mathematics

Article

On Two Problems Related to Divisibility Properties of $z(n)$

Pavel Trojovský

Department of Mathematics, Faculty of Science, University of Hradec Králové,
500 03 Hradec Králové, Czech Republic; pavel.trojovsky@uhk.cz; Tel.: +42-049-333-2860

Abstract: The order of appearance (in the Fibonacci sequence) function $z : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ is an arithmetic function defined for a positive integer n as $z(n) = \min\{k \geq 1 : F_k \equiv 0 \pmod n\}$. A topic of great interest is to study the Diophantine properties of this function. In 1992, Sun and Sun showed that Fermat's Last Theorem is related to the solubility of the functional equation $z(n) = z(n^2)$, where n is a prime number. In addition, in 2014, Luca and Pomerance proved that $z(n) = z(n+1)$ has infinitely many solutions. In this paper, we provide some results related to these facts. In particular, we prove that $\limsup_{n\to\infty} (z(n+1) - z(n))/(\log n)^{2-\epsilon} = \infty$, for all $\epsilon \in (0, 2)$.

Keywords: order of appearance; Fibonacci numbers; divisibility; functional equation; prime numbers

Citation: Trojovský, P. On Two Problems Related to Divisibility Properties of $z(n)$. *Mathematics* **2021**, *9*, 3273. https://doi.org/10.3390/math9243273

Academic Editor: Yang-Hui He

Received: 4 November 2021
Accepted: 15 December 2021
Published: 16 December 2021

1. Introduction

Let $(F_n)_n$ be the Fibonacci sequence. The arithmetic function $z : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ defined by $z(n) = \min\{k \geq 1 : n \mid F_k\}$ is known as the order of appearance (or rank of apparition) in the Fibonacci sequence. This function is well-defined (i.e., $z(n)$ is finite for all $n \geq 1$), as showed by Lucas ([1], p. 300). Furthermore, its sharpest upper bound is $z(n) \leq 2n$ as proved by Sallé [2] (the sharpness follows from $z(6 \cdot 5^k) = 12 \cdot 5^k$, for all $k \geq 0$).

The first 30 values of $z(n)$ are the following (see sequence A001177 in the Online Encyclopedia of Integer Sequences [3]):

$$1, 3, 4, 6, 5, 12, 8, 6, 12, 15, 10, 12, 7, 24, 20, 12, 9, 12, 18, 30, 8, 30, 24, 12, 25, 21, 36, 24, 14, 60.$$

In the last decades, many authors have considered, in varying degrees of generality, Diophantine problems involving the z-function (see, for instance, the recent works [4–10]). However, this function gained great interest in 1992, when Z. H. Sun and Z. W. Sun [6] proved that $z(p) \neq z(p^2)$, for all prime numbers p, implies the first case of Fermat's Last Theorem (i.e., that $x^p + y^p = z^p$ has no solution with $p \nmid xyz$). In fact, this is related to an old conjecture expressed by Wall [11] (see also [12]), that is, $e(p) := \nu_p(F_{z(p)})$ is equal to 1, for all prime numbers p. Here, $\nu_p(r)$ denotes the *p-adic valuation* (or *order*) of r, that is, the largest non-negative integer k for which p^k divides r (see [13–15] for more facts on p-adic valuation of the Fibonacci sequence and its generalizations). We remark that this conjecture was verified for all prime numbers $p < 3 \times 10^{17}$ (*PrimeGrid*—December 2020).

We point out to the existence of some conditional results relating Wall's conjecture to other Diophantine problems. For instance, Marques [16] proved that there is no non-trivial s-Cullen number (i.e., a number of the form $ms^m + 1$ with $m > 1$) in the Fibonacci sequence provided that $e(p) = 1$, for all prime factors p of s.

Let $P(n)$ be the *greatest prime factor* of n. Now, for any integer $k \geq 2$, let us provide the following weaker consequence of Wall's conjecture.

***k*-Weak Wall Conjecture.** *Let $n > 1$ be an integer. Then,*

$$\nu_{P(n)}(n) > \frac{e(P(n))}{k}. \tag{1}$$

For the sake of simplicity, we indicate the previous conjecture as (k-WWC). Clearly, Wall's conjecture implies (k-WWC), since $p \mid n$ implies $\nu_p(n) \geq 1 > 1/k = e(p)/k$. Note that, if ($k$-WWC) is true, then, for any prime p, one obtains $1 = \nu_p(p) > e(p)/k$; therefore, $e(p) \in \{1, 2, \ldots, k-1\}$. In particular, Wall's conjecture is equivalent to (2-WWC).

Our first result is a purely theoretical result on the counting function of positive integers satisfying (1).

Theorem 1. *Let $k \geq 2$ be an integer. Then, we have*

$$\#\{n \leq x : n \text{ satisfies } (k\text{-WWC})\} \geq \sum_{p \geq 2} \frac{p^{p/k}}{\pi(p)!} (\log(x/p^{(p+k)/k}))^{\pi(p)} \left(\prod_{q \leq p} \log q \right)^{-1}, \quad (2)$$

where p and q run over the set of prime numbers. Here, as usual, $\pi(z)$ is the prime counting function.

Remark 1. *We remark that the left-hand side of (2) is larger than any truncation of the series on the right-hand side. For instance, by truncating at $p = 3$, one obtains*

$$\#\{n \leq x : n \text{ satisfies } (2\text{-WWC})\} > 2.8 \log(0.25x) + 1.7 \log^2(0.06x).$$

Remark 2. *Let p_n be the nth prime number. We still point out that (2) can be written as*

$$\#\{n \leq x : n \text{ satisfies } (k\text{-WWC})\} \geq \sum_{j \geq 1} \frac{p_j^{p_j/k}}{j!} (\log(x/p_j^{(p_j+k)/k}))^j ((\log p_1) \cdots (\log p_j))^{-1}.$$

By using the weak (but enough) inequalities $j \log j < p_j < j^2$, $(\log p_1) \cdots (\log p_j) \leq (\log p_j)^j$ and $j! \leq (j/2)^j$, after a straightforward calculation, one arrives at the cleaner inequality

$$\#\{n \leq x : n \text{ satisfies } (k\text{-WWC})\} \geq \sum_{j=2}^{\infty} \left(\frac{(\log j)^{(\log j)/k} \cdot \log\left(\frac{x}{j^{3j^2/k}}\right)}{\log j} \right)^j.$$

The main goal of this paper is to study some analytic and Diophantine aspects of some functional equations involving $z(n)$. Our first result relates the k-Weak Wall's Conjecture to the Wall and Sun–Sun works. More precisely, see the following theorem.

Theorem 2. *Let us suppose that (ℓ-WWC) is true. Then, for any integer $k \geq \ell$, the functional equation $z(n) = z(n^k)$ has solution only if $k = \ell = 2$ and $n \in \{1, 6, 12\}$.*

Another interesting problem concerns the behavior of the order of appearance at consecutive arguments. In 2010, Han et al. [17] conjectured that $z(n) \neq z(n+1)$, for all positive integers n. However, in 2014, Luca and Pomerance [18] disproved this conjecture by proving that $z(n) = z(n+1)$ holds for infinitely many positive integers n. The first few positive integers with the previous property are following: Please leave it this way, it is necessary for the correct structure of the hypothesis

$$107, 493, 495, 600, 667, 1935, 1952, 2169, 2378, 2573, 2989, 3382.$$

Note that, in particular, their result implies that

$$\liminf_{n \to \infty} |z(n+1) - z(n)| = 0.$$

Therefore, an obvious question to ask concerns $\limsup_{n\to\infty} |z(n+1) - z(n)|$. Is it finite? If not, what is its order of growth?

In the next theorem, we prove that this lim sup is infinite; moreover, we partially answer the second question (about its growth order).

Theorem 3. *For any real number $\epsilon \in (0,2)$, we have that*

$$\limsup_{n\to\infty} \frac{|z(n+1) - z(n)|}{(\log n)^{2-\epsilon}} = \infty. \tag{3}$$

2. Auxiliary Results

In this section, we present some results which are essential tools in the proof.

The first ingredient is a kind of "closed formula" for $z(n)$ depending on $z(p^a)$ for all prime factors p of n. The proof of this fact may be found in [19].

Lemma 1 (Theorem 3.3 of [19])**.** *Let $n > 1$ be an integer with prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$. Then,*

$$z(n) = lcm(z(p_1^{a_1}), \ldots, z(p_k^{a_k})).$$

In general, one has that

$$z(lcm(m_1, \ldots, m_k)) = lcm(z(m_1), \ldots, z(m_k)).$$

Lemma 2. *Let p be a prime number and let n be a positive integer. We have the following:*

(a) (Theorem 2.4 of [20]) $z(p)$ divides $p - \left(\frac{5}{p}\right)$;

(b) (Theorem 2.4 of [20]) $z(p^n) = p^{\max\{n-e(p),0\}} z(p)$, if $p > 2$;

(c) (Theorem 1.1 of [21]) $z(2^n) = 3 \cdot 2^{n-2}$, for $n \geq 3$,

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol and $e(p) = \max\{k \geq 0 : p^k \mid F_{z(p)}\}$.

Remark 3. *We remark that, in the light of the previous lemma, the conjecture $z(p) \neq z(p^2)$, for all prime numbers p (which is discussed in the previous section) is equivalent to $e(p) = 1$. In fact, $z(p) \neq z(p^2)$ if and only if $z(p) \neq z(p^2) = p^{\max\{2-e(p),0\}} z(p)$, which holds if and only if $e(p) = 1$.*

We cannot go far in the lore of Fibonacci sequence without encountering its companion, *Lucas sequence* $(L_n)_{n\geq 0}$, which is defined by the same recursion as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$. The next lemma provides well-known arithmetic properties of Fibonacci and Lucas numbers.

Lemma 3. *We have the following:*

(a) $F_n \mid F_m$ if and only if $n \mid m$;

(b) $L_n \mid F_m$ if and only if $n \mid m$ and m/n is even;

(c) $L_n \mid L_m$ if and only if $n \mid m$ and m/n is odd;

(d) $F_{2n} = F_n L_n$;

(e) If $d = \gcd(m,n)$, then

$$\gcd(F_m, L_n) = \begin{cases} L_d, & \text{if } m/d \text{ is even and } n/d \text{ is odd;} \\ 1 \text{ or } 2, & \text{otherwise.} \end{cases}$$

(f) $2 \mid F_m$ if and only if $3 \mid m$; $3 \mid F_m$ if and only if $4 \mid m$;

(g) $2 \mid L_m$ if and only if $3 \mid m$.

The previous items can be proved by using *Binet's formulas*:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n \quad \text{for} \quad n \geq 0,$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ (indeed, they can be found in [22]).

Since Binet's formulas are still valid for Fibonacci and Lucas numbers with negative indices (and by using $\alpha = (-\beta)^{-1}$), one can deduce the following useful identity.

Lemma 4. *Let a and b be integers. Then,*

$$F_a L_b = F_{a+b} + (-1)^b F_{a-b}.$$

Our last tool is the following known bound for the nth Fibonacci number.

Lemma 5. *We have that*

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1},$$

for all $n \geq 1$.

We are now ready to proceed with the proof of the theorems.

3. The Proofs

3.1. Proof of Theorem 1

First, note that, by definition, $p^{e(p)}$ divides $F_{z(p)}$. In particular, $p^{e(p)} \leq F_{z(p)}$ and, by combining Lemma 2 (a) and Lemma 5, we obtain

$$p^{e(p)} \leq F_{z(p)} \leq \alpha^{z(p)-1} \leq \alpha^{(p+1)-1} = \alpha^p.$$

Thus,

$$e(p) \leq \frac{p}{\log p} \log \alpha. \tag{4}$$

Now, we provide a recipe to construct positive integers satisfying (k-WWC). For that, let us recall that an *m-smooth number* is a positive integer whose prime factors are all less than or equal to m, i.e., the set of m-smooth numbers can be written as $S(m) = \{n \geq 1 : P(n) \leq m\}$ (we adhere to the convention that $S(0) = \emptyset$). We claim that, for any prime number $p \geq 2$, the number $n = rp^{\eta_{p,k}}$ (where $\eta_{p,k} := \lfloor p/k \rfloor + 1$) satisfies ($k$-WWC), for any $r \in S(p)$, that is,

$$\{rp^{\eta_{p,k}} : r \in S(p)\} \cap [1, x] \subseteq \{n \leq x : n \text{ satisfies } (k\text{-WWC})\}. \tag{5}$$

In order to prove that, note that the largest prime factor of such an $n = rp^{\eta_{p,k}}$ is p (since $P(r) \leq p$). In addition, the p-adic order of n is at least $\eta_{p,k}$; then, by (4), one has

$$\frac{e(p)}{k} \leq \frac{p}{k \log p} \log \alpha \leq \frac{p}{k \log 2} \log \alpha < 0.7 \cdot \frac{p}{k} < \eta_{p,k} \leq \nu_{P(n)}(n).$$

Therefore, n satisfies (k-WWC); therefore, (by (5))

$$\#\{n \leq x : n \text{ satisfies } (k\text{-WWC})\} \geq \#\{rp^{\eta_{p,k}} : r \in S(p)\}.$$

Further, note that the set on the left-hand side of (5) can be written as the disjoint union

$$\{rp^{\eta_{p,k}} : r \in S(p)\} = \bigcup_{p \geq 2} \bigcup_{i \geq 1} \mathcal{W}_{p,i}^{(k)}, \tag{6}$$

where $\mathcal{W}_{p,i}^{(k)} := \{rp^{\eta_{p,k}+i} : r \in S(p-1)\}$.

It is known that the number of y-smooth numbers less than or equal to x, denoted by $\Psi(x,y)$, satisfies

$$\Psi(x,y) = \frac{1}{\pi(y)!}\prod_{p\le y}\left(\frac{\log x}{\log y}\right)\left(1+O\left(\frac{y^2}{\log x\log y}\right)\right)$$

whenever $y \le \sqrt{\log x \log\log x}$ (for this and more similar results, we refer the reader to [23] and its extensive annotated bibliography). In fact, by ([23], (1.24)), the lower bound

$$\Psi(x,y) \ge \frac{1}{\pi(y)!}\prod_{p\le y}\left(\frac{\log x}{\log y}\right) \tag{7}$$

holds. By noting that $\cup_{i\ge 1}\mathcal{W}_{p,i}^{(k)} = p^{\eta_{p,k}}S(p)$, we obtain

$$\#(\{rp^{\eta_{p,k}} : r\in S(p)\}\cap[1,x]) = \sum_{p\ge 2}\sum_{i\ge 1}\#\mathcal{W}_{p,i}^{(k)}(x) = \sum_{p\ge 2}p^{\eta_{p,k}}\Psi\left(\frac{x}{p^{\eta_{p,k}}},p\right)$$

as $x\to\infty$. Thus,

$$\#\{n\le x : n \text{ satisfies } (k\text{-WWC})\} \ge \sum_{p\ge 2}p^{\eta_{p,k}}\Psi\left(\frac{x}{p^{\eta_{p,k}}},p\right). \tag{8}$$

By using (7), we have

$$\Psi\left(\frac{x}{p^{\eta_{p,k}}},p\right) \ge \frac{1}{\pi(p)!}\prod_{q\le p}\left(\frac{\log(x/p^{\eta_{p,k}})}{\log q}\right),$$

where q runs over the set of prime numbers. In conclusion, we obtain

$$\#\{n\le x : n \text{ satisfies } (k\text{-WWC})\} \ge \sum_{p\ge 2}\frac{p^{\eta_{p,k}}}{\pi(p)!}(\log(x/p^{\eta_{p,k}}))^{\pi(p)}\left(\prod_{q\le p}\log q\right)^{-1}$$

which, combined with $p/k < \eta_{p,k} < (p+k)/k$, finishes the proof. □

3.2. *Proof of Theorem 2*

Clearly, if (ℓ-WWC) is true, then so is (k-WWC), for all $k\le\ell$. One has that $n=1$ is a solution of $z(n)=z(n^k)$. Therefore, for an integer $n>1$, let $n=p_1^{a_1}\cdots p_t^{a_t}$ be its prime factorization, where $p_1<\cdots<p_t$ and $a_i\ge 1$, for all $i\in[1,t]$.

By Lemma 1 and (2), we have that

$$z(n) = \operatorname{lcm}(p_1^{\alpha_1}z(p_1),\dots,p_t^{\alpha_t}z(p_t)),$$

where $\alpha_i := \max\{a_i - e(p_i),0\}$ (for $i\in[1,t]$) and

$$z(n^k) = \operatorname{lcm}(p_1^{\beta_1}z(p_1),\dots,p_t^{\beta_t}z(p_t)),$$

where $\beta_i := \max\{ka_i - e(p_i),0\}$ (for $i\in[1,t]$). Thus, $z(n)=z(n^k)$ implies

$$\operatorname{lcm}(p_1^{\alpha_1}z(p_1),\dots,p_t^{\alpha_t}z(p_t)) = \operatorname{lcm}(p_1^{\beta_1}z(p_1),\dots,p_t^{\beta_t}z(p_t)). \tag{9}$$

If $t=1$, then, supposing (k-WWC), one has that $a_1 > e(p_1)/k$; therefore,

$$z(p_1^{ka_1}) = p_1^{\max\{ka_1-e(p_1),0\}}z(p_1) = p_1^{ka_1-e(p_1)}z(p_1).$$

On the other hand,

$$z(p_1^{a_1}) = p_1^{\max\{a_1-e(p_1),0\}}z(p_1)$$

therefore, $z(p_1^{a_1}) \neq z(p_1^{2a_1})$, since $ka_1 - e(p_1) \geq 2a_1 - e(p_1) > \max\{a_1 - e(p_1), 0\}$. Thus, from now on, we may assume that $t > 1$.

Now, the proof conveniently splits into two cases.

The Case in which $P(n) = 5$.

In this case, we have that n can be written as $n = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3}$, with $(a_1, a_2, a_3) \in \mathbb{Z}^2_{\geq 0} \times \mathbb{Z}_{\geq 1}$. Thus, (9) becomes

$$\text{lcm}(3 \cdot 2^{\delta_1}, 4 \cdot 3^{\delta_2}, 5^{a_3}) = \text{lcm}(3 \cdot 2^{k\delta_1}, 4 \cdot 3^{k\delta_2}, 5^{ka_3}),$$

where $\delta_i := \max\{a_i - 1, 0\}$, for $i \in \{1, 2\}$. However, the above relation cannot be true, since its right-hand side is a multiple of 5^{ka_3}, while the left-hand side has a 5-adic valuation that equals to a_3 (note that $ka_3 \geq 2a_3 > a_3$). Therefore, there is no value of n (for which $z(n) = z(n^k)$) whose greatest prime factor is 5.

The Case in which $P(n) \neq 5$.

In this case, by assuming (k-WWC), we have that $a_t = \nu_{p_t}(n) > e(p_t)/k$; therefore, $\beta_t = ka_t - e(p_t)$ (since $ka_t > e(p_t)$). Thus $p_t^{ka_t - e(p_t)}$ divides $z(n^k) = z(n) = \text{lcm}(p_1^{\alpha_1} z(p_1), \ldots, p_t^{\alpha_t} z(p_t))$. Since $p_t \neq 5$, then, by Lemma 2 (a), $z(p_t)$ divides either $p_t - 1$ or $p_t + 1$. In particular, $\gcd(p_t, z(p_t)) = 1$; therefore, $p_t^{a_t}$ divides

$$\text{lcm}(p_1^{\alpha_1} z(p_1), \ldots, p_{t-1}^{\alpha_{t-1}} z(p_{t-1})),$$

because $t > 1$. Therefore, p_t must divide $z(p_i)$, for some $i \in [1, t-1]$. However $z(p_i) \mid p_i - (5/p_i)$ (Lemma 2 (b)) yields

$$p_i < p_t \leq z(p_i) \leq p_i + 1.$$

Thus, $p_t = p_i + 1$; therefore, $p_t = 3$ and $p_i = 2$ (the only two consecutive prime numbers), i.e., $P(n) \leq 3$. Thus, $n = 2^{a_1} \cdot 3^{a_2}$, with $a_1, a_2 \geq 0$. Clearly, if $(a_1, a_2) = (0, 0)$, we have $n = 1$ as a solution of $z(n) = z(n^2)$. In addition, if $a_1 \geq 1$ and $a_2 = 0$, we have (by Lemma 2 (c)) that $z(2^{a_1}) = 3, 6$ or $3 \cdot 2^{a_1 - 2}$ according to $a_1 = 1, 2$ or ≥ 3, respectively. On the other hand, $z(2^{2a_1}) \geq 6, 12$ or $3 \cdot 2^{ka_1 - 2}$ according to $a_1 = 1, 2$ or ≥ 3. By comparing the above respective values, we infer that there is no solution in these cases. Now, let us suppose that $a_1 = 0$ and $a_2 \geq 1$. Then, $z(3^{a_2}) = 4 \cdot 3^{a_2 - 1} \neq 4 \cdot 3^{ka_2 - 1} = z(3^{ka_2}))$, for all $k \geq 2$. Therefore, let us assume that $\min\{a_1, a_2\} \geq 1$; by splitting into some sub-cases, one has the following:

- If $a_1 = 1$, then $z(n) = \text{lcm}(3, 4 \cdot 3^{a_2 - 1}) = 4 \cdot 3^{\delta_2}$, where $\delta_2 := \max\{a_2 - 1, 1\}$. On the other hand,

$$z(n^k) = \text{lcm}(z(2^k), z(3^{ka_2})) = \text{lcm}(z(2^k), 4 \cdot 3^{ka_2 - 1}) \equiv 0 \pmod{3^{ka_2 - 1}},$$

therefore, $3^{ka_2 - 1}$ divides 3^{δ_2}. Thus, $ka_2 - 1 \leq \max\{a_2 - 1, 1\}$ and (since $k \geq 2$) the only possibility is $(k, a_2) = (2, 1)$ which correspond to $n = 6$ as the only solution to $z(n) = z(n^2)$ with $P(n) \leq 3$ and $\nu_2(n) = 1$.
- If $a_1 = 2$, then $z(n) = \text{lcm}(6, 4 \cdot 3^{a_2 - 1}) = 4 \cdot 3^{\delta_2}$. Therefore,

$$z(n^k) = \text{lcm}(z(2^{2k}), z(3^{ka_2})) = \text{lcm}(z(2^{2k}), 4 \cdot 3^{ka_2 - 1}) \equiv 0 \pmod{3^{ka_2 - 1}},$$

therefore, $3^{ka_2 - 1}$ divides 3^{δ_2}. As in the previous case, we infer that $(k, a_2) = (2, 1)$ which implies that $n = 12$ is the only solution of $z(n) = z(n^2)$ with $P(n) \leq 3$ and $\nu_2(n) = 2$.

Therefore, it remains to prove that the there is no solution when $a_1 \geq 3$. Indeed, by Lemma 2 (c), we have

$$\begin{aligned} z(n) &= \mathrm{lcm}(z(2^{a_1}), z(3^{a_2})) \\ &= \mathrm{lcm}(3 \cdot 2^{a_1-2}, 4 \cdot 3^{a_2-1}) \\ &= 2^{\delta_1} \cdot 3^{\delta_2}, \end{aligned} \tag{10}$$

where $\delta_1 := \max\{a_1 - 2, 2\}$. However,

$$\begin{aligned} z(n^2) &= \mathrm{lcm}(z(2^{ka_1}), z(3^{ka_2})) \\ &= \mathrm{lcm}(3 \cdot 2^{ka_1-2}, 4 \cdot 3^{ka_2-1}) \\ &= 2^{ka_1-2} \cdot 3^{ka_2-1}, \end{aligned} \tag{11}$$

where $ka_1 - 2 \geq 2a_1 - 2 \geq 2 \cdot 3 - 2 \geq 4$. By combining (10) and (11) in $z(n) = z(n^k)$, we arrive at the absurdity that $ka_1 - 2 = \delta_1 = \max\{a_1 - 2, 2\} < a_1 < 2a_1 - 2$. This finishes the proof of the theorem. □

3.3. *Proof of Theorem 3*

Let $m > 1$ be an integer and let us define $n = F_{4m}$. By definition, one has that $z(F_{4m}) = 4m$ (indeed, $z(F_j) = j$, for all $j > 2$). Now, by taking $(a, b) = (2m - 1, 2m + 1)$ in Lemma 4, we have

$$F_{2m-1}L_{2m-1} = F_{4m} + (-1)^{2m-1}F_{-2}.$$

Since $F_{-2} = -1$ (in fact, $F_{-j} = (-1)^{j+1}F_j$, for $j > 0$), we arrive at

$$F_{4m} + 1 = F_{2m-1}L_{2m+1}.$$

By Lemma 3 (e), one has $\gcd(F_{2m-1}, L_{2m+1}) = 1$ or 2 (since $2m - 1$ and $2m + 1$ are odd numbers). Moreover, by Lemma 3 (f) and (g), F_{2m-1} and L_{2m+1} are both even numbers if and only if 3 divides both $2m - 1$ and $2m + 1$. However, the last sentence cannot happen, since $(2m + 1) - (2m - 1) = 2$. Then, we infer that F_{2m-1} and L_{2m+1} are coprime. Therefore, Lemma 1 yields

$$z(F_{4m} + 1) = z(F_{2m-1}L_{2m+1}) = \mathrm{lcm}(z(F_{2m-1}), z(L_{2m+1})) = \mathrm{lcm}(2m - 1, z(L_{2m+1})), \tag{12}$$

since $z(F_{2m-1}) = 2m - 1$ (because $2m - 1 > 2$). The next step is to calculate $z(L_{2m+1})$. For that, by Lemma 3 (b), L_{2m+1} divides F_j if and only if $2m + 1 \mid j$ and $j/(2m + 1)$ is an even number. The minimal j with the required properties is $j = 2(2m + 1)$. Thus, $z(L_{2m+1}) = 2(2m + 1)$ and, by substituting in (12), we deduce that

$$z(F_{4m} + 1) = \mathrm{lcm}(2m - 1, z(L_{2m+1})) = \mathrm{lcm}(2m - 1, 2(2m + 1)) = 2(2m - 1)(2m + 1)$$

where $\gcd(2m - 1, 2(2m + 1)) = \gcd(2m - 1, 2(2m - 1) + 4) = 1$. Therefore,

$$\begin{aligned} z(n + 1) - z(n) &= z(F_{4m} + 1) - z(F_{4m}) \\ &= 2(4m^2 - 1) - 4m \\ &= 8m^2 - 4m - 2 \\ &> 2m^2, \end{aligned} \tag{13}$$

where $8m^2 - 4m - 2 > 2m^2$ if and only if $(3m + 1)(m - 1) = 3m^2 - 2m - 1 > 0$, which holds whenever $m < -1/3$ or $m > 1$.

On the other hand, by Lemma 5, we have that $n = F_{4m} \leq \alpha^{4m} < e^{2m}$ (since $\alpha < e^{1/2}$). Thus, $\log n < 2m$. By combining this inequality with (13), we obtain

$$\frac{z(n+1)-z(n)}{(\log n)^{2-\epsilon}} = \frac{z(F_{4m}+1)-z(F_{4m})}{(\log F_{4m})^{2-\epsilon}} > \frac{2m^2}{(2m)^{2-\epsilon}} > 2^{\epsilon-1}m^{\epsilon}.$$

Since m^{ϵ} tends to infinity as $m \to \infty$, we obtain that $(F_{4m})_{m\geq 2}$ is an infinity sequence of positive integers for which $|z(F_{4m}+1)-z(F_{4m})|/(\log F_{4m})^{2-\epsilon}$ tends to infinity as $m \to \infty$. In particular,

$$\limsup_{n\to\infty} \frac{|z(n+1)-z(n)|}{(\log n)^{2-\epsilon}} = \infty$$

as desired. The proof is complete. □

4. Further Comments and Some Questions

We close this paper by offering some questions for further research. The first natural question to ask is the following.

Question 1. *Is*

$$\limsup_{n\to\infty} \frac{|z(n+1)-z(n)|}{(\log n)^2} = \infty?$$

This question has a positive answer if one replaces 1 (in $z(n+1)$) by any non-Fibonacci number a. Indeed, a result due to Luca and Pomerance ([18], Proposition 3) is that

$$\lim_{k\to\infty} \frac{z(F_k+a)}{k^2} = \infty,$$

for all integers a such that $|a|$ is not a Fibonacci number. An immediate consequence is that, by taking $n_k := F_k$ and $a = 4$ (for example), we have

$$\lim_{k\to\infty} \frac{z(n_k+4)-z(n_k)}{(\log n_k)^2} = \lim_{k\to\infty} \frac{z(F_k+4)-z(F_k)}{(\log F_k)^2} \geq \frac{1}{(\log \alpha)^2} \lim_{k\to\infty} \frac{z(F_k+4)-k}{k^2} = \infty,$$

where $z(F_k) = k$ and $F_k \leq \alpha^k$ (by Lemma 5). In particular,

$$\limsup_{n\to\infty} \frac{z(n+4)-z(n)}{(\log n)^2} = \infty.$$

This leads us to consider the following general study. For any integer $a \neq 0$, let us set

$$\sigma_a := sup\left\{\nu \in \mathbb{R}_{>0} : \limsup_{n\to\infty} \frac{|z(n+1)-z(n)|}{(\log n)^{\nu}} = \infty\right\}.$$

Our previous discussion ensures that $\sigma_a \geq 2$, for all non-zero integer a. We then list some problems.

Question 2.

(i) $\sigma_1 = 2$?
(ii) $\sigma_a = 2$, *for any non-Fibonacci number* a?
(iii) *Is there a positive integer* a *for which* $\sigma_a = \infty$?
(iv) *Is there a positive integer* a *for which* $2 < \sigma_a < \infty$?
(v) *Determine some topological properties of the set* $\{\sigma_a : a \in \mathbb{Z}\backslash\{0\}\}$. *Is it an infinite set? Does it have a limit point? Is it dense in some open subset of* $[2,\infty)$?

5. Conclusions

In this paper, we study some Diophantine problems related to the order of appearance function $z(n) = \min\{k \geq 1 : n \mid F_k\}$. It is well known that, if $p > 2$ is a prime number for which $z(p) \neq z(p^2)$ (this is related to Wall's conjecture that the p-adic order of $F_{z(p)} = 1$, for all prime numbers p), then the equation $x^p + y^p = z^p$ does not have solution $(x,y,z) \in \mathbb{Z}^3$

with $p \nmid xyz$ (the is the first case of Fermat's Last Theorem). In this work, we state a weaker version of Wall's conjecture, named (k-WWC), and we first prove the existence of infinitely many positive integers satisfying this conjecture. After we show that, by supposing that (WWC) is true, the functional equation $z(n) = z(n^2)$ has, in the set of positive integers, only solutions $n \in \{1, 6, 12\}$. In addition, it was proved recently that $z(n+1) = z(n)$ has infinitely many solutions, which implies, in particular, that $\lim \inf_{n\to\infty} (z(n+1) - z(n)) = 0$. In this paper, we still prove that $\lim \sup_{n\to\infty} (z(n+1) - z(n))/(\log n)^{2-\epsilon} = +\infty$, for all $0 < \epsilon < 2$.

Funding: The research study was supported by the Excellence Project PřF UHK No. 2213/2021–2022, University of Hradec Králové, Czech Republic.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank anonymous referees for their careful corrections and their comments that helped to improve the quality of the paper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Lucas, E. Théorie des fonctions numériques simplement périodiques. *Am. J. Math.* **1878**, *1*, 289–321. [CrossRef]
2. Sallé, H.J.A. Maximum value for the rank of apparition of integers in recursive sequences. *Fibonacci Q.* **1975**, *13*, 159–161.
3. Sloane, N.J.A. *The On-Line Encyclopedia of Integer Sequences*. Available online: http://www.oeis.org/ (accessed on 20 August 2021).
4. Marques, D. Fixed points of the order of appearance in the Fibonacci sequence. *Fibonacci Q.* **2012**, *50*, 346–352.
5. Somer, L.; Křížek, M. Fixed points and upper bounds for the rank of appearance in Lucas sequences. *Fibonacci Q.* **2013**, *51*, 291–306.
6. Sun, Z.H.; Sun, Z.W. Fibonacci numbers and Fermat's last theorem. *Acta Arith.* **1992**, *60*, 371–388. [CrossRef]
7. Luca, F.; Tron, E. The distribution of self-Fibonacci divisors. In *Advances in the Theory of Numbers*; Springer: New York, NY, USA, 2015; pp. 149–158.
8. Trojovský, P. On Diophantine equations related to the order of appearance in Fibonacci sequence. *Mathematics* **2019**, *7*, 1073. [CrossRef]
9. Trojovská, E. On the diophantine equation $z(n) = (2 - 1/k)n$ involving the order of appearance in the Fibonacci Sequence. *Mathematics* **2020**, *8*, 124. [CrossRef]
10. Trojovský, P. On the Natural Density of Sets Related to Generalized Fibonacci Numbers of Order r. *Axioms* **2021**, *10*, 144. [CrossRef]
11. Wall, D.D. Fibonacci series modulo m. *Am. Math. Mon.* **1960**, *67*, 525–532. [CrossRef]
12. Klaška, J. Donald Dines Wall's Conjecture. *Fibonacci Q.* **2018**, *56*, 43–51.
13. Lengyel, T. The order of the Fibonacci and Lucas numbers. *Fibonacci Q.* **1995**, *33*, 234–239.
14. Sanna, C. The p-adic valuation of Lucas sequences. *Fibonacci Q.* **2016**, *54*, 118–224.
15. Kreutz, A.; Lelis, J.; Marques, D.; Silva, E.; Trojovský, P. The p-adic order of the k-Fibonacci and k-Lucas numbers. *p-Adic Numbers Ultrametric Anal. Appl.* **2017**, *9*, 15–21. [CrossRef]
16. Marques, D. On generalized Cullen and Woodall numbers which are also Fibonacci numbers. *J. Integer Seq.* **2014**, *17*, 14.9.4.
17. Han, J.S.; Kim, H.S.; Neggers, J. The Fibonacci-norm of a positive integer: Observations and conjectures. *Int. J. Number Theory* **2010**, *6*, 371–385. [CrossRef]
18. Luca, F.; Pomerance, C. On the local behavior of the order of appearance in the Fibonacci sequence. *Int. J. Number Theory* **2014**, *10*, 915–933. [CrossRef]
19. Renault, M. Properties of the Fibonacci Sequence under Various Moduli. Master's Thesis, Wake Forest University, Winston-Salem, NC, USA, 1996. Available online: http://webspace.ship.edu/msrenault/fibonacci/FibThesis.pdf (accessed on 20 October 2021).
20. Fulton, J.; Morris, W. On arithmetical functions related to the Fibonacci numbers. *Acta Arith.* **1969**, *16*, 105–110. [CrossRef]
21. Marques, D. Sharper upper bounds for the order of appearance in the Fibonacci sequence. *Fibonacci Q.* **2013**, *51*, 233–238.
22. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; Wiley: New York, NY, USA, 2001.
23. Granville, A. Smooth numbers: Computational number theory and beyond. In *Algorithmic Number Theory*; MSRI Publications: Cambridge, UK, 2008; Volume 44, pp. 267–323.