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Jensen-Type Inequalities for $(h, g; m)$-Convex Functions

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Abstract: Jensen-type inequalities for the recently introduced new class of $(h, g; m)$-convex functions are obtained, and certain special results are indicated. These results generalize and extend corresponding inequalities for the classes of convex functions that already exist in the literature. Schur-type inequalities are given.

Keywords: convex function; Schur inequality; Jensen inequality

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1. Introduction

A convex function is one whose epigraph is a convex set, or, as in the basic definition:

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if

$$f(\lambda x + (1-\lambda)y) \;\leq\; \lambda f(x) + (1-\lambda) f(y) \tag{1}$$

holds for all points x and y in I and all $\lambda \in [0,1]$.

It is called strictly convex if the inequality (1) holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.

The following lemma is equivalent to the definition of a convex function.

Lemma 1 ([1], p. 2)**.** *Let $x_1, x_2, x_3 \in I$ be such that $x_1 < x_2 < x_3$. The function $f : I \to \mathbb{R}$ is convex if and only if the following inequality holds*

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0.$$

By mathematical induction, we can extend the inequality (1) to the convex combinations of finitely many points in I and next to random variables associated to arbitrary probability spaces. These extensions are known as the discrete Jensen inequality and the integral Jensen inequality, respectively.

Theorem 1 (The discrete Jensen inequality)**.** *A real-valued function f defined on an interval I is convex if and only if for all $x_1, \ldots, x_n$ in I and all scalars $\lambda_1, \ldots, \lambda_n$ in $[0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ we have*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \;\leq\; \sum_{i=1}^{n} \lambda_i f(x_i).$$

The above inequality is strict if f is strictly convex, all the points x_i are distinct and all scalars λ_i are positive.

Proving Jensen's inequality in a more general setting is the main motivation for this paper. We will use the recently introduced new class of convexity:

Definition 1 ([2]). *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and g be a positive function on $I \subseteq \mathbb{R}$. Let $m \in (0,1]$. A function $f : I \to \mathbb{R}$ is said to be $(h,g;m)$-convex function if it is nonnegative and if*

$$f(\lambda x + m(1-\lambda)y) \ \leq \ h(\lambda)f(x)g(x) + m\,h(1-\lambda)f(y)g(y) \tag{2}$$

holds for all $x, y \in I$ and all $\lambda \in (0,1)$.

If (2) holds in the reversed sense, then f is said to be $(h,g;m)$-concave function.

This class generalizes quite a number of different convexities which exist in the literature. For different choices of functions h, g and parameter m in (2), an $(h,g;m)$-convex function becomes P-function [3], h-convex function [4], m-convex function [5], $(h-m)$-convex function [6], (s,m)-Godunova–Levin function of the second kind [7], exponentially convex function [8], exponentially s-convex in the second sense [9], and so on. For example, setting $h(\lambda) = \lambda^s$, $s \in (0,1]$, $g(x) = e^{-\alpha x}$, $\alpha \in \mathbb{R}$, the $(h,g;m)$-convexity reduces to exponentially (s,m)-convexity in the second sense from [10]:

$$f(\lambda x + m(1-\lambda)y) \leq \frac{\lambda^s}{e^{\alpha x}} f(x) + \frac{(1-\lambda)^s}{e^{\alpha y}} m f(y). \tag{3}$$

More on properties of $(h,g;m)$-convex functions can be found in [2]. Here are a few of them:

Lemma 2 ([2]). *If $f : I \to [0,\infty)$ is an $(h,g;m)$-convex function such that $f(0) = 0$, $g(x) \leq 1$ and $h(\lambda) \leq \lambda$, then f is a starshaped, that is $f(\lambda x) \leq \lambda f(x)$.*

Proposition 1 ([2]). *Let h_1, h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1, h_2 \not\equiv 0$, such that*

$$h_2(\lambda) \leq h_1(\lambda), \quad \lambda \in (0,1).$$

Let g be a positive function on $I \subseteq \mathbb{R}$ and $m \in (0,1]$. If $f : I \to [0,\infty)$ is an $(h_2,g;m)$-convex function, then f is an $(h_1,g;m)$-convex.

Proposition 2 ([2]). *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and g be a positive function on $I \subseteq \mathbb{R}$. Let $m \in (0,1]$ and $\alpha > 0$. If $f_1, f_2 : I \to [0,\infty)$ are $(h,g;m)$-convex function, then $f_1 + f_2$ and αf_1 are $(h,g;m)$-convex.*

Proposition 3 ([2]). *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and g be a positive increasing function on $I \subseteq \mathbb{R}$. Let $0 < n < m \leq 1$. If $f : I \to [0,\infty)$ is an $(h,g;m)$-convex function such that $f(0) = 0$, $g(x) \leq 1$ and $h(\lambda) \leq \lambda$, then f is $(h,g;n)$-convex.*

Proposition 4 ([2]). *Let h_1, h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1, h_2 \not\equiv 0$ and let*

$$h(t) = \max\{h_1(t), h_2(t)\}, \quad t \in J.$$

Let g_1, g_2 be positive functions on $I \subseteq \mathbb{R}$ and let $m_1, m_2 \in (0,1]$. For $i = 1,2$, let $f_i : I \to [0,\infty)$ be an $(h_i, g_i; m_i)$-convex function. If the functions $f_1 g_1$ and $f_2 g_2$ are monotonic in the same sense, i.e.,

$$[f_1(x)g_1(x) - f_1(y)g_1(y)][f_2(x)g_2(x) - f_2(y)g_2(y)] \geq 0, \quad x, y \in I,$$

and if $c > 0$ such that

$$h(\lambda) + m\,h(1-\lambda) \leq c, \quad \lambda \in (0,1),$$

where $m = \max\{m_1, m_2\}$, then $f_1 f_2$ is an $(ch, g_1 g_2; m)$-convex function.

In recent work, we investigated the Hermite–Hadamard inequality for $(h, g; m)$-convex functions [2], its weighted version—the Féjer inequality [11] and Lah–Ribarič inequality from which the inequalities of Giaccardi, Popoviciu and Petrović for $(h, g; m)$-convex functions are obtained [12]. Here, we will obtain Schur-type inequalities in Section 2 and Jensen-type inequalities in Section 3 for $(h, g; m)$-convex functions, which will generalize and extend corresponding inequalities for the classes of convex functions that already exist in the literature. For this, we need super(sub)multiplicative functions:

Definition 2. *A function $h : J \to \mathbb{R}$ is said to be a supermultiplicative function if*

$$h(xy) \geq h(x)h(y) \tag{4}$$

for all $x, y \in J$.

If inequality (4) is reversed, then h is said to be a submultiplicative function. If the equality holds in (4), then h is said to be a multiplicative function.

2. Schur Type Inequalities

We start with a result related to the definition of $(h, g; m)$-convex functions.

Proposition 5. *Let f be a nonnegative $(h, g; m)$-convex function on $I \subseteq \mathbb{R}$, where h is a nonnegative supermultiplicative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$, g is a positive function on I and $m \in (0, 1]$. Then, for $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$ with $x_3 - x_2, x_2 - x_1, x_3 - x_1 \in J$ the following inequality holds*

$$h(x_3 - x_2)f(x_1)g(x_1) - h(x_3 - x_1)f(x_2) + m\,h(x_2 - x_1)f\Big(\frac{x_3}{m}\Big)g\Big(\frac{x_3}{m}\Big) \geq 0. \tag{5}$$

If f is an $(h, g; m)$-concave function where h is a submultiplicative function, then inequality (5) is reversed.

Proof. Let f be an $(h, g; m)$-convex function and $x_1, x_2, x_3 \in I$. From the assumptions, we have

$$\frac{x_3 - x_2}{x_3 - x_1} \in (0, 1) \subseteq J, \quad \frac{x_2 - x_1}{x_3 - x_1} \in (0, 1) \subseteq J$$

and

$$\frac{x_3 - x_2}{x_3 - x_1} + \frac{x_2 - x_1}{x_3 - x_1} = 1.$$

Since h is a supermultiplicative function and $x_3 - x_2, x_2 - x_1, x_3 - x_1 \in J$, we obtain

$$h(x_3 - x_2) = h\left(\frac{x_3 - x_2}{x_3 - x_1} \cdot (x_3 - x_1)\right) \geq h\left(\frac{x_3 - x_2}{x_3 - x_1}\right)h(x_3 - x_1)$$

and also

$$h(x_2 - x_1) \geq h\left(\frac{x_2 - x_1}{x_3 - x_1}\right)h(x_3 - x_1).$$

Assume $h(x_3 - x_1) > 0$. If we set in (2) $\lambda = \frac{x_3 - x_2}{x_3 - x_1}$, $x = x_1$, $y = x_3$, then we obtain

$$\begin{aligned} f(x_2) &= f\Big(\lambda x + m(1-\lambda)\frac{y}{m}\Big) \\ &\leq h(\lambda)f(x)g(x) + m\,h(1-\lambda)f\Big(\frac{y}{m}\Big)g\Big(\frac{y}{m}\Big) \\ &= h\left(\frac{x_3 - x_2}{x_3 - x_1}\right)f(x_1)g(x_1) + m\,h\left(\frac{x_2 - x_1}{x_3 - x_1}\right)f\Big(\frac{x_3}{m}\Big)g\Big(\frac{x_3}{m}\Big) \\ &\leq \frac{h(x_3 - x_2)}{h(x_3 - x_1)}f(x_1)g(x_1) + m\frac{h(x_2 - x_1)}{h(x_3 - x_1)}f\Big(\frac{x_3}{m}\Big)g\Big(\frac{x_3}{m}\Big). \end{aligned} \tag{6}$$

Hence, (5) is proven.

Analogously follows reversed inequality (5) if f is an $(h, g; m)$-concave function where h is a submultiplicative function. □

Recall the Schur inequality:

If x, y, z are positive numbers and if λ is real, then

$$x^{\lambda}(x-y)(x-z)+y^{\lambda}(y-z)(y-x)+z^{\lambda}(z-x)(z-y) \geq 0$$

with equality if and only if $x = y = z$.

This inequality follows from (5) for $f(x) = x^{\lambda}$, $\lambda \in \mathbb{R}$, $h(x) = \frac{1}{x}$, $g \equiv 1$ and $m = 1$.

A related inequality was proved in [13] by Mitrinović and Pečarić:

$$(x_1-x_2)(x_1-x_3)f(x_1)+(x_2-x_1)(x_2-x_3)f(x_2)+(x_3-x_1)(x_3-x_2)f(x_3) \geq 0$$

where f is a Godunova–Levin function that is an $(h, g; m) \equiv (x^{-1}, 1; 1)$-convex function:

$$f(\lambda x+(1-\lambda)y) \ \leq \ \frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}.$$

Next inequality is of Schur type for $(x^{-k}, g; m)$-convex (and concave) functions, obtained for $h(x) = \frac{1}{x^k}$, $k \in \mathbb{R}$:

Corollary 1. *Let f be a positive $(x^{-k}, g; m)$-convex function on $I \subseteq \mathbb{R}$, where $k \in \mathbb{R}$, g is a positive function on I and $m \in (0, 1]$. Then, for $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$, the following inequality holds*

$$\begin{aligned} f(x_1)g(x_1)(x_3-x_1)^k(x_2-x_1)^k - f(x_2)(x_3-x_2)^k(x_2-x_1)^k \\ + mf\left(\frac{x_3}{m}\right)g\left(\frac{x_3}{m}\right)(x_3-x_1)^k(x_3-x_2)^k \geq 0. \end{aligned} \tag{7}$$

If the function f is a positive $(x^{-k}, g; m)$-concave function, then inequality (7) is reversed.

As an example of a special case, if we set $h(x) = x^s$, $s \in (0, 1]$, $g(x) = e^{-\alpha x}$, $\alpha \in \mathbb{R}$, then we obtain following Schur type inequality for convexity (3), i.e., exponentially (s, m)-convex functions in the second sense.

Corollary 2. *Let f be an exponentially (s, m)-convex function in the second sense on $I \subseteq \mathbb{R}$, where $s, m \in (0, 1]$. Then, for $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$ the following inequality holds*

$$\frac{(x_3-x_2)^s}{e^{\alpha x_1}}f(x_1)-(x_3-x_1)^s f(x_2)+\frac{m(x_2-x_1)^s}{e^{\frac{\alpha}{m}x_3}}f\left(\frac{x_3}{m}\right) \geq 0. \tag{8}$$

If the function f is an exponentially (s, m)-concave function in the second sense, then inequality (8) is reversed.

Remark 1. *Using special functions for h and/or g, as well as choosing a fixed parameter for m, Schur-type inequalities for different types of convexity can be derived. For instance, setting $g \equiv 1$ and $m = 1$ in (5) and (7), we obtain results for h-convex functions given in [4].*

3. Jensen-Type Inequalities for $(h, g; m)$-Convex Functions

We continue with Jensen-type inequalities for $(h, g; m)$-convex functions, where h is supermultiplicative function. In the following, for $n \in \mathbb{N}$, let

$$P_n = \sum_{i=1}^{n} p_i, \tag{9}$$

$$X_n = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \tag{10}$$

$$G_i^n = \prod_{j=i}^{n} g(X_j), \quad i \geq 1. \tag{11}$$

We will set empty products equal to 1, for example $G_{n+1}^n = \prod_{j=n+1}^{n} g(X_j) \equiv 1$.

Notice that $P_1 = p_1$, $X_1 = x_1$ and $G_n^n = g(X_n)$. The following recursive formulas hold

$$G_i^n = g(X_i) \cdot G_{i+1}^n, \quad i = 1, \ldots, n, \tag{12}$$

$$G_i^n = G_i^{n-1} \cdot g(X_n), \quad i = 1, \ldots, n. \tag{13}$$

Theorem 2 (The Jensen inequality for an $(h, g; m)$-convex function)**.** *Let $p_1, \ldots, p_n$ be positive real numbers. Let f be a nonnegative $(h, g; m)$-convex function on $[0, \infty)$ such that $I \subseteq [0, \infty)$, where h is a nonnegative supermultiplicative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Then, for $x_1, \ldots, x_n \in I$ the following inequality holds*

$$\begin{aligned} f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \;\; &\leq \;\; h\left(\frac{p_1}{P_n}\right) f(x_1) G_1^{n-1} \\ &+ m \sum_{i=2}^{n} h\left(\frac{p_i}{P_n}\right) f\left(\frac{x_i}{m}\right) g\left(\frac{x_i}{m}\right) G_i^{n-1}. \end{aligned} \tag{14}$$

If f is an $(h, g; m)$-concave function where h is a submultiplicative function, then inequality (14) is reversed.

Proof. We will prove the theorem by the mathematical induction.

If $n = 2$, then (14) is equivalent to (2) with $\lambda = \frac{p_1}{P_2}$, $1 - \lambda = \frac{p_2}{P_2}$, $x = x_1$ and $y = \frac{x_2}{m}$ (notice, $G_1^1 = g(X_1) = g(x_1)$ and $G_2^1 \equiv 1$).

Assume that (14) holds for $n - 1$. Then, for $p_1, \ldots, p_n$ and $x_1, \ldots, x_n$, we have

$$\begin{aligned} f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) &= f\left(m \frac{p_n}{P_n} \frac{x_n}{m} + \frac{P_{n-1}}{P_n} \sum_{i=1}^{n-1} \frac{p_i}{P_{n-1}} x_i\right) \\ &\leq m\, h\left(\frac{p_n}{P_n}\right) f\left(\frac{x_n}{m}\right) g\left(\frac{x_n}{m}\right) \\ &\quad + h\left(\frac{P_{n-1}}{P_n}\right) f\left(\sum_{i=1}^{n-1} \frac{p_i}{P_{n-1}} x_i\right) g\left(\sum_{i=1}^{n-1} \frac{p_i}{P_{n-1}} x_i\right) \\ &\leq m\, h\left(\frac{p_n}{P_n}\right) f\left(\frac{x_n}{m}\right) g\left(\frac{x_n}{m}\right) \\ &\quad + h\left(\frac{P_{n-1}}{P_n}\right) \left[h\left(\frac{p_1}{P_{n-1}}\right) f(x_1) G_1^{n-2} \right. \\ &\quad \left. + m \sum_{i=2}^{n-1} h\left(\frac{p_i}{P_{n-1}}\right) f\left(\frac{x_i}{m}\right) g\left(\frac{x_i}{m}\right) G_i^{n-2} \right] g(X_{n-1}). \end{aligned}$$

Since h is a supermultiplicative function, we obtain

$$
\begin{aligned}
f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq\ & m\, h\left(\frac{p_n}{P_n}\right) f\left(\frac{x_n}{m}\right) g\left(\frac{x_n}{m}\right) \\
& + h\left(\frac{p_1}{P_n}\right) f(x_1) g(X_{n-1}) G_1^{n-2} \\
& + m \sum_{i=2}^{n-1} h\left(\frac{p_i}{P_n}\right) f\left(\frac{x_i}{m}\right) g\left(\frac{x_i}{m}\right) g(X_{n-1}) G_i^{n-2}.
\end{aligned}
$$

Now, we apply the recursive formula (13) to find inequality (14). □

Remark 2. *As before, if we use special h, g and m in (14), then we obtain Jensen-type inequalities for different types of convexity. Hence, Theorem 2 is a generalization of Jensen's inequality for h-convex functions given in [4].*

The last result is a conversion of Jensen's inequality.

Theorem 3. *Let $p_1, \dots, p_n$ be a positive real numbers and $[\mu, M] \subseteq [0, \infty)$. Let f be a nonnegative $(h, g; m)$-convex function on $[0, \infty)$, where h is a nonnegative supermultiplicative function on $(0, \infty)$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Then, for $x_i \in (\mu, M)$ and $M - \frac{x_i}{m} > 0$ $(i = 1, \dots, n)$, the following inequalities hold*

$$
\begin{aligned}
& f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - h\left(\frac{p_1}{P_n}\right) f(x_1) G_1^{n-1} \\
\leq\ & m \sum_{i=2}^{n} h\left(\frac{p_i}{P_n}\right) f\left(\frac{x_i}{m}\right) g\left(\frac{x_i}{m}\right) G_i^{n-1} \\
\leq\ & m \sum_{i=2}^{n} h\left(\frac{p_i}{P_n}\right) \left[h\left(\frac{M - \frac{x_i}{m}}{M - \mu}\right) f(\mu) g(\mu) \right. \\
& \left. + m\, h\left(\frac{\frac{x_i}{m} - \mu}{M - \mu}\right) f\left(\frac{M}{m}\right) g\left(\frac{M}{m}\right) \right] g\left(\frac{x_i}{m}\right) G_i^{n-1}. \qquad (15)
\end{aligned}
$$

If f is an $(h, g; m)$-concave function where h is a submultiplicative function, then inequality (15) is reversed.

Proof. From (6) in Proposition 5, we have

$$
f(x_2) \leq h\left(\frac{x_3 - x_2}{x_3 - x_1}\right) f(x_1) g(x_1) + m\, h\left(\frac{x_2 - x_1}{x_3 - x_1}\right) f\left(\frac{x_3}{m}\right) g\left(\frac{x_3}{m}\right),
$$

which gives us for $x_1 = \mu$, $x_2 = \frac{x_i}{m}$ and $x_3 = M$ for $i = 2, \dots, n$

$$
f\left(\frac{x_i}{m}\right) \leq h\left(\frac{M - \frac{x_i}{m}}{M - \mu}\right) f(\mu) g(\mu) + m\, h\left(\frac{\frac{x_i}{m} - \mu}{M - \mu}\right) f\left(\frac{M}{m}\right) g\left(\frac{M}{m}\right).
$$

Notice, since $m \leq 1$, then $\mu < \frac{x_i}{m}$, and by that assumption, we have $\frac{x_i}{m} < M$. With this, the h function can be applied.

If we multiply the above with

$$
m\, h\left(\frac{p_i}{P_n}\right) g\left(\frac{x_i}{m}\right) G_i^{n-1},
$$

then, after adding all inequalities, from Theorem 2, (15) follows. □

Remark 3. *Corresponding conversions of Jensen's inequality for different types of convexity can be stated. However, the details are omitted.*

If, in (6), we let $x_1 = \mu$, $x_2 = x_i$, $x_3 = M$ and if we multiply such inequality with p_i, then after adding all inequalities for $i = 1, \dots, n$ we obtain the discrete Lah–Ribarič inequality for an $(h, g; m)$ convex function

$$\sum_{i=1}^{n} p_i f(x_i) \leq \frac{f(\mu)g(\mu)}{h(M-\mu)} \sum_{i=1}^{n} p_i h(M - x_i) + \frac{m}{h(M-\mu)} f\left(\frac{M}{m}\right) g\left(\frac{M}{m}\right) \sum_{i=1}^{n} p_i h(x_i - \mu).$$

Integral version of this inequality is given in [12] where one can also find more on how to obtain the inequalities of Giaccardi, Popoviciu and Petrović for $(h, g; m)$-convex functions.

4. Conclusions

This article is a continuation of our work given in paper [2], where we first introduced a new class of $(h, g; m)$-convex functions. Thus far, we have investigated inequalities of Hermite–Hadamard and Féjer type as well as the Lah–Ribarič inequality with inequalities of Giaccardi, Popoviciu and Petrovič [2,11,12]. In this paper, Schur- and Jensen-type inequalities for $(h, g; m)$-convex functions are obtained. The goal is to generalize and extend corresponding inequalities for the classes of convex functions that already exist in the literature. We have opened research that we will further explore in future work.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

References

1. Pečarić, J.E.; Proschan, F.; Tong, Y.C. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press, Inc.: San Diego, CA, USA, 1992.
2. Andrić, M.; Pečarić, J. On $(h, g; m)$-convexity and the Hermite–Hadamard inequality. *J. Convex Anal.* **2021**, accepted.
3. Dragomir, S.S.; Pečarić, J.E.; Persson, L.E. Some inequalities of Hadamard type. *Soochow J. Math.* **1995**, *21*, 335–341.
4. Varošanec, S. On h-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
5. Toader, G.H. Some generalizations of the convexity. *Proc. Colloq. Approx. Optim Cluj-Napoca (Rom.)* **1984**, *329*, 338.
6. Özdemir, M.E.; Akdemir, A.O.; Set, E. On $(h - m)$-convexity and Hadamard-type inequalities. *Transylv. J. Math. Mech.* **2016**, *8*, 51–58.
7. Noor, M.A.; Noor, K.I.; Awan, M.U. Fractional Ostrowski inequalities for (s, m)-Godunova–Levin functions. *Facta Univ. Ser. Math. Inform.* **2015**, *30*, 489–499.
8. Awan, M.U.; Noor, M.A.; Noor, K.I. Hermite–Hadamard inequalities for exponentially convex functions. *Appl. Math. Inf. Sci.* **2018**, *21*, 405–409. [CrossRef]
9. Mehreen, N.; Anwar, M. Hermite–Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications. *J. Inequal. Appl.* **2019**, *2019*, 92. [CrossRef]

10. Qiang, X.; Farid, G.; Pečarić, J.; Akbar, S.B. Generalized fractional integral inequalities for exponentially (s, m)-convex functions. *J. Inequal. Appl.* **2020**, *2020*, 70. [CrossRef]
11. Andrić, M. Fejér type inequalities for $(h, g; m)$-convex functions. *TWMS J. Pure Appl. Math.* **2021**, accepted.
12. Andrić, M.; Pečarić, J. Lah–Ribarič type inequalities for $(h, g; m)$-convex functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2022**, *116*, 39. [CrossRef]
13. Mitrinović, D.S.; Pečarić, J.E. Note on a class of fucntions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Can.* **1990**, *12*, 33–36.