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Laplace Transform and Semi-Hyers–Ulam–Rassias Stability of Some Delay Differential Equations

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Abstract: In this paper, we study semi-Hyers–Ulam–Rassias stability and generalized semi-Hyers–Ulam–Rassias stability of differential equations $x'(t) + x(t-1) = f(t)$ and $x''(t) + x'(t-1) = f(t)$, $x(t) = 0 \quad \text{if} \quad t \leq 0$, using the Laplace transform. Our results complete those obtained by S. M. Jung and J. Brzdek for the equation $x'(t) + x(t-1) = 0$.

Keywords: semi-Hyers–Ulam–Rassias stability; delay differential equations; Laplace transform

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1. Introduction

The study of Ulam stability began in 1940, when Ulam posed a problem concerning the stability of homomorphisms (see [1]). In 1941, Hyers [2] gave an answer, in the case of the additive Cauchy equation in Banach spaces, to the problem posed by Ulam [1].

In 1993, Obloza [3] started the study of Hyers–Ulam stability of differential equations. Later, in 1998, Alsina and Ger [4] studied the equation $y'(x) - y(x) = 0$. Many mathematicians have further studied the stability of various equations. For a collection of results regarding this problematic, see [5] or [6].

There are many methods for studying Hyers–Ulam stability of differential equations, such as the direct method, the Gronwall inequality method, the fixed point method, the integral transform method, etc.

We mention that the Laplace transform method was used by H. Rezaei, S. M. Jung and Th. M. Rassias [7] and by Q. H. Alqifiary and S. M. Jung [8] to study the differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t).$$

This method was also used in [9], where Laguerre differential equation

$$xy'' + (1-x)y' + ny = 0, \quad n \text{ positive integer}$$

and Bessel differential equation

$$xy'' + y' + xy = 0,$$

was studied. In [10], Mittag-Leffler–Hyers–Ulam stability of the following linear differential equation of first order was studied with this method:

$$u'(t) + lu(t) = r(t), \ t \in I, u, r \in C(I), I = [a, b].$$

In [11], the semi-Hyers–Ulam–Rassias stability of a Volterra integro-differential equation of order I with a convolution type kernel was studied via Laplace transform:

$$y'(t) + \int_0^t y(u)g(t-u)du - f(t) = 0, \quad t \in (0,\infty),$$

$f, g, y : (0,\infty) \to \mathbb{F}$ functions of exponential order and continuous and $\mathbb{F}$ the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

In [12], the semi-Hyers–Ulam–Rassias stability of the convection partial differential equation was also studied using Laplace transform:

$$\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} = 0,\ a > 0,\ x > 0,\ t > 0, y(0,t) = c,\ y(x,0) = 0.$$

In [13], the delay equation

$$y'(t) = \lambda y(t-\tau), \lambda \neq 0, \tau > 0,$$

was studied, using direct method.

In the following, we will study semi-Hyers–Ulam–Rassias stability and generalized semi-Hyers–Ulam–Rassias stability of some equations, with delay of order one and two, with Laplace transform. We complete the results obtained in [13]. Delay differential equations have many applications in various areas of engineering science, biology, physics, etc. The monograph [14] contains some modeling examples from mechanics, chemistry, ecology, biology, psychology, etc. For other applications, see also [15].

We first recall some notions and results regarding the Laplace transform.

Definition 1. *A function $x : \mathbb{R} \to \mathbb{R}$ is called an original function if the following conditions are satisfied:*

1. $x(t) = 0, t < 0$;
2. *x is piecewise continuous;*
3. $\exists M > 0$ *and* $\sigma_0 \geq 0$ *such that*

$$|x(t)| \leq M \cdot e^{\sigma_0 t}, \quad \forall t \in \mathbb{R}.$$

We denote by $\mathcal{O}$ the set of original functions. We denote by $M(x)$ the set of all numbers that satisfy the condition 3.

The number $\sigma_x = \inf\{\sigma_0 \mid \sigma_0 \in M(x)\}$ is called abscissa of convergence of x.

The functions that appear below are considered original functions. Hence, since in definition of Laplace transform are involved only the values of x on $[0,\infty)$, we may suppose that $x(t) = 0$ for $t < 0$. So by $x(t)$ we understand $x(t)u(t)$, where

$$u(t) = \begin{cases} 0, & \text{if} \quad t \leq 0 \\ 1, & \text{if} \quad t > 0 \end{cases}$$

is the unit step function of Heaviside. We write $x^{(n)}(0)$ instead the lateral limit $x^{(n)}(0^+)$ for $n \geq 0$.

We denote by $\mathcal{L}(x)$ the Laplace transform of the function x, defined by

$$\mathcal{L}(x)(s) = X(s) = \int_0^\infty x(t)e^{-st}dt,$$

on $\{s \in \mathbb{R} \mid s > \sigma_x\}$. It is well known that the Laplace transform is linear and one-to-one if the functions involved are continuous. The inverse Laplace transform will be denoted by $\mathcal{L}^{-1}(X)$ or by $\mathcal{L}^{-1}(\mathcal{L}(x))$.

The following properties are used in the paper:

$$\mathcal{L}(x^{(n)})(s) = s^n\mathcal{L}(x)(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \ldots - x^{(n-1)}(0),$$

$$\mathcal{L}(x(t-a))(s) = e^{-as}X(s),\ a > 0,$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^n}\right)(t) = \frac{t^{n-1}}{(n-1)!}u(t),$$

$$\mathcal{L}(f * g)(s) = \mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s),$$

where $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$ is the convolution product of f and g.

In the following, we consider the original functions $x, f : \mathbb{R} \to \mathbb{R}$.

The following Gronwall Lemma is also used in the paper ([16], p. 6):

Lemma 1 ([16])**.** *Let $x, v, h \in C[\mathbb{R}_+, \mathbb{R}_+]$, h nondecreasing. If*

$$x(t) \le h(t) + \int_{t_0}^{t} v(s)x(s)ds,\ t \ge t_0,$$

then

$$x(t) \le h(t)e^{\int_{t_0}^t v(s)ds},\ t \ge t_0.$$

2. Semi-Hyers–Ulam–Rassias Stability of a Delay Differential Equation of Order One

Let $f \in \mathcal{O}$. In what follows, we consider the equation

$$x'(t) + x(t-1) = f(t), \quad x(t) = 0 \quad \text{if} \quad t \le 0, \tag{1}$$

x continuous, piecewise differentiable.

Let $\varepsilon > 0$. We also consider the inequality

$$\left|x'(t) + x(t-1) - f(t)\right| \le \varepsilon, \quad t \in (0, \infty). \tag{2}$$

According to [17], we give the following definition:

Definition 2. *The Equation (1) is called semi-Hyers–Ulam–Rassias stable if there exists a function $k : (0, \infty) \to (0, \infty)$ such that for each solution x of the inequality (2), there exists a solution x_0 of the Equation (1) with*

$$|x(t) - x_0(t)| \le k(t), \forall t \in (0, \infty). \tag{3}$$

Remark 1. *A function $x : (0, \infty) \to \mathbb{R}$ is a solution of (2) if and only if there exists a function $p : (0, \infty) \to \mathbb{R}$ such that*

(1) $|p(t)| \le \varepsilon,\ \forall t \in (0, \infty)$,

(2) $x'(t) + x(t-1) - f(t) = p(t),\ \forall t \in (0, \infty)$.

Lemma 2. *For $s > 1$ we have*

$$\mathcal{L}^{-1}\left(\frac{1}{s + e^{-s}}\right)(t) = \sum_{n=0}^{[t]} (-1)^n \frac{(t-n)^n}{n!}.$$

Proof. As in [18] (p. 15), for $s > 1$ we have $\frac{e^{-s}}{s} < 1$, hence

$$\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{1+\frac{e^{-s}}{s}}\right)(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{e^{-ns}}{s^n}\right)(t)$$
$$= \sum_{n=0}^{\infty}(-1)^n\mathcal{L}^{-1}\left(\frac{e^{-ns}}{s^{n+1}}\right)(t) = \sum_{n=0}^{\infty}(-1)^n\frac{(t-n)^n}{n!}u(t-n)$$
$$= \sum_{n=0}^{[t]}(-1)^n\frac{(t-n)^n}{n!},$$

where $[t]$ denotes the integer part of the real number t. □

Applying a method used in [19], we prove now that the Laplace transform exist for the functions satisfying (1) and (2).

Theorem 1. *Let $f \in \mathcal{O}$. Let σ_f be abscissa of convergence of f and $M_f > 0$ such that $|f(t)| \leq M_f \cdot e^{\sigma_f t}$, $\forall t > 0$. Then the Laplace transform of x, which is the exact solution of* (1) *and of x' exist for all $s > \sigma$, where $\sigma = \max\left\{\sigma_f + 1, 2\right\}$.*

Proof. Integrating the relation (1) from 0 to t, we obtain

$$x(t) - x(0) + \int_0^t x(u-1)du = \int_0^t f(u)du,$$

hence

$$x(t) + \int_0^t x(u-1)du = \int_0^t f(u)du.$$

Changing the variable $v = u - 1$ in the first integral, we have

$$x(t) + \int_{-1}^{t-1} x(v)dv = \int_0^t f(u)du,$$

hence

$$x(t) + \int_0^{t-1} x(v)dv = \int_0^t f(u)du.$$

If $\sigma_f > 0$, we obtain

$$\begin{aligned} |x(t)| &\leq \left|\int_0^{t-1} x(v)dv\right| + \left|\int_0^t f(u)du\right| \leq \int_0^t |x(v)|dv + \int_0^t |f(u)|\,du \\ &\leq \int_0^t |x(v)|dv + \int_0^t M_f e^{\sigma_f u}du = \int_0^t |x(v)|dv + \frac{M_f}{\sigma_f}\left(e^{\sigma_f t} - 1\right) \leq \int_0^t |x(v)|dv + \frac{M_f}{\sigma_f}e^{\sigma_f t}. \end{aligned}$$

Applying now Gronwall Lemma 1, we obtain

$$|x(t)| \leq \frac{M_f}{\sigma_f}e^{\sigma_f t}e^{\int_0^t dv} = \frac{M_f}{\sigma_f}e^{\sigma_f t}e^t = \frac{M_f}{\sigma_f}e^{(\sigma_f+1)t},$$

that is the function x is of exponential order.

If $\sigma_f = 0$, we obtain

$$|x(t)| \leq \int_0^t |x(v)|dv + \int_0^t M_f du = \int_0^t |x(v)|dv + M_f t.$$

Applying now Gronwall Lemma, we obtain

$$|x(t)| \leq M_f t e^{\int_0^t dv} = M_f t e^t < M_f e^t e^t = M_f e^{2t}, t > 0$$

that is the function x is of exponential order.

From (1), we have

$$\left|x'(t)\right| \leq |x(t-1)| + |f(t)| \leq M_x e^{\sigma_x t} + M_f e^{\sigma_f t} \leq 2Me^{\sigma t},$$

where $M = \max\left\{M_x, M_f\right\}$ and $\sigma = \max\left\{\sigma_x, \sigma_f\right\}$. Hence, x' is of exponential order. □

Theorem 2. *Let $f \in \mathcal{O}$. Let σ_f be abscissa of convergence of f and $M_f > 0$ such that $|f(t)| \leq M_f \cdot e^{\sigma_f t}$, $\forall t > 0$. Then the Laplace transform of x (which is a solution of (2) and of x' exist for all $s > \sigma$, where $\sigma = \max\left\{2\sigma_f, 2\right\}$.*

Proof. From (2), we have

$$-\varepsilon \leq x'(t) + x(t-1) - f(t) \leq \varepsilon.$$

Integrating from 0 to t, we obtain

$$-\varepsilon t \leq x(t) + \int_0^t x(u-1)du - \int_0^t f(u)du \leq \varepsilon t,$$

hence

$$-\varepsilon t - \int_0^t x(u-1)du + \int_0^t f(u)du \leq x(t) \leq \varepsilon t - \int_0^t x(u-1)du + \int_0^t f(u)du,$$

Changing the variable $v = u - 1$ in the first integral, we have

$$-\varepsilon t - \int_0^{t-1} x(v)dv + \int_0^t f(u)du \leq x(t) \leq \varepsilon t - \int_0^{t-1} x(v)dv + \int_0^t f(u)du,$$

hence

$$|x(t)| \leq \varepsilon t + \int_0^t |x(v)|dv + \int_0^t |f(u)|du$$

If $\sigma_f > 0$, we obtain

$$|x(t)| \leq \varepsilon t + \int_0^t |x(v)|dv + \int_0^t M_f e^{\sigma_f u}du = \varepsilon t + \int_0^t |x(v)|dv + \frac{M_f}{\sigma_f}\left(e^{\sigma_f t} - 1\right) \leq \int_0^t |x(v)|dv + \varepsilon t + \frac{M_f}{\sigma_f}e^{\sigma_f t}.$$

Applying now Gronwall Lemma, we obtain

$$|x(t)| \leq \left(\varepsilon t + \frac{M_f}{\sigma_f}e^{\sigma_f t}\right)e^{\int_0^t dv} = \left(\varepsilon t + \frac{M_f}{\sigma_f}e^{\sigma_f t}\right)e^t \leq \left(\varepsilon e^{\sigma t} + \frac{M_f}{\sigma_f}e^{\sigma t}\right)e^{\sigma t} = \left(\varepsilon + \frac{M_f}{\sigma_f}\right)e^{2\sigma t},$$

where $\sigma = \max\left\{1, \sigma_f\right\}$, that is the function x is of exponential order.

If $\sigma_f = 0$, we obtain

$$|x(t)| \leq \varepsilon t + \int_0^t |x(v)|dv + \int_0^t M_f du = \varepsilon t + \int_0^t |x(v)|dv + M_f t,$$

or

$$|x(t)| \leq \left(\varepsilon + M_f\right)t + \int_0^t |x(v)|dv,$$

Applying now Gronwall Lemma, we obtain

$$|x(t)| \leq \left(\varepsilon + M_f\right)te^{\int_0^t dv} = \left(\varepsilon + M_f\right)te^t < \left(\varepsilon + M_f\right)e^t e^t = \left(\varepsilon + M_f\right)e^{2t}, t > 0$$

that is the function x is of exponential order.

From (2), we have

$$\left|x'(t)\right| \le \varepsilon + |x(t-1)| + |f(t)| \le \varepsilon + M_x e^{\sigma_x t} + M_f e^{\sigma_f t} \le (\varepsilon + 2M)e^{\sigma t},$$

where $M = \max\left\{M_x, M_f\right\}$ and $\sigma = \max\left\{\sigma_x, \sigma_f\right\}$. Hence, x' is of exponential order. □

Theorem 3. *If a function $x : (0,\infty) \to \mathbb{R}$ satisfies the inequality (2), where $f \in \mathcal{O}$, then there exists a solution $x_0 : (0,\infty) \to \mathbb{R}$ of (1) such that*

$$|x(t) - x_0(t)| \le \varepsilon\left(t + \frac{(t-1)^2}{2!} + \cdots + \frac{(t-[t])^{[t]+1}}{([t]+1)!}\right), \quad \forall t \in (0,\infty),$$

that is the Equation (1) is semi-Hyers–Ulam–Rassias stable.

Proof. Let $p : (0,\infty) \to \mathbb{R}$,

$$p(t) = x'(t) + x(t-1) - f(t), \quad t \in (0,\infty). \tag{4}$$

We have

$$\mathcal{L}(p) = s\mathcal{L}(x) - x(0) + e^{-s}\mathcal{L}(x) - \mathcal{L}(f),$$

hence

$$\mathcal{L}(x) = \frac{\mathcal{L}(p)}{s+e^{-s}} + \frac{\mathcal{L}(f)}{s+e^{-s}}.$$

Let

$$x_0(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{s+e^{-s}}\right)(t), \quad \forall t \in (0,\infty).$$

We remark that $x_0(0) = 0$.

Hence, we obtain

$$\begin{aligned} &\mathcal{L}\left[x_0'(t) + x_0(t-1) - f(t)\right] = s\mathcal{L}(x_0) - x_0(0) + e^{-s}\mathcal{L}(x_0) - \mathcal{L}(f) \\ &= s\frac{\mathcal{L}(f)}{s+e^{-s}} + e^{-s}\frac{\mathcal{L}(f)}{s+e^{-s}} - \mathcal{L}(f) = 0. \end{aligned}$$

Since $\mathcal{L}$ is one-to-one, it follows that

$$x_0'(t) + x_0(t-1) - f(t) = 0,$$

that is x_0 is a solution of (1).

We have

$$\mathcal{L}(x) - \mathcal{L}(x_0) = \frac{\mathcal{L}(p)}{s+e^{-s}},$$

hence

$$\begin{aligned} &|x(t) - x_0(t)| = \left|\mathcal{L}^{-1}\left(\frac{\mathcal{L}(p)}{s+e^{-s}}\right)\right| = \left|\mathcal{L}^{-1}(\mathcal{L}(p)) * \mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)\right| \\ &= \left|p * \mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)\right| = \left|\int_0^t p(\tau)\cdot\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)d\tau\right| \\ &\le \int_0^t |p(\tau)|\cdot\left|\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)\right|d\tau \le \varepsilon\int_0^t\left|\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)\right|d\tau. \end{aligned}$$

From Lemma 2, we obtain

$$\varepsilon \int_0^t \left| \mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)\right| d\tau = \varepsilon \int_0^t \left| \sum_{n=0}^{[t-\tau]} (-1)^n \frac{(t-\tau-n)^n}{n!} \right| d\tau$$
$$\leq \varepsilon \int_0^t \sum_{n=0}^{[t-\tau]} \left| (-1)^n \frac{(t-\tau-n)^n}{n!} \right| d\tau = \varepsilon \int_0^t \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^n}{n!} d\tau.$$

For $t > 1$, we have

$$[t-\tau] = \begin{cases} [t], \tau \in [0, t-[t]] \\ [t]-1, \tau \in (t-[t], t-[t]+1] \\ \quad \dots \\ 0, \tau \in (t-1, t] \end{cases}$$

hence

$$\int_0^t \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^n}{n!} d\tau = \int_0^{t-[t]} \sum_{n=0}^{[t]} \frac{(t-\tau-n)^n}{n!} d\tau + \int_{t-[t]}^{t-[t]+1} \sum_{n=0}^{[t]-1} \frac{(t-\tau-n)^n}{n!} d\tau + \dots + \int_{t-1}^{t} \sum_{n=0}^{0} \frac{(t-\tau-n)^n}{n!} d\tau$$
$$= \int_0^{t-[t]} \left(\frac{(t-\tau-0)^0}{0!} d\tau + \frac{(t-\tau-1)^1}{1!} + \dots + \frac{(t-\tau-[t])^{[t]}}{[t]!} \right) d\tau$$
$$+ \int_{t-[t]}^{t-[t]+1} \left(\frac{(t-\tau-0)^0}{0!} d\tau + \frac{(t-\tau-1)^1}{1!} + \dots + \frac{(t-\tau-[t]-1)^{[t]-1}}{([t]-1)!} \right) d\tau$$
$$\dots$$
$$+ \int_{t-1}^{t} \frac{(t-\tau-0)^0}{0!} d\tau.$$

We obtain

$$\int_0^t \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^n}{n!} d\tau = \int_0^t \frac{(t-\tau-0)^0}{0!} d\tau + \int_0^{t-1} \frac{(t-\tau-1)^1}{1!} d\tau + \dots + \int_0^{t-[t]} \frac{(t-\tau-[t])^{[t]}}{[t]!} d\tau$$
$$= t - \frac{(t-\tau-1)^2}{2!} \Big|_0^{t-1} - \dots - \frac{(t-\tau-[t])^{[t]+1}}{([t]+1)!} \Big|_0^{t-[t]}$$
$$= t + \frac{(t-1)^2}{2!} + \dots + \frac{(t-[t])^{[t]+1}}{([t]+1)!}$$

For $t \in [0,1)$, we have $[t-\tau] = 0$, hence

$$\int_0^t \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^n}{n!} d\tau = \int_0^t d\tau = t.$$

□

3. Semi-Hyers–Ulam–Rassias Stability of a Delay Differential Equation of Order Two

Let $f \in \mathcal{O}$. Next, we consider the equation

$$x''(t) + x'(t-1) = f(t), \quad x(t) = 0 \quad \text{if} \quad t \leq 0, \quad x'(0) = 0, \tag{5}$$

x continuous, piecewise twice differentiable.

Let $\varepsilon > 0$. We also consider the inequality

$$\left| x''(t) + x'(t-1) - f(t) \right| \leq \varepsilon, \quad t \in (0, \infty). \tag{6}$$

Definition 3. *The Equation* (5) *is called semi-Hyers–Ulam–Rassias stable if there exists a function* $k:(0,\infty)\to(0,\infty)$ *such that for each solution* x *of the inequality* (6)*, there exists a solution* x_0 *of the Equation* (5) *with*

$$|x(t)-x_0(t)|\le k(t), \forall t\in(0,\infty). \tag{7}$$

Remark 2. *A function* $x:(0,\infty)\to\mathbb{R}$ *is a solution of* (6) *if and only if there exists a function* $p:(0,\infty)\to\mathbb{R}$ *such that*

(1) $|p(t)|\le\varepsilon,\ \forall t\in(0,\infty)$,
(2) $x''(t)+x'(t-1)-f(t)=p(t),\ \forall t\in(0,\infty)$.

Lemma 3. *For* $s>1$*, we have*

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+se^{-s}}\right)(t)=\sum_{n=0}^{[t]}(-1)^n\frac{(t-n)^{n+1}}{(n+1)!}.$$

Proof. For $s>1$, we have $\frac{e^{-s}}{s}<1$, hence

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{s^2+se^{-s}}\right)(t)&=\mathcal{L}^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{1+\frac{e^{-s}}{s}}\right)(t)=\mathcal{L}^{-1}\left(\frac{1}{s^2}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{e^{-ns}}{s^n}\right)(t)\\&=\sum_{n=0}^{\infty}(-1)^n\mathcal{L}^{-1}\left(\frac{e^{-ns}}{s^{n+2}}\right)(t)=\sum_{n=0}^{\infty}(-1)^n\frac{(t-n)^{n+1}}{(n+1)!}u(t-n)\\&=\sum_{n=0}^{[t]}(-1)^n\frac{(t-n)^{n+1}}{(n+1)!}.\end{aligned}$$

□

Theorem 4. *Let* $f\in\mathcal{O}$*. Let* σ_f *be abscissa of convergence of* f *and* $M_f>0$ *such that* $|f(t)|\le M_f\cdot e^{\sigma_f t}$*,* $\forall t>0$*. Then the Laplace transform of* x*, which is the exact solution of* (5) *and of* x', x'' *exist for all* $s>\sigma_f$*.*

Proof. We can apply Theorem 3.1 from [19]. □

Theorem 5. *Let* $f\in\mathcal{O}$*. Let* σ_f *be abscissa of convergence of* f *and* $M_f>0$ *such that* $|f(t)|\le M_f\cdot e^{\sigma_f t}$*,* $\forall t>0$*. Then the Laplace transform of* x*, which is a solution of* (6) *and of* x', x'' *exist for a certain* $\sigma>\sigma_f$*, for all* $s>\sigma$*.*

Proof. The proof is similar to that of Theorem 2. □

Theorem 6. *Let* $f:\mathbb{R}\to\mathbb{R}$ *such that* $f\in\mathcal{O}$ *and* $\left(\mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{s^2+se^{-s}}\right)\right)'(0)=0$*. If a function* $x:(0,\infty)\to\mathbb{R}$ *satisfies the inequality* (6)*, then there exists a solution* $x_0:(0,\infty)\to\mathbb{R}$ *of* (5) *such that*

$$|x(t)-x_0(t)|\le\varepsilon\left(\frac{(t-1)^2}{2!}+\frac{(t-1)^3}{3!}+\cdots+\frac{(t-[t])^{[t]+2}}{([t]+2)!}\right),\quad\forall t\in(0,\infty),$$

that is the Equation (5) *is semi-Hyers–Ulam–Rassias stable.*

Proof. Let $p:(0,\infty)\to\mathbb{R}$,

$$p(t)=x''(t)+x'(t-1)-f(t),\quad t\in(0,\infty). \tag{8}$$

We have

$$\mathcal{L}(p) = s^2\mathcal{L}(x) - sx(0) - x'(0) + e^{-s}[s\mathcal{L}(x) - x(0)] - \mathcal{L}(f),$$

hence

$$\mathcal{L}(x) = \frac{\mathcal{L}(p)}{s^2 + se^{-s}} + \frac{\mathcal{L}(f)}{s^2 + se^{-s}}.$$

Let

$$x_0(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{s^2 + se^{-s}}\right)(t), \quad \forall t \in (0, \infty).$$

We remark that $x_0(0) = 0$ and $x_0'(0) = 0$

Hence, we obtain

$$\mathcal{L}\left[x_0''(t) + x_0'(t-1) - f(t)\right] = s^2\mathcal{L}(x_0) - sx_0(0) - x_0'(0) + e^{-s}[s\mathcal{L}(x_0) - x_0(0)] - \mathcal{L}(f)$$
$$= s^2\frac{\mathcal{L}(f)}{s^2 + se^{-s}} + se^{-s}\frac{\mathcal{L}(f)}{s^2 + se^{-s}} - \mathcal{L}(f) = 0.$$

Since $\mathcal{L}$ is one-to-one, it follows that

$$x_0''(t) + x_0'(t-1) - f(t) = 0,$$

that is x_0 is a solution of (5).

We have

$$\mathcal{L}(x) - \mathcal{L}(x_0) = \frac{\mathcal{L}(p)}{s^2 + se^{-s}},$$

hence

$$|x(t) - x_0(t)| = \left|\mathcal{L}^{-1}\left(\frac{\mathcal{L}(p)}{s^2 + se^{-s}}\right)\right| = \left|\mathcal{L}^{-1}(\mathcal{L}(p)) * \mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)\right|$$
$$= \left|p * \mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)\right| = \left|\int_0^t p(\tau) \cdot \mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau)d\tau\right|$$
$$\leq \int_0^t |p(\tau)| \cdot \left|\mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau)\right| d\tau \leq \varepsilon \int_0^t \left|\mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau)\right| d\tau.$$

From Lemma 3, we obtain

$$\varepsilon \int_0^t \left|\mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau)\right| d\tau = \varepsilon \int_0^t \left|\sum_{n=0}^{[t-\tau]} (-1)^n \frac{(t-\tau-n)^{n+1}}{(n+1)!}\right| d\tau$$
$$\leq \varepsilon \int_0^t \sum_{n=0}^{[t-\tau]} \left|(-1)^n \frac{(t-\tau-n)^{n+1}}{(n+1)!}\right| d\tau = \varepsilon \int_0^t \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^{n+1}}{(n+1)!} d\tau.$$

For $t > 1$, we have

$$\int_0^t \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^{n+1}}{(n+1)!} d\tau$$
$$= \int_0^{t-[t]} \sum_{n=0}^{[t]} \frac{(t-\tau-n)^{n+1}}{(n+1)!} d\tau + \int_{t-[t]}^{t-[t]+1} \sum_{n=0}^{[t]-1} \frac{(t-\tau-n)^{n+1}}{(n+1)!} d\tau + \cdots + \int_{t-1}^{t} \sum_{n=0}^{0} \frac{(t-\tau-n)^{n+1}}{(n+1)!} d\tau$$
$$= \int_0^{t-[t]} \left(\frac{(t-\tau-0)^1}{1!} d\tau + \frac{(t-\tau-1)^2}{2!} + \cdots + \frac{(t-\tau-[t])^{[t]+1}}{([t]+1)!} \right) d\tau$$
$$+ \int_{t-[t]}^{t-[t]+1} \left(\frac{(t-\tau-0)^1}{1!} d\tau + \frac{(t-\tau-1)^2}{2!} + \cdots + \frac{(t-\tau-[t]-1)^{[t]}}{[t]!} \right) d\tau$$
$$\ldots$$
$$+ \int_{t-1}^{t} \frac{(t-\tau-0)^1}{1!} d\tau.$$

We obtain

$$\int_0^t \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^{n+1}}{(n+1)!} d\tau = \int_0^t \frac{(t-\tau-0)^1}{1!} d\tau + \int_0^{t-1} \frac{(t-\tau-1)^2}{2!} d\tau + \cdots + \int_0^{t-[t]} \frac{(t-\tau-[t])^{[t]+1}}{([t]+1)!} d\tau$$
$$= -\frac{(t-\tau-0)^2}{2!} \Big|_0^t - \frac{(t-\tau-1)^3}{3!} \Big|_0^{t-1} - \cdots - \frac{(t-\tau-[t])^{[t]+2}}{([t]+2)!} \Big|_0^{t-[t]}$$
$$= \frac{(t-1)^2}{2!} + \frac{(t-1)^3}{3!} + \cdots + \frac{(t-[t])^{[t]+2}}{([t]+2)!}$$

For $t \in [0,1)$, we have $[t-\tau] = 0$, hence

$$\int_0^t \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^{n+1}}{(n+1)!} d\tau = \int_0^t \frac{(t-\tau-0)^1}{1!} d\tau = \frac{(t-1)^2}{2!}.$$

□

4. Generalized Semi-Hyers–Ulam–Rassias Stability of a Delay Differential Equation of Order One

We continue to study generalized semi-Hyers–Ulam–Rassias stability of the Equation (1). Let $\varphi \in \mathcal{O}$. We consider the inequality

$$\left| x'(t) + x(t-1) - f(t) \right| \le \varphi(t), \quad t \in (0,\infty). \tag{9}$$

Definition 4. *The Equation (1) is called generalized semi-Hyers–Ulam–Rassias stable if there exists a function $k : (0,\infty) \to (0,\infty)$ such that for each solution x of the inequality (9), there exists a solution x_0 of the Equation (1) with*

$$|x(t) - x_0(t)| \le k(t), \forall t \in (0,\infty). \tag{10}$$

Remark 3. *A function $x : (0,\infty) \to \mathbb{R}$ is a solution of (9) if, and only if, there exists a function $p : (0,\infty) \to \mathbb{R}$ such that*
(1) $|p(t)| \le \varphi(t)$, $\forall t \in (0,\infty)$,
(2) $x'(t) + x(t-1) - f(t) = p(t)$, $\forall t \in (0,\infty)$.

Theorem 7. *If a function $x : (0,\infty) \to \mathbb{R}$ satisfies the inequality (9), where $f, \varphi \in \mathcal{O}$, then there exists a solution $x_0 : (0,\infty) \to \mathbb{R}$ of (1) such that*

$$|x(t) - x_0(t)| \le \int_0^t \varphi(\tau) \left| \mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}} \right)(t-\tau) \right| d\tau, \tag{11}$$

that is the Equation (1) is generalized semi-Hyers–Ulam–Rassias stable.

Proof. Let $p:(0,\infty)\to\mathbb{R}$,

$$p(t)=x'(t)+x(t-1)-f(t),\quad t\in(0,\infty). \tag{12}$$

As in Theorem 3, for x that is a solution of (9) and Laplace transform of x, x' exists, we have

$$\mathcal{L}(x)=\frac{\mathcal{L}(p)}{s+e^{-s}}+\frac{\mathcal{L}(f)}{s+e^{-s}},$$

and

$$x_0(t)=\mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{s+e^{-s}}\right)(t),\quad \forall t\in(0,\infty),$$

is a solution of (1).

We have

$$\mathcal{L}(x)-\mathcal{L}(x_0)=\frac{\mathcal{L}(p)}{s+e^{-s}},$$

hence

$$\begin{aligned}|x(t)-x_0(t)| &= \left|\mathcal{L}^{-1}\left(\frac{\mathcal{L}(p)}{s+e^{-s}}\right)\right| = \left|\mathcal{L}^{-1}(\mathcal{L}(p))*\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)\right| \\ &= \left|p*\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)\right| = \left|\int_0^t p(\tau)\cdot\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)d\tau\right| \\ &\le \int_0^t |p(\tau)|\cdot\left|\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)\right|d\tau \le \int_0^t \varphi(\tau)\left|\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)\right|d\tau.\end{aligned}$$

□

Theorem 8. *Let $\varphi:(0,\infty)\to(0,\infty), \varphi(t)=t^n$. If a function $x:(0,\infty)\to\mathbb{R}$ satisfies the inequality (9), where $f\in\mathcal{O}$, then there exists a solution $x_0:(0,\infty)\to\mathbb{R}$ of (1) such that*

$$|x(t)-x_0(t)|\le\frac{t^{n+1}}{n+1}+\frac{(t-1)^{n+2}}{(n+1)(n+2)}+\frac{(t-2)^{n+3}}{(n+1)(n+2)(n+3)}+\cdots+\frac{(t-[t])^{n+[t]+1}}{(n+1)(n+2)\cdots(n+[t]+1)}.$$

Proof. From Theorem 7, we have that if $x:(0,\infty)\to\mathbb{R}$ satisfies the inequality (9), then there exists a solution $x_0:(0,\infty)\to\mathbb{R}$ of (1) such that

$$|x(t)-x_0(t)|\le\int_0^t \tau^n\left|\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)\right|d\tau$$

is satisfied. We have

$$\begin{aligned}&\int_0^t \tau^n\left|\mathcal{L}^{-1}\left(\frac{1}{s+e^{-s}}\right)(t-\tau)\right|d\tau=\int_0^t \tau^n\sum_{n=0}^{[t-\tau]}\frac{(t-\tau-n)^n}{n!} \\ &=\int_0^t \tau^n\frac{(t-\tau-0)^0}{0!}d\tau+\int_0^{t-1}\tau^n\frac{(t-\tau-1)^1}{1!}d\tau+\cdots+\int_0^{t-[t]}\tau^n\frac{(t-\tau-[t])^{[t]}}{[t]!}d\tau.\end{aligned}$$

We have

$$\int_0^t \tau^n\frac{(t-\tau-0)^0}{0!}d\tau=\frac{\tau^{n+1}}{n+1}\Big|_0^t=\frac{t^{n+1}}{n+1}.$$

Integrating by parts, we have

$$\int_0^{t-1} \tau^n \frac{(t-\tau-1)^1}{1!} d\tau = \int_0^{t-1} \left(\frac{\tau^{n+1}}{n+1}\right)'(t-\tau-1)d\tau = \underbrace{\frac{\tau^{n+1}}{n+1}(t-\tau-1)\Big|_0^{t-1}}_{0} + \int_0^{t-1} \frac{\tau^{n+1}}{n+1} d\tau$$

$$= \frac{\tau^{n+2}}{(n+1)(n+2)}\Big|_0^{t-1} = \frac{(t-1)^{n+2}}{(n+1)(n+2)},$$

$$\int_0^{t-2} \tau^n \frac{(t-\tau-2)^2}{2!} d\tau = \int_0^{t-2} \left(\frac{\tau^{n+1}}{n+1}\right)' \frac{(t-\tau-2)^2}{2!} d\tau = \underbrace{\frac{\tau^{n+1}}{n+1} \frac{(t-\tau-2)^2}{2!}\Big|_0^{t-2}}_{0} + \int_0^{t-2} \frac{\tau^{n+1}}{n+1} \frac{2(t-\tau-2)}{2!} d\tau$$

$$= \int_0^{t-2} \left(\frac{\tau^{n+2}}{(n+1)(n+2)}\right)' \frac{(t-\tau-2)}{1!} d\tau = \frac{\tau^{n+2}}{(n+1)(n+2)} \frac{(t-\tau-2)}{1!}\Big|_0^{t-2} + \int_0^{t-2} \frac{\tau^{n+2}}{(n+1)(n+2)} d\tau$$

$$= \frac{(t-2)^{n+3}}{(n+1)(n+2)(n+3)},$$

$$\cdots$$

$$\int_0^{t-[t]} \tau^n \frac{(t-\tau-[t])^{[t]}}{[t]!} d\tau = \int_0^{t-[t]} \left(\frac{\tau^{n+1}}{n+1}\right)' \frac{(t-\tau-[t])^{[t]}}{[t]!} d\tau$$

$$= \underbrace{\frac{\tau^{n+1}}{n+1} \frac{(t-\tau-[t])^{[t]}}{[t]!}\Big|_0^{t-[t]}}_{0} + \int_0^{t-[t]} \frac{\tau^{n+1}}{n+1} \frac{[t](t-\tau-[t])^{[t]-1}}{[t]!} d\tau$$

$$= \int_0^{t-[t]} \left(\frac{\tau^{n+2}}{(n+1)(n+2)}\right)' \frac{(t-\tau-[t])^{[t]-1}}{([t]-1)!} d\tau = \cdots = \frac{(t-[t])^{n+[t]+1}}{(n+1)(n+2)\cdots(n+[t]+1)}.$$

□

5. Generalized Semi-Hyers–Ulam–Rassias Stability of a Delay Differential Equation of Order Two

We are now studying the generalized semi-Hyers–Ulam–Rassias stability of the Equation (5). Let $\varphi \in \mathcal{O}$. We consider the inequality

$$\left|x''(t) + x'(t-1) - f(t)\right| \le \varphi(t), \quad t \in (0,\infty). \tag{13}$$

Definition 5. *The Equation (5) is called generalized semi-Hyers–Ulam–Rassias stable if there exists a function $k : (0,\infty) \to (0,\infty)$ such that for each solution x of the inequality (13), there exists a solution x_0 of the Equation (5) with*

$$|x(t) - x_0(t)| \le k(t), \forall t \in (0,\infty). \tag{14}$$

Remark 4. *A function $x : (0,\infty) \to \mathbb{R}$ is a solution of (13) if, and only if, there exists a function $p : (0,\infty) \to \mathbb{R}$ such that*

(1) $|p(t)| \le \varphi(t)$, $\forall t \in (0,\infty)$,

(2) $x''(t) + x'(t-1) - f(t) = p(t)$, $\forall t \in (0,\infty)$.

Theorem 9. *Let $f : \mathbb{R} \to \mathbb{R}$ such that $\left(\mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{s^2+se^{-s}}\right)\right)'(0) = 0$. If a function $x : (0,\infty) \to \mathbb{R}$ satisfies the inequality (13), where $f, \varphi \in \mathcal{O}$, then there exists a solution $x_0 : (0,\infty) \to \mathbb{R}$ of (5) such that*

$$|x(t) - x_0(t)| \le \int_0^t \varphi(\tau) \left|\mathcal{L}^{-1}\left(\frac{1}{s^2+se^{-s}}\right)(t-\tau)\right| d\tau, \tag{15}$$

that is the Equation (5) is generalized semi-Hyers–Ulam–Rassias stable.

Proof. Let $p : (0, \infty) \to \mathbb{R}$,

$$p(t) = x''(t) + x'(t-1) - f(t), \quad t \in (0, \infty). \tag{16}$$

As in Theorem 6, for x that is a solution of (13) and Laplace transform of x, x', x'' exists, we have

$$\mathcal{L}(x) = \frac{\mathcal{L}(p)}{s^2 + se^{-s}} + \frac{\mathcal{L}(f)}{s^2 + se^{-s}},$$

and

$$x_0(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{s^2 + se^{-s}}\right)(t), \quad \forall t \in (0, \infty),$$

is a solution of (5).

We have

$$\mathcal{L}(x) - \mathcal{L}(x_0) = \frac{\mathcal{L}(p)}{s^2 + se^{-s}},$$

hence

$$\begin{aligned}
|x(t) - x_0(t)| &= \left|\mathcal{L}^{-1}\left(\frac{\mathcal{L}(p)}{s^2 + se^{-s}}\right)\right| = \left|\mathcal{L}^{-1}(\mathcal{L}(p)) * \mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)\right| \\
&= \left|p * \mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)\right| = \left|\int_0^t p(\tau) \cdot \mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau) d\tau\right| \\
&\leq \int_0^t |p(\tau)| \cdot \left|\mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau)\right| d\tau \leq \int_0^t \varphi(\tau) \left|\mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau)\right| d\tau.
\end{aligned}$$

□

Theorem 10. *Let $f : \mathbb{R} \to \mathbb{R}$ such that $f \in \mathcal{O}$ and $\left(\mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{s^2 + se^{-s}}\right)\right)'(0) = 0$. Let $\varphi : (0, \infty) \to (0, \infty)$, $\varphi(t) = t^n$. If a function $x : (0, \infty) \to \mathbb{R}$ satisfies the inequality (13), then there exists a solution $x_0 : (0, \infty) \to \mathbb{R}$ of (5) such that*

$$|x(t) - x_0(t)| \leq \frac{t^{n+2}}{(n+1)(n+2)} + \frac{(t-1)^{n+3}}{(n+1)(n+2)(n+3)} + \cdots + \frac{(t-[t])^{n+[t]+2}}{(n+1)(n+2)\cdots(n+[t]+2)}.$$

Proof. From Theorem 9, we have that if $x : (0, \infty) \to \mathbb{R}$ satisfies the inequality (13), then there exists a solution $x_0 : (0, \infty) \to \mathbb{R}$ of (5) such that

$$|x(t) - x_0(t)| \leq \int_0^t \tau^n \left|\mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau)\right| d\tau$$

is satisfied. We have

$$\begin{aligned}
&\int_0^t \tau^n \left|\mathcal{L}^{-1}\left(\frac{1}{s^2 + se^{-s}}\right)(t-\tau)\right| d\tau = \int_0^t \tau^n \sum_{n=0}^{[t-\tau]} \frac{(t-\tau-n)^{n+1}}{(n+1)!} \\
&= \int_0^t \tau^n \frac{(t-\tau-0)^1}{1!} d\tau + \int_0^{t-1} \tau^n \frac{(t-\tau-1)^2}{2!} d\tau + \cdots + \int_0^{t-[t]} \tau^n \frac{(t-\tau-[t])^{[t]+1}}{([t]+1)!} d\tau.
\end{aligned}$$

Integrating by parts, we have

$$\int_0^t \tau^n \frac{(t-\tau-0)^1}{1!} d\tau = \int_0^t \left(\frac{\tau^{n+1}}{n+1}\right)' (t-\tau) d\tau = \underbrace{\frac{\tau^{n+1}}{n+1}(t-\tau)\Big|_0^t}_{0} + \int_0^t \frac{\tau^{n+1}}{n+1} d\tau$$

$$= \frac{\tau^{n+2}}{(n+1)(n+2)}\Big|_0^t = \frac{t^{n+2}}{(n+1)(n+2)},$$

$$\int_0^{t-1} \tau^n \frac{(t-\tau-1)^2}{2!} d\tau = \int_0^{t-1} \left(\frac{\tau^{n+1}}{n+1}\right)' \frac{(t-\tau-1)^2}{2!} d\tau = \underbrace{\frac{\tau^{n+1}}{n+1} \frac{(t-\tau-1)^2}{2!}\Big|_0^{t-1}}_{0} + \int_0^{t-1} \frac{\tau^{n+1}}{n+1} \frac{2(t-\tau-1)}{2!} d\tau$$

$$= \int_0^{t-1} \left(\frac{\tau^{n+2}}{(n+1)(n+2)}\right)' \frac{(t-\tau-2)}{1!} d\tau = \frac{\tau^{n+2}}{(n+1)(n+2)} \frac{(t-\tau-1)}{1!}\Big|_0^{t-1} + \int_0^{t-1} \frac{\tau^{n+2}}{(n+1)(n+2)} d\tau$$

$$= \frac{(t-1)^{n+3}}{(n+1)(n+2)(n+3)},$$

$$\cdots$$

$$\int_0^{t-[t]} \tau^n \frac{(t-\tau-[t])^{[t]+1}}{([t]+1)!} d\tau = \int_0^{t-[t]} \left(\frac{\tau^{n+1}}{n+1}\right)' \frac{(t-\tau-[t])^{[t]+1}}{([t]+1)!} d\tau$$

$$= \underbrace{\frac{\tau^{n+1}}{n+1} \frac{(t-\tau-[t])^{[t]+1}}{([t]+1)!}\Big|_0^{t-[t]}}_{0} + \int_0^{t-[t]} \frac{\tau^{n+1}}{n+1} \frac{([t]+1)!(t-\tau-[t])^{[t]}}{([t]+1)!} d\tau$$

$$= \int_0^{t-[t]} \left(\frac{\tau^{n+2}}{(n+1)(n+2)}\right)' \frac{(t-\tau-[t])^{[t]-1}}{[t]!} d\tau = \cdots = \frac{(t-[t])^{n+[t]+2}}{(n+1)(n+2)\cdots(n+[t]+2)}.$$

□

6. Conclusions

The use of the Laplace transform in the study of Hyers–Ulam stability of differential equations is relatively recent (2013, see [7]). This method was not used to study the stability of equations with delay. In this paper, we have studied semi-Hyers–Ulam–Rassias stability and generalized semi-Hyers–Ulam–Rassias stability of Equations (1) and (5) using the Laplace transform. Some examples were given. The results obtained complete those of S. M. Jung and J. Brzdek from [13]. This method can be used successfully in the case of other equations with delay, integro-differential equations, partial differential equations or for fractional calculus. In [11], we have already studied a Volterra integro-differential equation of order I with a convolution type kernel and, in [12], the convection partial differential equation. In [20], the Poisson partial differential equation was studied via the double Laplace transform method. We intend to further study other equations.

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