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# Steffensen Type Inequalities for Convex Functions on Borel $\sigma$ -Algebra

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**Abstract:** In the paper, we prove Steffensen type inequalities for positive finite measures by using functions which are convex in point. Further, we prove Steffensen type inequalities on Borel  $\sigma$ -algebra for the function of the form  $f/h$  which is convex in point. We conclude the paper by showing that these results also hold for convex functions.

**Keywords:** Steffensen's inequality; positive measures; weaker conditions; convexity in point; convex functions

## 1. Introduction

In 1918, Steffensen proved an inequality that intrigued many mathematicians. Over the years, various generalizations and refinements have been proven not only in the classical sense, but also in the theory of measures, time scales, quantum calculus, fractional calculus, special functions, functional equations and many others. Some of the results dealing with the topic can be found in [1–12]. A complete overview of the results related to Steffensen's inequality can be found in monographs [13,14].

**Theorem 1** (Steffensen's inequality, [15]). *Suppose that  $f$  is non-increasing and  $g$  is integrable on  $[a, b]$  with  $0 \leq g \leq 1$  and  $\lambda = \int_a^b g(t)dt$ . Then, we have*

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt.$$

*The inequalities are reversed for  $f$  non-decreasing.*

Steffensen's inequality is closely related to many other well-known inequalities such as Gauss', Gauss–Steffensen's, Hölder's, Jenssen–Steffensen's, Iyengar's and other inequalities. Therefore, the Steffensen type inequalities proved in this paper can be used to obtain different generalizations and refinements of these inequalities. For more details about the connection of Steffensen's inequality to other inequalities see in [14].

In [16], Pečarić and Smoljak introduced the following class of functions.

**Definition 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function and  $c \in (a, b)$ . We say that  $f$  belongs to class  $\mathcal{M}_1^c[a, b]$  ( $\mathcal{M}_2^c[a, b]$ ) if there exists a constant  $A$  such that the function  $F(x) = f(x) - Ax$  is non-increasing (non-decreasing) on  $[a, c]$  and non-decreasing (non-increasing) on  $[c, b]$ .*

Further, in [16] Pečarić and Smoljak proved the following.

**Theorem 2.** *If  $f \in \mathcal{M}_1^c[a, b]$  ( $f \in \mathcal{M}_2^c[a, b]$ ) for every  $c \in (a, b)$ , then  $f$  is convex (concave).*

**Remark 1.** *As noted in [16], the class of functions introduced in Definition 1 can be described as a class of functions convex in point  $c$ . Therefore, the function  $f$  is convex on  $[a, b]$  if and only if it is convex in every  $c \in (a, b)$ .*



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Throughout the paper, we will use the notation  $\mathcal{B}([a, b])$  for the Borel  $\sigma$ -algebra which is generated on segment  $[a, b]$ .

In the following two theorems, we recall Steffensen’s inequality for positive measures on  $\mathcal{B}([a, b])$  proved in [9].

**Theorem 3.** *Let  $\mu$  be a positive, finite, measure on  $\mathcal{B}([a, b])$  and let  $f$  and  $g$  be measurable functions such that  $f$  is non-increasing and  $0 \leq g \leq 1$ . If there exists  $\lambda \in \mathbb{R}_+$  such that*

$$\mu([a, a + \lambda]) = \int_{[a,b]} g(t)d\mu(t),$$

then

$$\int_{[a,a+\lambda]} f(t)d\mu(t) \geq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{1}$$

If  $f$  is non-decreasing, the reverse inequality in (1) holds.

**Theorem 4.** *Let  $\mu$  be a positive, finite, measure on  $\mathcal{B}([a, b])$  and let  $f$  and  $g$  be measurable functions such that  $f$  is non-increasing and  $0 \leq g \leq 1$ . If there exists  $\lambda \in \mathbb{R}_+$  such that*

$$\mu((b - \lambda, b]) = \int_{[a,b]} g(t)d\mu(t),$$

then

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{2}$$

If  $f$  is non-decreasing, the reverse inequality in (2) holds.

Let us also recall weaker conditions for Steffensen’s inequality for positive finite measures on  $\mathcal{B}([a, b])$  given in [10].

**Theorem 5.** *Let  $\mu$  be a finite, positive measure on  $\mathcal{B}([a, b])$ , let  $g : [a, b] \rightarrow \mathbb{R}$  be a  $\mu$ -integrable function.*

(a) *Let  $\lambda$  be a positive constant such that  $\mu([a, a + \lambda]) = \int_{[a,b]} g(t)d\mu(t)$ . The inequality (1) holds for every non-increasing, right-continuous function  $f : [a, b] \rightarrow \mathbb{R}$  if and only if*

$$\int_{[a,x]} g(t)d\mu(t) \leq \mu([a, x]) \quad \text{and} \quad \int_{[x,b]} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a, b]. \tag{3}$$

(b) *Let  $\lambda$  be a positive constant such that  $\mu((b - \lambda, b]) = \int_{[a,b]} g(t)d\mu(t)$ . The inequality (2) holds for every non-increasing, right-continuous function  $f : [a, b] \rightarrow \mathbb{R}$  if and only if*

$$\int_{[x,b]} g(t)d\mu(t) \leq \mu([x, b]) \quad \text{and} \quad \int_{[a,x]} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a, b]. \tag{4}$$

The generalization of the classical Steffensen inequality proved by Pečarić [17] was intensively used by many mathematicians. The following generalization in the theory of measures was proved in [11].

**Theorem 6.** *Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g$  and  $h$  be measurable functions on  $[a, b]$  such that  $h$  is positive,  $f/h$  is non-increasing and  $0 \leq g \leq 1$ .*

(a) *If there exists  $\lambda \in \mathbb{R}_+$  such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t),$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)d\mu(t). \tag{5}$$

(b) If there exists  $\lambda \in \mathbb{R}_+$  such that

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t),$$

then

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{6}$$

If  $f/h$  is non-decreasing, the reverse inequalities in (5) and (6) hold.

In addition to generalization obtained by Pečarić in [17], another generalization of Steffensen’s inequality which was proved by Mercer in [18] is often used. As the original Mercer generalization is wrong, there are many corrected versions of it. One corrected version was not only proved but also refined by Wu and Srivastava in [19]. Their refinement extended to the theory of measures was proved in [11]. Here, we recall the refinement of Pečarić generalization which can easily be obtained from the refinement proved in [11].

**Theorem 7.** Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g$  and  $h$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq 1$  and  $f/h$  is non-increasing.

(a) If there exists  $\lambda \in \mathbb{R}_+$  such that

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t), \tag{7}$$

then

$$\begin{aligned} \int_{[a,b]} f(t)g(t)d\mu(t) &\leq \int_{[a,a+\lambda]} \left( f(t) - \left[ \frac{f(t)}{h(t)} - \frac{f(a+\lambda)}{h(a+\lambda)} \right] h(t)[1-g(t)] \right) d\mu(t) \\ &\leq \int_{[a,a+\lambda]} f(t)d\mu(t). \end{aligned} \tag{8}$$

(b) If there exists  $\lambda \in \mathbb{R}_+$  such that

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t), \tag{9}$$

then

$$\begin{aligned} \int_{(b-\lambda,b]} f(t)d\mu(t) &\leq \int_{(b-\lambda,b]} \left( f(t) - \left[ \frac{f(t)}{h(t)} - \frac{f(b-\lambda)}{h(b-\lambda)} \right] h(t)[1-g(t)] \right) d\mu(t) \\ &\leq \int_{[a,b]} f(t)g(t)d\mu(t). \end{aligned} \tag{10}$$

If  $f/h$  is non-decreasing, the reverse inequalities in (8) and (10) hold.

In [19], Wu and Srivastava also proved that the correction of Mercer’s generalization can be sharper. Their result was extended to Borel  $\sigma$ -algebra in [11]. We recall this extension in the following theorem.

**Theorem 8.** Let  $\mu$  be a positive finite measure on  $\mathcal{B}([a, b])$ , let  $f, g, h$  and  $\psi$  be measurable functions on  $[a, b]$  such that  $0 \leq \psi \leq g \leq h - \psi$  and  $f$  is non-increasing.

(a) If there exists  $\lambda \in \mathbb{R}_+$  such that

$$\int_{[a, a+\lambda]} h(t) d\mu(t) = \int_{[a, b]} g(t) d\mu(t),$$

then

$$\int_{[a, b]} f(t)g(t) d\mu(t) \leq \int_{[a, a+\lambda]} f(t)h(t) d\mu(t) - \int_{[a, b]} |f(t) - f(a + \lambda)|\psi(t) d\mu(t). \tag{11}$$

(b) If there exists  $\lambda \in \mathbb{R}_+$  such that

$$\int_{(b-\lambda, b]} h(t) d\mu(t) = \int_{[a, b]} g(t) d\mu(t),$$

then

$$\int_{(b-\lambda, b]} f(t)h(t) d\mu(t) + \int_{[a, b]} |f(t) - f(b - \lambda)|\psi(t) d\mu(t) \leq \int_{[a, b]} f(t)g(t) d\mu(t). \tag{12}$$

If  $f/h$  is non-decreasing, the reverse inequalities in (11) and (12) hold.

In [16,20], the authors proved the classical Steffensen type inequalities for functions which are convex in point and also extended their result to convex functions. In this paper, we prove measure theoretic generalization of the above mentioned Steffensen type inequalities. For that purpose, we use measures on Borel  $\sigma$ -algebra  $\mathcal{B}([a, b])$ .

### 2. Main Results

Let us begin by proving Steffensen type inequalities for the functions from  $\mathcal{M}_1^c[a, b]$  on  $\mathcal{B}([a, b])$ .

**Theorem 9.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $\lambda_1$  be a positive constant such that

$$\mu([a, a + \lambda_1]) = \int_{[a, c]} g(t) d\mu(t) \tag{13}$$

and let  $\lambda_2$  be a positive constant such that

$$\mu((b - \lambda_2, b]) = \int_{[c, b]} g(t) d\mu(t). \tag{14}$$

If  $f \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a, b]} tg(t) d\mu(t) = \int_{[a, a+\lambda_1]} td\mu(t) + \int_{(b-\lambda_2, b]} td\mu(t), \tag{15}$$

then

$$\int_{[a, b]} f(t)g(t) d\mu(t) \leq \int_{[a, a+\lambda_1]} f(t) d\mu(t) + \int_{(b-\lambda_2, b]} f(t) d\mu(t). \tag{16}$$

If  $f \in \mathcal{M}_2^c[a, b]$  and (15) holds, the inequality in (16) is reversed.

**Proof.** Let  $A$  be the constant as in the Definition 1. Assume that  $f \in \mathcal{M}_1^c[a, b]$ . We consider the function  $F(x) = f(x) - Ax$ .

The function  $F$  is non-increasing on  $[a, c]$  so we can apply the inequality (1) to the function  $F$  and obtain

$$\begin{aligned} 0 &\leq \int_{[a, a+\lambda_1]} F(t) d\mu(t) - \int_{[a, c]} F(t)g(t) d\mu(t) \\ &= \int_{[a, a+\lambda_1]} f(t) d\mu(t) - \int_{[a, c]} f(t)g(t) d\mu(t) - A \left( \int_{[a, a+\lambda_1]} t d\mu(t) - \int_{[a, c]} t g(t) d\mu(t) \right). \end{aligned} \tag{17}$$

Similarly, the function  $F$  is non-decreasing on  $[c, b]$  so we can apply the reverse inequality (2) to the function  $F$  and obtain

$$\begin{aligned} 0 &\geq \int_{[c, b]} F(t)g(t) d\mu(t) - \int_{(b-\lambda_2, b]} F(t) d\mu(t) \\ &= \int_{[c, b]} f(t)g(t) d\mu(t) - \int_{(b-\lambda_2, b]} f(t) d\mu(t) - A \left( \int_{[c, b]} t g(t) d\mu(t) - \int_{(b-\lambda_2, b]} t d\mu(t) \right). \end{aligned} \tag{18}$$

Combining (17) and (18), we have

$$\begin{aligned} &\int_{[a, a+\lambda_1]} f(t) d\mu(t) + \int_{(b-\lambda_2, b]} f(t) d\mu(t) - \int_{[a, b]} f(t)g(t) d\mu(t) \\ &\geq A \left( \int_{[a, a+\lambda_1]} t d\mu(t) + \int_{(b-\lambda_2, b]} t d\mu(t) - \int_{[a, b]} t g(t) d\mu(t) \right). \end{aligned}$$

Therefore, if  $\int_{[a, b]} t g(t) d\mu(t) = \int_{[a, a+\lambda_1]} t d\mu(t) + \int_{(b-\lambda_2, b]} t d\mu(t)$ , then (16) holds. Similarly, for  $f \in \mathcal{M}_2^c[a, b]$  we obtain the reversed inequality in (16).  $\square$

**Theorem 10.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $\lambda_1$  be a positive constant such that

$$\mu((c - \lambda_1, c]) = \int_{[a, c]} g(t) d\mu(t) \tag{19}$$

and let  $\lambda_2$  be a positive constant such that

$$\mu([c, c + \lambda_2]) = \int_{[c, b]} g(t) d\mu(t). \tag{20}$$

If  $f \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a, b]} t g(t) d\mu(t) = \int_{(c-\lambda_1, c+\lambda_2]} t d\mu(t), \tag{21}$$

then

$$\int_{[a, b]} f(t)g(t) d\mu(t) \geq \int_{(c-\lambda_1, c+\lambda_2]} f(t) d\mu(t). \tag{22}$$

If  $f \in \mathcal{M}_2^c[a, b]$  and (21) holds, the inequality in (22) is reversed.

**Proof.** Let  $A$  be the constant as in the Definition 1. Assume that  $f \in \mathcal{M}_1^c[a, b]$ . We consider the function  $F(x) = f(x) - Ax$ .

The function  $F$  is non-increasing on  $[a, c]$  so we can apply the inequality (2) to the function  $F$  and obtain

$$0 \leq \int_{[a, c]} f(t)g(t) d\mu(t) - \int_{(c-\lambda_1, c]} f(t) d\mu(t) - A \left( \int_{[a, c]} t g(t) d\mu(t) - \int_{(c-\lambda_1, c]} t d\mu(t) \right). \tag{23}$$

Similarly, the function  $F$  is non-decreasing on  $[c, b]$  so we can apply the reverse inequality (1) to the function  $F$  and obtain

$$0 \geq \int_{[c, c+\lambda_2]} f(t) d\mu(t) - \int_{[c, b]} f(t)g(t) d\mu(t) - A \left( \int_{[c, c+\lambda_2]} t d\mu(t) - \int_{[c, b]} t g(t) d\mu(t) \right). \tag{24}$$

Combining (23) and (24) we have

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t) \geq A \left( \int_{[a,b]} tg(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} td\mu(t) \right).$$

Therefore, if  $\int_{[a,b]} tg(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} td\mu(t)$ , then (22) holds.

Similarly, for  $f \in \mathcal{M}_2^c[a, b]$  we obtain the reversed inequality in (22)  $\square$

Replacing the condition  $0 \leq g \leq 1$  by the weaker one in Theorems 9 and 10 we obtain the following theorems.

**Theorem 11.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be  $\mu$ -integrable functions such that

$$\int_{[a,x)} g(t)d\mu(t) \leq \mu([a, x)) \quad \text{and} \quad \int_{[x,c]} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a, c] \quad (25)$$

and

$$\int_{[x,b]} g(t)d\mu(t) \leq \mu([x, b]) \quad \text{and} \quad \int_{[c,x)} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [c, b]. \quad (26)$$

Let  $\lambda_1$  be a positive constant such that (13) holds and let  $\lambda_2$  be a positive constant such that (14) holds. If  $f \in \mathcal{M}_1^c[a, b]$  and (15) holds, then (16) holds.

If  $f \in \mathcal{M}_2^c[a, b]$  and (15) holds, the inequality in (16) is reversed.

**Proof.** Let  $A$  be the constant from Definition 1. Assume that  $f \in \mathcal{M}_1^c[a, b]$ . We consider the function  $F(x) = f(x) - Ax$ .

The function  $F$  is non-increasing on  $[a, c]$  and the condition (25) holds, so we can apply Theorem 5(a) and obtain (17).

Similarly, the function  $F$  is non-decreasing on  $[c, b]$  and the condition (26) holds so we can apply Theorem 5(b) to the function  $F$  and obtain (18). By similar reasoning as in the proof of Theorem 9 we obtain (16).  $\square$

**Theorem 12.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be  $\mu$ -integrable functions such that

$$\int_{[x,c]} g(t)d\mu(t) \leq \mu([x, c]) \quad \text{and} \quad \int_{[a,x)} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a, c] \quad (27)$$

and

$$\int_{[c,x)} g(t)d\mu(t) \leq \mu([c, x)) \quad \text{and} \quad \int_{[x,b]} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [c, b]. \quad (28)$$

Let  $\lambda_1$  be a positive constant such that (19) holds and let  $\lambda_2$  be a positive constant such that (20) holds. If  $f \in \mathcal{M}_1^c[a, b]$  and (21) holds, then (22) holds.

If  $f \in \mathcal{M}_2^c[a, b]$  and (21) holds, the inequality in (22) is reversed.

**Proof.** Let  $A$  be the constant from Definition 1. Assume that  $f \in \mathcal{M}_1^c[a, b]$ . We consider the function  $F(x) = f(x) - Ax$ .

The function  $F$  is non-increasing on  $[a, c]$  and the condition (27) holds so we can apply Theorem 5(b) and obtain (23).

Similarly, the function  $F$  is non-decreasing on  $[c, b]$  and the condition (28) holds so we can apply Theorem 5(a) and obtain (24). By similar reasoning as in the proof of Theorem 10 we obtain (22).  $\square$

2.1. Results for the Function of the Form  $f/h$

In [16], Pečarić and Smoljak Kalamir observed Steffensen type inequalities for the functions of the form  $f/h$  from the class  $\mathcal{M}_1^c[a, b]$ . Similar to the Definition 1 they used the following definition of the class  $\mathcal{M}_1^c[a, b]$  for the functions of the form  $f/h$ :

**Definition 2.** Let  $h : [a, b] \rightarrow \mathbb{R}$  be a positive function,  $f : [a, b] \rightarrow \mathbb{R}$  be a function and  $c \in (a, b)$ . We say that  $f/h$  belongs to the class  $\mathcal{M}_1^c[a, b]$  ( $\mathcal{M}_2^c[a, b]$ ) if there exists a constant  $A$  such that the function  $\frac{F(x)}{h(x)} = \frac{f(x)}{h(x)} - Ax$  is non-increasing (non-decreasing) on  $[a, c]$  and non-decreasing (non-increasing) on  $[c, b]$ .

Now, let us prove Steffensen type inequalities for the function of the form  $f/h$ .

**Theorem 13.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$  and let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{[a, a+\lambda_1]} h(t)d\mu(t) = \int_{[a, c]} h(t)g(t)d\mu(t) \tag{29}$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{[b-\lambda_2, b]} h(t)d\mu(t) = \int_{[c, b]} h(t)g(t)d\mu(t) \tag{30}$$

If  $f/h \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a, b]} th(t)g(t)d\mu(t) = \int_{[a, a+\lambda_1]} th(t)d\mu(t) + \int_{[b-\lambda_2, b]} th(t)d\mu(t), \tag{31}$$

then

$$\int_{[a, b]} f(t)g(t)d\mu(t) \leq \int_{[a, a+\lambda_1]} f(t)d\mu(t) + \int_{[b-\lambda_2, b]} f(t)d\mu(t). \tag{32}$$

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (31) holds, the inequality in (32) is reversed.

**Proof.** Let  $A$  be the constant as in the Definition 2. Assume that  $f/h \in \mathcal{M}_1^c[a, b]$ . We consider the function  $F(x) = f(x) - Axh(x)$ .

As  $F/h$  is non-increasing on  $[a, c]$ , from Theorem 6(a) we obtain

$$\begin{aligned} 0 &\leq \int_{[a, a+\lambda_1]} F(t)d\mu(t) - \int_{[a, c]} F(t)g(t)d\mu(t) \\ &= \int_{[a, a+\lambda_1]} f(t)d\mu(t) - \int_{[a, c]} f(t)g(t)d\mu(t) - A \left( \int_{[a, a+\lambda_1]} th(t)d\mu(t) - \int_{[a, c]} th(t)g(t)d\mu(t) \right). \end{aligned} \tag{33}$$

As  $F/h$  is non-decreasing on  $[c, b]$ , from Theorem 6(b) we obtain

$$\begin{aligned} 0 &\geq \int_{[c, b]} F(t)g(t)d\mu(t) - \int_{[b-\lambda_2, b]} F(t)d\mu(t) \\ &= \int_{[c, b]} f(t)g(t)d\mu(t) - \int_{[b-\lambda_2, b]} f(t)d\mu(t) - A \left( \int_{[c, b]} th(t)g(t)d\mu(t) - \int_{[b-\lambda_2, b]} th(t)d\mu(t) \right). \end{aligned} \tag{34}$$

Combining (33) and (34) we have

$$\begin{aligned} &\int_{[a, a+\lambda_1]} f(t)d\mu(t) + \int_{[b-\lambda_2, b]} f(t)d\mu(t) - \int_{[a, b]} f(t)g(t)d\mu(t) \\ &\geq A \left( \int_{[a, a+\lambda_1]} th(t)d\mu(t) + \int_{[b-\lambda_2, b]} th(t)d\mu(t) - \int_{[a, b]} th(t)g(t)d\mu(t) \right). \end{aligned}$$

Therefore, if (31) is satisfied, then (32) holds.  
 For  $f/h \in \mathcal{M}_2^c[a, b]$  the proof is similar.  $\square$

**Theorem 14.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$  and let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{(c-\lambda_1, c]} h(t)d\mu(t) = \int_{[a, c]} h(t)g(t)d\mu(t) \tag{35}$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{[c, c+\lambda_2]} h(t)d\mu(t) = \int_{[c, b]} h(t)g(t)d\mu(t). \tag{36}$$

If  $f/h \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a, b]} th(t)g(t)d\mu(t) = \int_{(c-\lambda_1, c+\lambda_2]} th(t)d\mu(t), \tag{37}$$

then

$$\int_{[a, b]} f(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1, c+\lambda_2]} f(t)d\mu(t). \tag{38}$$

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (37) holds, the inequality in (38) is reversed.

**Proof.** Let  $A$  be the constant as in the Definition 2. Assume that  $f/h \in \mathcal{M}_1^c[a, b]$ . We consider the function  $F(x) = f(x) - Axh(x)$ .

As  $F/h$  is non-increasing on  $[a, c]$ , from Theorem 6(b) we obtain

$$0 \leq \int_{[a, c]} f(t)g(t)d\mu(t) - \int_{(c-\lambda_1, c]} f(t)d\mu(t) - A \left( \int_{[a, c]} th(t)g(t)d\mu(t) - \int_{(c-\lambda_1, c]} th(t)d\mu(t) \right). \tag{39}$$

As  $F/h$  is non-decreasing on  $[c, b]$ , from Theorem 6(a) we obtain

$$0 \geq \int_{[c, c+\lambda_2]} f(t)d\mu(t) - \int_{[c, b]} f(t)g(t)d\mu(t) - A \left( \int_{[c, c+\lambda_2]} th(t)d\mu(t) - \int_{[c, b]} th(t)g(t)d\mu(t) \right). \tag{40}$$

The rest of the proof is the same as the proof of Theorem 13.  $\square$

Now, let us prove generalizations of corrected Mercer’s results for the functions which are convex in point.

**Theorem 15.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$ , and let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq h$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{[a, a+\lambda_1]} h(t)d\mu(t) = \int_{[a, c]} g(t)d\mu(t),$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{(b-\lambda_2, b]} h(t)d\mu(t) = \int_{[c, b]} g(t)d\mu(t).$$

If  $f \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a, b]} tg(t)d\mu(t) = \int_{[a, a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2, b]} th(t)d\mu(t), \tag{41}$$



then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda_1]} f(t)h(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)h(t)d\mu(t). \tag{42}$$

If  $f \in \mathcal{M}_2^c[a, b]$  and (41) holds, the inequality in (42) is reversed.

**Proof.** Taking substitutions  $g \mapsto g/h$  and  $f \mapsto fh$  in Theorem 13 we obtain the statement of this theorem.  $\square$

**Theorem 16.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$ , and let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq h$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{(c-\lambda_1,c]} h(t)d\mu(t) = \int_{[a,c]} g(t)d\mu(t),$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{[c,c+\lambda_2]} h(t)d\mu(t) = \int_{[c,b]} g(t)d\mu(t).$$

If  $f \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a,b]} tg(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t), \tag{43}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1,c+\lambda_2]} f(t)h(t)d\mu(t). \tag{44}$$

If  $f \in \mathcal{M}_2^c[a, b]$  and (43) holds, the inequality in (44) is reversed.

**Proof.** Taking substitutions  $g \mapsto g/h$  and  $f \mapsto fh$  in Theorem 14 we obtain the statement of this theorem.  $\square$

We continue with another generalization of the Mercer type for the functions which are convex in point. It can be proved that it is equivalent to Theorems 9 and 10. For the details about this equivalence in the classical sense, the interested reader can see in [21].

**Theorem 17.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$ , and let  $f, g$  and  $k$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq k$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{[a,a+\lambda_1]} k(t)h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t) \tag{45}$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{(b-\lambda_2,b]} k(t)h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t). \tag{46}$$

If  $f/h \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{[a,a+\lambda_1]} tk(t)h(t)d\mu(t) + \int_{(b-\lambda_2,b]} tk(t)h(t)d\mu(t), \tag{47}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda_1]} f(t)k(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)k(t)d\mu(t). \tag{48}$$

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (47) holds, the inequality in (48) is reversed.

**Proof.** Taking substitutions  $h \mapsto kh$ ,  $g \mapsto g/k$  and  $f \mapsto fk$  in Theorem 13 we obtain the statement of this theorem.  $\square$

**Theorem 18.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$  and let  $f$ ,  $g$  and  $k$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq k$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{(c-\lambda_1, c]} k(t)h(t)d\mu(t) = \int_{[a, c]} h(t)g(t)d\mu(t) \tag{49}$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{(c, c+\lambda_2]} k(t)h(t)d\mu(t) = \int_{[c, b]} h(t)g(t)d\mu(t). \tag{50}$$

If  $f/h \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a, b]} th(t)g(t)d\mu(t) = \int_{(c-\lambda_1, c+\lambda_2]} tk(t)h(t)d\mu(t), \tag{51}$$

then

$$\int_{[a, b]} f(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1, c+\lambda_2]} f(t)k(t)d\mu(t). \tag{52}$$

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (51) holds, the inequality in (52) is reversed.

**Proof.** Taking substitutions  $h \mapsto kh$ ,  $g \mapsto g/k$  and  $f \mapsto fk$  in Theorem 14 we obtain the statement of this theorem.  $\square$

In the following theorems, we prove refinements of Theorems 13 and 14.

**Theorem 19.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$ , and let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{[a, a+\lambda_1]} h(t)d\mu(t) = \int_{[a, c]} h(t)g(t)d\mu(t),$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{(b-\lambda_2, b]} h(t)d\mu(t) = \int_{[c, b]} h(t)g(t)d\mu(t).$$

If  $f/h \in \mathcal{M}_1^c[a, b]$  and

$$\begin{aligned} \int_{[a, b]} th(t)g(t)d\mu(t) &= \int_{[a, a+\lambda_1]} (th(t) - [t - a - \lambda_1]h(t)[1 - g(t)])d\mu(t) \\ &+ \int_{(b-\lambda_2, b]} (th(t) - [t - b + \lambda_2]h(t)[1 - g(t)])d\mu(t), \end{aligned} \tag{53}$$

then

$$\begin{aligned} \int_{[a, b]} f(t)g(t)d\mu(t) &\leq \int_{[a, a+\lambda_1]} \left( f(t) - \left[ \frac{f(t)}{h(t)} - \frac{f(a + \lambda_1)}{h(a + \lambda_1)} \right] h(t)[1 - g(t)] \right) d\mu(t) \\ &+ \int_{(b-\lambda_2, b]} \left( f(t) - \left[ \frac{f(t)}{h(t)} - \frac{f(b - \lambda_2)}{h(b - \lambda_2)} \right] h(t)[1 - g(t)] \right) d\mu(t). \end{aligned} \tag{54}$$

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (53) holds, the inequality in (54) is reversed.

**Proof.** Similar to the proof of Theorem 13 applying Theorem 7(a) for  $F/h$  non-increasing on  $[a, c]$  and Theorem 7(b) for  $F/h$  non-decreasing on  $[c, b]$ .  $\square$

**Theorem 20.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$ , and let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{(c-\lambda_1, c]} h(t) d\mu(t) = \int_{[a, c]} h(t) g(t) d\mu(t),$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{[c, c+\lambda_2]} h(t) d\mu(t) = \int_{[c, b]} h(t) g(t) d\mu(t).$$

If  $f/h \in \mathcal{M}_1^c[a, b]$  and

$$\begin{aligned} \int_{[a, b]} th(t)g(t)d\mu(t) &= \int_{(c-\lambda_1, c]} (th(t) - [t - c + \lambda_1]h(t)[1 - g(t)])d\mu(t) \\ &\quad + \int_{[c, c+\lambda_2]} (th(t) - [t - c - \lambda_2]h(t)[1 - g(t)])d\mu(t), \end{aligned} \tag{55}$$

then

$$\begin{aligned} \int_{[a, b]} f(t)g(t)d\mu(t) &\geq \int_{(c-\lambda_1, c]} \left( f(t) - \left[ \frac{f(t)}{h(t)} - \frac{f(c - \lambda_1)}{h(c - \lambda_1)} \right] h(t)[1 - g(t)] \right) d\mu(t) \\ &\quad + \int_{[c, c+\lambda_2]} \left( f(t) - \left[ \frac{f(t)}{h(t)} - \frac{f(c + \lambda_2)}{h(c + \lambda_2)} \right] h(t)[1 - g(t)] \right) d\mu(t). \end{aligned} \tag{56}$$

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (55) holds, the inequality in (56) is reversed.

**Proof.** Similar to the proof of Theorem 14 applying Theorem 7(b) for  $F/h$  non-increasing on  $[a, c]$  and Theorem 7(a) for  $F/h$  non-decreasing on  $[c, b]$ .  $\square$

Let us prove sharpened and generalized Steffensen type inequalities on  $\mathcal{B}([a, b])$ .

**Theorem 21.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$ , and let  $f, g$  and  $\psi$  be measurable functions on  $[a, b]$  such that  $0 \leq \psi \leq g \leq 1 - \psi$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{[a, a+\lambda_1]} h(t) d\mu(t) = \int_{[a, c]} h(t) g(t) d\mu(t),$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{(b-\lambda_2, b]} h(t) d\mu(t) = \int_{[c, b]} h(t) g(t) d\mu(t).$$

If  $f/h \in \mathcal{M}_1^c[a, b]$  and

$$\begin{aligned} \int_{[a, b]} th(t)g(t)d\mu(t) &= \int_{[a, a+\lambda_1]} th(t)d\mu(t) - \int_{[a, c]} |t - a - \lambda_1|h(t)\psi(t)d\mu(t) \\ &\quad + \int_{(b-\lambda_2, b]} th(t)d\mu(t) + \int_{[c, b]} |t - b + \lambda_2|h(t)\psi(t)d\mu(t), \end{aligned} \tag{57}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda_1]} f(t)d\mu(t) - \int_{[a,c]} \left| \frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)} \right| h(t)\psi(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)d\mu(t) + \int_{[c,b]} \left| \frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)} \right| h(t)\psi(t)d\mu(t). \tag{58}$$

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (57) holds, the inequality in (58) is reversed.

**Proof.** Similar to the proof of Theorem 13 applying Theorem 8(a) for  $F/h$  non-increasing on  $[a, c]$  and Theorem 8(b) for  $F/h$  non-decreasing on  $[c, b]$ .  $\square$

**Theorem 22.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h$  be positive measurable function on  $[a, b]$ , and let  $f, g$  and  $\psi$  be measurable functions on  $[a, b]$  such that  $0 \leq \psi \leq g \leq 1 - \psi$ . Let  $\lambda_1$  be a positive constant such that

$$\int_{(c-\lambda_1,c]} h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t),$$

and let  $\lambda_2$  be a positive constant such that

$$\int_{(c,c+\lambda_2]} h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t).$$

If  $f/h \in \mathcal{M}_1^c[a, b]$  and

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t) - \int_{[a,c]} |t - c + \lambda_1| h(t)\psi(t)d\mu(t) + \int_{[c,b]} |t - c - \lambda_2| h(t)\psi(t)d\mu(t), \tag{59}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t) + \int_{[a,c]} \left| \frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)} \right| h(t)\psi(t)d\mu(t) - \int_{[c,b]} \left| \frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)} \right| h(t)\psi(t)d\mu(t). \tag{60}$$

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (59) holds, the inequality in (60) is reversed.

**Proof.** Similar to the proof of Theorem 14 applying Theorem 8(b) for  $F/h$  non-increasing on  $[a, c]$  and Theorem 8(a) for  $F/h$  non-decreasing on  $[c, b]$ .  $\square$

### 2.2. Weaker Conditions for the Function of the Form $f/h$

Replacing the condition  $0 \leq g \leq k$  in Theorems 17 and 18 by the weaker one we obtain the following theorems.

**Theorem 23.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $f, g, k, h : [a, b] \rightarrow \mathbb{R}$  be  $\mu$ -integrable functions such that  $k$  is positive,  $h$  is non-negative and

$$\int_{[a,x]} k(t)g(t)d\mu(t) \leq \int_{[a,x]} k(t)h(t)d\mu(t) \quad \text{and} \quad \int_{[x,c]} k(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a, c] \tag{61}$$

and

$$\int_{[x,b]} k(t)g(t)d\mu(t) \leq \int_{[x,b]} k(t)h(t)d\mu(t) \quad \text{and} \quad \int_{[c,x]} k(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [c, b].$$

Let  $\lambda_1$  be a positive constant such that (45) holds and let  $\lambda_2$  be a positive constant such that (46) holds. If  $f/h \in \mathcal{M}_1^c[a, b]$  and (47) holds, then (48) holds.  
 If  $f/h \in \mathcal{M}_2^c[a, b]$  and (47) holds, the inequality in (48) is reversed.

**Proof.** Similar to the proof of Theorem 11 using the weaker conditions proved in [11] (Theorem 3.1).  $\square$

**Theorem 24.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $f, g, k, h : [a, b] \rightarrow \mathbb{R}$  be  $\mu$ -integrable functions such that  $k$  is positive,  $h$  is non-negative and

$$\int_{[x,c]} k(t)g(t)d\mu(t) \leq \int_{[x,c]} k(t)h(t)d\mu(t) \quad \text{and} \quad \int_{[a,c]} k(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a, c] \quad (62)$$

and

$$\int_{[c,x]} k(t)g(t)d\mu(t) \leq \int_{[c,x]} k(t)h(t)d\mu(t) \quad \text{and} \quad \int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [c, b].$$

Let  $\lambda_1$  be a positive constant such that (49) holds and let  $\lambda_2$  be a positive constant such that (50) holds. If  $f/h \in \mathcal{M}_1^c[a, b]$  and (51) holds, then (52) holds.

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (51) holds, the inequality in (52) is reversed.

**Proof.** Similar to the proof of Theorem 12 using the weaker conditions proved in ([11], Theorem 3.1).  $\square$

Replacing the condition  $0 \leq g \leq 1$  in Theorems 13 and 14 by the weaker one we obtain the following theorems.

**Theorem 25.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h : [a, b] \rightarrow \mathbb{R}$  be a positive  $\mu$ -integrable function, and let  $f, g : [a, b] \rightarrow \mathbb{R}$  be  $\mu$ -integrable functions such that

$$\int_{[a,x]} h(t)g(t)d\mu(t) \leq \int_{[a,x]} h(t)d\mu(t) \quad \text{and} \quad \int_{[x,c]} h(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a, c] \quad (63)$$

and

$$\int_{[x,b]} h(t)g(t)d\mu(t) \leq \int_{[x,b]} h(t)d\mu(t) \quad \text{and} \quad \int_{[c,x]} h(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [c, b].$$

Let  $\lambda_1$  be a positive constant such that (29) holds and let  $\lambda_2$  be a positive constant such that (30) holds. If  $f/h \in \mathcal{M}_1^c[a, b]$  and (31) holds, then (32) holds.

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (31) holds, the inequality in (32) is reversed.

**Proof.** Similar to the proof of Theorem 11 using modification of the weaker conditions proved in [11].  $\square$

**Theorem 26.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $h : [a, b] \rightarrow \mathbb{R}$  be a positive  $\mu$ -integrable function and let  $f, g : [a, b] \rightarrow \mathbb{R}$  be  $\mu$ -integrable functions such that

$$\int_{[x,c]} h(t)g(t)d\mu(t) \leq \int_{[x,c]} h(t)d\mu(t) \quad \text{and} \quad \int_{[a,x]} h(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a, c] \quad (64)$$

and

$$\int_{[c,x]} h(t)g(t)d\mu(t) \leq \int_{[c,x]} h(t)d\mu(t) \quad \text{and} \quad \int_{[x,b]} h(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [c, b]. \quad (65)$$

Let  $\lambda_1$  be a positive constant such that (35) holds, and let  $\lambda_2$  be a positive constant such that (36) holds. If  $f/h \in \mathcal{M}_1^c[a, b]$  and (37) holds then (38) holds.

If  $f/h \in \mathcal{M}_2^c[a, b]$  and (37) holds, the inequality in (38) is reversed.

**Proof.** Similar to the proof of Theorem 12 using modification of the weaker conditions proved in [11]. □

**Remark 2.** In a similar way we can obtain the weaker conditions for refinements given in Theorems 19 and 20. Further, we can obtain weaker conditions for sharpened and generalized Steffensen type inequalities given in Theorems 21 and 22.

### 3. Concluding Remarks

Results obtained in this paper hold not only for the functions which are convex in point, but also for convex functions.

If  $f$  is a convex function on  $[a, b]$ , from Remark 1 we have that  $f$  is convex in every point  $c \in (a, b)$ , i.e.,  $f \in \mathcal{M}_1^c[a, b]$  for every  $c \in (a, b)$ . Therefore, from Theorem 9 we obtain that for convex functions the following theorem holds.

**Corollary 1.** Let  $\mu$  be a positive, finite measure on  $\mathcal{B}([a, b])$  and let  $c \in (a, b)$ . Let  $f$  and  $g$  be measurable functions on  $[a, b]$  such that  $0 \leq g \leq 1$ . Let  $\lambda_1$  be a positive constant such that

$$\mu([a, a + \lambda_1]) = \int_{[a,c]} g(t) d\mu(t)$$

and let  $\lambda_2$  be a positive constant such that

$$\mu((b - \lambda_2, b]) = \int_{[c,b]} g(t) d\mu(t).$$

If  $f$  is convex function and

$$\int_{[a,b]} tg(t) d\mu(t) = \int_{[a,a+\lambda_1]} td\mu(t) + \int_{(b-\lambda_2,b]} td\mu(t), \tag{66}$$

then

$$\int_{[a,b]} f(t)g(t) d\mu(t) \leq \int_{[a,a+\lambda_1]} f(t) d\mu(t) + \int_{(b-\lambda_2,b]} f(t) d\mu(t). \tag{67}$$

If  $f$  is concave function and (66) holds, the inequality in (67) is reversed.

Therefore, taking convex function  $f$  (or convex function  $f/h$ ) instead of  $f \in \mathcal{M}_1^c[a, b]$  (or  $f/h \in \mathcal{M}_1^c[a, b]$ ) in all theorems proved in Section 2 we can obtain Steffensen type inequalities for convex functions on Borel  $\sigma$ -algebra.

Further, let us show that the condition (15) in Theorem 9 can be weakened.

For  $f \in \mathcal{M}_1^c[a, b]$  from the proof of Theorem 9 we see that we can replace the condition (15) by

$$A \left( \int_{[a,a+\lambda_1]} td\mu(t) + \int_{(b-\lambda_2,b]} td\mu(t) - \int_{[a,b]} tg(t) d\mu(t) \right) \geq 0, \tag{68}$$

where  $A$  is the constant from the Definition 1.

In [20], the following property was proved:

**Remark 3.** If  $f \in \mathcal{M}_1^c[a, b]$  or  $f \in \mathcal{M}_2^c[a, b]$  and  $f'(c)$  exists, then  $f'(c) = A$

Using Remark 3, for a non-decreasing function  $f$  it holds that  $A = f'(c) \geq 0$ , so the condition (68) can be further weakened to

$$\int_{[a,a+\lambda_1]} td\mu(t) + \int_{(b-\lambda_2,b]} td\mu(t) \geq \int_{[a,b]} tg(t) d\mu(t). \tag{69}$$

Therefore, taking non-decreasing function  $f \in \mathcal{M}_1^c[a, b]$  in Theorem 9 the condition (15) can be replaced by the weaker condition (69).

By similar reasoning, taking non-decreasing functions  $f \in \mathcal{M}_1^c[a, b]$  (or non-decreasing functions  $f/h \in \mathcal{M}_1^c[a, b]$ ) in other theorems proved in Section 2 we can obtain results with weaker conditions.

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## References

1. Anderson, D.R. Time-scale integral inequalities. *J. Inequal. Pure Appl. Math.* **2005**, *6*, 16.
2. Cerone, P. Special functions: approximations and bounds. *Appl. Anal. Discret. Math.* **2007**, *1*, 72–91.
3. Choczewski, B.; Corovei, I.; Matkowska, A. On some functional equations related to Steffensen's inequality. *Ann. Univ. Paedagog. Crac. Stud. Math.* **2004**, *4*, 31–37.
4. El-Deeb, A.A.; Ahmed, H.; Awrejcewicz, J. New Fractional Dynamic Inequalities via Conformable Delta Derivative on Arbitrary Time Scales. *Symmetry* **2021**, *13*, 2049. [[CrossRef](#)]
5. Evard, J.C.; Gauchman, H. Steffensen type inequalities over general measure spaces. *Analysis* **1997**, *17*, 301–322. [[CrossRef](#)]
6. Fink, A.M. Steffensen type inequalities. *Rocky Mt. J. Math.* **1982**, *12*, 785–793. [[CrossRef](#)]
7. Gauchman, H. A Steffensen type inequality. *J. Inequal. Pure Appl. Math.* **2000**, *1*, 3.
8. Gauchman, H. On further generalization of Steffensen's inequality. *J. Inequal. Appl.* **2000**, *5*, 505–513. [[CrossRef](#)]
9. Jakšetić, J.; Pečarić, J. Steffensen's inequality for positive measures. *Math. Inequal. Appl.* **2015**, *18*, 1159–1170. [[CrossRef](#)]
10. Jakšetić, J.; Pečarić, J.; Smoljak Kalamir, K. Some measure theoretic aspects of Steffensen's and reversed Steffensen's inequality. *J. Math. Inequal.* **2016**, *10*, 459–469. [[CrossRef](#)]
11. Jakšetić, J.; Pečarić, J.; Smoljak Kalamir, K. Measure theoretic generalization of Pečarić, Mercer and Wu-Srivastava results. *Sarajevo J. Math.* **2016**, *12*, 33–49. [[CrossRef](#)]
12. Ozkan, U.M.; Yildirim, H. Steffensen's integral inequality on time scales. *J. Inequal. Appl.* **2007**, *2007*, 46524. [[CrossRef](#)]
13. Jakšetić, J.; Pečarić, J.; Perušić Pribanić, A.; Smoljak Kalamir, K. *Weighted Steffensen's Inequality (Recent Advances in Generalizations of Steffensen's Inequality)*; Monographs in Inequalities 17; Element: Zagreb, Croatia, 2020.
14. Pečarić, J.; Smoljak Kalamir, K.; Varošaneć, S. *Steffensen's and Related Inequalities (A Comprehensive Survey and Recent Advances)*; Monographs in Inequalities 7; Element: Zagreb, Croatia, 2014.
15. Steffensen, J.F. On certain inequalities between mean values and their application to actuarial problems. *Skand. Aktuarietids* **1918**, *1918*, 82–97. [[CrossRef](#)]
16. Pečarić, J.; Smoljak, K. Steffensen type inequalities involving convex functions. *Math. Inequal. Appl.* **2015**, *18*, 363–378.
17. Pečarić, J.E. Notes on some general inequalities. *Publ. Inst. Math.* **1982**, *32*, 131–135.
18. Mercer, P.R. Extensions of Steffensen's inequality. *J. Math. Anal. Appl.* **2000**, *246*, 325–329. [[CrossRef](#)]
19. Wu, S.H.; Srivastava, H.M. Some improvements and generalizations of Steffensen's integral inequality. *Appl. Math. Comput.* **2007**, *192*, 422–428. [[CrossRef](#)]
20. Pečarić, J.; Smoljak Kalamir, K. Generalized Steffensen type inequalities involving convex functions. *J. Funct. Spaces* **2014**, *2014*, 428030. [[CrossRef](#)]
21. Pečarić, J.; Perušić, A.; Smoljak, K. Mercer and Wu-Srivastava generalisations of Steffensen's inequality. *Appl. Math. Comput.* **2013**, *219*, 10548–10558. [[CrossRef](#)]