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η-$*$-Ricci Solitons and Almost co-Kähler Manifolds

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Abstract: The subject of the present paper is the investigation of a new type of solitons, called η-$*$-Ricci solitons in (k,μ)-almost co-Kähler manifold (briefly, *ackm*), which generalizes the notion of the η-Ricci soliton introduced by Cho and Kimura . First, the expression of the $*$-Ricci tensor on *ackm* is obtained. Additionally, we classify the η-$*$-Ricci solitons in (k,μ)-*ackm*s. Next, we investigate (k,μ)-*ackm*s admitting gradient η-$*$-Ricci solitons. Finally, we construct two examples to illustrate our results.

Keywords: $*$-Ricci tensor; η-$*$-Ricci solitons; gradient η-$*$-Ricci solitons; almost co-Kähler manifolds; (k,μ)-almost co-Kähler manifolds

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1. Introduction

The investigation of Ricci solitons on Riemannian and semi-Riemannian manifolds is an important topic in the area of differential geometry and in physics as well. Over the last few years, Ricci solitons and their generalizations have been enjoying quick growth by providing new techniques in understanding the geometry and topology of Riemannian manifolds. Interest in studying Ricci solitons and their generalizations in different geometrical contexts has also considerably increased due to their connection to general relativity.

The Ricci soliton generalizes the notion of the Einstein metric. A Ricci soliton of a Riemannian metric g on a manifold N is defined as follows:

$$\pounds_V g + 2\alpha g + 2\mathcal{S} = 0, \tag{1}$$

where $\pounds$, V, and α indicate the Lie-derivative operator, a smooth vector field, and a constant, respectively, and $\mathcal{S}$ is the Ricci tensor. Ricci solitons are the special solutions of the Ricci flow equation:

$$\frac{\partial}{\partial t} g = -2\mathcal{S}, \tag{2}$$

which was introduced by Hamilton [1]. Kumara and Venkatesha [2] investigated Ricci solitons in perfect fluid spacetime with the torse-forming vector field. Several authors [3–7] and many others have investigated Ricci solitons.

Cho and Kimura [8] studied real hypersurfaces in a complex space form and extended the idea of Ricci solitons to η-Ricci solitons. These solitons on a Riemannian manifold (N,g) satisfy the following equation:

$$\pounds_V g + 2\alpha g + 2\beta\eta\otimes\eta + 2\mathcal{S} = 0, \tag{3}$$

where β is a constant. If $\beta = 0$, then this soliton becomes a Ricci soliton, and for $\beta \neq 0$, it is called proper.

The soliton is reduced to a gradient η-Ricci soliton if V is the gradient of a smooth function $f : N \to \mathbb{R}$ (called the potential function). Then, Equation (3) becomes the following:

$$Hessf + \mathcal{S} + \alpha g + \beta\eta \otimes \eta = 0, \tag{4}$$

where $Hessf$ is the Hessian of f. The soliton is named shrinking, steady, or expanding when $\alpha < 0$, $\alpha = 0$, or $\alpha > 0$, respectively.

Recently, many works have been dedicated to η-Ricci solitons and gradient η-Ricci solitons in the context of Riemannian and contact manifolds. Furthermore, geometric flows have been initiated in the investigation of the cosmological model such as perfect fluid spacetimes. In [9], Blaga studied the η-Ricci soliton in perfect fluid spacetimes and obtained the Poisson equation from the soliton equation when the potential vector field ξ was of the gradient type. In [10], the η-Ricci solitons have also been studied in generalized Robertson–Walker spacetimes. The η-Ricci solitons have been investigated by many authors such as [11–16] and many others.

Tachibana [17] introduced the concept of the $*$-Ricci tensor in almost Hermitian manifolds, and this idea was applied to the almost contact manifold by Hamada [18], which was defined by the following equation:

$$\mathcal{S}^*(G, H) = \frac{1}{2} trace(K \to \tilde{R}(G, \varphi H)\varphi K), \tag{5}$$

for any G, H, K.

In 2014, Kaimakamis and Panagiotidou [19] introduced the concept of the $*$-Ricci soliton within the background of real hypersurfaces of a complex space form, and given by the following equation:

$$\pounds_V g + 2\alpha g + 2\mathcal{S}^* = 0. \tag{6}$$

In this article, we are interested in investigating a new type of soliton that combines the η-Ricci soliton and the $*$-Ricci soliton, which is named the η-$*$-Ricci soliton.

Definition 1. *g, the Riemannian metric is named the η-$*$-Ricci soliton if*

$$\pounds_V g + 2\alpha g + 2\beta\eta \otimes \eta + 2\mathcal{S}^* = 0. \tag{7}$$

The above-mentioned soliton is called gradient if V is the gradient of f, while Equation (7) becomes:

$$Hessf + 2\mathcal{S}^* + 2\alpha g + 2\beta\eta \otimes \eta = 0. \tag{8}$$

Very recently, Majhi, De, and Suh [20] undertook the study of $*$-Ricci solitons on Sasakian manifolds. Here, they proved that if the metric of a Sasakian manifold is a $*$-Ricci soliton, then it has a constant scalar curvature. The case of a $*$-Ricci soliton in the (ϵ)-Kenmotsu manifold was treated by De, Blaga, and De in [21]. In this line, it is suitable to mention that Venkatesha, Naik, and Kumara studied the $*$-Ricci soliton on η-Einstein Kenmotsu and three-dimensional Kenmotsu manifolds; they proved that if the metric of a η-Einstein Kenmotsu manifold is a $*$-Ricci soliton, then it is Einstein (see [22], Theorem 3.2). For the three-dimensional case, it was proved that if M admits a $*$-Ricci soliton, then it is of a constant sectional curvature -1 (see [22], Theorem 3.3).

In a recent paper Wang [23] studied Ricci solitons and gradient Ricci solitons in (k, μ)-*ackm*s. In recent years, some researchers have also studied $*$-Ricci solitons in the frame work of contact and paracontact manifolds [21,24–26]. To our knowledge, there are no results in the literature regarding $*$-Ricci solitons in *ackm*s or (k, μ)-*ackm*s in particular, nor in perfect fluid spacetimes.

Motivated by the studies above, in this article, we are interested in investigating η-$*$-Ricci solitons and gradient η-$*$-Ricci solitons in (k, μ)-*ackm*s.

The rest of the article is organized as follows:

In Section 2, after preliminaries, we obtain the $*$-Ricci tensor on (k,μ)-*ackm*s. We then classify the η-$*$-Ricci solitons in $(2m+1)$-dimensional (k,μ)-*ackm*s in Section 3. Next, we investigate (k,μ)-*ackm*s that admit gradient η-$*$-Ricci solitons. Finally, we provide an example to illustrate our result.

2. Preliminaries

A smooth manifold N^{2m+1} together with the triple (φ,ρ,η), where η, ρ, and φ indicate a 1-form, a global vector field, and a $(1,1)$-tensor field, is said to be an almost contact manifold [27] if:

$$\varphi^2 = -I + \eta \otimes \rho, \quad \eta(\rho) = 1, \tag{9}$$

where I is the identity automorphism. From (9), we can obtain $\varphi\rho = 0$ and $\eta \circ \varphi = 0$. An almost contact structure (φ,ρ,η) is called normal [27] if the Nijenhuis tensor of φ vanishes.

An almost contact manifold N^{2m+1} is called an almost contact metric manifold if the metric g obeys the following equation:

$$g(\varphi G, \varphi H) = g(G,H) - \eta(G)\eta(H) \tag{10}$$

for any vector fields $G, H \in \chi(N)$. Φ, the fundamental 2-form on N^{2m+1}, is defined by $\Phi(G,H) = g(G,\varphi H)$ for all $G, H \in \chi(N)$.

An almost contact metric manifold N^{2m+1} is named an *ackm* if both Φ and η are closed, that is, $d\Phi = 0$ and $d\eta = 0$. In addition, if N^{2m+1} is normal, then the manifold N^{2m+1} is called a co-Kähler manifold. An *ackm* is the same as an almost cosymplectic manifold [28] and investigated by several geometers [7,23,29–40].

On any *ackm*, we can define a (1,1)-tensor field $h = \pounds_\rho \varphi$. According to [41–43], it is known that h and $h'(= h \circ \varphi)$ are symmetric tensors and satisfy the following equations:

$$h\rho = 0, \quad h\varphi + \varphi h = 0, \quad trh = trh' = 0, \tag{11}$$

$$\nabla_\rho \varphi = 0, \quad \nabla\rho = h', \tag{12}$$

$$\varphi l \varphi - l = 2h^2, \tag{13}$$

$$\nabla_\rho h = -h^2\varphi - \varphi l, \tag{14}$$

$$\mathcal{S}(\rho,\rho) + trh^2 = 0, \tag{15}$$

where $l = \tilde{R}(.,\rho)\rho$ indicates the Jacobi operator along the Reeb vector field, $\tilde{R}$ is the Riemannian curvature tensor, tr denotes trace, and ∇ is the Riemannian connection.

The relation $(\pounds_\rho g)(G,H) = 2g(h'G,H)$, which holds on an *ackm* can be obtained by the Lie-derivative of g along the Reeb vector ρ and (12). This shows that the Reeb vector field ρ of N^{2m+1} is Killing if and only if h vanishes on N^{2m+1}. Hence, we get the subsequent definition.

Definition 2. *An ackm is named a K-ackm if ρ is Killing.*

In [42], Olszak established that an associated almost Kähler structure is integrable if and only if

$$(\nabla_G \varphi)(H) = g(hG,H)\rho - \eta(H)hG \tag{16}$$

for all vector fields $G, H \in \chi(N)$. From the equation above, it follows that an *ackm* is co-Kahler if and only if it is *K-ackm*. Moreover, a 3-dimensional *ackm* is co-Kähler if and only if it is *K-ackm*.

A (k,μ)-*ackm* N^{2m+1}, introduced by Endo [44], is an *ackm* whose structure vector field ρ belongs to the (k,μ)-nullity distribution, that is, $\tilde{R}$ obeys the equation below:

$$\tilde{R}(G,H)\rho = k(\eta(H)G - \eta(G)H) + \mu(\eta(H)hG - \eta(G)hH) \tag{17}$$

for any $G, H \in \chi(N)$ and $k, \mu \in \mathbb{R}$. Taking ρ instead of φ in (17), we have $l = -k\varphi^2 + \mu h$. Using this value of l in (13), it follows that:

$$h^2 = k\varphi^2. \tag{18}$$

Equation (18) infers that $k \leq 0$ and $k = 0$ if and only if N^{2m+1} is a K-*ackm*. In particular, if $\mu = 0$, then the manifold is said to be $N(k)$-*ackm* [45]. Any co-Kähler manifold satisfies (17) with $k = \mu = 0$. Dacko and Olszak [46] defined almost co-Kähler (k, μ, v)-spaces. An *ackm* is named a (k, μ, v)-space if $\tilde{R}$ obeys the following:

$$\begin{aligned} \tilde{R}(G, H)\rho &= k(\eta(H)G - \eta(G)H) + \mu(\eta(H)hG - \eta(G)hH) \\ &\quad - v(\eta(H)h'G - \eta(G)h'H) \end{aligned}$$

for all $G, H \in \chi(N)$.

In a (k, μ)-*ackm*, the subsequent relations hold [47]:

$$\nabla_\rho h = \mu h', \tag{19}$$

$$\nabla_\rho h^2 = 0, \tag{20}$$

$$l\varphi - \varphi l = 2\mu h'. \tag{21}$$

Lemma 1 ([38])**.** *The Ricci operator $\mathcal{Q}$ of a $(2m+1)$-dimensional (k, μ)-ackm, $m \geq 1$, is written as follows:*

$$\mathcal{Q} = \mu h + 2mk\eta \otimes \rho. \tag{22}$$

Lemma 2. *In a (k, μ)-ackm, the subsequent relations hold:*

$$\begin{aligned} (\nabla_G h)H - (\nabla_H h)G &= k[\eta(H)\varphi G - \eta(G)\varphi H + 2g(\varphi G, H)\rho] \\ &\quad + \mu[\eta(H)\varphi hG - \eta(G)\varphi hH]. \end{aligned} \tag{23}$$

$$\tilde{R}(\rho, G)H = k[g(G, H)\rho - \eta(H)G] + \mu[g(hG, H)\rho - \eta(H)hG], \tag{24}$$

$$\mathcal{S}(G, \rho) = 2mk\eta(G), \tag{25}$$

Lemma 3. *In a (k, μ)-ackm, the $*$-Ricci tensor is written as follows:*

$$\mathcal{S}^*(H, K) = 2\mu g(hH, K) - k[g(H, K) - \eta(H)\eta(K)]. \tag{26}$$

Proof. From (16), we have the following equation:

$$\nabla_H \varphi K = g(hH, K)\rho - \eta(K)hH + \varphi \nabla_H K. \tag{27}$$

Taking the covariant derivative of (27), we infer the following:

$$\begin{aligned} \nabla_G \nabla_H \varphi K &= \nabla_G g(hH, K)\rho + g(hH, K)h\varphi G - \nabla_H \eta(K)hG \\ &\quad - \eta(K)\nabla_H hG + g(hH, \nabla_G K)\rho - \eta(\nabla_G K)hH \\ &\quad + \varphi \nabla_H \nabla_G K. \end{aligned} \tag{28}$$

The foregoing equation entails that:

$$\begin{aligned} \nabla_H \nabla_G \varphi K &= \nabla_H g(hG, K)\rho + g(hG, K)h\varphi H - \nabla_G \eta(K)hH \\ &\quad - \eta(K)\nabla_G hH + g(hG, \nabla_H K)\rho - \eta(\nabla_H K)hG \\ &\quad + \varphi \nabla_G \nabla_H K. \end{aligned} \tag{29}$$

Equation (27) implies the following:

$$\nabla_{[G,H]}\varphi K = g(h([G,H]),K)\rho - \eta(K)h([G,H]) + \varphi\nabla_{[G,H]}K. \tag{30}$$

In view of (28)–(30), we get the equations below:

$$\begin{aligned}\tilde{R}(G,H)\varphi K &= g((\nabla_G h)H - (\nabla_H h)G, K)\rho + g(hH,K)h\varphi G \\ &\quad - g(hG,K)h\varphi H - g(h\varphi G,K)hH + g(h\varphi H,K)hG \\ &\quad - \eta(K)[(\nabla_G h)H - (\nabla_H h)G] + \varphi\tilde{R}(G,H)K.\end{aligned} \tag{31}$$

Using (23) in (31) gives the following:

$$\begin{aligned}\tilde{R}(G,H)\varphi K &= k[g(\varphi G,K)\eta(H) - g(\varphi H,K)\eta(G) + 2g(\varphi G,H)\eta(K)]\rho \\ &\quad + \mu[g(\varphi hG,K)\eta(H) - g(\varphi hH,K)\eta(G)]\rho + g(hH,K)h\varphi G \\ &\quad - g(hG,K)h\varphi H - g(h\varphi G,K)hH + g(h\varphi H,K)hG \\ &\quad - k[\eta(H)\eta(K)\varphi G - \eta(G)\eta(K)\varphi H + 2g(\varphi G,H)\eta(K)\rho] \\ &\quad - \mu[\eta(H)\eta(K)\varphi hG - \eta(G)\eta(K)\varphi hH] + \varphi\tilde{R}(G,H)K.\end{aligned} \tag{32}$$

From the equation above, we obtain:

$$\begin{aligned}g(\tilde{R}(G,H)\varphi K,\varphi L) &= -g(hH,K)g(\varphi hG,\varphi L) + g(hG,K)g(\varphi hH,\varphi L) \\ &\quad - g(\varphi hG,K)g(\varphi hH,L) + g(\varphi hH,K)g(\varphi hG,L) \\ &\quad - k[\eta(H)\eta(K)g(\varphi G,\varphi L) - \eta(G)\eta(K)g(\varphi H,\varphi L)] \\ &\quad - \mu[\eta(H)\eta(K)g(hG,L) - \eta(G)\eta(K)g(hH,L)] \\ &\quad + g(\tilde{R}(G,H)K,L) - \eta(\tilde{R}(G,H)K)\eta(L).\end{aligned} \tag{33}$$

Contracting G and L in (33) gives the following equation:

$$\mathcal{S}^*(H,K) = \mathcal{S}(H,K) - kg(H,K) - (2m-1)k\eta(H)\eta(K) + \mu g(hH,K). \tag{34}$$

Using Lemma 3 in (34), we then get:

$$\mathcal{S}^*(H,K) = 2\mu g(hH,K) - k[g(H,K) - \eta(H)\eta(K)]. \tag{35}$$

□

3. η-$*$-Ricci Solitons on (k,μ)-*ackms*

Let us assume that N^{2m+1} admit an η-$*$-Ricci soliton (g,ρ,α,β). Hence, from (7), we get the following:

$$(\pounds_\rho g)(G,H) + 2\mathcal{S}^*(G,H) + 2\alpha g(G,H) + 2\beta\eta(G)\eta(H) = 0, \tag{36}$$

which implies

$$g(\nabla_G\rho,H) + g(G,\nabla_H\rho) + 2\mathcal{S}^*(G,H) + 2\alpha g(G,H) + 2\beta\eta(G)\eta(H) = 0. \tag{37}$$

Using (12) and (26) in (37) reveals that

$$g(h\varphi G,H) + 2\mu g(hG,H) + (\alpha - k)g(G,H) + (\beta + k)\eta(G)\eta(H) = 0. \tag{38}$$

Substituting H by ρ in (38), we infer the following:

$$(\alpha+\beta)\eta(G) = 0, \tag{39}$$

which implies that $\alpha + \beta = 0$. Hence, we have:

Theorem 1. *If a (k,μ)-ackm admits an η-$*$-Ricci soliton, then the constants α and β are related by $\alpha+\beta=0$.*

In particular, if $\beta=0$, then $\alpha=0$. Therefore, we have:

Corollary 1. *If a (k,μ)-ackm admits a $*$-Ricci soliton, then the constant $\alpha=0$; hence, the soliton is steady.*

Let V be pointwise collinear with ρ, that is, $V=b\rho$, where b is a smooth function. Then (7) implies the following:

$$\begin{aligned}&(Gb)\eta(H)+(Hb)\eta(G)+b[g(\nabla_G\rho,H)+g(G,\nabla_H\rho)]\\&+2\mathcal{S}^*(G,H)+2\alpha g(G,H)+2\beta\eta(G)\eta(H)=0.\end{aligned} \tag{40}$$

Using (12) and (26) in (40) reveals that

$$\begin{aligned}&(Gb)\eta(H)+(Hb)\eta(G)+2bg(h\varphi G,H)+4\mu g(hG,H)\\&+2(\alpha-k)g(G,H)+2(\beta+k)\eta(G)\eta(H)=0.\end{aligned} \tag{41}$$

Setting $H=\rho$ in the foregoing equation entails that

$$Gb+(\rho b)\eta(G)=-2(\alpha+\beta)\eta(G). \tag{42}$$

Putting $G=\rho$ in (42), we obtain the following equation:

$$\rho b=-(\alpha+\beta). \tag{43}$$

With the help of (42) and (43), we infer that

$$Gb=-(\alpha+\beta)\eta(G), \tag{44}$$

which implies that *grad* b is a constant multiple of ρ.
If we take $\alpha+\beta=0$, then (44) implies the following:

$$Gb=0, \tag{45}$$

which implies that b is constant. Hence, we have:

Theorem 2. *In a (k,μ)-ackm admitting an η-$*$-Ricci soliton, if V is pointwise collinear with ρ, then grad b is a constant multiple of ρ, and for $\alpha+\beta=0$, V is a constant multiple of ρ.*

In particular, if $\beta=0$, then $\alpha=0$. Hence, we have:

Corollary 2. *In a (k,μ)-ackm admitting a steady $*$-Ricci soliton, if V is pointwise collinear with ρ, then V is a constant multiple of ρ.*

4. Gradient η-$*$-Ricci Solitons on (k,μ)-*ackms*

We consider a (k,μ)-*ackm* that admits a gradient η-$*$-Ricci soliton. Then (8) implies the following:

$$\nabla_G\mathcal{D}f+\mathcal{Q}^*G+\alpha G+\beta\eta(G)\rho=0. \tag{46}$$

Using Lemma 5 in (46) reveals that

$$\nabla_G\mathcal{D}f=-2\mu hG+(k-\alpha)G-(k+\beta)\eta(G)\rho. \tag{47}$$

Taking the covariant derivative of (47), we obtain the following:

$$\nabla_H \nabla_G \mathcal{D}f = -2\mu \nabla_H hG + (k-\alpha)\nabla_H G - (k+\beta)[\nabla_H \eta(G)\rho + \eta(G)h\varphi H]. \tag{48}$$

Interchanging G and H in (48) gives the equations below:

$$\nabla_G \nabla_H \mathcal{D}f = -2\mu \nabla_G hH + (k-\alpha)\nabla_G H - (k+\beta)[\nabla_G \eta(H)\rho + \eta(H)h\varphi G]. \tag{49}$$

Equation (47) implies:

$$\nabla_{[G,H]}\mathcal{D}f = -2\mu h([G,H]) + (k-\alpha)([G,H]) - (k+\beta)\eta([G,H])\rho. \tag{50}$$

In view of (48)–(50), we infer the following:

$$\tilde{R}(G,H)\mathcal{D}f = -2\mu[(\nabla_G h)H - (\nabla_H h)G] - (k+\beta)[(\nabla_G \eta)H - (\nabla_H \eta)G]\rho - (k+\beta)[\eta(H)h\varphi G - \eta(G)h\varphi H]. \tag{51}$$

Using Lemma 4 in (51) entails that

$$\tilde{R}(G,H)\mathcal{D}f = -2\mu k[\eta(H)\varphi G - \eta(G)\varphi H + 2g(\varphi G, H)\rho] + (2\mu^2 - k - \beta)[\eta(H)h\varphi G - \eta(G)h\varphi H]. \tag{52}$$

Taking the inner product of (52) with ρ and using (17), we get the following:

$$k[\eta(G)Hf - \eta(H)Gf] + \mu[\eta(G)hHf - \eta(H)hGf] = -4\mu k g(\varphi G, H). \tag{53}$$

Contracting G from Equation (52), we infer the following:

$$\mathcal{S}(H, Df) = 0. \tag{54}$$

With the help of (22) and (54), we obtain the equation below:

$$\mu g(h\mathcal{D}f, H) + 2mk(\rho f)\eta(H) = 0. \tag{55}$$

Substituting H with ρ in (55) gives:

$$k(\rho f) = 0, \tag{56}$$

which entails either $k = 0$ or $\rho f = 0$.

Case I: If $k = 0$, then it is a *K-ackm*.

Case II: If $\rho f = 0$, then by putting $G = \rho$ in (53) we get the following equation:

$$k(Hf) + \mu(hHf) = 0. \tag{57}$$

Once again, Equation (55) with $\rho f = 0$ implies the following:

$$\mu(hHf) = 0, \tag{58}$$

which entails either $\mu = 0$ or $hHf = 0$. If $\mu = 0$, the manifold is reduced to an $N(k)$-*ackm*. If $hHf = 0$, then either it is a *K-ackm* or f is constant. Therefore, we provide the following:

Theorem 3. *If a (k,μ)-ackm admits a gradient η-$*$-Ricci soliton, then one of the subsequent cases occur:*

(i) The manifold is an $N(k)$-ackm;
(ii) Either the manifold is a K-ackm or the potential function is constant.

5. Examples

Example 1. *We consider a 3-dimensional manifold $N^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard coordinate of $\mathbb{R}^3$. Let $\{w_1, w_2, w_3\}$ be a linearly independent vector fields on N^3 given by*

$$w_1 = e^z \frac{\partial}{\partial x} + e^{-z} \frac{\partial}{\partial y}, \quad w_2 = -e^z \frac{\partial}{\partial x} + e^{-z} \frac{\partial}{\partial y}, \quad w_3 = \frac{\partial}{\partial z}.$$

Let the metric g be defined by

$$g(w_1, w_1) = g(w_2, w_2) = g(w_3, w_3) = 1$$

and

$$g(w_1, w_2) = g(w_1, w_3) = g(w_2, w_3) = 0.$$

Let the one-form η, the Reeb vector field ρ and the (1,1)-tensor field φ are defined by

$$\eta = dz, \quad \rho = w_3, \quad \varphi w_1 = w_2, \quad \varphi w_2 = -w_1, \quad \varphi w_3 = 0.$$

From the above, we get

$$\eta(w_3) = 1, \quad \varphi^2 G = -G + \eta(G) w_3$$

and

$$g(\varphi G, \varphi H) = g(G, H) - \eta(G)\eta(H)$$

for any G, H. Also, we have

$$h w_1 = w_1, \quad h w_2 = -w_2, \quad h w_3 = 0.$$

The 2-form Φ is given by

$$\Phi = -dx \wedge dy.$$

Since $d\eta = 0$ and $d\Phi = 0$, N^3 is an almost co-Kähler manifold.

The Riemannian connection ∇ is given by

$$\nabla_{w_1} w_1 = 0, \quad \nabla_{w_1} w_2 = -w_3, \quad \nabla_{w_1} w_3 = w_2,$$
$$\nabla_{w_2} w_1 = -w_3, \quad \nabla_{w_2} w_2 = 0, \quad \nabla_{w_2} w_3 = w_1,$$
$$\nabla_{w_3} w_1 = 0, \quad \nabla_{w_3} w_2 = 0, \quad \nabla_{w_3} w_3 = 0.$$

From the above expressions, the following components of the Riemannian curvature tensor $\tilde{R}$ are obtained

$$\tilde{R}(w_1, w_2)w_3 = 0, \quad \tilde{R}(w_2, w_3)w_3 = -w_2, \quad \tilde{R}(w_1, w_3)w_3 = w_1,$$
$$\tilde{R}(w_1, w_2)w_2 = w_1, \quad \tilde{R}(w_2, w_3)w_2 = w_3, \quad \tilde{R}(w_1, w_3)w_2 = 0,$$
$$\tilde{R}(w_1, w_2)w_1 = -w_2, \quad \tilde{R}(w_2, w_3)w_1 = 0, \quad \tilde{R}(w_1, w_3)w_1 = w_3.$$

Using the above expressions of the curvature tensor $\tilde{R}$, we have

$$\tilde{R}(G, H)\rho = -[\eta(H)G - \eta(G)H]$$

for any G, H. Therefore N^3 is a $N(-1)$-almost co-Kähler manifold.
The Ricci tensor are given by

$$\mathcal{S}(w_1, w_1) = 0, \quad \mathcal{S}(w_2, w_2) = 0, \quad \mathcal{S}(w_3, w_3) = -2.$$

Suppose that f = constant and $\alpha = -1$, $\beta = 1$. Hence Equation (47) is satisfied. Thus g is a gradient η-$*$-Ricci soliton with the soliton vector field $V = \mathcal{D}f$, where f = constant and $\alpha = -1$, $\beta = 1$. Thus, Theorem 3 is verified.

Example 2. *Let us consider $N^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard coordinates of $\mathbb{R}^3$.*

Let $v_1 = \frac{\partial}{\partial x}$, $v_2 = \frac{\partial}{\partial y}$, $v_3 = 2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y} + e^z\frac{\partial}{\partial z}$ be three linearly independent vector fields in N^3.

Let the metric g be defined by the following:

$$g(v_1, v_1) = g(v_2, v_2) = g(v_3, v_3) = 1$$

and

$$g(v_1, v_2) = g(v_1, v_3) = g(v_2, v_3) = 0.$$

Let the 1-form η, the Reeb vector field ρ, and $(1,1)$-tensor field φ be defined by the following:

$$\eta = e^{-z}dz, \quad \rho = v_3, \quad \varphi v_1 = v_2, \quad \varphi v_2 = -v_1, \quad \varphi v_3 = 0.$$

Moreover, the (1,1)-tensor field h is defined by the following equation:

$$hv_1 = v_2, \quad hv_2 = -v_1, \quad hv_3 = 0.$$

The 2-form Φ is given by the following:

$$\Phi = -2dx \wedge dy - 4ye^{-z}dy \wedge dz - 4xe^{-z}dz \wedge dx.$$

Since $d\eta = 0$ and $d\Phi = 0$, N^3 is an almost co-Kähler manifold.
The Riemannian connection ∇ is given by the following:

$$\nabla_{v_1}v_1 = 0, \quad \nabla_{v_1}v_2 = -2v_3, \quad \nabla_{v_1}v_3 = 2v_2,$$

$$\nabla_{v_2}v_1 = -2v_3, \quad \nabla_{v_2}v_2 = 0, \quad \nabla_{v_2}v_3 = 2v_1,$$

$$\nabla_{v_3}v_1 = 0, \quad \nabla_{v_3}v_2 = 0, \quad \nabla_{v_3}v_3 = 0.$$

From the expressions above, the following components of the Riemannian curvature tensor $\tilde{R}$ are obtained:

$$\tilde{R}(v_1, v_2)v_1 = -4v_2, \quad \tilde{R}(v_1, v_2)v_2 = 4v_1, \quad \tilde{R}(v_1, v_2)v_3 = 0,$$

$$\tilde{R}(v_1, v_3)v_1 = 4v_3, \quad \tilde{R}(v_1, v_3)v_2 = 0, \quad \tilde{R}(v_1, v_3)v_3 = -4v_1,$$

$$\tilde{R}(v_2, v_3)v_1 = 0, \quad \tilde{R}(v_2, v_3)v_2 = 4v_3, \quad \tilde{R}(v_2, v_3)v_3 = -4v_2.$$

Using the expressions of the curvature tensor $\tilde{R}$ above, it follows that

$$\tilde{R}(G, H)\rho = -4\{\eta(H)G - \eta(G)H\}$$

for all $G, H \in \chi(N)$. Hence, N^3 is an $N(-4)$-almost co-Kähler manifold.
The components of the Ricci tensor $\mathcal{S}$ are given by the following:

$$\mathcal{S}(v_1, v_1) = \mathcal{S}(v_2, v_2) = 0, \quad \mathcal{S}(v_3, v_3) = -8.$$

In Equation (38), by substituting $G = H = v_i (i = 1, 2)$ and using the above results, we have $\alpha = k + 1$; similarly, if we take $G = H = v_3$, we get $\alpha + \beta = 0$. Since $k = -4$, we then infer that $\alpha = -3$, and from $\alpha + \beta = 0$, we get $\beta = 3$. Therefore, the data (g, ρ, α, β) for $\alpha = -3$ and $\beta = 3$ define an η-$*$-Ricci soliton on $(N, \varphi, \rho, \eta, g)$.

Again, suppose that $f = e^{-z}$ and $\alpha = k$, $\alpha + \beta = -1$. Therefore, $\mathcal{D}f = -e^{-z}v_3$. Hence, we get the following equations:

$$\nabla_{v_1}\mathcal{D}f = -2e^{-z}v_2,$$

$$\nabla_{v_2}\mathcal{D}f = -2e^{-z}v_1,$$

$$\nabla_{v_3}\mathcal{D}f = v_3.$$

Therefore, for $\alpha = k$, $\alpha + \beta = -1$, Equation (47) is satisfied. Thus, g is a gradient η-$*$-Ricci soliton with the soliton vector field $V = \mathcal{D}f$, where $f = e^{-z}$ and $\alpha = k$, $\alpha + \beta = -1$. Hence, Theorem 3 is verified.

6. Discussion

In reality, solitons are physically the waves that propagate with little loss of energy and hold their shape and speed after colliding with another such wave. Solitons are significant in the insightful treatment of initial-value problems for nonlinear partial differential equations that describe wave propagation. They additionally clarify the recurrence in the Fermi–Pasta–Ulam system.

In this current investigation, η-$*$-Ricci solitons and gradient η-$*$-Ricci solitons in (k,μ)-$ackm$s are studied. As far as our knowledge goes, the properties of $*$-Ricci solitons and η-$*$-Ricci solitons in perfect fluid spacetimes have not been studied by researchers. To fill this gap, in the near future, we or perhaps other authors could study the properties of η-$*$-Ricci solitons and gradient η-$*$-Ricci solitons in the general theory of relativity and cosmology, or in particular, in perfect fluid spacetimes.

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