mathematics

Article

Blow-Up and Global Existence of Solutions for the Time Fractional Reaction–Diffusion Equation

Linfei Shi [1,*], Wenguang Cheng [2], Jinjin Mao [1] and Tianzhou Xu [1,*]

[1] Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China; 3120205703@bit.edu.cn
[2] Department of Mathematics, Yuxi Normal University, Yuxi 653100, China; wgchengmath@yahoo.com
* Correspondence: 3120205702@bit.edu.cn (L.S.); xutianzhou@bit.edu.cn (T.X.)

Abstract: In this paper, we investigate a reaction–diffusion equation with a Caputo fractional derivative in time and with boundary conditions. According to the principle of contraction mapping, we first prove the existence and uniqueness of local solutions. Then, under some conditions of the initial data, we obtain two sufficient conditions for the blow-up of the solutions in finite time. Moreover, the existence of global solutions is studied when the initial data is small enough. Finally, the long-time behavior of bounded solutions is analyzed.

Keywords: caputo derivative; reaction–diffusion equation; blow-up; global existence

check for updates

Citation: Shi, L.; Cheng, W.; Mao, J.; Xu, T. Blow-Up and Global Existence of Solutions for the Time Fractional Reaction–Diffusion Equation. *Mathematics* **2021**, *9*, 3248. https://doi.org/10.3390/math9243248

Academic Editor: Alberto Ferrero

Received: 18 November 2021
Accepted: 13 December 2021
Published: 15 December 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1. Introduction

The purpose of this paper is to study the Cauchy problem for the following time fractional reaction–diffusion equation:

$$ {}^{c}D_t^{\alpha} u - d\Delta u = -u(1-u), \quad x \in \Omega,\ t > 0, \tag{1}$$

supplemented with a boundary condition:

$$ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{2}$$

and the following initial condition:

$$ u(x,0) = u_0(x), \quad x \in \Omega. \tag{3}$$

Here, $\Omega \subseteq \mathbb{R}^N (N \geq 1)$ is an open bounded domain with the Dirichlet boundary values $\partial\Omega$, $d > 0$ is the diffusion coefficient, and ${}^{c}D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1]$ defined by the following equation:

$$ {}^{c}D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(x,s)ds, \tag{4}$$

where $\Gamma(\cdot)$ is the Gamma function.

Fractional calculus is a generalization of ordinary differential as well as arbitrary non-integer orders. In recent years, it has achieved considerable development and has been widely used for modeling in various fields of science and engineering such as in diffusion process, signal processing, porous media, economics, physics and chemistry, etc. It is also considered to be an excellent tool for describing the hereditary properties and diffusion process of various materials. For more details, we refer the reader to the monographs of Samko, Kilbas, and Marichev [1], which is an encyclopedic treatment of fractional calculus and Podlubny [2], Kilbas et al. [3], and papers [4–9] and the references therein. In particular, differential equations with fractional derivatives are widely used to simulate the reaction–diffusion phenomenon. A powerful impetus for scholars to study such equations comes from physics. Reaction–diffusion on fractals is described by fractional diffusion equations (for example, strong porous materials or amorphous semiconductors can be found in [10] and the references therein); hence, scholars worked to obtain the so-called fractional diffusion equation.

Fractional diffusion theory has developed rapidly in the past few decades because it has successfully described many important phenomena. It is well known that many physical and biological phenomena can be described by the following reaction–diffusion equation:

$$u_t = \Delta u + F(u(t,x)),$$

where the variable $u(t,x)$ can be seen as the temperature in a chemical reaction or the population density of a biological species (see [11,12]). Meanwhile, since the important conception of "critical exponent" was defined in Fujita's work [13], many scholars have investigated the blow-up and global existence issues and have obtained some rich results in the past decades [14–18].

From the introduction above, it is easy to guess that researchers are interested in the blow-up problem of the solution for fractional diffusion equations. Many relevant papers have been published in recent years, such as [19–21]. Particularly, Hnaien et al. [22] studied the blow-up time and the large-time behavior of the global existence for fractional equations. In [23], the authors considered a reaction–diffusion equation with a Caputo fractional derivative in time and with various boundary conditions. In [24], the blow-up phenomenon and conditions of its appearance were proved by Xu.

This paper is motivated by the recent work of [25], in which they proved the dissipativity of the time fractional-order sub-diffusion equation:

$$^{c}D_t^{\alpha} u - d\Delta u + f(u) = 0, \quad x \in \Omega,\ \alpha \in (0,1), \tag{5}$$

where $f(u) = \sum_{j=0}^{2p-1} b_j u^j,\ b_{2p-1} > 0,\ p \in \mathbb{N}^+$.

Recently, Cao et al. [26] studied the following reaction–diffusion equation with a weak spatial source:

$$^{c}D_t^{\alpha} u - d\Delta u + a(x) + u^p = 0, \quad x \in \Omega,\ \alpha \in (0,1). \tag{6}$$

In this paper, we are interested in the blow-up phenomenon of solutions to the initial boundary value problem (1)–(3). Since the solution of the problem (1)–(3) may blow up in finite time, we shall use the following notation:

$$T_{\max} = T_{\max}(u_0) := \sup\{T > 0 : \text{the classical solution exists on } [0,T] \text{for the initial data } u_0\}.$$

If $T_{\max} < \infty$ and

$$\lim_{t \to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty,$$

where we call the solution u that blows up in finite time and say $T_{\max}$ is the blow-up time; otherwise, the solution exists globally.

We need to overcome some difficulties when we modify $f(u)$ based on the above-mentioned literature and give an appropriate $f(u)$. We decided to analyze the blow-up and global existence of solutions to the fractional Fisher–KPP (Kolmogorov–Petrovski–Piskunov) equations because most of the literature on Fisher–KPP equations is of integer order. This paper is mainly divided into the following parts: By using the contraction mapping principle, Theorem 1 in Section 2 proves the existence and uniqueness of the local solution of problems (1)–(3). In Section 3, two sufficient conditions for the blow-up of solutions in finite time are given in Theorem 2. According to the conditions of the initial data, we analyze the existence of globally bounded solutions and the long-time behavior of global solutions in Section 4. Finally, conclusions and a brief discussion are presented in Section 5.

2. Existence of a Local Solution

In this section, we prove the existence and uniqueness of the solution to problems (1)–(3).

Lemma 1. *Let $T_{\max} > 0$, we say that $u \in C([0, T_{\max}], C(\mathcal{D}))$ is a mild solution to (1)–(3) if u satisfies the following integral equation:*

$$u(t) = E_{\alpha,1}(-dt^{\alpha}\mathcal{A})u_0 + \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha}\mathcal{A})f(u(t-s))ds, \tag{7}$$

where $\mathcal{D} = \Omega$ or $\mathbb{R}^N$, $f(u(s)) = -u(s)(1-u(s))$, and $\mathcal{A}$ is the L^2 realization of the Laplacian $-\Delta$. $E_{\alpha}(z)$ and $E_{\alpha,\,\beta}(z)$ are the Mittag–Leffler functions defined by the following equation (see [3]):

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \beta > 0, \quad z \in \mathbb{C}. \tag{8}$$

Proof. Similar to the proof of [3], let us now solve (1) by the method of Laplace transforms. Assuming that:

$$I^{\alpha c}D_t^{\alpha}(u(t)) = u(t),$$

where $I^{\alpha}u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau,\ t > 0$. Then, applying I^{α} to both sides of (1), we obtain the following:

$$u(t) = dI^{\alpha}(\Delta u(t)) + I^{\alpha}f(u(t)). \tag{9}$$

The Laplace transform of a function f is defined as follows:

$$\mathcal{L}\{f(t)\} := \int_0^{\infty} e^{-st}f(t)dt,\ s \in \mathbb{C}.$$

The application of the Laplace transform yields the following:

$$U(s) = \frac{(-d\mathcal{A})}{s^{\alpha}}U(s) + s^{-\alpha}F(u(s)),$$

where $U = \mathcal{L}u$, $F = \mathcal{L}f$. Hence,

$$U(s) = \frac{1}{s^{\alpha}+d\mathcal{A}}F(u(s)). \tag{10}$$

We find that from the inversion of the Laplace transforms in (10),

$$u(t) = E_{\alpha,1}(-dt^{\alpha}\mathcal{A})u_0 + \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha}\mathcal{A})f(u(t-s))ds.$$

This completes the proof. □

Theorem 1. *Suppose that u_0 is continuous, then there exists a unique local mild solution $u \in C([0, T_{\max}], C(\mathcal{D}))$ for the problem (1)–(3), with the following alternative:*

- *either $T_{\max} = +\infty$,*
- *or $T_{\max} < +\infty$ and $\lim\limits_{t \to T_{\max}} \|u(\cdot,t)\|_{L^{\infty}} = +\infty$.*

Proof. Firstly, two important inequalities (see [19]) are given:

$$\|E_{\alpha,1}(-dt^{\alpha}\mathcal{A})u_0\|_{L^{\infty}} \leq \|u_0\|_{L^{\infty}}, \tag{11}$$

and

$$\|E_{\alpha,\alpha}(-dt^{\alpha}\mathcal{A})u_0\|_{L^{\infty}} \leq \frac{1}{\Gamma(\alpha)}\|u_0\|_{L^{\infty}}. \tag{12}$$

The existence of a local solution is derived by the contraction mapping principle. For $r \in [2, +\infty)$, let us define the Banach space as follows:

$$\mathcal{B} = \left\{ u \in C([0,\tau], C(\mathcal{D})) : \sup_{t \in [0,\tau]} \|u(t)\|_{L^\infty} \le r\|u_0\|_{L^\infty} \right\},$$

where τ is determined later. We define the following:

$$Tu(t) = E_{\alpha,1}(-dt^\alpha \mathcal{A})u_0 + \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-dt^\alpha \mathcal{A}) f(u(t-s))ds, \ t \in [0,\tau].$$

Observe that the nonlinear term $f(s) = -u(s)(1-u(s))$ is a locally Lipschitzian function; hence, there exists a constant $L > 0$ such that:

$$\|f(u(t-s)) - f(v(t-s))\|_{L^\infty} \le L\|u(t-s) - v(t-s)\|_{L^\infty}. \tag{13}$$

We first need to show that $T : \mathcal{B} \to \mathcal{B}$. If $u \in \mathcal{B}$, then by (11), (12), and (13), we see that:

$$\begin{aligned} \|Tu(t)\|_{L^\infty} &\le \|E_{\alpha,1}(-dt^\alpha \mathcal{A})u_0\|_{L^\infty} + \frac{\tau^\alpha}{\alpha}\|E_{\alpha,\alpha}(-dt^\alpha \mathcal{A})f(u)\|_{L^\infty} \\ &\le \|u_0\|_{L^\infty} + \frac{L\tau^\alpha}{\Gamma(\alpha+1)}\|u_0\|_{L^\infty}. \end{aligned}$$

Then $T : \mathcal{B} \to \mathcal{B}$ whenever $\tau^\alpha \le \frac{(r-1)\Gamma(\alpha+1)}{L}$.

Next, we prove that T is a contraction mapping. Assume that $u, v \in \mathcal{B}$, we derive the following equation:

$$\begin{aligned} \|Tu(t) - Tv(t)\|_{L^\infty} &\le \frac{t^\alpha}{\Gamma(\alpha+1)} \sup_{0 \le s \le t} \|f(u(t-s)) - f(v(t-s))\|_{L^\infty} \\ &\le \frac{t^\alpha L}{\Gamma(\alpha+1)} \sup_{0 \le s \le t} \|u(t-s) - v(t-s)\|_{L^\infty}. \end{aligned}$$

Furthermore, if we choose a τ that is small enough, such that $\tau^\alpha \le \frac{\Gamma(\alpha+1)}{2L}$, then:

$$\sup_{0 \le s \le t} \|Tu(t) - Tv(t)\|_{L^\infty} \le \frac{1}{2} \sup_{0 \le s \le t} \|u(t-s) - v(t-s)\|_{L^\infty}.$$

As a result, T is the contraction mapping in $\mathcal{B}$. Then, by the contraction mapping principle, problems (1)–(3) admit a unique mild solution $u \in \mathcal{B}$. □

3. Blow-Up of Solution

From the current literature, there are three main methods of considering the blow-up phenomenon: the comparison method [27], the concavity method [28], and the Kaplan's first eigenvalue method [29]. The Kaplan's first eigenvalue method is simpler than the other two methods, so we chose to use it for this paper in order to analyze the the blow-up phenomenon of the reaction–diffusion equation with time fractional derivatives. Problems (1)–(3) are reduced to the classic reaction–diffusion equation when $\alpha = 1$, which was studied in [11].

Firstly, we get two sufficient conditions for the blow-up of the solutions for (1)–(3) in finite time. Now, let us consider the following eigenvalue problem:

$$\begin{cases} -d\phi_{xx} = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{14}$$

Let ϕ_1 be the first eigenfunction associated with the first eigenvalue λ_1 of (14), which is normalized such that $\int_\Omega \phi_1(x)dx = 1$.

The following Jensen's inequality is important in Section 3.

Lemma 2 ([30]). *Let χ be a real-valued convex function defined on Ω, and let ψ and φ be non-negative Riemann integrable functions on Ω, where $\int_\Omega \varphi_1(x)dx = 1$. Then:*

$$\chi\left(\int_\Omega \psi(x)\varphi(x)dx\right) \leq \int_\Omega \chi(\psi(x))\varphi(x)dx. \tag{15}$$

Next, we state our main theoretical result.

Theorem 2. *(Finite-Time Blow-up) Let λ_1 and ϕ_1 be the principal eigenvalue and associated positive eigenfunction of the boundary value problem (14), with $\int_\Omega \phi_1(x)dx = 1$. Then, there exist solutions for problems (1)–(3), with $u_0(x) = \sigma\varphi(x)$ blowing up in finite time if one of the following conditions holds true:*

(i) $\lambda_1 < -1$ *and* $\sigma > 0$;

(ii) $\lambda_1 > -1$ *and* $\sigma > \sigma^* := (1+\lambda_1)A_0^{-1}$, *where* $A_0 = \int_\Omega \varphi\phi_1 dx$.

Furthermore, $T_{\max} \leq \bar{C}\sigma^{-1}$ has a positive constant $\bar{C} > 0$ that is dependent on the value of α, φ.

Proof. Define the following:

$$A(t) = \int_\Omega u(x,t)\phi_1 dx. \tag{16}$$

Then, let $\chi(\varphi(x)) = \varphi^2(x)$ in Lemma 2, which leads to:

$$\frac{\partial^\alpha A}{\partial t^\alpha} = \int_\Omega \frac{\partial^\alpha u}{\partial t^\alpha}\phi_1 dx = \int_\Omega (d\Delta u + u(u-1)\phi_1 dx \geq -(1+\lambda_1)A + A^2. \tag{17}$$

Now, according to the monotonicity and maximum principle of the diffusion equation, one gets the following:

$$-(1+\lambda_1)A + A^2 \leq \frac{\partial^\alpha A}{\partial t^\alpha} \leq \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}A'(t), \tag{18}$$

which implies that:

$$A'(t) \geq \frac{\Gamma(2-\alpha)}{t^{1-\alpha}}[-(1+\lambda_1)A + A^2]. \tag{19}$$

Now, we try to solve (19). Let us denote $B(t) = A^{-1}$, then $B'(t) = -\frac{A'(t)}{A^2}$, we then have the following equation:

$$\begin{aligned} \frac{dB(t)}{dt} &\leq -\frac{1}{A^2}\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}[-(1+\lambda_1)A + A^2] \\ &= (1+\lambda_1)\Gamma(2-\alpha)t^{\alpha-1}B(t) - \Gamma(2-\alpha)t^{\alpha-1}. \end{aligned} \tag{20}$$

It follows that when $\lambda_1 \neq -1$,

$$\begin{aligned} B(t) &\leq e^{(1+\lambda_1)t^\alpha\frac{\Gamma(2-\alpha)}{\alpha}}\left\{\left(B(0) - \frac{1}{1+\lambda_1}\right) + \frac{1}{1+\lambda_1}e^{-(1+\lambda_1)t^\alpha\frac{\Gamma(2-\alpha)}{\alpha}}\right\} \\ &= e^{(1+\lambda_1)t^\alpha\frac{\Gamma(2-\alpha)}{\alpha}}\left\{\delta^{-1}A_0^{-1} - \frac{1}{1+\lambda_1}\left(1 - e^{-(1+\lambda_1)t^\alpha\frac{\Gamma(2-\alpha)}{\alpha}}\right)\right\}, \end{aligned} \tag{21}$$

where $A_0 = \int_\Omega \varphi\phi_1 dx$. This yields the equation below:

$$A(t) \geq \frac{1}{\left(\sigma^{-1}A_0^{-1} - \frac{1}{1+\lambda_1}\right)e^{(1+\lambda_1)t^{\alpha}\frac{\Gamma(2-\alpha)}{\alpha}} + \frac{1}{1+\lambda_1}}. \tag{22}$$

Combining the above inequalities, we then deduce the following:
(i) if $\lambda_1 < -1$, then $u(x,t)$ will eventually blow up in finite time for any $\sigma > 0$;
(ii) if $\lambda_1 > -1$, it follows that when $\sigma^{-1}A_0^{-1} - \frac{1}{1+\lambda_1} < 0$, i.e., $\sigma > (1+\lambda_1)A_0^{-1}$, $u(x,t)$ blows up in finite time.

In particular, since (22) implies that $\widetilde{T}_{\max} \leq \bar{C}\sigma^{-1}$ with a constant $\bar{C} > 0$ dependent on α, φ, the comparison principle derives that same estimate holds for $T_{\max}$. This completes the proof. □

4. Global Existence and Long-Time Asymptotic Behavior

In this section, we obtain the global existence and long-time behavior of the global solutions of (1)–(3) for small initial data.

4.1. Existence of a Global Solution

Theorem 3. *Let* $0 \leq u(x,0) \leq 1$. *Then, problems* (1)–(3) *admit a global solution u that satisfies* $0 \leq u(x,t) \leq 1$.

Proof. Firstly, we show that $u \geq 0$. Multiplying scalarly in $L^2(\Omega)$ Equation (1) by $\tilde{u} := \min(u,0)$ and integrating over Ω, we obtain the following:

$$\int_{\Omega} {}^{c}D_t^{\alpha}\tilde{u}(x,t)\cdot\tilde{u}(x,t)dx - \int_{\Omega} d\Delta\tilde{u}(x,t)\cdot\tilde{u}(x,t)dx = \int_{\Omega}\tilde{u}^2(x,t)(\tilde{u}(x,t)-1)dx. \tag{23}$$

Then, using the maximum principle and the inequality (see [31]):

$$2v(t)^{c}D_t^{\alpha}v(t) \geq {}^{c}D_t^{\alpha}v^2(t),\ v \in C^1([0,T]),$$

we have the following:

$${}^{c}D_t^{\alpha}\int_{\Omega}\tilde{u}^2(x,t)dx \lesssim \int_{\Omega}\tilde{u}^2(x,t)dx. \tag{24}$$

By denoting $\int_{\Omega}\tilde{u}^2(x,t)dx = E(t)$ in (24), we can deduce the equation below:

$$\begin{cases} {}^{c}D_t^{\alpha}E(t) \lesssim E(t), \\ E(0) = 0, \end{cases} \tag{25}$$

which implies that $\int_{\Omega}\tilde{u}^2(x,t)dx = 0$ as $\tilde{u}(x,0) = 0$. Consequently, $u \geq 0$.

Now, we show the upper estimate $u \leq 1$. We multiply scalarly in $L^2(\Omega)$ Equation (1) by $\hat{u} := \min(1-u,0)$, we then obtain the following:

$$\int_{\Omega} {}^{c}{D_t}^{\alpha}\hat{u}(x,t)\cdot\hat{u}(x,t)dx - \int_{\Omega} d\Delta\hat{u}(x,t)\cdot\hat{u}(x,t)dx = \int_{\Omega}\hat{u}^2(x,t)(\hat{u}(x,t)-1)dx. \tag{26}$$

As in the calculations above, for the function $\hat{u} := \min(1-u,0)$ we have the following:

$${}^{c}D_t^{\alpha}\int_{\Omega}\hat{u}^2(x,t)dx \lesssim \int_{\Omega}\hat{u}^2(x,t)dx. \tag{27}$$

Hence, $\int_{\Omega}\hat{u}^2(x,t)dx = 0$, which implies $u \leq 1$. The result follows as $0 \leq u(x,0) \leq 1$. This completes the proof. □

4.2. Long-Time Behavior of the Global Solution

Lemma 3 ([3])**.** *Let $\omega(t) \geq 0$ be a locally integrable non-negative function on $[0, +\infty)$, such that ${}^cD_t^\alpha \omega(t) \leq \lambda\omega(t) + b$. Then, we have the following equation:*

$$\omega(t) \leq \omega_0 E_\alpha(\lambda t^\alpha) + bt^\alpha E_{\alpha,\, \alpha+1}(\lambda t^\alpha). \tag{28}$$

where $E_\alpha(z)$ and $E_{\alpha,\, \beta}(z)$ are defined in (8), which are the fractional generalizations of the exponential function and play an important role in fractional calculus.

Theorem 4. *Assume that $0 \leq u(x, 0) \leq 1$. Then, the solution of problems (1)–(3) satisfies the estimate below:*

$$\|u\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} E_\alpha(-zt^\alpha) + C|\Omega| E_{\alpha,\alpha+1}(-zt^\alpha), \quad t > 0, \tag{29}$$

where $z = \frac{2d}{c_0}$ and $c_0 = c_0(\Omega) > 0$ are constant, and C is dependent on t.

Proof. Similar to Theorem 1 in [25], let $f(u) = u(1-u)$, we can verify that there exists a constant $c_1 > 0$, such that:

$$\langle f(u), u \rangle \geq c_1, \tag{30}$$

where $\langle\ ,\ \rangle$ denotes the L^2-inner product. By multiplying scalarly in $L^2(\Omega)$ Equation (1) by $u = u(x,t)$ and integrating over Ω, we get the following equation:

$$\langle u, {}^c D_t^\alpha \rangle - d \int_\Omega u \Delta u dx + \langle f(u), u \rangle = 0. \tag{31}$$

According to the energy inequality:

$$\langle u, {}^cD_t^\alpha u \rangle = \int_\Omega u\, {}^cD_t^\alpha u dx \geq \frac{1}{2} {}^cD_t^\alpha \|u\|^2_{L^2(\Omega)}$$

and from the estimation in (30), it follows that:

$$\frac{1}{2} {}^cD_t^\alpha \|u\|^2_{L^2(\Omega)} + d\|\nabla u\|^2_{L^2(\Omega)} + c_1|\Omega| \leq 0, \tag{32}$$

where $|\Omega|$ denotes the measure of Ω. By applying Poincaré inequality, there exists a constant $c_0 = c_0(\Omega) > 0$, such that $\|u\|_{L^2(\Omega)} \leq c_0\|\nabla u\|_{L^2(\Omega)}$. Hence, we further infer that:

$${}^cD_t^\alpha \|u\|^2_{L^2(\Omega)} + \frac{2d}{c_0}\|u\|^2_{L^2(\Omega)} \leq -2c_1|\Omega|. \tag{33}$$

Via the fractional Grönwall inequality in Lemma 3, we get the following equation:

$$\|u\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} E_\alpha(-zt^\alpha) + C|\Omega| E_{\alpha,\alpha+1}(-zt^\alpha), \tag{34}$$

where $z = \frac{2d}{c_0}$. The asymptotic decay of $\|u\|^2_{L^2(\Omega)}$ can be deduced from the above-mentioned inequality. By the asymptotic property of the Mittag–Leffler function, for any $\varepsilon > 0$, there exists $t_1 > 0$, such that $\|u_0\|^2_{L^2(\Omega)} E_\alpha(-zt^\alpha) < \varepsilon$ for any $t > t_1$. On the other hand, we know that $E_{\alpha,\alpha+1}(-zt^\alpha) \leq \frac{1}{t^\alpha z}$. This completes the proof. □

5. Conclusions

In the present paper, we analyze a reaction–diffusion equation with a Caputo fractional derivative in time and with boundary conditions. Firstly, the existence and uniqueness of a local solution are obtained in Theorem 1 by using the contraction mapping principle. Then, by Jensen's inequality and the Kaplan's first eigenvalue method, we obtain some sufficient conditions for a finite-time blow-up where a principal eigenvalue problem plays a crucial role. Meanwhile, based on the work of Cheng et al., after some modifications to the equation, we prove the existence of global solutions under small initial conditions by using the maximum principle. Finally, the long-time behavior of bounded solutions are analyzed in Theorem 4. This enriches the application of fractional reaction–diffusion equations in the field of blow-up problems.

By reading the relevant literature, we found that there are few studies on the free boundary problem of fractional reaction–diffusion equations. Therefore, we will consider doing some work in this regard in the future. Inspired by Fujita's paper on critical exponent, we will also consider adding an exponential term to $F(u(t,x))$ in order to analyze the blow-up and global existence of solutions under the Dirichlet or Neumann condition and the Stefan condition.

Author Contributions: Conceptualization and methodology, T.X.; formal analysis, software, and validation, L.S.; visualization, J.M.; writing—review and editing, W.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the Joint Special Funds of the Basic Research Foundation of Yunnan Province (No. 2019FH001-080).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: All authors are very grateful to the reviewers for the invaluable and expert comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives, Theory and Applications*; Gordon & Breach: Yverdon, Switzerland, 1993.
2. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1998.
3. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Inc.: New York, NY, USA, 2006.
4. Wang, G.; Xu, T. Invariant analysis and explicit solutions of the time fractional nonlinear perturbed Burgers equation. *Nonlinear Anal-Model.* **2015**, *20*, 570–584. [CrossRef]
5. Yang, Z.; Xu, T. On new existence results for fractional differential equations in quaternionic analysis. *Adv. Appl. Cliffoed. Algebr.* **2015**, *25*, 733–740. [CrossRef]
6. Wang, C.; Xu, T. Hyers-Ulam stability of fractional linear differential equations involving Caputo fractional derivatives. *Appl. Math-Czech.* **2015**, *60*, 383–393. [CrossRef]
7. Wang, G.; Xu, T. Symmetry properties and explicit solutions of the nonlinear time fractional KdV equation. *Bound. Value. Probl.* **2013**, *2013*, 1–13. [CrossRef]
8. Wang, G.; Xu, T. The modified fractional sub-equation method and its applications to nonlinear fractional partial differential equations. *Rom. J. Phys.* **2014**, *59*, 636–645.
9. Jleli, M.; Samet, B. Sufficient criteria for the absence of global solutions for an inhomogeneous system of fractional differential equations. *Mathematics* **2020**, *8*, 9. [CrossRef]
10. Anh. V.V.; Leonenko. N.N. Spectral analysis of fractional kenetic equations with random data. *J. Statist. Phys.* **2001**, *104*, 1349–1387. [CrossRef]
11. Du, Y.; Lin, Z. Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary. *SIAM J. Math. Anal.* **2010**, *42*, 377–405. [CrossRef]
12. Du, Y.; Guo, Z. Spreading-vanishing dichotomy in a diffusive logistic model with a free boundary. II. *J. Differ. Equ.* **2011**, *250*, 4336–4366. [CrossRef]
13. Fujita, H. On the blowing up of solution of Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Univ. Tokyo Sect. I* **1966**, *13*, 109–405.
14. Meier, P. On the critical exponent for reaction-diffusion equations. *Arch. Ration. Mech. Anal.* **1990**, *109*, 63–71. [CrossRef]
15. Zhou, P.; Bao, J.; Lin, Z. Global existence and blowup of a localized problem with free boundary. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 2523–2533. [CrossRef]
16. Zhang, G.; Wang, Y. Critical exponent for nonlocal diffusion equations with Dirichlet boundary condition. *Math. Comput. Model.* **2011**, *54*, 203–209. [CrossRef]

17. Wang, J.; Cao, J. Fujita type critical exponent for a free boundary problem with spatial-temporal source. *Nonlinear Anal. Real World Appl.* **2020**, *51*, 103004. [CrossRef]
18. Wang, J.; Wang, J.; Cao, J. Blow up and global existence of a free boundary problem with weak spatial source. *Appl. Anal.* **2021**, *100*, 964–974. [CrossRef]
19. Tapdigoğlu, R.; Orebek, B. Global existence and blow-up of solutions of the time-fractional space-involution reaction-diffusion equation. *Turkish J. Math.* **2020**, *44*, 960–969. [CrossRef]
20. Ahmad, B.; Kirane, M. Blowing-up solutions of distributed fractional differential systems. *Chaos Solitons Fractals* **2021**, *145*, 110747. [CrossRef]
21. Alsaedi, A.; Ahmad, B.; Kirane, M.; Torebek, B.T. Blowing-up solutions of the time-fractional dispersive equations. *Adv. Nonlinear Anal.* **2021**, *10*, 952–971. [CrossRef]
22. Hnaien, D.; Kellil, F.; Lassoued, R. On a problem of J. Nakagawa, K. Sakamoto, M. Yamamoto. *arXiv* **2012**, arXiv:1207.1235.
23. Ahmad, B.; Alhothuali, M.S.; Alsulami, H.H.; Kirane, M.; Timoshin, S. On a time fractional reaction diffusion equation. *Appl. Math. Comput.* **2015**, *257*, 199–204. [CrossRef]
24. Xu, Q.; Xu, Y. Extremely low order time-fractional differential equation and application in combustion process. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *64*, 135–148. [CrossRef]
25. Cheng, B.; Guo, Z.; Wang, D. Dissipativity of semilinear time fractional subdiffusion equations and numerical approximations. *Appl. Math. Lett.* **2018**, *86*, 276–283. [CrossRef]
26. Cao, J.; Song, G.; Wang, J.; Shi, Q.; Sun, S. Blow-up and global solutions for a class of time fractional nonlinear reaction–diffusion equation with weakly spatial source. *Appl. Math. Lett.* **2019**, *91*, 201–206. [CrossRef]
27. Evans, L.C. *Partial Differential Equations*; American Mathematical Society: New York, NY, USA, 2010.
28. Levine, H.A.; Payne, L.E. Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time. *J. Differ. Equ.* **1974**, *16*, 319–334. [CrossRef]
29. Kaplan, S. On the growth of solutions of quasi-linear parabolic equations.*Commun. Pure Appl. Math.* **1963**, *16*, 305–330. [CrossRef]
30. Borwein, p.; Erdlyi, T. *Polynomials and Polynomial Inequalities*; Springer: New York, NY, USA, 1995.
31. Alsaedi, A.; Ahmad, B.; Kirane, M. Maximum principle for certain generalized time and space fractional diffusion equations. *Q. Appl. Math.* **2015**, *73*, 163–175. [CrossRef]