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Article

An Improved Alternating CQ Algorithm for Solving Split Equality Problems

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Abstract: The CQ algorithm is widely used in the scientific field and has a significant impact on phase retrieval, medical image reconstruction, signal processing, etc. Moudafi proposed an alternating CQ algorithm to solve the split equality problem, but he only obtained the result of weak convergence. The work of this paper is to improve his algorithm so that the generated iterative sequence can converge strongly.

Keywords: split equality problem; alternating CQ algorithm; strong convergence; split feasibility problem

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1. Introduction

Let $C \subseteq \mathcal{H}_1$, $Q \subseteq \mathcal{H}_2$ be two nonempty closed convex subsets, $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces. For all $b \in \mathcal{H}_1$, $d \in \mathcal{H}_2$, the equality $\langle Ab, d\rangle = \langle b, A^*d\rangle$ is true, A^* is called the adjoint operator of A. The split feasibility problem (SFP) can be described as finding $b^\dagger \in C$ such that

$$Ab^\dagger \in Q, \tag{1}$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear bounded operator.

The SFP was first proposed by Censor and Elfving [1]. It is used to model the inverse problems of phase retrieval and medical image reconstruction in finite-dimensional Hilbert spaces. It has a significant impact on signal processing, image reconstruction and radiation therapy, see [2–4]. The following CQ algorithm proposed by Byrne [4] is an important method to solve the SFP

$$u_{n+1} = P_C(u_n + \rho A^*(P_Q - I)Au_n), n \geq 0 \tag{2}$$

where $\rho \in (0, \frac{2}{\lambda})$, λ represents the largest eigenvalue of the operator A^*A. Recently, many other algorithms have appeared to solve problem (1), for example, [5–8].

Let $\{C_i\}_i^p \subseteq \mathcal{H}_1$ and $\{Q_j\}_{j=1}^r \subseteq \mathcal{H}_2$ be nonempty closed convex subsets, $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces, $p \geq 1$ and $r \geq 1$ are two non-negative integers. $\mathcal{H}_3$ is also a real Hilbert space. The multiple-sets split equality problem (MSSEP) can be described as finding $b^\dagger \in C := \bigcap_{i=1}^p C_i$, $d^\dagger \in Q := \bigcap_{j=1}^r Q_j$ such that

$$Ab^\dagger = Bd^\dagger \tag{3}$$

where $B : \mathcal{H}_2 \to \mathcal{H}_3$ and $A : \mathcal{H}_1 \to \mathcal{H}_3$ are two linear bounded operators.

Remark 1. *When $B = I$, the MSSEP is reduced to an MSSFP. The MSSFP is widely used in intensity-modulated radiation therapy (IMRT) [9–13], image reconstruction [14–16], signal processing [17–21]. Recently, many other algorithms have appeared to solve the MSSFP, see [22–24].*

Remark 2. *When $p = r = 1$, the MSSEP is reduced to a split equality problem (SEP).*

The SEP can be described as finding $b^\dagger \in C$, $d^\dagger \in Q$ such that

$$Ab^\dagger = Bd^\dagger \tag{4}$$

The SEP is applied to optimal control and approximation theory [25], in intensity-modulated radiation therapy (IMRT) [26] and game theory [27]. Byrne [28] proposed the following Landweber projection algorithm to study the SEP:

$$(PLA)\begin{cases} u_{n+1} = P_C(u_n - \rho_n A^*(Au_n - Bv_n)) \\ v_{n+1} = P_Q(v_n + \rho_n B^*(Au_n - Bv_n)). \end{cases} \tag{5}$$

Different from Byrne's algorithm, Moudafi [29] proposed the following alternating CQ algorithm

$$(ACQA)\begin{cases} u_{n+1} = P_C(u_n - \rho_n A^*(Au_n - Bv_n)) \\ v_{n+1} = P_Q(v_n + \rho_n B^*(Au_{n+1} - Bv_n)). \end{cases} \tag{6}$$

However, Moudafi only obtained the result of weak convergence. Inspired by this work, we propose an improved alternating CQ algorithm to solve the SEP. This improved method changes the iterative sequence from weak to strong convergence.

The structure of this article is as follows. In Section 2, we review some of the definitions, properties, and lemmas used to prove the convergence of the method. In Section 3, we propose a new algorithm and prove its strong convergence. In Section 4, at the end of the article, we reach a conclusion.

2. Preliminaries

We define the strong convergence of sequence $\{u_n\}_{n\in N}$ as $u_n \to b$ and weak convergence as $u_n \rightharpoonup b$, $b \in \mathcal{H}$. Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset, $\mathcal{H}$ is a real Hilbert space, $\forall b \in \mathcal{H}$, the orthogonal projection from $\mathcal{H}$ to C is defined by

$$P_C(b) = \arg\min_{z\in C} \|b - z\|.$$

Definition 1 ([30])**.** *Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset, $\mathcal{H}$ is a real Hilbert space, for all $b^\dagger, d^\dagger \in \mathcal{H}$ and $z^\dagger \in C$, we have*

1. $\langle b^\dagger - P_C b^\dagger, z^\dagger - P_C b^\dagger\rangle \le 0$;
2. $\|P_C b^\dagger - P_C d^\dagger\|^2 \le \langle P_C b^\dagger - P_C d^\dagger, b^\dagger - d^\dagger\rangle$;
3. $\|P_C b^\dagger - z^\dagger\|^2 \le \|b^\dagger - z^\dagger\|^2 - \|P_C b^\dagger - b^\dagger\|^2$.
4. $\|P_C b^\dagger - P_C d^\dagger\|^2 \le \|b^\dagger - d^\dagger\|^2 - \|(I - P_C)(b^\dagger) - (I - P_C)(d^\dagger)\|^2$.

Lemma 1 ([31])**.** *For all $\tilde{b}, d^\dagger \in \mathcal{H}$, $\mathcal{H}$ is a real Hilbert space, we have*

$$\|\tilde{b} + d^\dagger\|^2 = \|\tilde{b}\|^2 + \|d^\dagger\|^2 + 2\langle \tilde{b}, d^\dagger\rangle$$

$$\|\tilde{b} - d^\dagger\|^2 = \|\tilde{b}\|^2 + \|d^\dagger\|^2 - 2\langle \tilde{b}, d^\dagger\rangle$$

$$\|\tilde{b} + d^\dagger\|^2 \le \| \tilde{b} \|^2 + 2\langle d^\dagger, \tilde{b} + d^\dagger\rangle$$

$$\|\tilde{b}\| - \|d^\dagger\| \le \|\tilde{b} + d^\dagger\| \le \|\tilde{b}\| + \|d^\dagger\|.$$

Lemma 2 ([32])**.** *For all $n \ge 0$, assume that the three sequences $\{\alpha_n\}$, $\{\rho_n\}$, $\{\delta_n\}$ satisfy the following conditions:*

1. $\alpha_n \geq 0$;
2. $\{\delta_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \delta_n = \infty$;
3. $\limsup_{n\to\infty} \rho_n \leq 0$;
4. $\alpha_{n+1} \leq (1-\delta_n)\alpha_n + \delta_n \rho_n$.

Then, the following conclusion holds:

$$lim_{n\to\infty}\alpha_n = 0.$$

3. Main Results

Let the solution set of problem (4) given by $\Omega = \{b^\dagger \in C, d^\dagger \in Q; Ab^\dagger = Bd^\dagger\}$. We propose the following new alternating CQ algorithm to solve problem (4):

$$\begin{cases} u_{n+1} = P_C((1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n))) \\ v_{n+1} = P_Q((1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n))) \end{cases} \tag{7}$$

Assume that a_0 and b_0 are two given points, the sequence $\{\alpha_n\}$ satisfies $\{\alpha_n\}_{n\geq 0} \in (0,1)$, $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$ and $\lim_{n\to\infty} \alpha_n = 1$. Below, we prove the strong convergence of the sequence generated by Equation (7).

Theorem 1. *The sequence $\{(u_n, v_n)\}$ is generated by Equation (7), the sequence $\{\rho_n\}$ is positive and non-increasing, for a sufficiently small $\varepsilon > 0$, $\rho_n \in (\varepsilon, min(\frac{1}{\rho(A^*A)}, \frac{1}{\rho(B^*B)}) - \varepsilon)$. $\rho(A^*A)$, $\rho(B^*B)$ are the spectral radius of A^*A and B^*B, respectively. Then, the sequence $\{(u_n, v_n)\}$ strongly converges to a solution $(b^\ddagger, d^\ddagger)$ of Equation (4).*

Proof. Let $(b^\dagger, d^\dagger) \in \Omega$, which is, $b^\dagger \in C$, $d^\dagger \in Q$, $Ab^\dagger = Bd^\dagger$. According to (4) of Definition 1 and Lemma 1, on the one hand, we have

$$\begin{aligned}
&\|u_{n+1} - b^\dagger\|^2 \\
&\leq \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - b^\dagger\|^2 \\
&- \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - u_{n+1}\|^2 \\
&= \|(1-\alpha_n)(a_0 - b^\dagger) + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n) - b^\dagger)\|^2 \\
&- \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - u_{n+1}\|^2 \\
&\leq (1-\alpha_n)\|(a_0 - b^\dagger)\|^2 + \alpha_n\|u_n - \rho_n A^*(Au_n - Bv_n)) - b^\dagger\|^2 \\
&- \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - u_{n+1}\|^2
\end{aligned} \tag{8}$$

It follows that

$$\begin{aligned}
&\|u_{n+1} - b^\dagger\|^2 \\
&\leq (1-\alpha_n)\|(a_0 - b^\dagger)\|^2 + \alpha_n\|u_n - b^\dagger\|^2 \\
&+ \alpha_n\rho_n^2\|A^*(Au_n - Bv_n)\|^2 - 2\alpha_n\rho_n\langle A^*(Au_n - Bv_n), u_n - b^\dagger\rangle \\
&- \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - u_{n+1}\|^2
\end{aligned} \tag{9}$$

We consider first

$$\begin{aligned}
\alpha_n\rho_n^2\|A^*(Au_n - Bv_n)\|^2 &= \alpha_n\rho_n^2\langle Au_n - Bv_n, AA^*(Au_n - Bv_n)\rangle \\
&\leq \alpha_n\rho(A^*A)\rho_n^2\langle Au_n - Bv_n, Au_n - Bv_n\rangle \\
&= \alpha_n\rho(A^*A)\rho_n^2\|Au_n - Bv_n\|^2
\end{aligned} \tag{10}$$

Then, we consider

$$\begin{aligned}&2\alpha_n\rho_n\langle A^*(Au_n - Bv_n), u_n - b^\dagger\rangle \\&= 2\alpha_n\rho_n\langle Au_n - Bv_n, Au_n - Ab^\dagger\rangle \\&= 2\alpha_n\rho_n(\|Au_n - Bv_n\|^2 + \langle Au_n - Bv_n, Bv_n - Ab^\dagger\rangle)\end{aligned} \tag{11}$$

Then, Equation (9) becomes

$$\begin{aligned}\|u_{n+1} - b^\dagger\|^2 \leq{}& (1-\alpha_n)\|(a_0 - b^\dagger)\|^2 + \alpha_n\|u_n - b^\dagger\|^2 \\&- 2\alpha_n\rho_n\langle Au_n - Bv_n, Bv_n - Ab^\dagger\rangle \\&- \alpha_n\rho_n(2 - \rho_n\rho(A^*A))\|Au_n - Bv_n\|^2 \\&- \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - u_{n+1}\|^2\end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned}&\|v_{n+1} - d^\dagger\|^2 \\&\leq \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - d^\dagger\|^2 \\&- \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - v_{n+1}\|^2 \\&= \|(1-\alpha_n)(b_0 - d^\dagger) + \alpha_n(v_n - d^\dagger + \rho_n B^*(Au_{n+1} - Bv_n))\|^2 \\&- \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - v_{n+1}\|^2 \\&\leq (1-\alpha_n)\|(b_0 - d^\dagger)\|^2 + \alpha_n\|v_n - d^\dagger + \rho_n B^*(Au_{n+1} - Bv_n)\|^2 \\&- \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - v_{n+1}\|^2\end{aligned} \tag{13}$$

It follows that

$$\begin{aligned}&\|v_{n+1} - d^\dagger\|^2 \\&\leq (1-\alpha_n)\|(b_0 - d^\dagger)\|^2 + \alpha_n\|v_n - d^\dagger\|^2 \\&+ \alpha_n\rho_n^2\|B^*(Au_{n+1} - Bv_n)\|^2 + 2\alpha_n\rho_n\langle B^*(Au_{n+1} - Bv_n), v_n - d^\dagger\rangle \\&- \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - v_{n+1}\|^2\end{aligned} \tag{14}$$

We have

$$\begin{aligned}\alpha_n\rho_n^2\|B^*(Au_{n+1} - Bv_n)\|^2 &= \alpha_n\rho_n^2\langle Au_{n+1} - Bv_n, BB^*(Au_{n+1} - Bv_n)\rangle \\&\leq \alpha_n\rho(B^*B)\rho_n^2\langle Au_{n+1} - Bv_n, Au_{n+1} - Bv_n\rangle \\&= \alpha_n\rho(B^*B)\rho_n^2\|Au_{n+1} - Bv_n\|^2\end{aligned} \tag{15}$$

At the same time, we have

$$\begin{aligned}2\alpha_n\rho_n\langle B^*(Au_{n+1} - Bv_n), v_n - d^\dagger\rangle &= 2\alpha_n\rho_n\langle Au_{n+1} - Bv_n, Bv_n - Bd^\dagger\rangle \\&= -2\alpha_n\rho_n(\|Au_{n+1} - Bv_n\|^2 \\&- \langle Au_{n+1} - Bv_n, Au_{n+1} - Bd^\dagger\rangle)\end{aligned} \tag{16}$$

Then, Equation (14) becomes

$$\begin{aligned}\|v_{n+1} - d^\dagger\|^2 \leq{}& (1-\alpha_n)\|(b_0 - d^\dagger)\|^2 + \alpha_n\|v_n - d^\dagger\|^2 \\&+ 2\alpha_n\rho_n\langle Au_{n+1} - Bv_n, Au_{n+1} - Bd^\dagger\rangle \\&- \alpha_n\rho_n(2 - \rho_n\rho(B^*B))\|Au_{n+1} - Bv_n\|^2 \\&- \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - v_{n+1}\|^2\end{aligned} \tag{17}$$

We have

$$\begin{aligned} &2\langle Au_n - Bv_n, Bv_n - Ab^\dagger\rangle \\ &= -\|Au_n - Bv_n\|^2 - \|Bv_n - Ab^\dagger\|^2 + \|Au_n - Ab^\dagger\|^2 \end{aligned} \tag{18}$$

and

$$\begin{aligned} &2\langle Bv_n - Au_{n+1}, Au_{n+1} - Bd^\dagger\rangle \\ &= -\|Bv_n - Au_{n+1}\|^2 - \|Au_{n+1} - Bd^\dagger\|^2 + \|Bv_n - Bd^\dagger\|^2 \end{aligned} \tag{19}$$

In the light of $Ab^\dagger = Bd^\dagger$, combining equalities (18) and (19), adding Equations (12) and (17) together, we finally obtain

$$\begin{aligned} &\|u_{n+1} - b^\dagger\|^2 + \|v_{n+1} - d^\dagger\|^2 \\ &\le (1-\alpha_n)\|(a_0 - b^\dagger)\|^2 + (1-\alpha_n)\|(b_0 - d^\dagger)\|^2 \\ &+ \alpha_n\|u_n - b^\dagger\|^2 + \alpha_n\|v_n - d^\dagger\|^2 - \alpha_n\rho_n\|Au_n - Ab^\dagger\|^2 \\ &+ \alpha_n\rho_{n+1}\|Au_{n+1} - Ab^\dagger\|^2 \\ &- \alpha_n\rho_n(1 - \rho_n\rho(A^*A))\|Au_n - Bv_n\|^2 \\ &- \alpha_n\rho_n(1 - \rho_n\rho(B^*B))\|Au_{n+1} - Bv_n\|^2 \\ &- \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - u_{n+1}\|^2 \\ &- \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - v_{n+1}\|^2 \end{aligned} \tag{20}$$

It follows that

$$\begin{aligned} &\|u_{n+1} - b^\dagger\|^2 + \|v_{n+1} - d^\dagger\|^2 \\ &\le (1-\alpha_n)(\|(a_0 - b^\dagger)\|^2 + \|(b_0 - d^\dagger)\|^2) \\ &+ \alpha_n(\|u_n - b^\dagger\|^2 + \|v_n - d^\dagger\|^2 - \rho_n\|Au_n - Ab^\dagger\|^2) \\ &+ \rho_{n+1}\|Au_{n+1} - Ab^\dagger\|^2 \\ &- \alpha_n\rho_n(1 - \rho_n\rho(A^*A))\|Au_n - Bv_n\|^2 \\ &- \alpha_n\rho_n(1 - \rho_n\rho(B^*B))\|Au_{n+1} - Bv_n\|^2 \\ &- \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - u_{n+1}\|^2 \\ &- \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - v_{n+1}\|^2 \end{aligned} \tag{21}$$

We assume $\Omega_n(b^\dagger, d^\dagger) := \|u_n - b^\dagger\|^2 + \|v_n - d^\dagger\|^2 - \rho_n\|Au_n - Ab^\dagger\|^2$, in view of (21), we then obtain the following result

$$\begin{aligned} \Omega_{n+1}(b^\dagger, d^\dagger) &\le \alpha_n\Omega_n(b^\dagger, d^\dagger) + (1-\alpha_n)(\|(a_0 - b^\dagger)\|^2 + \|(b_0 - d^\dagger)\|^2) \\ &- \alpha_n\rho_n(1 - \rho_n\rho(A^*A))\|Au_n - Bv_n\|^2 \\ &- \alpha_n\rho_n(1 - \rho_n\rho(B^*B))\|Au_{n+1} - Bv_n\|^2 \\ &- \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - u_{n+1}\|^2 \\ &- \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - v_{n+1}\|^2 \end{aligned} \tag{22}$$

According to the conditions of sequence $\{\rho_n\}$, we deduced

$$\begin{aligned} \Omega_{n+1}(b^\dagger, d^\dagger) &\le \alpha_n\Omega_n(b^\dagger, d^\dagger) + (1-\alpha_n)(\|(a_0 - b^\dagger)\|^2 + \|(b_0 - d^\dagger)\|^2) \\ &\le \max\{\Omega_n(b^\dagger, d^\dagger), \|(a_0 - b^\dagger)\|^2 + \|(b_0 - d^\dagger)\|^2\} \\ &\le \cdots \\ &\le \max\{\Omega_0(b^\dagger, d^\dagger), \|(a_0 - b^\dagger)\|^2 + \|(b_0 - d^\dagger)\|^2\} \end{aligned} \tag{23}$$

We note that

$$\begin{aligned} &\rho_n\|Au_n - Ab^\dagger\|^2 \\ &= \rho_n\langle u_n - b^\dagger, A^*A(u_n - b^\dagger)\rangle \\ &\leq \rho_n\rho(A^*A)\|u_n - b^\dagger\|^2 \end{aligned} \tag{24}$$

According to the condition of the sequence $\{\rho_n\}$, we have

$$\begin{aligned} \Omega_n(b^\dagger, d^\dagger) &= \|u_n - b^\dagger\|^2 + \|v_n - d^\dagger\|^2 - \rho_n\|Au_n - Ab^\dagger\|^2 \\ &\geq (1 - \rho_n\rho(A^*A))\|u_n - b^\dagger\|^2 + \|v_n - d^\dagger\|^2 \\ &> \varepsilon\rho(A^*A)\|u_n - b^\dagger\|^2 + \|v_n - d^\dagger\|^2 \\ &\geq 0 \end{aligned} \tag{25}$$

According to Equation (23), we obtain that the sequence $\{\Omega_n(b^\dagger, d^\dagger)\}$ is bounded. Therefore, in view of Equation (25), the sequences $\{u_n\}$ and $\{v_n\}$ are bounded.

Let $b^\ddagger$ and $d^\ddagger$ be the convergence points of sequences $\{u_n\}$ and $\{v_n\}$, respectively. We obtain

$$\begin{aligned} &\|u_{n+1} - b^\ddagger\|^2 + \|v_{n+1} - d^\ddagger\|^2 \\ &\leq \|(1-\alpha_n)a_0 + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n)) - b^\ddagger\|^2 \\ &+ \|(1-\alpha_n)b_0 + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n)) - d^\ddagger\|^2 \\ &= \|(1-\alpha_n)(a_0 - b^\ddagger) + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n) - b^\ddagger)\|^2 \\ &+ \|(1-\alpha_n)(b_0 - d^\ddagger) + \alpha_n(v_n - d^\ddagger + \rho_n B^*(Au_{n+1} - Bv_n))\|^2 \\ &\leq \alpha_n\|u_n - \rho_n A^*(Au_n - Bv_n) - b^\ddagger\|^2 \\ &+ 2(1-\alpha_n)\langle a_0 - b^\ddagger, u_{n+1} - b^\ddagger\rangle \\ &+ \alpha_n\|v_n + \rho_n B^*(Au_{n+1} - Bv_n) - d^\ddagger\|^2 \\ &+ 2(1-\alpha_n)\langle b_0 - d^\ddagger, v_{n+1} - d^\ddagger\rangle \end{aligned} \tag{26}$$

It follows that

$$\begin{aligned} &\|u_{n+1} - b^\ddagger\|^2 + \|v_{n+1} - d^\ddagger\|^2 \\ &\leq \alpha_n\|u_n - b^\ddagger\|^2 + \alpha_n\|\rho_n A^*(Au_n - Bv_n)\|^2 + \alpha_n\|v_n - d^\ddagger\|^2 \\ &- 2\alpha_n\langle u_n - b^\ddagger, \rho_n A^*(Au_n - Bv_n)\rangle + \alpha_n\|\rho_n B^*(Au_{n+1} - Bv_n)\|^2 \\ &+ 2\alpha_n\langle v_n - d^\ddagger, \rho_n B^*(Au_{n+1} - Bv_n)\rangle \\ &+ 2(1-\alpha_n)(\langle a_0 - b^\ddagger, u_{n+1} - b^\ddagger\rangle + \langle b_0 - d^\ddagger, v_{n+1} - d^\ddagger\rangle) \end{aligned} \tag{27}$$

Combining Equations (10), (11), (15), (16), (18) and (19), we obtain

$$\begin{aligned} &\|u_{n+1} - b^\ddagger\|^2 + \|v_{n+1} - d^\ddagger\|^2 \\ &\leq \alpha_n\|u_n - b^\ddagger\|^2 + \alpha_n\|v_n - d^\ddagger\|^2 - \alpha_n\rho_n\|Au_n - Ab^\ddagger\|^2 \\ &+ \alpha_{n+1}\rho_{n+1}\|Au_{n+1} - Ab^\ddagger\|^2 \\ &- \alpha_n\rho_n(1 - \rho_n\rho(A^*A))\|Au_n - Bv_n\|^2 \\ &- \alpha_n\rho_n(1 - \rho_n\rho(B^*B))\|Au_{n+1} - Bv_n\|^2 \\ &+ 2(1-\alpha_n)(\langle a_0 - b^\ddagger, u_{n+1} - b^\ddagger\rangle + \langle b_0 - d^\ddagger, v_{n+1} - d^\ddagger\rangle) \end{aligned} \tag{28}$$

It follows that

$$\begin{aligned}
&\|u_{n+1}-b^{\ddagger}\|^2+\|v_{n+1}-d^{\ddagger}\|^2-\rho_{n+1}\|Au_{n+1}-Ab^{\ddagger}\|^2\\
&\leq \alpha_n(\|u_n-b^{\ddagger}\|^2+\|v_n-d^{\ddagger}\|^2-\rho_n\|Au_n-Ab^{\ddagger}\|^2)\\
&-\alpha_n\rho_n(1-\rho_n\rho(A^*A))\|Au_n-Bv_n\|^2\\
&-\alpha_n\rho_n(1-\rho_n\rho(B^*B))\|Au_{n+1}-Bv_n\|^2\\
&+2(1-\alpha_n)(\langle a_0-b^{\ddagger},u_{n+1}-b^{\ddagger}\rangle+\langle b_0-d^{\ddagger},v_{n+1}-d^{\ddagger}\rangle)
\end{aligned} \tag{29}$$

This implies

$$\Omega_{n+1}(b^{\ddagger},d^{\ddagger})\leq \alpha_n\Omega_n(b^{\ddagger},d^{\ddagger})+(1-\alpha_n)b_n \tag{30}$$

where

$$\begin{aligned}
b_n&=2(\langle a_0-b^{\ddagger},u_{n+1}-b^{\ddagger}\rangle+\langle b_0-d^{\ddagger},v_{n+1}-d^{\ddagger}\rangle)\\
&-\frac{\alpha_n\rho_n(1-\rho_n\rho(A^*A))}{(1-\alpha_n)}\|Au_n-Bv_n\|^2\\
&-\frac{\alpha_n\rho_n(1-\rho_n\rho(B^*B))}{(1-\alpha_n)}\|Au_{n+1}-Bv_n\|^2
\end{aligned} \tag{31}$$

Because $\{u_n\}$ and $\{v_n\}$ are bounded, we obtain

$$\begin{aligned}
b_n&\leq 2(\langle a_0-b^{\ddagger},u_{n+1}-b^{\ddagger}\rangle+\langle b_0-d^{\ddagger},v_{n+1}-d^{\ddagger}\rangle)\\
&\leq 2(\|a_0-b^{\ddagger}\|\|u_{n+1}-b^{\ddagger}\|+\|b_0-d^{\ddagger}\|\|v_{n+1}-d^{\ddagger}\|)\\
&<\infty
\end{aligned} \tag{32}$$

It follows that $\limsup_{n\to\infty} b_n<\infty$. Let $\alpha_n=1-t_n$, $t_n\in(0,1)$, we assume that $\limsup_{n\to\infty} b_n<-1$, for all $n\leq n_0$, there exists n_0 such that $b_n\leq -1$. Then, in view of Equation (30), we have

$$\begin{aligned}
\Omega_{n+1}(b^{\ddagger},d^{\ddagger})&\leq \alpha_n\Omega_n(b^{\ddagger},d^{\ddagger})+(1-\alpha_n)b_n\\
&=(1-t_n)\Omega_n(b^{\ddagger},d^{\ddagger})+t_nb_n\\
&\leq(1-t_n)\Omega_n(b^{\ddagger},d^{\ddagger})-t_n\\
&=\Omega_n(b^{\ddagger},d^{\ddagger})-t_n(\Omega_n(b^{\ddagger},d^{\ddagger})+1)\\
&\leq\Omega_n(b^{\ddagger},d^{\ddagger})-t_n\\
&\leq\Omega_{n-1}(b^{\ddagger},d^{\ddagger})-t_{n-1}-t_n\\
&\leq\cdots\\
&\leq\Omega_{n_0}(b^{\ddagger},d^{\ddagger})-\sum_{i=n_0}^{n}t_i
\end{aligned} \tag{33}$$

Since $\sum_{i=n_0}^{\infty}t_i>\Omega_{n_0}(b^{\ddagger},d^{\ddagger})$, there exists $N>n_0$ such that $\sum_{i=n_0}^{N}t_i=\infty$. We deduced that

$$\Omega_{N+1}(b^{\ddagger},d^{\ddagger})\leq\Omega_{n_0}(b^{\ddagger},d^{\ddagger})-\sum_{i=n_0}^{N}t_i<0 \tag{34}$$

In view of Equation (25), we know that $\Omega_n(b^{\ddagger},d^{\ddagger})$ is a non-negative real sequence, the inequality in Equation (34) contradicts the fact, hence, $\limsup_{n\to\infty} b_n\geq -1$. Since $\limsup_{n\to\infty} b_n$ has a finite limit, we take a subsequence $\{n_k\}$ such that

$$\begin{aligned}\limsup_{n\to\infty} b_n = \lim_{k\to\infty} b_{n_k} &= \lim_{k\to\infty} 2(\langle a_0 - b^\ddagger, u_{n_k+1} - b^\ddagger\rangle + \langle b_0 - d^\ddagger, v_{n_k+1} - d^\ddagger\rangle) \\ &- \lim_{k\to\infty} \frac{\alpha_{n_k}\rho_{n_k}(1-\rho_{n_k}\rho(A^*A))}{(1-\alpha_n)}\|Au_{n_k} - Bv_{n_k}\|^2 \\ &- \lim_{k\to\infty} \frac{\alpha_{n_k}\rho_{n_k}(1-\rho_{n_k}\rho(B^*B))}{(1-\alpha_{n_k})}\|Au_{n_k+1} - Bv_{n_k}\|^2\end{aligned} \tag{35}$$

We assume that $\lim_{k\to\infty}\langle a_0 - b^\ddagger, u_{n_k+1} - b^\ddagger\rangle$ and $\lim_{k\to\infty}\langle b_0 - d^\ddagger, v_{n_k+1} - d^\ddagger\rangle$ have finite limits, then the following limit exists

$$\lim_{k\to\infty} \frac{\alpha_{n_k}\rho_{n_k}(1-\rho_{n_k}\rho(A^*A))}{(1-\alpha_{n_k})}\|Au_{n_k} - Bv_{n_k}\|^2 \tag{36}$$

and

$$\lim_{k\to\infty} \frac{\alpha_{n_k}\rho_{n_k}(1-\rho_{n_k}\rho(B^*B))}{(1-\alpha_{n_k})}\|Au_{n_k+1} - Bv_{n_k}\|^2 \tag{37}$$

Since $\lim_{k\to\infty}\frac{\alpha_{n_k}}{(1-\alpha_{n_k})} = \infty$, we deduce that

$$\lim_{k\to\infty}\|Au_{n_k} - Bv_{n_k}\| = 0 \tag{38}$$

and

$$\lim_{k\to\infty}\|Au_{n_k+1} - Bv_{n_k}\| = 0 \tag{39}$$

From Equation (38), we obtain that any weak cluster point of $\{(u_{n_k}, v_{n_k})\}$ belongs to Ω. Hence, it follows that

$$\begin{aligned}&\lim_{k\to\infty}\|u_{n_k+1} - u_{n_k}\| \\ &= \lim_{k\to\infty}\|(1-\alpha_{n_k})(a_0 - u_{n_k}) + \alpha_{n_k}(u_{n_k} - \rho_{n_k}A^*(Au_{n_k} - Bv_{n_k}) - u_{n_k})\| \\ &\leq \lim_{k\to\infty}(\|(1-\alpha_{n_k})(a_0 - u_{n_k})\| + \|\rho_{n_k}A^*(Au_{n_k} - Bv_{n_k})\|) \\ &= 0\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\lim_{k\to\infty}\|v_{n_k+1} - v_{n_k}\| \\ &= \lim_{k\to\infty}\|(1-\alpha_{n_k})(b_0 - v_{n_k}) + \alpha_{n_k}(v_{n_k} + \rho_{n_k}B^*(Au_{n_k+1} - Bv_{n_k}) - v_{n_k})\| \\ &\leq \lim_{k\to\infty}(\|(1-\alpha_{n_k})(b_0 - v_{n_k})\| + \|\rho_{n_k}B^*(Au_{n_k+1} - Bv_{n_k})\|) \\ &= 0\end{aligned} \tag{41}$$

This implies that any weak cluster point of $\{(u_{n_k+1}, v_{n_k+1})\}$ belongs to Ω. We assume that $\{(u_{n_k+1}, v_{n_k})\}$ weakly converges to $(\tilde{b}, \tilde{d})$, then, we have

$$\begin{aligned}\limsup_{n\to\infty} b_n &\leq \lim_{k\to\infty} 2(\langle a_0 - b^\ddagger, u_{n_k+1} - b^\ddagger\rangle + \langle b_0 - d^\ddagger, v_{n_k+1} - d^\ddagger\rangle) \\ &= 2(\langle a_0 - b^\ddagger, \tilde{b} - b^\ddagger\rangle + \langle b_0 - d^\ddagger, \tilde{d} - d^\ddagger\rangle) \\ &\leq 0\end{aligned} \tag{42}$$

In the light of Lemma 2, we have $\lim_{n\to\infty}\Omega_n(b^\ddagger, d^\ddagger) = 0$. From Equation (25), we obtain

$$\Omega_n(b^\ddagger, d^\ddagger) \geq \varepsilon\rho(A^*A)\|u_n - b^\ddagger\|^2 + \|v_n - d^\ddagger\|^2 \geq 0 \tag{43}$$

Therefore, we obtain

$$\lim_{n\to\infty} \|u_n - b^\ddagger\| = 0 \quad (44)$$

$$\lim_{n\to\infty} \|v_n - d^\ddagger\| = 0 \quad (45)$$

Then,

$$\|Ab^\ddagger - Bd^\ddagger\| = \lim_{n\to+\infty} \|Au_n - Bv_n\| = 0 \quad (46)$$

Hence, $(b^\ddagger, d^\ddagger) \in \Omega$. We obtain that $u_n \to b^\ddagger$ and $v_n \to d^\ddagger$. This proof has been completed.

Let f_1 and f_2 be two strict contraction mappings with contraction coefficients of $c_1 \in [0,1)$ and $c_2 \in [0,1)$, respectively.

$$\begin{cases} u_{n+1} = P_C((1-\alpha_n)f_1(u_n) + \alpha_n(u_n - \rho_n A^*(Au_n - Bv_n))) \\ v_{n+1} = P_Q((1-\alpha_n)f_2(v_n) + \alpha_n(v_n + \rho_n B^*(Au_{n+1} - Bv_n))) \end{cases} \quad (47)$$

□

Corollary 1. *Let Ω_2 be the solution set of Equation (4) and assume that the solution set Ω_2 is not empty. Then, in the light of Theorem 1, the sequence $\{(u_n, v_n)\}$ generated by Equation (47) exists $(\tilde{b}, \tilde{d}) \in \Omega_2$ such that $u_n \to \tilde{b}$ and $v_n \to \tilde{d}$.*

4. Conclusions

In this paper, we proposed an improved alternating CQ algorithm to solve the SEP. This improved method changes the generated iterative sequence from weak to strong convergence.

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