{u} := (\vartheta_1 u_1, \vartheta_2 u_2)$, *and*

$$\xi(t) = (\xi_1(t), \xi_2(t)) \in \partial_1 J(\mathbf{x}(t), M\mathbf{u}(t)) \times \partial_2 J(\mathbf{x}(t), M\mathbf{u}(t)).$$

Next, we prove the existence of a mild solution to the RAS by using the Rothe rule along with the feedback iteration technique.

For $N \geq 1$, we put $\tau = \frac{T}{N}$ and $t_i = i\tau$ for $i = 0, 1, ..., N$, and formulate the HIS below.

HIS. Find $\{\mathbf{u}_\tau^k\}_{k=0}^N \subset \mathrm{V}$, $\mathbf{x}_\tau \in C(0, T; \mathrm{E})$ and $\{\xi_\tau^k\}_{k=1}^N \subset \mathrm{X}^*$, with $\mathbf{u}_\tau^k = (u_{1,\tau}^k, u_{2,\tau}^k)$, $\mathbf{x}_\tau = (x_{1,\tau}, x_{2,\tau})$ and $\xi_\tau^k = (\xi_{1,\tau}^k, \xi_{2,\tau}^k)$, such that $\mathbf{u}_\tau^0 = \mathbf{u}_0$ and

$$\begin{cases} x_{1,\tau}(t) = e^{A_1 t} x_1^0 + \int_0^t e^{A_1(t-s)} f_1(s, \mathbf{x}_\tau(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) ds, \ \text{ a.e. } t \in (0, t_k), \\ x_{2,\tau}(t) = e^{A_2 t} x_2^0 + \int_0^t e^{A_2(t-s)} f_2(s, \mathbf{x}_\tau(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) ds, \ \text{ a.e. } t \in (0, t_k), \end{cases} \tag{16}$$

$$\begin{cases} \frac{u_{1,\tau}^k - u_{1,\tau}^{k-1}}{\tau} + \mathcal{N}_1(u_{1,\tau}^k) + M_1^* \xi_{1,\tau}^k = F_{1,\tau}^k, \\ \frac{u_{2,\tau}^k - u_{2,\tau}^{k-1}}{\tau} + \mathcal{N}_2(u_{2,\tau}^k) + M_2^* \xi_{2,\tau}^k = F_{2,\tau}^k, \end{cases} \tag{17}$$

$$\xi_\tau^k = (\xi_{1,\tau}^k, \xi_{2,\tau}^k) \in \partial_1 J(\mathbf{x}_\tau(t_k), M\mathbf{u}_\tau^k) \times \partial_2 J(\mathbf{x}_\tau(t_k), M\mathbf{u}_\tau^k),$$

for $k = 1, 2, ..., N$, where for $j \neq i = 1, 2$, $F_{i,\tau}^k$ and $\widehat{\mathbf{u}}_\tau(t) = (\widehat{u}_{1,\tau}(t), \widehat{u}_{2,\tau}(t))$ for $t \in (0, t_k)$ are defined by

$$F_{i,\tau}^k := \frac{1}{\tau} \int_{t_{k-1}}^{t_k} F_i(s, x_{j,\tau}(s)) ds,$$

$$\widehat{u}_{i,\tau}(t) = \begin{cases} u_{i,\tau}^{k-1} + \frac{t - t_k}{\tau}(u_{i,\tau}^{k-1} - u_{i,\tau}^{k-2}) & \text{for } t \in (t_{k-1}, t_k],\ 2 \leq k \leq N, \\ u_i^0, & \text{for } t \in [0, t_1]. \end{cases} \tag{18}$$

Obviously, this system is constituted by a stationary system of PCGS inclusions along with a system of abstract integral equations.

Here, we first establish the existence lemma of solutions to the HIS.

Lemma 4. *If $H(f)$, $H(0)$, $H(\vartheta)$, $H(M)$, $H(J)$, $H(\mathcal{N})$, $H(F)$ and $H(A)$ are valid, then, $\exists \tau_0 > 0$ s.t., $\forall \tau \in (0, \tau_0)$, the HIS has at least one solution.*

Proof. Given elements $\mathbf{u}_\tau^0, \mathbf{u}_\tau^1, ..., \mathbf{u}_\tau^{k-1}$, it follows from the definition of $\widehat{\mathbf{u}}_\tau$ (see (18)) that $\widehat{\mathbf{u}}_\tau$ is well-defined and $\widehat{\mathbf{u}}_\tau \in C(0, t_k; \mathrm{V})$. For $l = 1, 2$, one formulates $\mathcal{F}_{l,\tau} : (0, T) \times \mathrm{E} \to E_l$ below

$$\mathcal{F}_{l,\tau}(t, \mathbf{x}) = f_l(t, \mathbf{x}, \vartheta\widehat{\mathbf{u}}_\tau(t)) \ \text{ for } \mathbf{x} \in \mathrm{E}.$$

Note that $t \mapsto f_l(t, \mathbf{x}, \mathbf{u})$ is of measurability on $(0, T)$ to $(\mathbf{x}, \mathbf{u}) \in \mathrm{E} \times \mathrm{Y}$, $(\mathbf{x}, \mathbf{u}) \mapsto f_l(t, \mathbf{x}, \mathbf{u})$ is of continuity, and $\widehat{\mathbf{u}}_\tau \in C(0, t_k; \mathrm{V})$. So we have

$$t \mapsto \mathcal{F}_{l,\tau}(t, \mathbf{x}) \text{ is of measurability to } \mathbf{x} \in \mathrm{E}.$$

By the condition $H(f)$ (iii), we know that for $l = 1, 2$, $\mathcal{F}_{l,\tau}$ satisfies the following properties

$$\begin{cases} \|\mathcal{F}_{l,\tau}(t, \mathbf{0})\|_{E_l} \leq \varphi_l(t)(1 + \|\vartheta_l \widehat{u}_{l,\tau}(t)\|_{Y_l}), \ \textit{a.e. } t \in (0, t_k), \\ \|\mathcal{F}_{l,\tau}(t, \mathbf{x}_1) - \mathcal{F}_{l,\tau}(t, \mathbf{x}_2)\|_{E_l} \leq \varphi_l(t)\|x_l^1 - x_l^2\|_{E_l}, \ \textit{a.e. } t \in (0, t_k). \end{cases}$$

These along with [7], (Proposition 5.3, p. 66), [23], (Section 4) and [26], (Section 4) ensure that $\exists |\ \mathbf{x}_\tau = (x_{1,\tau}, x_{2,\tau}) \in C(0, t_k; \mathrm{E})$ s.t.

$$\begin{cases} x_{1,\tau}(t) = e^{A_1 t} x_1^0 + \int_0^t e^{A_1(t-s)} f_1(s, \mathbf{x}_\tau(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) ds, \ \text{ a.e. } t \in (0, t_k), \\ x_{2,\tau}(t) = e^{A_2 t} x_2^0 + \int_0^t e^{A_2(t-s)} f_2(s, \mathbf{x}_\tau(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) ds, \ \text{ a.e. } t \in (0, t_k). \end{cases}$$

Further, from hypothesis $H(F)$ and $\mathbf{x}_\tau = (x_{1,\tau}, x_{2,\tau}) \in C(0, t_k; \mathrm{E})$ we can easily check

$$\begin{cases} F^k_{1,\tau} = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} F_1(s, x_{2,\tau}(s))ds \in V_1^*, \\ F^k_{2,\tau} = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} F_2(s, x_{1,\tau}(s))ds \in V_2^*. \end{cases}$$

It remains to find elements $\mathbf{u}^k_\tau = (u^k_{1,\tau}, u^k_{2,\tau}) \in \mathrm{V}$ and $\xi^k_\tau = (\xi^k_{1,\tau}, \xi^k_{2,\tau}) \in \partial_1 J(\mathbf{x}_\tau(t_k), M\mathbf{u}^k_\tau) \times \partial_2 J(\mathbf{x}_\tau(t_k), M\mathbf{u}^k_\tau)$ such that for $l = 1, 2$,

$$\frac{u^k_{l,\tau} - u^{k-1}_{l,\tau}}{\tau} + \mathcal{N}_l(u^k_{l,\tau}) + M_l^*\xi^k_{l,\tau} = F^k_{l,\tau}.$$

Next, it is sufficient to show that $S : \mathrm{V} \to 2^{\mathrm{V}^*}$ formulated below is surjective

$$\begin{cases} S\mathrm{v} = (S_1 v_1, S_2 v_2), \ \ \forall \mathrm{v} = (v_1, v_2) \in \mathrm{V}, \\ S_1 v_1 = \frac{\iota_1^*\iota_1 v_1}{\tau} + \mathcal{N}_1(v_1) + M_1^*\partial_1 J(\mathbf{x}_\tau(t_k), M\mathbf{v}), \\ S_2 v_2 = \frac{\iota_2^*\iota_2 v_2}{\tau} + \mathcal{N}_2(v_2) + M_2^*\partial_2 J(\mathbf{x}_\tau(t_k), M\mathbf{v}). \end{cases}$$

According to condition $H(J)$ (ii), we get the estimation for $l = 1, 2$,

$$\begin{aligned} |\langle \xi_l, M_l v_l\rangle_{X_l^*\times X_l}| &\le \|\xi_l\|_{X_l^*}\|M_l v_l\|_{X_l} \le c_{l,J}(1 + \|M_l v_l\|_{X_l})\|M_l v_l\|_{X_l} \\ &\le c_{l,J}\|M_l\|^2\|v_l\|^2_{V_l} + c_{l,J}\|M_l\|\|v_l\|_{V_l} \end{aligned} \tag{19}$$

for all $\mathbf{v} = (v_1, v_2) \in \mathrm{V}$ and $\xi = (\xi_1, \xi_2) \in \partial_1 J(\mathbf{x}_\tau(t_k), M\mathbf{v}) \times \partial_2 J(\mathbf{x}_\tau(t_k), M\mathbf{v})$. Moreover, hypotheses $H(\mathcal{N})$ (l) reveals that for $l = 1, 2$,

$$\begin{aligned} \langle S_l v_l, v_l\rangle_{V_l^*\times V_l} &= \tfrac{1}{\tau}(v_l, v_l)_{H_l} + \langle \mathcal{N}_l(v_l), v_l\rangle_{V_l^*\times V_l} + \langle \partial_l J(\mathbf{x}_\tau(t_k), M\mathbf{v}), M_l v_l\rangle_{X_l^*\times X_l} \\ &\ge \tfrac{1}{\tau}\|v_l\|^2_{H_l} + a_{l,0}\|v_l\|^2_{V_l} - a_{l,1}\|v_l\|^2_{H_l} - \sup_{\xi_l\in\partial_l J(\mathbf{x}_\tau(t_k), M\mathbf{v})} |\langle \xi_l, M_l v_l\rangle_{X_l^*\times X_l}|. \end{aligned}$$

After inserting (19) into the above inequality, we obtain that for $l = 1, 2$,

$$\begin{aligned} \langle S_l v_l, v_l\rangle_{V_l^*\times V_l} &\ge (\tfrac{1}{\tau} - a_{l,1})\|v_l\|^2_{H_l} + (a_{l,0} - c_{l,J}\|M_l\|^2)\|v_l\|^2_{V_l} - c_{l,J}\|M_l\|\|v_l\|_{V_l} \\ &\ge (\tfrac{1}{\tau} - \max_{l\in\{1,2\}} a_{l,1})\|v_l\|^2_{H_l} + (\min_{l\in\{1,2\}} a_{l,0} - \max_{l\in\{1,2\}} c_{l,J}\|M_l\|^2)\|v_l\|^2_{V_l} \\ &\quad - (\max_{l\in\{1,2\}} c_{l,J}\|M_l\|)\|v_l\|_{V_l} \end{aligned}$$

for all $\mathbf{v} = (v_1, v_2) \in \mathrm{V}$. Choosing $\tau_0 = (\max_{l\in\{1,2\}} a_{l,1})^{-1}$ and noticing the smallness hypothesis $H(0)$, we deduce that for $l = 1, 2$, S_l is coercive for all $\tau \in (0, \tau_0)$. According to [3], Proposition 3.59, whenever all components of S_l are of pseudomonotonicity, S_l is also of pseudomonotonicity. Because $v_l \mapsto \frac{\iota_l^*\iota_l v_l}{\tau}$ is nonnegative, linear and continuous, it is of pseudomonotonicity. Besides, conditions $H(M)$, $H(J)$ (i), $H(J)$ (ii) and Proposition 2.2 guarantee that

$$v_l \mapsto M_l^*\partial_l J(\mathbf{x}_\tau(t_k), M\mathbf{v}) \text{ is of pseudomonotonicity as well.}$$

Because $\mathcal{N}_l$ is of pseudomonotonicity (due to $H(\mathcal{N})$), we deduce from [[3], Proposition 3.59] that S_l is of pseudomonotonicity. From Theorem 2.1, we obtain that for $l = 1, 2$, there exist $u^k_{l,\tau} \in V_l$ and $\xi^k_{l,\tau} \in X_l^*$ such that $\xi^k_{l,\tau} \in \partial_l J(\mathbf{x}_\tau(t_k), M\mathbf{u}^k_\tau)$ and (17) holds, for all $\tau \in (0, \tau_0)$. That is, there exist $\mathbf{u}^k_\tau = (u^k_{1,\tau}, u^k_{2,\tau}) \in \mathrm{V}$ and $\xi^k_\tau = (\xi^k_{1,\tau}, \xi^k_{2,\tau}) \in \mathrm{X}^*$ such that $\xi^k_\tau = (\xi^k_{1,\tau}, \xi^k_{2,\tau}) \in \partial_1 J(\mathbf{x}_\tau(t_k), M\mathbf{u}^k_\tau) \times \partial_2 J(\mathbf{x}_\tau(t_k), M\mathbf{u}^k_\tau)$ and (17) holds, for all $\tau \in (0, \tau_0)$. $\square$

Now, we present a lemma on the priori estimation for solutions to the HIS.

Lemma 5. *If $H(f)$, $H(0)$, $H(\vartheta)$, $H(M)$, $H(J)$, $H(\mathcal{N})$, $H(F)$ and $H(A)$ are valid, then, $\exists\tau_0, C > 0$ (independent of τ) s.t., $\forall\tau \in (0, \tau_0)$, the solutions to the HIS, satisfy*

$$\max_{1\le k\le N} \|\mathrm{u}_\tau^k\|_{\mathrm{H}} \le C, \tag{20}$$

$$\sum_{k=1}^{N} \|\mathrm{u}_\tau^k - \mathrm{u}_\tau^{k-1}\|_{\mathrm{H}} \le C, \tag{21}$$

$$\tau\sum_{k=1}^{N} \|\mathrm{u}_\tau^k\|_{\mathrm{V}}^2 \le C, \tag{22}$$

Proof. Let $\xi_\tau^k = (\xi_{1,\tau}^k, \xi_{2,\tau}^k) \in \partial_1 J(\mathbf{x}_\tau(t_k), M\mathbf{u}_\tau^k) \times \partial_2 J(\mathbf{x}_\tau(t_k), M\mathbf{u}_\tau^k)$ be such that (17) holds. Multiplying the equalities in (17) by $u_{l,\tau}^k, l = 1, 2$, one has

$$\begin{cases} \left(\frac{u_{1,\tau}^k - u_{1,\tau}^{k-1}}{\tau}, u_{1,\tau}^k\right)_{H_1} + \langle \mathcal{N}_1(u_{1,\tau}^k), u_{1,\tau}^k\rangle_{V_1^*\times V_1} + \langle \xi_{1,\tau}^k, M_1 u_{1,\tau}^k\rangle_{X_1^*\times X_1} = \langle F_{1,\tau}^k, u_{1,\tau}^k\rangle_{V_1^*\times V_1} \\ \left(\frac{u_{2,\tau}^k - u_{2,\tau}^{k-1}}{\tau}, u_{2,\tau}^k\right)_{H_2} + \langle \mathcal{N}_2(u_{2,\tau}^k), u_{2,\tau}^k\rangle_{V_2^*\times V_2} + \langle \xi_{2,\tau}^k, M_2 u_{2,\tau}^k\rangle_{X_2^*\times X_2} = \langle F_{2,\tau}^k, u_{2,\tau}^k\rangle_{V_2^*\times V_2}. \end{cases} \tag{23}$$

From $H(\mathcal{N})$ (i), one gets

$$\langle \mathcal{N}_l(u_{l,\tau}^k), u_{l,\tau}^k\rangle_{V_l^*\times V_l} \ge a_{l,0}\|u_{l,\tau}^k\|_{V_l}^2 - a_{l,1}\|u_{l,\tau}^k\|_{H_l}^2. \tag{24}$$

Moreover, hypothesis $H(J)$ (ii) guarantees that

$$\begin{aligned} \langle \xi_\tau^k, M\mathbf{u}_\tau^k\rangle_{\mathrm{X}^*\times\mathrm{X}} &= \langle \xi_{1,\tau}^k, M_1 u_{1,\tau}^k\rangle_{X_1^*\times X_1} + \langle \xi_{2,\tau}^k, M_2 u_{2,\tau}^k\rangle_{X_2^*\times X_2} \\ &\ge -\|\xi_{1,\tau}^k\|_{X_1^*}\|M_1 u_{1,\tau}^k\|_{X_1} - \|\xi_{2,\tau}^k\|_{X_2^*}\|M_2 u_{2,\tau}^k\|_{X_2} \\ &\ge -\sum_{l=1}^{2} c_{l,J}\|M_l\|(1 + \|M_l u_{l,\tau}^k\|_{X_l})\|u_{l,\tau}^k\|_{V_l} \\ &\ge -\sum_{l=1}^{2} (c_{l,J}\|M_l\|^2\|u_{l,\tau}^k\|_{V_l}^2 + c_{l,J}\|M_l\|\|u_{l,\tau}^k\|_{V_l}). \end{aligned} \tag{25}$$

Inserting (24) and (25) into (23), and taking into account the identity

$$(v_l - w_l, v_l)_{H_l} = \frac{1}{2}(\|v_l\|_{H_l}^2 + \|v_l - w_l\|_{H_l}^2 - \|w_l\|_{H_l}^2), \quad \forall v_l, w_l \in H_l,$$

we obtain that for $l = 1, 2$,

$$\begin{aligned} \|F_{l,\tau}^k\|_{V_l^*}\|u_{l,\tau}^k\|_{V_l} &\ge \langle F_{l,\tau}^k, u_{l,\tau}^k\rangle_{V_l^*\times V_l} \\ &= \left(\frac{u_{l,\tau}^k - u_{l,\tau}^{k-1}}{\tau}, u_{l,\tau}^k\right)_{H_l} + \langle \mathcal{N}_l(u_{l,\tau}^k), u_{l,\tau}^k\rangle_{V_l^*\times V_l} + \langle \xi_{l,\tau}^k, M_l u_{l,\tau}^k\rangle_{X_l^*\times X_l} \\ &\ge \frac{1}{2\tau}(\|u_{l,\tau}^k\|_{H_l}^2 + \|u_{l,\tau}^k - u_{l,\tau}^{k-1}\|_{H_l}^2 - \|u_{l,\tau}^{k-1}\|_{H_l}^2) \\ &\quad + a_{l,0}\|u_{l,\tau}^k\|_{V_l}^2 - a_{l,1}\|u_{l,\tau}^k\|_{H_l}^2 - c_{l,J}\|M_l\|^2\|u_{l,\tau}^k\|_{V_l}^2 - c_{l,J}\|M_l\|\|u_{l,\tau}^k\|_{V_l}. \end{aligned}$$

Using Cauchy's inequality with $\varepsilon > 0$, one has

$$\begin{aligned} \varepsilon\|u_{l,\tau}^k\|_{V_l}^2 + \frac{\|F_{l,\tau}^k\|_{V_l^*}^2}{4\varepsilon} &\ge \frac{1}{2\tau}(\|u_{l,\tau}^k\|_{H_l}^2 + \|u_{l,\tau}^k - u_{l,\tau}^{k-1}\|_{H_l}^2 - \|u_{l,\tau}^{k-1}\|_{H_l}^2) \\ &\quad + a_{l,0}\|u_{l,\tau}^k\|_{V_l}^2 - a_{l,1}\|u_{l,\tau}^k\|_{H_l}^2 - c_{l,J}\|M_l\|^2\|u_{l,\tau}^k\|_{V_l}^2 - \frac{c_{l,J}^2\|M_l\|^2}{4\varepsilon} - \varepsilon\|u_{l,\tau}^k\|_{V_l}^2, \end{aligned}$$

that is,

$$\begin{aligned} &\frac{\tau}{2\varepsilon}\|F_{l,\tau}^k\|_{V_l^*}^2 + \frac{c_{l,J}^2\|M_l\|^2\tau}{2\varepsilon} + 2\tau a_{l,1}\|u_{l,\tau}^k\|_{H_l}^2 \\ &\ge \|u_{l,\tau}^k\|_{H_l}^2 + \|u_{l,\tau}^k - u_{l,\tau}^{k-1}\|_{H_l}^2 - \|u_{l,\tau}^{k-1}\|_{H_l}^2 + 2\tau(a_{l,0} - c_{l,J}\|M_l\|^2 - 2\varepsilon)\|u_{l,\tau}^k\|_{V_l}^2. \end{aligned}$$

Adding up the above inequalities for $k = 1, ..., n$, we obtain that for $n = 1, ..., N$,

$$\begin{aligned} &2\tau(a_{l,0} - c_{l,J}\|M_l\|^2 - 2\varepsilon)\sum_{k=1}^{n}\|u_{l,\tau}^k\|_{V_l}^2 + \sum_{k=1}^{n}\|u_{l,\tau}^k - u_{l,\tau}^{k-1}\|_{H_l}^2 + \|u_{l,\tau}^n\|_{H_l}^2 - \|u_{l,\tau}^0\|_{H_l}^2 \\ &\leq \tfrac{\tau}{2\varepsilon}\sum_{k=1}^{n}\|F_{l,\tau}^k\|_{V_l^*}^2 + \tfrac{c_{l,J}^2\|M_l\|^2 T}{2\varepsilon} + 2\tau a_{l,1}\sum_{k=1}^{n}\|u_{l,\tau}^k\|_{H_l}^2. \end{aligned}$$

It follows from hypothesis $H(F)$ that $\|F_{l,\tau}^k\|_{V_l^*} \leq m_{F_l}$ for $k = 1, 2, ..., N$. Using $a_{l,0} > c_{l,J}\|M_l\|^2$ (due to $H(0)$) and choosing $\varepsilon = \frac{a_{l,0} - c_{l,J}\|M_l\|^2}{4}$, we obtain

$$\begin{aligned} &\tau(a_{l,0} - c_{l,J}\|M_l\|^2)\sum_{k=1}^{n}\|u_{l,\tau}^k\|_{V_l}^2 + \sum_{k=1}^{n}\|u_{l,\tau}^k - u_{l,\tau}^{k-1}\|_{H_l}^2 + \|u_{l,\tau}^n\|_{H_l}^2 \\ &\leq \tfrac{2Tm_{F_l}^2}{a_{l,0} - c_{l,J}\|M_l\|^2} + \|u_{l,\tau}^0\|_{H_l}^2 + \tfrac{2c_{l,J}^2\|M_l\|^2 T}{a_{l,0} - c_{l,J}\|M_l\|^2} + 2\tau a_{l,1}\sum_{k=1}^{n}\|u_{l,\tau}^k\|_{H_l}^2. \end{aligned}$$

Applying the discrete Gronwall inequality and Lemma 2, We know that for $l = 1, 2$, $\exists \tau_{l,0}, C_l > 0$ (independent of τ) s.t., $\forall \tau \in (0, \tau_{l,0})$, the solutions to the HIS, satisfy

$$\begin{cases} \max\limits_{1\leq k\leq N}\|u_{1,\tau}^k\|_{H_1} \leq C_1,\ \sum\limits_{k=1}^{N}\|u_{1,\tau}^k - u_{1,\tau}^{k-1}\|_{H_1} \leq C_1 \text{ and } \tau\sum\limits_{k=1}^{N}\|u_{1,\tau}^k\|_{V_1}^2 \leq C_1; \\ \max\limits_{1\leq k\leq N}\|u_{2,\tau}^k\|_{H_2} \leq C_2,\ \sum\limits_{k=1}^{N}\|u_{2,\tau}^k - u_{2,\tau}^{k-1}\|_{H_2} \leq C_2 \text{ and } \tau\sum\limits_{k=1}^{N}\|u_{2,\tau}^k\|_{V_2}^2 \leq C_2. \end{cases}$$

Putting $C = 2(C_1 + C_2)$ and $\tau_0 = \min_{l\in\{1,2\}} \tau_{l,0}$, we can readily see that for all $\tau \in (0, \tau_0)$, (20) and (21) hold. Observe that for all $\tau \in (0, \tau_0)$,

$$\begin{aligned} \tau\sum_{k=1}^{N}\|\mathbf{u}_\tau^k\|_{\mathrm{V}}^2 &= \tau\sum_{k=1}^{N}(\|u_{1,\tau}^k\|_{V_1} + \|u_{2,\tau}^k\|_{V_2})^2 \\ &\leq 2\tau\sum_{k=1}^{N}(\|u_{1,\tau}^k\|_{V_1}^2 + \|u_{2,\tau}^k\|_{V_2}^2) \\ &\leq 2(C_1 + C_2) = C. \end{aligned}$$

That is, (22) is valid. □

Subsequently, for a given $\tau > 0$, we define the piecewise affine function $\mathbf{u}_\tau = (u_{1,\tau}, u_{2,\tau})$ and the piecewise constant interpolant functions $\bar{\mathbf{u}}_\tau = (\bar{u}_{1,\tau}, \bar{u}_{2,\tau})$, $\xi_\tau = (\xi_{1,\tau}, \xi_{2,\tau})$, $\hbar_\tau = (\hbar_{1,\tau}, \hbar_{2,\tau})$ as follows: for $l = 1, 2$,

$$\begin{aligned} &u_{l,\tau}(t) = u_{l,\tau}^k + \tfrac{t - t_k}{\tau}(u_{l,\tau}^k - u_{l,\tau}^{k-1}),\ \ \forall t \in (t_{k-1}, t_k], \\ &\xi_{l,\tau}(t) = \xi_{l,\tau}^k,\ \ \forall t \in (t_{k-1}, t_k], \\ &\bar{u}_{l,\tau}(t) = \begin{cases} u_{l,\tau}^k,\ t \in (t_{k-1}, t_k], \\ u_{l,0},\ t = 0, \end{cases} \\ &\hbar_{l,\tau}(t) = F_{l,\tau}^k,\ \ \forall t \in (t_{k-1}, t_k]. \end{aligned}$$

Lemma 6. *If $H(f)$, $H(0)$, $H(\vartheta)$, $H(M)$, $H(J)$, $H(F)$, $H(\mathcal{N})$ and $H(A)$ are valid, then, $\exists \tau_0, C > 0$ (independent of τ) s.t. $\forall \tau \in (0, \tau_0)$, the $\mathbf{u}_\tau$, $\bar{\mathbf{u}}_\tau$ and ξ_τ satisfy*

$$\|\mathbf{u}_\tau\|_{C(0,T;\mathrm{H})} \leq C, \tag{26}$$

$$\|\bar{\mathbf{u}}_\tau\|_{L^\infty(0,T;\mathrm{H})} \leq C, \tag{27}$$

$$\|\bar{\mathbf{u}}_\tau\|_{\mathcal{V}} \leq C, \tag{28}$$

$$\|\mathbf{u}_\tau\|_{\mathcal{V}} \leq C, \tag{29}$$

$$\|\xi_\tau\|_{\mathcal{X}^*} \le C, \tag{30}$$

$$\|\mathbf{u}'_\tau\|_{\mathcal{V}^*} \le C, \tag{31}$$

$$\|\bar{\mathbf{u}}_\tau\|_{M^{2,2}(0,T;\mathrm{V},\mathrm{V}^*)} \le C. \tag{32}$$

Proof. From estimation (20), one has

$$\begin{aligned} \|\mathbf{u}_\tau(t)\|_{\mathrm{H}} &= \|u_{1,\tau}(t)\|_{H_1} + \|u_{2,\tau}(t)\|_{H_2} \le \sum_{l=1}^{2}(\|u^k_{l,\tau}\|_{H_l} + \tfrac{|t-t_k|}{\tau}\|u^k_{l,\tau} - u^{k-1}_{l,\tau}\|_{H_l}) \\ &\le \sum_{l=1}^{2}(2\|u^k_{l,\tau}\|_{H_l} + \|u^{k-1}_{l,\tau}\|_{H_l}) = 2\|\mathbf{u}^k_\tau\|_{\mathrm{H}} + \|\mathbf{u}^{k-1}_\tau\|_{\mathrm{H}} \le C, \end{aligned}$$

$\forall t \in (t_{k-1}, t_k]$, $k \in \{1, ..., N\}$, thus estimation (26) is valid. Besides, (27) is checked via estimation (20).

Furthermore, the bound in (22) guarantees that

$$\begin{aligned} \|\bar{\mathbf{u}}_\tau\|^2_{\mathcal{V}} &= (\|\bar{u}_{1,\tau}\|_{\mathcal{V}_1} + \|\bar{u}_{2,\tau}\|_{\mathcal{V}_2})^2 \le 2(\|\bar{u}_{1,\tau}\|^2_{\mathcal{V}_1} + \|\bar{u}_{2,\tau}\|^2_{\mathcal{V}_2}) \\ &= 2(\textstyle\int_0^T \|\bar{u}_{1,\tau}(t)\|^2_{V_1}dt + \int_0^T \|\bar{u}_{2,\tau}(t)\|^2_{V_2}dt) = 2(\tau\displaystyle\sum_{k=1}^{N}\|u^k_{1,\tau}\|^2_{V_1} + \tau\sum_{k=1}^{N}\|u^k_{2,\tau}\|^2_{V_2}) \\ &\le 2\tau\sum_{k=1}^{N}\|\mathbf{u}^k_\tau\|^2_{\mathrm{V}} \le C, \end{aligned}$$

$$\begin{aligned} \|\mathbf{u}_\tau\|^2_{\mathcal{V}} &= (\|u_{1,\tau}\|_{\mathcal{V}_1} + \|u_{2,\tau}\|_{\mathcal{V}_2})^2 \le 2(\|u_{1,\tau}\|^2_{\mathcal{V}_1} + \|u_{2,\tau}\|^2_{\mathcal{V}_2}) \\ &= 2(\textstyle\int_0^T \|u_{1,\tau}(t)\|^2_{V_1}dt + \int_0^T \|u_{2,\tau}(t)\|^2_{V_2}dt) \\ &= 2(\displaystyle\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k}\|u^k_{1,\tau} + \tfrac{t-t_k}{\tau}(u^k_{1,\tau} - u^{k-1}_{1,\tau})\|^2_{V_1}dt + \sum_{k=1}^{N}\int_{t_{k-1}}^{t_k}\|u^k_{2,\tau} + \tfrac{t-t_k}{\tau}(u^k_{2,\tau} - u^{k-1}_{2,\tau})\|^2_{V_2}dt) \\ &\le 2(10\tau\sum_{k=1}^{N}\|u^k_{1,\tau}\|^2_{V_1} + 10\tau\sum_{k=1}^{N}\|u^k_{2,\tau}\|^2_{V_2}) \\ &\le 20\tau\sum_{k=1}^{N}\|\mathbf{u}^k_\tau\|^2_{\mathrm{V}} \le C. \end{aligned}$$

Hence, (28) and (29) both are valid. In addition, combining condition $H(J)$ (ii) and bound in (22) yields

$$\begin{aligned} \|\xi_\tau\|^2_{\mathcal{X}^*} &= (\|\xi_{1,\tau}\|_{\mathcal{X}_1^*} + \|\xi_{2,\tau}\|_{\mathcal{X}_2^*})^2 \le 2(\|\xi_{1,\tau}\|^2_{\mathcal{X}_1^*} + \|\xi_{2,\tau}\|^2_{\mathcal{X}_2^*}) \\ &= 2(\textstyle\int_0^T \|\xi_{1,\tau}(t)\|^2_{\mathrm{X}_1^*}dt + \int_0^T \|\xi_{2,\tau}(t)\|^2_{\mathrm{X}_2^*}dt) \\ &\le 2(\tau\displaystyle\sum_{k=1}^{N}\|\xi^k_{1,\tau}\|^2_{\mathrm{X}_1^*} + \tau\sum_{k=1}^{N}\|\xi^k_{2,\tau}\|^2_{\mathrm{X}_2^*}) \\ &\le 2\tau\sum_{k=1}^{N}[c^2_{1,J}(1 + \|M_1u^k_{1,\tau}\|_{\mathrm{X}_1})^2 + c^2_{2,J}(1 + \|M_2u^k_{2,\tau}\|_{\mathrm{X}_2})^2] \\ &\le 4\tau\sum_{k=1}^{N}[c^2_{1,J}(1 + \|M_1\|^2\|u^k_{1,\tau}\|^2_{\mathrm{X}_1}) + c^2_{2,J}(1 + \|M_2\|^2\|u^k_{2,\tau}\|^2_{\mathrm{X}_2})] \\ &\le 4(c^2_{1,J} + c^2_{2,J})T + 4(c^2_{1,J}\|M_1\|^2 + c^2_{2,J}\|M_2\|^2)\tau\sum_{k=1}^{N}(\|u^k_{1,\tau}\|^2_{\mathrm{X}_1} + \|u^k_{2,\tau}\|^2_{\mathrm{X}_2}) \\ &\le 4(c^2_{1,J} + c^2_{2,J})T + 4(c^2_{1,J}\|M_1\|^2 + c^2_{2,J}\|M_2\|^2)\tau\sum_{k=1}^{N}\|\mathbf{u}^k_\tau\|^2_{\mathrm{X}} \le C. \end{aligned}$$

Thus, (30) is valid as well.

It is clear that (17) is equivalent to the following

$$\begin{cases} u'_{1,\tau}(t) + \mathcal{N}_1(\bar{u}_{1,\tau}(t)) + M_1^*\xi_{1,\tau}(t) = \hbar_{1,\tau}(t), \\ u'_{2,\tau}(t) + \mathcal{N}_2(\bar{u}_{2,\tau}(t)) + M_2^*\xi_{2,\tau}(t) = \hbar_{2,\tau}(t). \end{cases}$$

Take $\mathbf{v} = (v_1, v_2) \in \mathcal{V}$ arbitrarily. Then we multiply the above equalities with $v_i, i = 1,2$ to obtain that for $i = 1,2$,

$$\langle \hbar_{i,\tau}, v_i\rangle_{\mathcal{V}_i^*\times\mathcal{V}_i} - \langle \widetilde{\mathcal{N}_i}(\bar{u}_{i,\tau}), v_i\rangle_{\mathcal{V}_i^*\times\mathcal{V}_i} - \langle \xi_{i,\tau}, \mathcal{M}_i v_i\rangle_{\mathcal{X}_i^*\times\mathcal{X}_i} = (u'_{i,\tau}, v_i)_{\mathcal{H}_i} = \langle u'_{i,\tau}, v_i\rangle_{\mathcal{V}_i^*\times\mathcal{V}_i}.$$

Hence, we deduce that for $i = 1,2$,

$$\|u'_{i,\tau}\|_{\mathcal{V}_i^*} \le \|\hbar_{i,\tau}\|_{\mathcal{V}_i^*} + \|\widetilde{\mathcal{N}_i}(\bar{u}_{i,\tau})\|_{\mathcal{V}_i^*} + \|M_i\|\|\xi_{i,\tau}\|_{\mathcal{X}_i^*}. \tag{33}$$

Note that $\widetilde{\mathcal{N}_i}$ is of boundedness in $\mathcal{V}_i \cap L^\infty(0,T;H_i)$. Then this condition along with bounds on $\{\bar{u}_{i,\tau}\}$ in $\mathcal{V}_i \cap L^\infty(0,T;H_i)$ (due to (27) and (28)), ensures that for $i = 1,2$, $\|\widetilde{\mathcal{N}_i}(\bar{u}_{i,\tau})\|_{\mathcal{V}_i^*} \le m_{i,0}\ \forall\tau > 0$ with $m_{i,0} > 0$ (independent of τ). This along with (31), (28), (29) and $H(F)$ implies that $\|u'_{i,\tau}\|_{\mathcal{V}_i^*} \le C_i$ for some $C_i > 0$. Therefore, one has

$$\|\mathbf{u}'_\tau\|^2_{\mathcal{V}^*} = (\|u'_{1,\tau}\|_{\mathcal{V}_1^*} + \|u'_{2,\tau}\|_{\mathcal{V}_2^*})^2 \le 2(\|u'_{1,\tau}\|^2_{\mathcal{V}_1^*} + \|u'_{2,\tau}\|^2_{\mathcal{V}_2^*}) \le 2(C_1^2 + C_2^2).$$

That is, (31) is valid.

Finally, it is sufficient to show that $\{\bar{\mathbf{u}}_\tau\}$ is bounded in $M^{2,2}(0,T;\mathrm{V},\mathrm{V}^*)$. But, using (28), we only know the boundedness of $\{\bar{\mathbf{u}}_\tau\}$ in $BV^2(0,T;\mathrm{V}^*)$. To the aim, we make a division $0 = b_0 < b_1 < ... < b_n = T$ with $b_j \in ((m_j-1)\tau, m_j\tau]$. Hence $\bar{u}_{i,\tau}(b_j) = u_{i,\tau}^{m_j}$ with $m_0 = 0$, $m_n = N$ and $m_{j+1} > m_j$ for $i = 1,2$ and $j = 1,2,...,N-1$. Thus, one has that for $i = 1,2$,

$$\begin{aligned}
\|\bar{u}_{i,\tau}\|^2_{BV^2(0,T;V_i^*)} &= \sum_{j=1}^n \|u_{i,\tau}^{m_j} - u_{i,\tau}^{m_{j-1}}\|^2_{V_i^*} \le \sum_{j=1}^n (m_j - m_{j-1}) \sum_{l=m_{j-1}+1}^{m_j} \|u_{i,\tau}^l - u_{i,\tau}^{l-1}\|^2_{V_i^*} \\
&\le \sum_{j=1}^n (m_j - m_{j-1}) \sum_{l=1}^N \|u_{i,\tau}^l - u_{i,\tau}^{l-1}\|^2_{V_i^*} \le N \sum_{l=1}^N \|u_{i,\tau}^l - u_{i,\tau}^{l-1}\|^2_{V_i^*} \\
&= T\tau \sum_{l=1}^N \|\tfrac{u_{i,\tau}^l - u_{i,\tau}^{l-1}}{\tau}\|^2_{V_i^*} = T\|u'_{i,\tau}\|^2_{\mathcal{V}_i^*}.
\end{aligned}$$

So it follows that

$$\begin{aligned}
\|\bar{\mathbf{u}}_\tau\|^2_{BV^2(0,T;\mathrm{V}^*)} &= (\|\bar{u}_{1,\tau}\|_{BV^2(0,T;V_1^*)} + \|\bar{u}_{2,\tau}\|_{BV^2(0,T;V_2^*)})^2 \\
&\le 2(\|\bar{u}_{1,\tau}\|^2_{BV^2(0,T;V_1^*)} + \|\bar{u}_{2,\tau}\|^2_{BV^2(0,T;V_2^*)}) \\
&= 2(\sum_{j=1}^n \|u_{1,\tau}^{m_j} - u_{1,\tau}^{m_{j-1}}\|^2_{V_1^*} + \sum_{j=1}^n \|u_{2,\tau}^{m_j} - u_{2,\tau}^{m_{j-1}}\|^2_{V_2^*}) \\
&= 2\sum_{j=1}^n (\|u_{1,\tau}^{m_j} - u_{1,\tau}^{m_{j-1}}\|^2_{V_1^*} + \|u_{2,\tau}^{m_j} - u_{2,\tau}^{m_{j-1}}\|^2_{V_2^*}) \\
&\le 2T(\|u'_{1,\tau}\|^2_{\mathcal{V}_1^*} + \|u'_{2,\tau}\|^2_{\mathcal{V}_2^*}) \le 2T\|\mathbf{u}'_\tau\|^2_{\mathcal{V}^*}.
\end{aligned}$$

Consequently, (32) is valid due to the bound in (31). □

Next, for the convenience, let the $\to$ and $\rightharpoonup$ denote the strong convergence and weak convergence, respectively.

Theorem 2. *Suppose that $H(f)$, $H(0)$, $H(\vartheta)$, $H(M)$, $H(J)$, $H(\mathcal{N})$, $H(F)$ and $H(A)$ hold. Let $\{\tau_n\}$ be a sequence such that $\lim_{n\to\infty}\tau_n = 0$. Then, for a subsequence, still denoted by $\{\tau_n\}$, one has*

$$\bar{\mathbf{u}}_\tau \rightharpoonup \mathbf{u} \text{ in } \mathcal{V} \text{ and } \mathcal{H}, \tag{34}$$

$$\mathbf{u}_\tau \rightharpoonup \mathbf{u} \text{ in } \mathcal{V}, \tag{35}$$

$$\mathbf{u}'_\tau \rightharpoonup \mathbf{u}' \text{ in } \mathcal{V}^*, \tag{36}$$

$$\xi_\tau \rightharpoonup \xi \text{ in } \mathcal{X}^*, \tag{37}$$

$$\mathbf{x}_\tau \to \mathbf{x} \text{ in } C(0,T;\mathrm{E}), \tag{38}$$

where $(\mathbf{x},\mathbf{u},\xi) \in C(0,T;\mathrm{E}) \times \mathcal{W} \times \mathcal{X}^*$ *is a solution to the RAS (in terms of Definition 1).*

Proof. Since $\mathcal{V}$ and $\mathcal{H}$ are reflexive, using (27)–(29) we might assume that, $\exists \mathbf{u}, \widehat{\mathbf{u}} \in \mathcal{V}$ s.t., (34) is valid and $\mathbf{u}_\tau \rightharpoonup \widehat{\mathbf{u}}$ in $\mathcal{V}$, as $\tau \to 0$. Meanwhile, simple calculations yield

$$\begin{aligned}
\|\bar{\mathbf{u}}_\tau - \mathbf{u}_\tau\|^2_{\mathcal{V}^*} &= (\|\bar{u}_{1,\tau} - u_{1,\tau}\|_{\mathcal{V}_1^*} + \|\bar{u}_{2,\tau} - u_{2,\tau}\|_{\mathcal{V}_2^*})^2 \le 2(\|\bar{u}_{1,\tau} - u_{1,\tau}\|^2_{\mathcal{V}_1^*} + \|\bar{u}_{2,\tau} - u_{2,\tau}\|^2_{\mathcal{V}_2^*}) \\
&= 2(\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - s)^2 \|\tfrac{u^k_{1,\tau} - u^{k-1}_{1,\tau}}{\tau}\|^2_{V_1^*} ds + \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - s)^2 \|\tfrac{u^k_{2,\tau} - u^{k-1}_{2,\tau}}{\tau}\|^2_{V_2^*} ds) \\
&\le 2(\tfrac{\tau^2}{3}\|u'_{1,\tau}\|^2_{\mathcal{V}_1^*} + \tfrac{\tau^2}{3}\|u'_{2,\tau}\|^2_{\mathcal{V}_2^*}) \le \tfrac{2\tau^2}{3}\|\mathbf{u}'_\tau\|^2_{\mathcal{V}^*}.
\end{aligned}$$

This along with the bound in (31) implies

$$\bar{\mathbf{u}}_\tau - \mathbf{u}_\tau \to 0_{\mathcal{V}^*} \text{ in } \mathcal{V}^*, \text{ as } \tau \to 0.$$

Noticing $\mathbf{u}_\tau \rightharpoonup \widehat{\mathbf{u}}$ in $\mathcal{V}$ and utilizing (34), one has $\bar{\mathbf{u}}_\tau - \mathbf{u}_\tau \rightharpoonup \mathbf{u} - \widehat{\mathbf{u}}$ in $\mathcal{V}$ as $\tau \to 0$. Besides, since the embedding $\mathcal{V} \hookrightarrow \mathcal{V}^*$ is continuous, one gets $\bar{\mathbf{u}}_\tau - \mathbf{u}_\tau \rightharpoonup \mathbf{u} - \widehat{\mathbf{u}}$ in $\mathcal{V}^*$. Hence, one has $\mathbf{u} = \widehat{\mathbf{u}}$, that is, (35) is valid.

Because the $\widehat{\mathbf{u}}_\tau$ given in (Lemma 3.6) are of boundedness in $\mathcal{V}$, we know that $\exists \mathbf{u}^* \in \mathcal{V}$ s.t. $\widehat{\mathbf{u}}_\tau \rightharpoonup \mathbf{u}^*$ in $\mathcal{V}$ as $\tau \to 0$. Meanwhile, simple calculations lead to

$$\begin{aligned}
\|\widehat{\mathbf{u}}_\tau - \mathbf{u}_\tau\|^2_{\mathcal{V}^*} &= (\|\widehat{u}_{1,\tau} - u_{1,\tau}\|_{\mathcal{V}_1^*} + \|\widehat{u}_{2,\tau} - u_{2,\tau}\|_{\mathcal{V}_2^*})^2 \le 2(\|\widehat{u}_{1,\tau} - u_{1,\tau}\|^2_{\mathcal{V}_1^*} + \|\widehat{u}_{2,\tau} - u_{2,\tau}\|^2_{\mathcal{V}_2^*}) \\
&= 2\sum_{i=1}^{2}\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \|\tfrac{t - t_k + \tau}{\tau}(u^{k-1}_{i,\tau} - u^k_{i,\tau}) + \tfrac{t - t_k}{\tau}(u^{k-1}_{i,\tau} - u^{k-2}_{i,\tau})\|^2_{V_i^*} dt \\
&\le 4\sum_{i=1}^{2}\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t - t_{k-1})^2 \|\tfrac{u^k_{i,\tau} - u^{k-1}_{i,\tau}}{\tau}\|^2_{V_i^*} + (t_k - t)^2 \|\tfrac{u^{k-1}_{i,\tau} - u^{k-2}_{i,\tau}}{\tau}\|^2_{V_i^*} dt \\
&\le 4(\tfrac{1}{3}\tau^2\|u'_{1,\tau}\|^2_{\mathcal{V}_1^*} + \tfrac{1}{3}\tau^2\|u'_{2,\tau}\|^2_{\mathcal{V}_2^*}) \\
&\le \tfrac{4}{3}\tau^2\|\mathbf{u}'_\tau\|^2_{\mathcal{V}^*}.
\end{aligned} \tag{39}$$

This implies that $\widehat{\mathbf{u}}_\tau - \mathbf{u}_\tau \to \mathbf{0}_{\mathcal{V}^*}$ as $\tau \to 0$. In a similar way, we can derive $\mathbf{u}^* = \mathbf{u}$. Besides, (31) ensures that, $\exists \mathbf{w}^* \in \mathcal{V}^*$ s.t.

$$\mathbf{u}'_\tau \rightharpoonup \mathbf{w}^* \text{ in } \mathcal{V}^*,$$

which along with (35), from [24], Proposition 23.19 guarantees that $\mathbf{w}^* = \mathbf{u}'$, that is, (36) is valid. In addition, estimation (29) ensures that $\exists \xi \in \mathcal{X}^*$ s.t. (37) is valid.

Applying [7], Proposition 5.3, p. 66 and [20], Section 4 (see, also [19], Section 4), we infer from $\mathbf{u} \in \mathcal{V}$ that $\exists |$ (mild solution) $\mathbf{x} = (x_1, x_2) \in C(0,T;\mathrm{E})$ formulated below

$$\begin{cases}
x_1(t) = e^{A_1(t)}x_1^0 + \int_0^t e^{A_1(t-s)} f_1(s, \mathbf{x}(s), \vartheta\mathbf{u}(s)) ds, \\
x_2(t) = e^{A_2(t)}x_2^0 + \int_0^t e^{A_2(t-s)} f_2(s, \mathbf{x}(s), \vartheta\mathbf{u}(s)) ds,
\end{cases}$$

to the problem

$$\begin{cases}
x_1'(t) = A_1 x_1(t) + f_1(t, \mathbf{x}(t), \vartheta\mathbf{u}(t)), \\
x_2'(t) = A_2 x_2(t) + f_2(t, \mathbf{x}(t), \vartheta\mathbf{u}(t)), \\
\mathbf{x}(0) = \mathbf{x}_0 \text{ and } \mathbf{u}(0) = \mathbf{u}_0.
\end{cases}$$

Next, we pay attention to $\mathbf{x}_\tau$ and $\mathbf{x}$. For each $t \in [0, T]$ one has

$$\begin{aligned} \|\mathbf{x}_\tau(t) - \mathbf{x}(t)\|_{\mathrm{E}} &= \|x_{1,\tau}(t) - x_1(t)\|_{E_1} + \|x_{2,\tau}(t) - x_2(t)\|_{E_2} \\ &\le M_{A_1} \int_0^t \|f_1(s, \mathbf{x}_\tau(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) - f_1(s, \mathbf{x}(s), \vartheta\mathbf{u}(s))\|_{E_1} ds \\ &\quad + M_{A_2} \int_0^t \|f_2(s, \mathbf{x}_\tau(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) - f_2(s, \mathbf{x}(s), \vartheta\mathbf{u}(s))\|_{E_2} ds \\ &\le \sum_{i=1}^{2} (M_{A_i} \int_0^t \|f_i(s, \mathbf{x}_\tau(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) - f_i(s, \mathbf{x}(s), \vartheta\widehat{\mathbf{u}}(s))\|_{E_i} ds \\ &\quad + M_{A_i} \int_0^t \|f_i(s, \mathbf{x}(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) - f_i(s, \mathbf{x}(s), \vartheta\mathbf{u}(s))\|_{E_i} ds) \\ &\le \sum_{i=1}^{2} (M_{A_i} \int_0^t \varphi_i(s) \|x_{i,\tau}(s) - x_i(s)\|_{E_i} ds \\ &\quad + M_{A_i} \int_0^t \|f_i(s, \mathbf{x}(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) - f_i(s, \mathbf{x}(s), \vartheta\mathbf{u}(s))\|_{E_i} ds), \end{aligned} \tag{40}$$

where $M_{A_i} := \max_{t \in [0,T]} \|e^{A_i(t)}\|$ for $i = 1, 2$. For $i = 1, 2$, one puts

$$g_i(t) = \int_0^t \|f_i(s, \mathbf{x}(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) - f_i(s, \mathbf{x}(s), \vartheta\mathbf{u}(s))\|_{E_i} ds, \ \ \forall t \in [0, T].$$

Using Gronwall's inequality and $g_i(s) \le g_i(t)$ $\forall s \le t$, one gets

$$\begin{aligned} \|\mathbf{x}_\tau(t) - \mathbf{x}(t)\|_{\mathrm{E}} &\le \sum_{i=1}^{2} [M_{A_i} g_i(t) + M_{A_i}^2 \int_0^t g_i(s)\varphi_i(s) \exp(M_{A_i} \int_0^r \varphi_i(r) dr) ds] \\ &\le \sum_{i=1}^{2} M_{A_i} g_i(t)(1 + M_{A_i} \int_0^t \varphi_i(s) \exp(M_{A_i} \int_0^r \varphi_i(r) dr) ds) \\ &\le \sum_{i=1}^{2} M_{A_i} g_i(t)(1 + M_{A_i} \|\varphi_i\|_{L^1} \exp(M_{A_i} \|\varphi_i\|_{L^1})), \end{aligned}$$

for all $t \in [0, T]$. Since $\mathbf{u}_\tau \rightharpoonup \mathbf{u}$ in $\mathcal{V}$ and $\mathbf{u}'_\tau \rightharpoonup \mathbf{u}'$ in $\mathcal{V}^*$, by the continuity of embedding $\mathcal{W} \hookrightarrow C(0, T; \mathrm{H})$, we deduce that $\mathbf{u}_\tau \rightharpoonup \mathbf{u}$ in $C(0, T; \mathrm{H})$. By [26], Lemma 4, one has

$$\mathbf{u}_\tau(t) \rightharpoonup \mathbf{u}(t) \text{ in H}, \ \ \forall t \in [0, T].$$

Since ϑ is compact, one gets

$$\vartheta(\widehat{\mathbf{u}}_\tau(t)) \to \vartheta(\mathbf{u}(t)) \text{ in Y}, \ \ \forall t \in [0, T].$$

Thus, using condition $H(f)$ (ii) and Lebesgue's dominated convergent theorem (i.e., [3], Theorem 1.65), one deduces from the last inequality that

$$\begin{aligned} \lim_{\tau \to 0} \|\mathbf{x}_\tau - \mathbf{x}\|_{C(0,T;\mathrm{E})} &\le \sum_{i=1}^{2} m_{i,1} \lim_{\tau \to 0} \int_0^T \|f_i(s, \mathbf{x}(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) - f_i(s, \mathbf{x}(s), \vartheta\mathbf{u}(s))\|_{E_i} ds \\ &\le \sum_{i=1}^{2} m_{i,1} \int_0^T \lim_{\tau \to 0} \|f_i(s, \mathbf{x}(s), \vartheta\widehat{\mathbf{u}}_\tau(s)) - f_i(s, \mathbf{x}(s), \vartheta\mathbf{u}(s))\|_{E_i} ds \to 0, \end{aligned}$$

where $m_{i,1} := M_{A_i}(1 + M_{A_i} \|\varphi_i\|_{L^1} \exp(M_{A_i} \|\varphi_i\|_{L^1}))$. Hence

$$\mathbf{x}_\tau \to \mathbf{x} \text{ in } C(0, T; \mathrm{E}),$$

that is, (38) is valid. So, from condition $H(F)$ one has

$$\begin{aligned} &\|\tfrac{1}{\tau} \int_{t_{k-1}}^{t_k} F_i(s, x_{j,\tau}(s)) ds - \tfrac{1}{\tau} \int_{t_{k-1}}^{t_k} F_i(s, x_j(s)) ds\|_{V_i^*} \\ &\le \tfrac{1}{\tau} \int_{t_{k-1}}^{t_k} \|F_i(s, x_{j,\tau}(s)) - F_i(s, x_j(s))\|_{V_i^*} ds \\ &\le \max_{s \in [0,T]} \|F_i(s, x_{j,\tau}(s)) - F_i(s, x_j(s))\|_{V_i^*} \to 0, \ \text{ as } \tau \to 0 \end{aligned} \tag{41}$$

for $j \neq i = 1,2$. Again from Lebesgue's dominated convergent theorem, we deduce that for $j \neq i = 1,2$, $\hbar_{i,\tau}(\cdot) - \tilde{h}_{i,\tau}(\cdot) \to 0_{\mathcal{V}_i^*}$ in $\mathcal{V}_i^*$ as $\tau \to 0$, with $\tilde{h}_{i,\tau}(t) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} F_i(s, x_j(s))ds$ for $t \in [t_{k-1}, t_k), k \in \{1, ..., N\}$. From $\mathbf{x} \in C(0,T;\mathrm{E})$, condition $H(f)$ and [27], Lemma 3.3, one has that for $j \neq i = 1,2$,

$$\hbar_{i,\tau}(\cdot) \to \hbar_i(\cdot) := F_i(\cdot, x_j(\cdot)) \text{ in } \mathcal{V}_i^*. \tag{42}$$

Next, it is sufficient to show that $(\mathbf{x}, \mathbf{u}, \xi)$ is a mild solution to the RAS. From (36) one has that for $i = 1,2$,

$$(u'_{i,\tau}, v_i)_{\mathcal{H}_i} = \langle u'_{i,\tau}, v_i\rangle_{\mathcal{V}_i^* \times \mathcal{V}_i} \to \langle u'_i, v_i\rangle_{\mathcal{V}_i^* \times \mathcal{V}_i} = (u'_i, v_i)_{\mathcal{H}_i}, \ \forall \mathbf{v} = (v_1, v_2) \in \mathcal{V}. \tag{43}$$

For Nemytskii's mapping $\widetilde{\mathcal{N}_i}$, in the case of $H(\mathcal{N})$ (ii)$_a$, using the uniform bound of $\{\bar{u}_{i,\tau}\} \subset M^{2,2}(0,T;V_i,V_i^*)$ (due to (32)), $\bar{u}_{i,\tau} \rightharpoonup u_i$ in $\mathcal{V}_i$ (due to (34)), and [23], Lemma 1, we obtain

$$\widetilde{\mathcal{N}_i} u_{i,\tau} \rightharpoonup \widetilde{\mathcal{N}_i} u_i \text{ in } \mathcal{V}_i^*, \ \text{ as } \tau \to 0.$$

Clearly, the above relation is still valid, in the case of $H(\mathcal{N})$ (ii)$_b$, because $u_{i,\tau} \rightharpoonup u_i$ in $\mathcal{V}_i$ (due to (35)). Therefore, we conclude that for $i = 1,2$,

$$\langle \widetilde{\mathcal{N}_i} \bar{u}_{i,\tau}, v_i\rangle_{\mathcal{V}_i^* \times \mathcal{V}_i} \to \langle \widetilde{\mathcal{N}_i} u_i, v_i\rangle_{\mathcal{V}_i^* \times \mathcal{V}_i}, \ \forall \mathbf{v} = (v_1, v_2) \in \mathcal{V}. \tag{44}$$

The convergence (37) implies that for $i = 1,2$,

$$\langle \xi_{i,\tau}, \mathcal{M}_i v_i\rangle_{\mathcal{X}_i^* \times \mathcal{X}_i} \to \langle \xi_i, \mathcal{M}_i v_i\rangle_{\mathcal{X}_i^* \times \mathcal{X}_i}, \ \forall \mathbf{v} = (v_1, v_2) \in \mathcal{V}. \tag{45}$$

In addition, from (3.30), one has that for $i = 1,2$,

$$\langle \hbar_{i,\tau}, v_i\rangle_{\mathcal{V}_i^* \times \mathcal{V}_i} = \langle \hbar_i, v_i\rangle_{\mathcal{V}_i^* \times \mathcal{V}_i}, \ \forall \mathbf{v} = (v_1, v_2) \in \mathcal{V}.$$

From (3.31)–(3.33), we obtain that for $i = 1,2$,

$$(u'_i, v_i)_{\mathcal{H}_i} + \langle \widetilde{\mathcal{N}_i} u_i, v_i\rangle_{\mathcal{V}_i^* \times \mathcal{V}_i} + \langle \xi_i, \mathcal{M}_i v_i\rangle_{\mathcal{X}_i^* \times \mathcal{X}_i} = \langle F_i, v_i\rangle_{\mathcal{V}_i^* \times \mathcal{V}_i}, \ \forall \mathbf{v} = (v_1, v_2) \in \mathcal{V}.$$

Finally, we need to show that $\xi(t) = (\xi_1(t), \xi_2(t)) \in \partial_1 J(\mathbf{x}(t), M\mathbf{u}(t)) \times \partial_2 J(\mathbf{x}(t), M\mathbf{u}(t))$. From (32), (34) and hypothesis $H(M)$, one has that for $i = 1,2$, $\mathcal{M}_i(\bar{u}_{i,\tau}) \to \mathcal{M}_i(u_i)$ in $\mathcal{X}_i^*$. Thus, we might assume that for $i = 1,2$, $M_i\bar{u}_{i,\tau}(t) \to M_i u_i(t)$ in X_i^*. Then it immediately follows that $M\bar{\mathbf{u}}_\tau(t) \to M\mathbf{u}(t)$ in X^*. On the other hand, since $\xi_\tau(t) = (\xi_{1,\tau}(t), \xi_{2,\tau}(t)) \in \partial_1 J(\mathbf{x}_\tau(t), M\bar{\mathbf{u}}_\tau(t)) \times \partial_2 J(\mathbf{x}_\tau(t), M\bar{\mathbf{u}}_\tau(t))$, by Lemma 1 (v) we conclude that for $i = 1,2$,

$$\langle \xi_{i,\tau}(t), M_i v_i\rangle_{X_i^* \times X_i} \leq J_i^\circ(\mathbf{x}_\tau(t), M\bar{\mathbf{u}}_\tau(t); M_i v_i), \ \forall \mathbf{v} = (v_1, v_2) \in \mathrm{V}, \tag{46}$$

which hence yields

$$\limsup_{\tau \to 0} \langle \xi_{i,\tau}(t), M_i v_i\rangle_{X_i^* \times X_i} \leq \limsup_{\tau \to 0} J_i^\circ(\mathbf{x}_\tau(t), M\bar{\mathbf{u}}_\tau(t); M_i v_i), \tag{47}$$

for all $\mathbf{v} = (v_1, v_2) \in \mathrm{V}$. Clearly, (38) ensures that $\mathbf{x}_\tau(t) \to \mathbf{x}(t)$ in E as $\tau \to 0$. In addition, since $\xi_{i,\tau} \to \xi_i$ weakly in $\mathcal{X}_i^*$ (due to (37)), it is easy to see from Proposition 4 that for $i = 1,2$,

$$\langle \xi_i(t), M_i v_i\rangle_{X_i^* \times X_i} \leq J_i^\circ(\mathbf{x}(t), M\mathbf{u}(t); M_i v_i), \ \forall \mathbf{v} = (v_1, v_2) \in \mathrm{V}. \tag{48}$$

Therefore, by the definition of $\partial_i J(\mathbf{x}(t), M\mathbf{u}(t)), i = 1,2$, we conclude that

$$\xi(t) = (\xi_1(t), \xi_2(t)) \in \partial_1 J(\mathbf{x}(t), M\mathbf{u}(t)) \times \partial_2 J(\mathbf{x}(t), M\mathbf{u}(t)). \tag{49}$$

Hence, $(\mathbf{x}, \mathbf{u}, \xi) \in C(0, T; \mathrm{E}) \times \mathcal{W} \times \mathcal{X}^*$ is a mild solution to the RAS in terms of Definition 3.1. $\square$

It is remarkable that the class of differential hemivariational inequalities (DHVIs) in [18] is extended to develop our general class of differential hemivariational inequalities systems (SDHVIs) by virtue of the partial Clarke's generalized subgradient operator. We first establish the upper semicontinuity of the partial Clarke's generalized directional derivatives (see our Proposition 4), and then extend the results for DHVI in [18] to the setting of SDHVI. Our main results can be applied to a special case of our abstract system (AS), where locally Lipschitz J and functions $F_l, l = 1, 2$ are supposed to be independent of $\mathbf{x}$. Thus, the AS reduces to the parabolic-type SHVI (i.e., problem (2.11)), which is a generalization of the parabolic-type HVI in [14] (i.e., problem (2.12)). In this case, the main results in [18] can not be applied to problem (2.11) because the criteria are not valid for it.

It is worth pointing out that there are the obvious disadvantages of the method based on the KKM approach for studying generalized parabolic or evolutionary SHVIs. In fact, if the operators in the method based on the KKM approach are not the KKM mappings, there are several possibilities which happen in the demonstrating process, e.g., in particular, whenever studying generalized parabolic or evolutionary SHVIs. In this article, when we deal with the parabolic-type SHVI in the demonstration process, the surjective theorem for pseudomonotonicity mappings, instead of the KKM theorems exploited by other authors in recent literature for a SHVI, ensures the successful continuation of our demonstration. This overcomes the drawback of the KKM-based approach. Hence, this shows that the surjective theorem for pseudomonotonicity mappings enjoys a highlighted contribution to the study of SDHVI from the viewpoint of methodology.

The unique findings of the article are specified below. First, we make use of the backward Euler difference formula (i.e., the Rothe rule) to investigate the parabolic-type SHVI driven by the abstract SEE. It is worth mentioning that, for the first time, the Rothe rule was used in [18] to study the parabolic-type HVI driven by the abstract EE. Up to now, there have been a few papers devoted to the Rothe rule for HVIs, see [21]. It is worth emphasizing that these were focused on only a single HVI via the Rothe rule.

Second, the main results can be applied to a special case of the AS, where locally Lipschitz J and functions $F_l, l = 1, 2$ are supposed to be independent of $\mathbf{x}$. Thus, the AS reduces to the parabolic-type SHVI (i.e., problem (2.11)), which is a generalization of the parabolic-type HVI (i.e., problem (2.12)). Without question, the main results in [18] can not be applied to problem (2.11). This is exactly the utility of our obtained results.

Third, to the best of our knowledge, except for the DHVI considered in[18], many works on the DVIs were promoted only by elliptic-type VIs/HVIs. For the first time, we consider the SDHVI driven by the parabolic-type SHVI. In addition, except for the DHVI considered in [18], in comparison with the previous works [11,16,17,19], we assume no convexity condition on the functions $\mathbf{u} \mapsto f_l(t, \mathbf{x}, \mathbf{u}), l = 1, 2$ and no compactness condition on C_0-semigroups $e^{A_l(t)}, l = 1, 2$.

4. Conclusions

In this article, under very suitable conditions, we take advantage of the Rothe rule to deal with the parabolic-type SHVI driven by the abstract SEE. For the first time, the Rothe rule was applied in [18] to study the parabolic-type HVI driven by the abstract EE. It is worth emphasizing that there have been a few papers devoted to the Rothe rule for HVIs, see [20]. However, these paid attention to only a single HVI by means of the Rothe rule.

As mentioned above, a particular case of our main theorem is an extension of ([18], Theorem 3.1) for the parabolic-type HVI driven by the abstract EE. Moreover, a special case of the one in [18] is also an extension of only a single parabolic-type HVI in [23]. An HVI is known as parabolic or evolutionary HVI if it involves the time derivative of unknown function. To the best of our knowledge, it will be quite extraordinary and very interesting to explore under what conditions the results in this article are still true for a generalized parabolic or evolutionary SHVI driven by the abstract SEE.

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