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A Dependent Lindeberg Central Limit Theorem for Cluster Functionals on Stationary Random Fields

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Abstract: In this paper, we provide a central limit theorem for the finite-dimensional marginal distributions of empirical processes $(Z_n(f))_{f\in\mathcal{F}}$ whose index set $\mathcal{F}$ is a family of cluster functionals valued on blocks of values of a stationary random field. The practicality and applicability of the result depend mainly on the usual Lindeberg condition and on a sequence T_n which summarizes the dependence between the blocks of the random field values. Finally, in application, we use the previous result in order to show the Gaussian asymptotic behavior of the proposed iso-extremogram estimator.

Keywords: central limit theorem; cluster functional; weak dependence; Lindeberg method; extremogram

MSC: 60G60; 60F05; 60G70

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1. Introduction

Recent developments in massive data processing lead us to think differently about certain problems in statistics. In particular, it is interesting to develop the construction of statistics as functions of data blocks and to study their inference. On the other hand, in some applications, only very little data are relevant to the estimates, not to mention that the estimates are also hidden among a large mass of "raw data". We can refer the reader to Davis and Mikosch [1] for examples in extremes and to Long and De Sousa [2] for examples in astronomy. This leads us to think of clusters of data deemed "relevant" (or extremal type, within the framework of extreme value theory), where we say that two relevant values belong to two different clusters if they belong to two different blocks. Moreover, these relevant values are in the cores of blocks, where the core of a block B is defined as the smaller sub-block $\mathcal{C}(B)$ of B which contains all relevant values of B, if they exist.

In this context, we consider functionals that act on these clusters of relevant values and we develop useful lemmas in order to simplify the essential step to establish a Lindeberg central limit theorem (CLT) for these "cluster functionals" on stationary random fields, inspired by the definitions of Drees and Rootzén [3] and the approach of Bardet et al. [4] and Gómez-García [5].

The mathematical background is as follows. Let $d \in \mathbb{N} := \{1,2,\ldots\}$, and denote $\mathbf{n} := (n_1,\ldots,n_d)$, $\mathbf{1} := (1,\ldots,1) \in \mathbb{N}^d$ and $[j] := [1:j]$, where $[i:j] := \{i,i+1,\ldots,j\} \subset \mathbb{Z}$. Let $X = \left\{X_{\mathbf{t}} \;:\; \mathbf{t} \in \mathbb{N}^d\right\}$ be a $\mathbb{R}^k-$valued stationary random field and let $\mathbb{X} = \{X_{\mathbf{n},\mathbf{t}} \;:\; \mathbf{t} \in [n_1] \times \cdots \times [n_d]\}_{\mathbf{n}\in\mathbb{N}^d}$ be the corresponding normalized random observations from the random field X, defined by $X_{\mathbf{n},\mathbf{t}} = L_{\mathbf{n}}(X_{\mathbf{t}})\mathbb{I}_A(L_{\mathbf{n}}(X_{\mathbf{t}}))$ for some measurable functions $L_{\mathbf{n}} : \mathbb{R}^k \longrightarrow \mathbb{R}^k$, such that

$$\mathbb{P}(X_{\mathbf{n},\mathbf{1}} \in \cdot \,|X_{\mathbf{n},\mathbf{1}} \in A) \underset{\mathbf{n}\to\infty}{\longrightarrow} G(\cdot), \tag{1}$$

where G is a non-degenerate distribution and $A \subseteq \mathbb{R}^k \setminus \{0\}$ is the so-called relevance set. Here, $\mathbb{I}_A(\cdot)$ denotes the usual indicator function of a subset A and the tendency $\mathbf{n} \to \infty$ means that $n_i \to \infty$ for all $i \in [d]$. In particular, the convergence (1) is fulfilled if the random vector $X_{\mathbf{1}}$ is regularly varying. For more details on regularly varying vectors, one can refer to Resnick [6,7].

For each $i \in [d]$, let $r_i := r_{n_i,i}$ be an integer value such that $r_i = o(n_i)$ and $m_i := \lfloor n_i/r_i \rfloor := \max\{k \in \mathbb{N} : k \leq n_i/r_i\}$. We define the d-blocks (or simply blocks) of $\mathbb{X}$ by

$$Y_{\mathbf{n},j_1\ldots j_d} := (X_{\mathbf{n},\mathbf{t}})_{\mathbf{t}\in\prod_{i=1}^d[(j_i-1)r_i+1\,:\,j_ir_i]}, \tag{2}$$

where $(j_1,\ldots,j_d) \in D_{\mathbf{n},d} := \prod_{i=1}^d [m_i]$. Thus, we have $m_1m_2\cdots m_d$ complete blocks $Y_{\mathbf{n},j_1\ldots j_d}$, and no more than $(m_1+1)(m_2+1)\cdots(m_d+1) - m_1m_2\cdots m_d$ incomplete ones which we ignore because we consider m_i large enough. Moreover, as usual, $\prod_{i=1}^d A_i$ denotes the Cartesian product $A_1 \times \cdots \times A_d$ and, by stationarity, we denote $Y_{\mathbf{n}} \overset{\mathcal{D}}{=} Y_{\mathbf{n},\mathbf{1}}$ as a generic block of $\mathbb{X}$.

We are now going to formally define the core of a block, cluster functional and the empirical process of cluster functionals, which are generalizations of the definitions of Yun [8], Segers [9] and Drees and Rootzén [3] to d-blocks.

Let $y = (x_{\mathbf{t}})_{\mathbf{t}\in\prod_{i=1}^d[r_i]}$ be a d-block. The core of the block y with respect to the relevance set A is defined as

$$\mathcal{C}(y) = \begin{cases} (x_{\mathbf{t}})_{\mathbf{t}\in\prod_{i=1}^d[r_{i,I}\,:\,r_{i,S}]}, & \text{if } x_{\mathbf{t}} \in A \text{ for some } \mathbf{t} \in \prod_{i=1}^d[r_i]; \\ 0, & \text{otherwise,} \end{cases}$$

where, for each $i \in [d]$, $r_{i,I} := \min P_i$ and $r_{i,S} := \max P_i$ with

$$P_i = \left\{ j_i \in [r_i] : \; x_{(j_1,\ldots,j_i,\ldots,j_d)} \in A, \text{for some } (j_1,\ldots,j_{i-1},j_{i+1},\ldots,j_d) \in \prod_{k\in[d]\setminus\{i\}} [r_k] \right\}.$$

Let $(E,\mathcal{E})$ be a measurable subspace of $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ for some $k \geq 1$ such that $0 \in E$ and let $\mathbb{B}_{l_1,\ldots,l_d}(E)$ be the set of E-valued blocks (or arrays) of size $l_1 \times l_2 \times \cdots \times l_d$, with $l_1,\ldots,l_d \in \mathbb{N}$. Consider now the set

$$E_\cup := \bigcup_{l_1,\ldots,l_d=1}^{\infty} \mathbb{B}_{l_1,\ldots,l_d}(E),$$

which is equipped with the σ-field $\mathcal{E}_\cup$ induced by the Borel-σ-fields on $\mathbb{B}_{l_1,\ldots,l_d}(E)$, for $l_1,\ldots,l_d \in \mathbb{N}$. A cluster functional is a measurable map $f : (E_\cup, \mathcal{E}_\cup) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$f(y) = f(\mathcal{C}(y)), \text{ for all } y \in E_\cup, \quad \text{and} \quad f(0) = 0. \tag{3}$$

Let $\mathcal{F}$ be a class of cluster functionals and let $\{Y_{\mathbf{n},j_1j_2\ldots j_d} : (j_1,\ldots,j_d) \in D_{\mathbf{n},d}\}$ be the family of blocks of size $r_1 \times r_2 \times \cdots \times r_d$ defined in (2). The empirical process $Z_{\mathbf{n}}$ of cluster functionals in $\mathcal{F}$, is the process $(Z_{\mathbf{n}}(f))_{f\in\mathcal{F}}$ defined by

$$Z_{\mathbf{n}}(f) := \frac{1}{\sqrt{n_{\mathbf{n}}v_{\mathbf{n}}}} \sum_{(j_1,\ldots,j_d)\in D_{\mathbf{n},d}} (f(Y_{\mathbf{n},j_1\ldots j_d}) - \mathbb{E}f(Y_{\mathbf{n},j_1\ldots j_d})), \tag{4}$$

where $n_{\mathbf{n}} = n_1\cdots n_d$ and $v_{\mathbf{n}} := \mathbb{P}(X_{\mathbf{n},\mathbf{1}} \in A)$ with $A \subseteq E \setminus \{0\}$ denoting the relevance set.

Under the Lindeberg condition and the convergence to zero of a sequence $T_{\mathbf{n}}$ that summarizes the dependence between the blocks of values of the random field, we prove that the finite-dimensional marginal distributions (fidis) of the empirical process (4) converge to a Gaussian process. The proof basically consists of the "Lindeberg method" for a CLT of stationary time series as in Bardet et al. [4], but adapted here to stationary random fields.

Since Bardet et al. [4] gave a Lindeberg CLT for time series, Gómez-García [5] used this approach in order to obtain a Lindeberg CLT for cluster functionals on time series whose convergence depends mainly on the Lindeberg condition and the convergence to zero of $T_{\mathbf{n}}$ that summarizes the dependence. Moreover, Gómez-García [5] simplified $T_{\mathbf{n}}$ by using the coefficients of weak-dependence of Doukhan and Louhichi [10]. This allowed the attainment of partially more general results than Drees and Rootzén [3] which are established under mixing. Note that the family of weakly dependent processes of Doukhan and Louhichi [10] is more general that the family of mixing processes, see Andrews [11].

In the context of random fields, the approach is not very simple. In fact, we must first generalize the results of Bardet et al. [4] within the framework of random fields, then we could simplify the term of dependence by fixing short range dependence conditions on the random field X like convenient conditions for the decay rates of the weak-dependence coefficients of Doukhan and Louhichi [10]. In this work, we concentrate on the first part and we introduce a measure (and its estimator) which motivates the choice of this generalization: the iso-extremogram, which can be viewed as a correlogram for extreme values of space–time processes.

The rest of the paper consists of three complementary sections. In Section 2, we provide useful lemmas in order to establish the CLT for the fidis of the cluster functionals empirical process (4). Then, in Section 3, we introduce the iso-extremogram and we use the CLT of Section 2 in order to show that, under appropriate additional conditions, the iso-extremogram estimator has an asymptotically Gaussian behavior. Section 4 is dedicated to the conclusions and perspectives of this approach.

2. Results

In this section, we provide useful lemmas which notably simplify the essential step to establish a CLT for the fidis of the empirical process defined in (4). The proof consists of the same techniques as Bardet et al. [4] used in the demonstrations of their dependent and independent Lindeberg lemmas, but generalized here to random fields.

In order to establish the CLT, firstly, consider the following basic assumption:

(Bas) The vector $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{N}^d$ is such that $r_i \ll n_i$ for each $i \in [d]$.
In addition, denoting $r_{\mathbf{n}} = r_1 \cdots r_d$, we have $r_{\mathbf{n}} v_{\mathbf{n}} \longrightarrow \tau < \infty$ and $n_{\mathbf{n}} v_{\mathbf{n}} \longrightarrow \infty$, as $\mathbf{n} \to \infty$.

Secondly, consider the following essential convergence assumptions:

(Lin) $(r_{\mathbf{n}} v_{\mathbf{n}})^{-1} \mathbb{E}\Big[\big(f(Y_{\mathbf{n}}) - \mathbb{E}f(Y_{\mathbf{n}})\big)^2 \mathbb{I}_{\{|f(Y_{\mathbf{n}}) - \mathbb{E}f(Y_{\mathbf{n}})| > \epsilon \sqrt{n_{\mathbf{n}} v_{\mathbf{n}}}\}} \Big] = o(1),\ \forall \epsilon > 0,\ \forall f \in \mathcal{F};$

(Cov) $(r_{\mathbf{n}} v_{\mathbf{n}})^{-1} \operatorname{Cov}(f(Y_{\mathbf{n}}), g(Y_{\mathbf{n}})) \longrightarrow c(f,g),\ \forall f, g \in \mathcal{F}.$

Consider now the random blocks $Y_{\mathbf{n}, j_1 \dots j_d}$, with $(j_1, \dots, j_d) \in D_{\mathbf{n},d}$ defined in (2). For each k-tuple of cluster functionals $\mathbf{f}_k = (f_1, \dots, f_k)$ and each $(j_1, \dots, j_d) \in D_{\mathbf{n},d}$, we define the following random vector:

$$W_{\mathbf{n}, j_1 \dots j_d} := \frac{1}{\sqrt{n_{\mathbf{n}} v_{\mathbf{n}}}} \big(f_1(Y_{\mathbf{n}, j_1 \dots j_d}) - \mathbb{E}f_1(Y_{\mathbf{n}, j_1 \dots j_d}), \dots, f_k(Y_{\mathbf{n}, j_1 \dots j_d}) - \mathbb{E}f_k(Y_{\mathbf{n}, j_1 \dots j_d}) \big). \quad (5)$$

Without loss of generality and in order to simplify the writing, we consider $d = 2$ in the rest of this section.

Let $(W'_{\mathbf{n},ij})_{(i,j) \in D_{\mathbf{n},2}}$ be a sequence of zero mean independent $\mathbb{R}^k$-valued random variables, independent of the sequence $(W_{\mathbf{n},ij})_{(i,j) \in D_{\mathbf{n},2}}$, such that $W'_{\mathbf{n},ij} \sim \mathcal{N}_k\big(\mathbf{0}, \operatorname{Cov}(W_{\mathbf{n},ij})\big)$, for all $(i,j) \in D_{\mathbf{n},2}$. Denote by $\mathcal{C}_b^3$ the set of bounded functions $h : \mathbb{R}^k \longrightarrow \mathbb{R}$ with bounded and continuous partial derivatives up to order 3 and, for $h \in \mathcal{C}_b^3$ and $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$, define

$$\Delta_{\mathbf{n}} := \left| \mathbb{E}\left[h\left(\sum_{(i,j)\in D_{\mathbf{n},2}} W_{\mathbf{n},ij} \right) - h\left(\sum_{(i,j)\in D_{\mathbf{n},2}} W'_{\mathbf{n},ij} \right) \right] \right|. \tag{6}$$

The following assumption allows us to present, in a useful and simplified form, lemmas of Lindeberg under independence and dependence.

(Lin′) There exists $\delta \in (0,1]$ such that, for any $(i,j) \in D_{\mathbf{n},2}$, we have

$$\mathbb{E}\left\| W_{\mathbf{n},ij} \right\|^{2+\delta} < \infty$$

for all $\mathbf{n} \in \mathbb{N}^2$ and all $k-$tuple of cluster functionals $(f_1, \ldots, f_k) \in \mathcal{F}^k$.

Moreover, denote

$$A_{\mathbf{n}} := \sum_{(i,j)\in D_{\mathbf{n},2}} \mathbb{E}\left\| W_{\mathbf{n},ij} \right\|^{2+\delta}.$$

Lemma 1 (Lindeberg under independence)**.** *Suppose that the blocks $(Y_{\mathbf{n},ij})_{(i,j)\in D_{\mathbf{n},2}}$ are independent and that the random variables $(W_{\mathbf{n},ij})_{(i,j)\in D_{\mathbf{n},2}}$ defined in (5) satisfy Assumption (Lin′). Then, for all $\mathbf{n} \in \mathbb{N}^2$, we have*

$$\Delta_{\mathbf{n}} \le 6\, \|h^{(2)}\|_\infty^{1-\delta}\, \|h^{(3)}\|_\infty^{\delta}\, A_{\mathbf{n}}.$$

Proof. First, notice that

$$\Delta_{\mathbf{n}} \le \sum_{(i,j)\in D_{\mathbf{n},2}} \Delta_{\mathbf{n},ij}, \tag{7}$$

where

$$\Delta_{\mathbf{n},ij} := \left| \mathbb{E}\left[h_{ij}(V_{\mathbf{n},ij} + W_{\mathbf{n},ij}) - h_{ij}(V_{\mathbf{n},ij} + W'_{\mathbf{n},ij}) \right] \right|, \quad \forall (i,j) \in D_{\mathbf{n},2}\,;$$

$$V_{\mathbf{n},ij} := \sum_{(u,v)\in D_{\mathbf{n},2} \setminus \left(\bigcup_{l=0}^{i-1} L_l^{m_2} \cup L_i^j \right)} W_{\mathbf{n},uv}, \quad \forall (i,j) \in D_{\mathbf{n},2} \setminus \{(m_1, m_2)\},$$

$V_{\mathbf{n},m_1 m_2} = 0$; and

$$h_{ij}(x) := \mathbb{E}\left[h\left(x + \sum_{u=0}^{i-1} \sum_{v=1}^{m_2} W'_{\mathbf{n},uv} + \sum_{v=0}^{j-1} W'_{\mathbf{n},iv} \right) \right].$$

Furthermore, we adopt the convention $W_{\mathbf{n},ij} = 0$, if either $i = 0$ or $j = 0$.

Now, we use some lines of the proof of Lemma 1 in Bardet et al. [4].

Let $v, w \in \mathbb{R}^k$. From Taylor's formula, there exist vectors $v_{1,w}, v_{2,w} \in \mathbb{R}^k$ such that

$$\begin{aligned} h(v+w) &= h(v) + h^{(1)}(v)(w) + \frac{1}{2} h^{(2)}(v_{1,w})(w,w) \\ &= h(v) + h^{(1)}(v)(w) + \frac{1}{2} h^{(2)}(v)(w,w) + \frac{1}{6} h^{(3)}(v_{2,w})(w,w,w), \end{aligned}$$

where, for $j = 1,2,3$, $h^{(j)}(v)(w_1, w_2, \ldots, w_j)$ stands for the value of the symmetric $j-$linear form from $h^{(j)}$ of $(w_1, \ldots, w_j)$ at v. Moreover, denote

$$\|h^{(j)}(v)\|_1 = \sup_{\|w_1\|,\ldots,\|w_j\| \le 1} |h^{(j)}(v)(w_1, \ldots, w_j)| \quad \text{and} \quad \|h^{(j)}\|_\infty = \sup_{v\in\mathbb{R}^k} \|h^{(j)}(v)\|_1.$$

Thus, for $v, w, w' \in \mathbb{R}^k$, there exist some suitable vectors $v_{1,w}, v_{2,w}, v_{1,w'}, v_{2,w'} \in \mathbb{R}^k$ such that

$$h(v+w) - h(v+w') = h^{(1)}(v)(w-w') + \frac{1}{2}\Big(h^{(2)}(v)(w,w) - h^{(2)}(v)(w',w')\Big) + \frac{1}{2}\Big(\Big(h^{(2)}(v_{1,w}) - h^{(2)}(v)\Big)(w,w) - \Big(h^{(2)}(v_{1,w'}) - h^{(2)}(v)\Big)(w',w')\Big),$$

by using the approximation of Taylor of order 2, and

$$h(v+w) - h(v+w') = h^{(1)}(v)(w-w') + \frac{1}{2}\Big(h^{(2)}(v)(w,w) - h^{(2)}(v)(w',w')\Big) + \frac{1}{6}\Big(h^{(3)}(v_{2,w})(w,w,w) - h^{(3)}(v_{2,w'})(w',w',w')\Big),$$

by using the approximation of Taylor of order 3.

Thus, $\gamma = h(v+w) - h(v+w') - h^{(1)}(v)(w-w') - \frac{1}{2}\Big(h^{(2)}(v)(w,w) - h^{(2)}(v)(w',w')\Big)$ satisfies

$$\begin{aligned}
|\gamma| &\le \Big(\Big(\|w\|^2 + \|w'\|^2\Big)\|h^{(2)}\|_\infty\Big) \wedge \Big(\frac{1}{6}\Big(\|w\|^3 + \|w'\|^3\Big)\|h^{(3)}\|_\infty\Big) \\
&\le \Big(\|w\|^2\|h^{(2)}\|_\infty\Big) \wedge \Big(\frac{1}{6}\|w\|^3\|h^{(3)}\|_\infty\Big) + \Big(\|w\|^2\|h^{(2)}\|_\infty\Big) \wedge \Big(\frac{1}{6}\|w'\|^3\|h^{(3)}\|_\infty\Big) \\
&\quad + \Big(\|w'\|^2\|h^{(2)}\|_\infty\Big) \wedge \Big(\frac{1}{6}\|w\|^3\|h^{(3)}\|_\infty\Big) + \Big(\|w'\|^2\|h^{(2)}\|_\infty\Big) \wedge \Big(\frac{1}{6}\|w'\|^3\|h^{(3)}\|_\infty\Big) \\
&\le \frac{1}{6^\delta}\|h^{(2)}\|_\infty^{1-\delta}\|h^{(3)}\|_\infty^{\delta}\Big(\|w\|^{2+\delta} + \|w\|^{2(1-\delta)}\|w'\|^{3\delta} + \|w\|^{3\delta}\|w'\|^{2(1-\delta)} + \|w'\|^{2+\delta}\Big), \qquad (8)
\end{aligned}$$

where (8) is given by using the inequality $1 \wedge a \le a^\delta$, with $a \ge 0$ and $\delta \in [0,1]$.

Substituting $h_{ij}, V_{\mathbf{n},ij}, W_{\mathbf{n},ij}$ and $W'_{\mathbf{n},ij}$ for h, v, w and w' in the preceding inequality (8) and taking expectations, we obtain a bound for $\Delta_{\mathbf{n},ij}$. Indeed, we have

$$\begin{aligned}
&\mathbb{E}\Big[h_{ij}(V_{\mathbf{n},ij} + W_{\mathbf{n},ij}) - h_{ij}(V_{\mathbf{n},ij} + W'_{\mathbf{n},ij})\Big] \\
&\quad = \mathbb{E}\Big[h_{ij}(V_{\mathbf{n},ij} + W_{\mathbf{n},ij}) - h_{ij}(V_{\mathbf{n},ij} + W'_{\mathbf{n},ij})\Big] + 0 \\
&\quad = \mathbb{E}\Big[h_{ij}(V_{\mathbf{n},ij} + W_{\mathbf{n},ij}) - h_{ij}(V_{\mathbf{n},ij} + W'_{\mathbf{n},ij})\Big] - \mathbb{E}\Big[h_{ij}^{(1)}(V_{\mathbf{n},ij})(W_{\mathbf{n},ij} - W'_{\mathbf{n},ij})\Big] \\
&\qquad - \frac{1}{2}\mathbb{E}\Big[h_{ij}^{(2)}(V_{\mathbf{n},ij})(W_{\mathbf{n},ij}, W_{\mathbf{n},ij}) - h_{ij}^{(2)}(V_{\mathbf{n},ij})(W'_{\mathbf{n},ij}, W'_{\mathbf{n},ij})\Big],
\end{aligned}$$

because $V_{\mathbf{n},ij}$ is independent of $W_{\mathbf{n},ij}$ and $W'_{\mathbf{n},ij}$, and because $\mathbb{E}W_{\mathbf{n},ij} = \mathbb{E}W'_{\mathbf{n},ij} = 0$ and $\mathrm{Cov}(W_{\mathbf{n},ij}) = \mathrm{Cov}(W'_{\mathbf{n},ij})$ for all $(i,j) \in D_{\mathbf{n},2}$.

On the other hand, using Jensen's inequality, we derive $\mathbb{E}\|W'_{\mathbf{n},ij}\|^{2+\delta} \le \Big(\mathbb{E}\|W'_{\mathbf{n},ij}\|^4\Big)^{\frac{1}{2}+\frac{\delta}{4}}$, and $\mathbb{E}\|W'_{\mathbf{n},ij}\|^4 \le 3\big(\mathbb{E}\|W_{\mathbf{n},ij}\|^2\big)^2$ because $W'_{\mathbf{n},ij}$ is a Gaussian random variable with the same covariance as $W_{\mathbf{n},ij}$.

Therefore,

$$\mathbb{E}\|W'_{\mathbf{n},ij}\|^{2+\delta} \le \Big(3\Big(\mathbb{E}\|W_{\mathbf{n},ij}\|^2\Big)^2\Big)^{\frac{1}{2}+\frac{\delta}{4}} = 3^{\frac{1}{2}+\frac{\delta}{4}}\Big(\mathbb{E}\|W_{\mathbf{n},ij}\|^2\Big)^{1+\frac{\delta}{2}} \le 3^{\frac{1}{2}+\frac{\delta}{4}}\mathbb{E}\|W_{\mathbf{n},ij}\|^{2+\delta} \qquad (9)$$

and

$$\mathbb{E}\|W'_{\mathbf{n},ij}\|^{2(1-\delta)}\mathbb{E}\|W_{\mathbf{n},ij}\|^{3\delta} \leq \left(\mathbb{E}\|W'_{\mathbf{n},ij}\|^2\right)^{1-\delta}\mathbb{E}\|W_{\mathbf{n},ij}\|^{3\delta}$$
$$\leq \left(\mathbb{E}\|W_{\mathbf{n},ij}\|^2\right)^{1-\delta}\mathbb{E}\|W_{\mathbf{n},ij}\|^{3\delta} \leq \mathbb{E}\|W_{\mathbf{n},ij}\|^{2+\delta}. \tag{10}$$

In addition, for $3\delta < 2$,

$$\mathbb{E}\|W_{\mathbf{n},ij}\|^{2(1-\delta)}\mathbb{E}\|W'_{\mathbf{n},ij}\|^{3\delta} \leq \mathbb{E}\|W_{\mathbf{n},ij}\|^{2(1-\delta)}\left(\mathbb{E}\|W'_{\mathbf{n},ij}\|^2\right)^{\frac{3\delta}{2}} \leq \mathbb{E}\|W_{\mathbf{n},ij}\|^{2+\delta}, \tag{11}$$

else

$$\mathbb{E}\|W_{\mathbf{n},ij}\|^{2(1-\delta)}\mathbb{E}\|W'_{\mathbf{n},ij}\|^{3\delta} \leq \mathbb{E}\|W_{\mathbf{n},ij}\|^{2(1-\delta)}\left(\mathbb{E}\|W'_{\mathbf{n},ij}\|^4\right)^{\frac{3\delta}{4}}, \quad \text{because } 3\delta \leq 4$$
$$\leq 3^{\frac{3\delta}{4}}\mathbb{E}\|W_{\mathbf{n},ij}\|^{2(1-\delta)}\left(\mathbb{E}\|W_{\mathbf{n},ij}\|^2\right)^{\frac{3\delta}{2}} \leq 3^{\frac{1}{2}+\frac{\delta}{4}}\mathbb{E}\|W_{\mathbf{n},ij}\|^{2+\delta}. \tag{12}$$

The inequalities (9)–(12) allow to simplify the terms between parentheses in the last inequality in (8). Recall that $\|h^{(k)}_{ij}\|_\infty \leq \|h^{(k)}\|_\infty$ for all $(i,j) \in D_{\mathbf{n},2}$ and $0 \leq k \leq 3$. Therefore, we obtain

$$\Delta_{\mathbf{n},ij} \leq \frac{2(1+3^{\frac{1}{2}+\frac{\delta}{4}})}{6^\delta}\|h^{(2)}\|^{1-\delta}_\infty\|h^{(3)}\|^\delta_\infty\mathbb{E}\|W_{\mathbf{n},ij}\|^{2+\delta} \leq 6\|h^{(2)}\|^{1-\delta}_\infty\|h^{(3)}\|^\delta_\infty\mathbb{E}\|W_{\mathbf{n},ij}\|^{2+\delta},$$

because, for all $\delta \in [0,1]$, $C(\delta) = \frac{2(1+3^{\frac{1}{2}+\frac{\delta}{4}})}{6^\delta} \leq C(0) = 2(1+\sqrt{3}) < 6$.

As a consequence, from Assumption (Lin′), we obtain $\Delta_{\mathbf{n}} \leq 6\,\|h^{(2)}\|^{1-\delta}_\infty\,\|h^{(3)}\|^\delta_\infty\,A_{\mathbf{n}}$. The proof of Lemma 1 ends. □

Remark 1. *By taking $\epsilon < 6\|h^{(2)}\|_\infty\,(\|h^{(3)}\|_\infty)^{-1}$ and suitably using the second inequality of (8) in the proof of Lemma 1, the classical Lindeberg conditions can be used:*

$$\Delta_{\mathbf{n}} \leq 2\,\|h^{(2)}\|_\infty\,B_{\mathbf{n}}(\epsilon) + \|h^{(3)}\|_\infty\,a_{\mathbf{n}}\left(\frac{4}{3}\epsilon + \sqrt{B_{\mathbf{n}}(\epsilon)}\right), \tag{13}$$

where

$$B_{\mathbf{n}}(\epsilon) = \sum_{(i,j)\in D_{\mathbf{n},2}}\mathbb{E}\left[\|W_{\mathbf{n},ij}\|^2\mathbb{I}_{\{\|W_{\mathbf{n},ij}\|>\epsilon\}}\right], \quad \epsilon > 0,\ \mathbf{n} \in \mathbb{N}^2\,;$$
$$a_{\mathbf{n}} = \sum_{(i.j)\in D_{\mathbf{n},2}}\mathbb{E}\|W_{\mathbf{n},ij}\|^2 < \infty, \quad \mathbf{n} \in \mathbb{N}^2.$$

Moreover, these classical Lindeberg conditions imply the conditions from Lemma 1. Indeed, we have

$$\Delta_{\mathbf{n}} \leq 2\,\|h^{(2)}\|_\infty\,\epsilon^{-\delta}A_{\mathbf{n}} + \|h^{(3)}\|_\infty\,a_{\mathbf{n}}\left(\frac{4}{3}\epsilon + \epsilon^{-\delta/2}\sqrt{A_{\mathbf{n}}}\right),$$

for $\delta \in (0,1)$ and $\epsilon > 0$.

The proof of this remark for general independent random vectors is given in (Bardet et al. [4], p. 165) .

Remark 2. *Observe that Assumptions (Lin) and (Cov) imply that $B_{\mathbf{n}}(\epsilon) \underset{\mathbf{n}\to\infty}{\longrightarrow} 0$ and that $a_{\mathbf{n}} = \sum_{i=1}^k (r_{\mathbf{n}}v_{\mathbf{n}})^{-1}\,\mathrm{Cov}(f_i(Y_{\mathbf{n}}), f_i(Y_{\mathbf{n}})) \underset{\mathbf{n}\to\infty}{\longrightarrow} \sum_{i=1}^k c(f_i, f_i) < \infty$, respectively. Therefore, if the blocks $(Y_{\mathbf{n},ij})_{(i,j)\in D_{\mathbf{n},2}}$ are independent and if Assumptions (Lin) and (Cov) hold, then from Lemma 1 and*

Remark 1, the fidis of the empirical process $(Z_{\mathbf{n}}(f))_{f\in\mathcal{F}}$ *of cluster functionals converges to the fidis of a Gaussian process* $(Z(f))_{f\in\mathcal{F}}$ *with covariance function c.*

For the dependent case, we need to consider more notations:
Let $L_i^j := \{(i,v) : v \in [j]\} \subset D_{\mathbf{n},2}$, for all $(i,j) \in D_{\mathbf{n},2}$. We set $L_i^0 = L_0^j = \emptyset$ for any $i \in [m_1]$ and any $j \in [m_2]$. For each $k \in \mathbb{N}$, $\mathbf{f}_k = (f_1,\dots,f_k) \in \mathcal{F}^k$, $\mathbf{t} \in \mathbb{R}^k$ and $\mathbf{n} \in \mathbb{N}^2$, we define

$$T_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k) := \sum_{(j_1,j_2)\in D_{\mathbf{n},2}} \left|\operatorname{Cov}\left(\exp\left(\mathrm{i}\langle \mathbf{t}, \sum_{(u_1,u_2)\in D_{\mathbf{n},2}\setminus(\cup_{l=0}^{j_1-1} L_l^{m_2}\cup L_{j_1}^{j_2})} W_{\mathbf{n},u_1u_2}\rangle\right), \exp(\mathrm{i}\langle \mathbf{t}, W_{\mathbf{n},j_1j_2}\rangle)\right)\right|.$$

Lemma 2 (Dependent Lindeberg lemma)**.** *Suppose that the random variables* $(W_{\mathbf{n},ij})_{(i,j)\in D_{\mathbf{n},2}}$ *defined in (5) satisfy Assumption (Lin'). Consider the special case of complex exponential functions* $h(\mathbf{w}) = \exp(\mathrm{i}\langle \mathbf{t}, \mathbf{w}\rangle)$ *with* $\mathbf{t} \in \mathbb{R}^k$*. Then, for each* $k \in \mathbb{N}$ *and each* $k-$*tuple* $\mathbf{f}_k = (f_1,\dots,f_k)$ *of cluster functionals, the following inequality holds:*

$$\Delta_{\mathbf{n}} \le T_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k) + 6\|\mathbf{t}\|^{2+\delta} A_{\mathbf{n}}, \qquad \mathbf{n} \in \mathbb{N}^2.$$

Proof. Consider $(W^*_{\mathbf{n},j_1j_2})_{(j_1,j_2)\in D_{\mathbf{n},2}}$ an array of independent random variables satisfying Assumption (Lin') and such that $(W^*_{\mathbf{n},j_1j_2})_{(j_1,j_2)\in D_{\mathbf{n},2}}$ is independent of $(W_{\mathbf{n},j_1j_2})_{(j_1,j_2)\in D_{\mathbf{n},2}}$ and $(W'_{\mathbf{n},j_1j_2})_{(j_1,j_2)\in D_{\mathbf{n},2}}$. Moreover, assume that $W^*_{\mathbf{n},j_1j_2}$ has the same distribution as $W_{\mathbf{n},j_1j_2}$ for $(j_1,j_2) \in D_{\mathbf{n},2}$.

Then, using the same decomposition (7) in the proof of the previous lemma, one can also write

$$\Delta_{\mathbf{n},j_1j_2} \le \left|\mathbb{E}\left[h_{j_1j_2}(V_{\mathbf{n},j_1j_2} + W_{\mathbf{n},j_1j_2}) - h_{j_1j_2}(V_{\mathbf{n},j_1j_2} + W^*_{\mathbf{n},j_1j_2})\right]\right| + \left|\mathbb{E}\left[h_{j_1j_2}(V_{\mathbf{n},j_1j_2} + W^*_{\mathbf{n},j_1j_2}) - h_{j_1j_2}(V_{\mathbf{n},j_1j_2} + W'_{\mathbf{n},j_1j_2})\right]\right|. \quad (14)$$

Then, from the previous lemma, the second term of the right-hand side (RHS) of the inequality (14) is bounded by

$$6\,\|h^{(2)}\|_\infty^{1-\delta}\,\|h^{(3)}\|_\infty^{\delta}\,\mathbb{E}\|W_{\mathbf{n},j_1j_2}\|^{2+\delta} \le 6\,\|\mathbf{t}\|^{2+\delta}\,\mathbb{E}\|W_{\mathbf{n},j_1j_2}\|^{2+\delta}.$$

For the first term of the RHS of the inequality (14), first notice that, for a $\mathbb{R}^k-$valued random vector X independent from $(W'_{\mathbf{n},j_1j_2})_{(j_1,j_2)\in D_{\mathbf{n},2}}$, we have

$$\begin{aligned}\mathbb{E}h_{j_1j_2}(X) &= \mathbb{E}\left[h\left(X + \sum_{u=0}^{j_1-1}\sum_{v=1}^{m_2} W'_{\mathbf{n},uv} + \sum_{v=0}^{j_2-1} W'_{\mathbf{n},j_1v}\right)\right] \\ &= \exp\left(-\frac{1}{2}\mathbf{t}^T\left(\sum_{u=0}^{j_1-1}\sum_{v=1}^{m_2} C_{\mathbf{n},uv} + \sum_{v=0}^{j_2-1} C_{\mathbf{n},j_1v}\right)\mathbf{t}\right)\mathbb{E}[\exp(\mathrm{i}\langle \mathbf{t}, X\rangle)],\end{aligned}$$

because $W'_{\mathbf{n},j_1j_2} \sim \mathcal{N}_k(\mathbf{0}, C_{\mathbf{n},j_1j_2})$, where $C_{\mathbf{n},j_1j_2} := \operatorname{Cov}(W_{\mathbf{n},j_1j_2})$ is the covariance matrix of the vector $W_{\mathbf{n},j_1j_2}$, for $(j_1,j_2) \in D_{\mathbf{n},2}$. For $j_1 = 0$ or $j_2 = 0$, recall that $W_{\mathbf{n},j_1j_2} = 0$. In this case, we also set $C_{\mathbf{n},j_1j_2} = 0$.

Thus,

$$
\begin{aligned}
&\left|\mathbb{E}\left[h_{j_1j_2}(V_{\mathbf{n},j_1j_2}+W_{\mathbf{n},j_1j_2})-h_{j_1j_2}(V_{\mathbf{n},j_1j_2}+W^*_{\mathbf{n},j_1j_2})\right]\right| \\
&\quad=\left|\exp\left(-\frac{1}{2}\mathbf{t}^T\left(\sum_{u=0}^{j_1-1}\sum_{v=1}^{m_2}C_{\mathbf{n},uv}+\sum_{v=0}^{j_2-1}C_{\mathbf{n},j_1v}\right)\mathbf{t}\right)\right. \\
&\qquad\left.\times\,\mathbb{E}\left[\exp\left(\mathrm{i}\langle\mathbf{t},V_{\mathbf{n},j_1j_2}\rangle\right)\left(\exp\left(\mathrm{i}\langle\mathbf{t},W_{\mathbf{n},j_1j_2}\rangle\right)-\exp\left(\mathrm{i}\langle\mathbf{t},W^*_{\mathbf{n},j_1j_2}\rangle\right)\right)\right]\right| \\
&\quad=\left|\exp\left(-\frac{1}{2}\mathbf{t}^T\left(\sum_{u=0}^{j_1-1}\sum_{v=1}^{m_2}C_{\mathbf{n},uv}+\sum_{v=0}^{j_2-1}C_{\mathbf{n},j_1v}\right)\mathbf{t}\right)\right| \\
&\qquad\times\left|\mathrm{Cov}\left(\exp\left(\mathrm{i}\langle\mathbf{t},V_{\mathbf{n},j_1j_2}\rangle\right),\exp\left(\mathrm{i}\langle\mathbf{t},W_{\mathbf{n},j_1j_2}\rangle\right)\right)\right| \\
&\qquad\le\left|\mathrm{Cov}\left(\exp\left(\mathrm{i}\langle\mathbf{t},V_{\mathbf{n},j_1j_2}\rangle\right),\exp\left(\mathrm{i}\langle\mathbf{t},W_{\mathbf{n},j_1j_2}\rangle\right)\right)\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta_{\mathbf{n}}&=\sum_{(j_1,j_2)\in D_{\mathbf{n},2}}\Delta_{\mathbf{n},j_1j_2} \\
&\le\sum_{(j_1,j_2)\in D_{\mathbf{n},2}}\left(\left|\mathrm{Cov}\left(\exp\left(\mathrm{i}\langle\mathbf{t},V_{\mathbf{n},j_1j_2}\rangle\right),\exp\left(\mathrm{i}\langle\mathbf{t},W_{\mathbf{n},j_1j_2}\rangle\right)\right)\right|+6\|\mathbf{t}\|^{2+\delta}\mathbb{E}\|W_{\mathbf{n},j_1j_2}\|^{2+\delta}\right) \\
&=T_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k)+6\|\mathbf{t}\|^{2+\delta}A_{\mathbf{n}}.
\end{aligned}
$$

This completes the proof of Lemma 2. □

The previous lemma together with Remark 1 imply the following theorem.

Theorem 1 (CLT for cluster functionals on random fields)**.** *Suppose that the basic Assumption (Bas) holds and that Assumptions (Lin) and (Cov) are satisfied. Then, if for each $k\in\mathbb{N}$, $T_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k)$ converges to zero as $\mathbf{n}\to\infty$, for all $\mathbf{t}\in\mathbb{R}^k$ and all $k-$tuple $\mathbf{f}_k=(f_1,\ldots,f_k)\in\mathcal{F}^k$ of cluster functionals, the fidis of the empirical process $(Z_{\mathbf{n}}(f))_{f\in\mathcal{F}}$ of cluster functionals converges to the fidis of a Gaussian process $(Z(f))_{f\in\mathcal{F}}$ with covariance function c defined in (Cov).*

Proof. The assumptions (Lin) and (Cov) imply that, as $\mathbf{n}\to\infty$, $B_{\mathbf{n}}(\epsilon)\longrightarrow 0$ and $a_{\mathbf{n}}\longrightarrow\sum_{s=1}^k c(f_s,f_s)<\infty$, respectively. Therefore, taking into account Remark 1, we obtain from Lemma 2 that, for each $k\in\mathbb{N}$,

$$\Delta_{\mathbf{n}}=\left|\mathbb{E}\left[h\left(\sum_{(i,j)\in D_{\mathbf{n},2}}W_{\mathbf{n},ij}\right)-h\left(\sum_{(i,j)\in D_{\mathbf{n},2}}W'_{\mathbf{n},ij}\right)\right]\right|\underset{\mathbf{n}\to\infty}{\longrightarrow}0,$$

for all $\mathbf{t}\in\mathbb{R}^k$, with $h(\mathbf{w})=\exp(\mathrm{i}\langle\mathbf{t},\mathbf{w}\rangle)$, because by hypothesis, $T_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k)\underset{\mathbf{n}\to\infty}{\longrightarrow}0$ for all $\mathbf{t}\in\mathbb{R}^k$ and all $\mathbf{f}_k=(f_1,\ldots,f_k)\in\mathcal{F}^k$.
Notice that

$$W'_{\mathbf{n}}:=\sum_{(i,j)\in D_{\mathbf{n},2}}W'_{\mathbf{n},ij}\sim\mathcal{N}_k(\mathbf{0},m_1m_2\,\mathrm{Cov}(W_{\mathbf{n},11}))$$

and that $|\mathbb{E}(h(W'_{\mathbf{n}})-h(W))|\underset{\mathbf{n}\to\infty}{\longrightarrow}0$, where $W\sim\mathcal{N}_k(\mathbf{0},\Sigma_k)$, with $\Sigma_k=(c(f_i,f_j))_{(i,j)\in[k]^2}$.

Using triangular inequality, we deduce that

$$\left|\mathbb{E}\left[h\left(\sum_{(i,j)\in D_{\mathbf{n},2}}W_{\mathbf{n},ij}\right)-h(W)\right]\right|\underset{\mathbf{n}\to\infty}{\longrightarrow}0,$$

and therefore $(Z_{\mathbf{n}}(f_1),\ldots,Z_{\mathbf{n}}(f_k))=\sum_{(i,j)\in D_{\mathbf{n},2}}W_{\mathbf{n},ij}\xrightarrow[\mathbf{n}\to\infty]{\mathcal{D}}W$. The proof of Theorem 1 is complete. □

Remark 3. *The previous theorem can be formulated for $d = 3$ as follows. Define $S_i = \{(u, v, w) : u \in [i], v \in [m_2], w \in [m_3]\} \subseteq D_{\mathbf{n},3}$, for $i \in [m_1]$, with the convention $S_0 = \emptyset$. Moreover, $L^k_{ij} = \{(i, j, w) : w \in [k]\}$, for $(i, j, k) \in D_{\mathbf{n},3}$, and $L^k_{ij} = \emptyset$ if i, j or k is zero. Then, if Assumptions (Bas), (Lin), (Cov) are satisfied (for $d = 3$), and if for each $k \in \mathbb{N}$,*

$$T^*_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k) = \sum_{(j_1,j_2,j_3)\in D_{\mathbf{n},3}} \left|\mathrm{Cov}\big(\exp(\mathrm{i}\langle \mathbf{t}, V_{\mathbf{n},j_1j_2j_3}\rangle), \exp\big(\mathrm{i}\langle \mathbf{t}, W_{\mathbf{n},j_1j_2j_3}\rangle\big)\big)\right| \tag{15}$$

converges to zero as $\mathbf{n} \to \infty$ for all $\mathbf{t} \in \mathbb{R}^k$ and all $k-$tuple $\mathbf{f}_k = (f_1, \ldots, f_k) \in \mathcal{F}^k$ of cluster functionals, with

$$V_{\mathbf{n},j_1j_2j_3} := \sum_{(u_1,u_2,u_3)\in D_{\mathbf{n},3}\setminus\left(S_{j_1-1} \cup \bigcup_{l=0}^{j_2-1} L^{m_3}_{j_1 l} \cup L^{j_3}_{j_1 j_2}\right)} W_{\mathbf{n},u_1u_2u_3},$$

the fidis of the empirical process $(Z_{\mathbf{n}}(f))_{f\in\mathcal{F}}$ of cluster functionals converges to the fidis of a Gaussian process $(Z(f))_{f\in\mathcal{F}}$ with covariance function c.

Remark 4. *We have mentioned earlier that $\mathbf{n} = (n_1, \ldots, n_d) \to \infty$ means $n_i \to \infty$ for each $i \in [d]$. However, the limits of the sequences indexed with $\mathbf{n}$, as $\mathbf{n} \to \infty$, could be reformulated in terms of the limits of such sequences as "$\mathbf{n} \to \infty$ along a monotone path on the lattice $\mathbb{N}^d$", i.e., along $\mathbf{n} = (\lceil\vartheta_1(n)\rceil, \ldots, \lceil\vartheta_d(n)\rceil)$ for some strictly increasing continuous functions $\vartheta_i : [1, \infty) \longrightarrow [1, \infty)$, with $i \in [d]$, such that $\vartheta_i(n) \longrightarrow \infty$ as $n \to \infty$, for $i \in [d]$.*

Suppose that from each block $Y_{\mathbf{n}}$ we extract a sub-block $Y'_{\mathbf{n}}$ and that the remaining parts $R_{\mathbf{n}} = Y_{\mathbf{n}} - Y'_{\mathbf{n}}$ of the blocks $Y_{\mathbf{n}}$ do not influence the process $Z_{\mathbf{n}}(f)$. In particular, this last statement is fulfilled if

$$(r_{\mathbf{n}} v_{\mathbf{n}})^{-1}\mathbb{E}|\Delta_{\mathbf{n}}(f) - \mathbb{E}\Delta_{\mathbf{n}}(f)|^2 \mathbb{I}_{\{|\Delta_{\mathbf{n}}(f) - \mathbb{E}\Delta_{\mathbf{n}}(f)| \le \sqrt{n_{\mathbf{n}} v_{\mathbf{n}}}\}} = o(1)$$

and $\mathbb{P}(|\Delta_{\mathbf{n}}(f) - \mathbb{E}\Delta_{\mathbf{n}}(f)| > \sqrt{n_{\mathbf{n}} v_{\mathbf{n}}}) = o(r_{\mathbf{n}}/n_{\mathbf{n}})$, where $\Delta_{\mathbf{n}}(f) := f(Y_{\mathbf{n}}) - f(Y'_{\mathbf{n}})$. This assumption would allow us to consider $T_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k)$ (or $T^*_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k)$) as a function of the blocks $Y'_{\mathbf{n}}$ (separated by $l_{\mathbf{n}}$) instead of the blocks $Y_{\mathbf{n}}$, in order to provide them bounds based on either the strong mixing coefficient of Rosenblatt [12] or the weak-dependence coefficients of Doukhan and Louhichi [10] for stationary random fields. These bounds are developed in Gómez-García [5] for the case of weakly-dependent time series. However, we do not develop them in the random field context as this is not the aim of this work. This topic will be addressed in a forthcoming applied statistics paper with numerical simulations.

3. Asymptotic Behavior of the Extremogram for Space–Time Processes

In this section, we propose a measure (in two versions) of serial dependence on space and time of extreme values of space–time processes. We provide an estimator for this measure and we use Theorem 1 in order to establish an asymptotic result. This work is inspired by the extremogram for time series defined in Davis and Mikosch [13].

Let $X = \{X_t(\mathbf{s}) : \mathbf{s} \in \mathbb{Z}^d, t \ge 0\}$ be a $\mathbb{R}^k$-valued space–time process, which is stationary in both space and time. We define the extremogram of X for two sets A and B both bounded away from zero by

$$\rho_{A,B}(\mathbf{s}, h_t) := \lim_{x\to\infty} \mathbb{P}\Big(x^{-1}X_{h_t}(\mathbf{s}) \in B \Big| x^{-1}X_0(\mathbf{0}) \in A\Big), \tag{16}$$

with $(\mathbf{s}, h_t) \in \mathbb{Z}^d \times [0, \infty)$, provided that the limit exists.

In estimating the extremogram, the limit on x in (16) is replaced by a high quantile u_n of the process. Defining u_n as the $(1-1/k_n)-$quantile of the stationary distribution of $\|X_t(\mathbf{s})\|$ or related quantity, with $k_n = o(n) \longrightarrow \infty$, as $n \to \infty$, one can redefine (16) by

$$\rho_{A,B}(\mathbf{s}, h_t) = \lim_{n\to\infty} \mathbb{P}\Big(u_n^{-1}X_{h_t}(\mathbf{s}) \in B \Big| u_n^{-1}X_0(\mathbf{0}) \in A\Big), \tag{17}$$

with $(\mathbf{s}, h_t) \in \mathbb{Z}^d \times [0, \infty)$.

The choice of such a sequence of quantiles $(u_n)_{n\in\mathbb{N}}$ is not arbitrary. The main condition to guarantee the existence of the limit (17) for any two sets A and B bounded away from zero, is that it must satisfy the following convergence

$$k_n\mathbb{P}\Big(u_n^{-1}\Big(X_{t_1}(\mathbf{s}_1), \ldots, X_{t_p}(\mathbf{s}_p)\Big) \in \cdot\Big) \xrightarrow[n\to\infty]{vague} m_{(\mathbf{s}_1,t_1),\ldots,(\mathbf{s}_p,t_p)}(\,\cdot\,), \tag{18}$$

for all $(\mathbf{s}_i, t_i) \in \mathbb{Z}^d \times [0, \infty)$, $i \in [p]$, $p \in \mathbb{N}$, where

$$\Big(m_{(\mathbf{s}_1,t_1),\ldots,(\mathbf{s}_p,t_p)}\Big)_{(\mathbf{s_i},t_i)\in\mathbb{Z}^d\times[0,\infty),\ i\in[p],\ p\in\mathbb{N}}$$

is a collection of Radon measures on the Borel σ-field $\mathcal{B}(\overline{\mathbb{R}}^{kp} \setminus \{\mathbf{0}\})$, not all of them being the null measure, with $m_{(\mathbf{s}_1,t_1),\ldots,(\mathbf{s}_p,t_p)}(\overline{\mathbb{R}}^{kp} \setminus \mathbb{R}^{kp}) = 0$. In this case, we have

$$\begin{aligned}\mathbb{P}\Big(u_n^{-1}X_{h_t}(\mathbf{s}) \in B \Big| u_n^{-1}X_0(\mathbf{0}) \in A\Big) &= \frac{k_n\mathbb{P}\big(u_n^{-1}\big(X_0(\mathbf{0}), X_{h_t}(\mathbf{s})\big) \in A \times B\big)}{k_n\mathbb{P}\Big(u_n^{-1}X_0(\mathbf{0}) \in A\Big)} \\ &\longrightarrow \frac{m_{(\mathbf{0},0),(\mathbf{s},h_t)}(A \times B)}{m_{(\mathbf{0},0)}(A)} = \rho_{A,B}(\mathbf{s}, h_t),\end{aligned}$$

provided that $m_{(\mathbf{0},0)}(A) > 0$.

Remark 5. *The condition (18) is particularly satisfied if the space–time process X is regularly varying. For details and examples of regularly varying space–time processes and time series, see Davis and Mikosch [1] and Basrak and Segers [14], respectively.*

Note that the extremogram (17) is a function of two lags: a spatial-lag $\mathbf{s} \in \mathbb{Z}^d$ and a non-negative time-lag h_t. Due to all the spatial values that the spatial-lag $\mathbf{s}$ takes, in practice, it is very complicated to analyze the results of estimating such an extremogram. Moreover, the calculation would be very slow in terms of computation. To obtain a simpler interpretation and to simplify the calculations, we assume that the space–time process X satisfies the following "isotropy" condition:

(I) For each pair of non-negative integers h_t and h_s,

$$\mathbb{P}\big(X_0(\mathbf{0}) \in A, X_{h_t}(\mathbf{s}) \in B\big) = \mathbb{P}\big(X_0(\mathbf{0}) \in A, X_{h_t}(\mathbf{s}') \in B\big), \quad \forall \mathbf{s}, \mathbf{s}' \in \mathbb{S}_{h_s}^{d-1},$$

where $\mathbb{S}_h^{d-1} := \Big\{\mathbf{s} \in \mathbb{Z}^d : \|\mathbf{s}\|_\infty = h\Big\}$ with $h \geq 0$ and $\|(s_1, \ldots, s_d)\|_\infty = \max_{i=1,\ldots,d} |s_i|$.

Under this condition, the extremogram (17) can be redefined using only two non-negative integer lags: a spatial-lag h_s and a time-lag h_t. Indeed, under Condition (I), we define the iso-extremogram of X for two sets A and B both bounded away from zero by

$$\rho^*_{A,B}(h_s, h_t) = \rho_{A,B}(h_s\,\vec{e}_1,\ h_t), \qquad h_s, h_t \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}, \tag{19}$$

where $\vec{e}_1 = (1, 0, 0, 0, \ldots, 0) \in \mathbb{R}^d$ is the first element of the canonical basis of $\mathbb{R}^d$.

We now propose an estimator for the iso-extremogram. For this, without loss of generality, consider $d = 2$ because the case $d > 2$ can be treated in the same way.

Let $X_{\mathbf{n}} := \{X_t(i,j) : (i,j,t) \in [n_1] \times [n_2] \times [n_3]\}$ be the observations from a $\mathbb{R}^k$-valued space–time process X, stationary in both space and time, and which satisfies Condition (I). Let us set $n = n_1 n_2 n_3$. The sample iso-extremogram based on the observations $X_{\mathbf{n}}$ is given by

$$\widehat{\rho^*}_{A,B}(h_s,h_t) \quad := \quad \frac{\displaystyle\sum_{(j_1,j_2)\in[m_1]\times[m_2]} \sum_{t=1}^{n_3-h_t} \sum_{(i_1,i_2)\in\mathbb{S}_{h_s}(c_{j_1 j_2})} \frac{\mathbb{I}_{\left\{\frac{X_{t+h_t}(i_1,i_2)}{u_n}\in B,\ \frac{X_t(c_{j_1 j_2})}{u_n}\in A\right\}}}{\#\mathbb{S}_{h_s}(c_{j_1 j_2})}}{\displaystyle\sum_{(j_1,j_2)\in[m_1]\times[m_2]} \sum_{t=1}^{n_3} \mathbb{I}_{\left\{\frac{X_t(c_{j_1 j_2})}{u_n}\in A\right\}}}, \tag{20}$$

for $h_s = 0,1,2,\ldots,\lceil 2^{-1}\min\{r_1,r_2\}\rceil - 1$, and $h_t = 0,\ldots,n-1$, where

$$c_{ij} := \left(\left\lceil \frac{(2i-1)r_1+1}{2} \right\rceil, \left\lceil \frac{(2j-1)r_2+1}{2} \right\rceil\right)$$

denotes the "center" of the block $B_{ij} = [(i-1)r_1+1 : ir_1] \times [(j-1)r_2+1 : jr_2]$, for $(i,j) \in [m_1] \times [m_2]$. Moreover, $\mathbb{S}_h(u,v) := \{(i,j) \in [n_1] \times [n_2] : \|(u,v)-(i,j)\|_\infty = h\}$ with $h \geq 0$ and $\#E$ denotes the cardinality of the set E. We recall that $r_i = r_{n_i,i}$ and $m_i = \lceil n_i/r_i \rceil$, for $i = 1,2,3$.

Defining the cluster functional

$$f_{A,B,h_1,h_2} : \left(\bigcup_{l_1,l_2,l_3=1}^{\infty} \mathbb{B}_{l_1 l_2 l_3}(\mathbb{R}^k), \mathcal{R}_\cup \right) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

for $h_1, h_2 = 0,1,2,\ldots$, such that

$$f_{A,B,h_1,h_2}\Big((x_{(i_1,i_2,i_3)})_{(i_1,i_2,i_3)\in[l_1]\times[l_2]\times[l_3]} \Big) = \sum_{(i_1,i_2)\in\mathbb{S}_{h_1}(c)} \sum_{i_3=1}^{l_3-h_2} \frac{\mathbb{I}_{A\times B}(x_{(c,i_3)}, x_{(i_1,i_2,i_3+h_2)})}{\#\mathbb{S}_{h_1}(c)}, \tag{21}$$

with $c = (\lceil (l_1+1)/2 \rceil, \lceil (l_2+1)/2 \rceil) \in [l_1] \times [l_2]$ (the "center" of the block $B = [l_1] \times [l_2]$), we can rewrite the estimator (20) as

$$\widehat{\rho^*}_{A,B}(h_s,h_t) = \frac{\sum_{(j_1,j_2,j_3)\in D_{\mathbf{n},3}} f_{A,B,h_s,h_t}(Y_{\mathbf{n},j_1 j_2 j_3}) + \delta_{\mathbf{n}} + R_{A,B,h_s,h_t}}{\sum_{(j_1,j_2,j_3)\in D_{\mathbf{n},3}} f_{A,A,0,0}(Y_{\mathbf{n},j_1 j_2 j_3}) + R_{A,A,0,0}}, \tag{22}$$

where

$$\delta_{\mathbf{n}} := \sum_{(j_1,j_2,j_3)\in D_{\mathbf{n},3}} \sum_{(i_1,i_2)\in\mathbb{S}_{h_s}(c_{j_1 j_2})} \sum_{t=j_3 r_3-h_t+1}^{j_3 r_3} \frac{\mathbb{I}_{\left\{\frac{X_{t+h_t}(i_1,i_2)}{u_n}\in B,\ \frac{X_t(c_{j_1 j_2})}{u_n}\in A\right\}}}{\#\mathbb{S}_{h_s}(c_{j_1 j_2})},$$

$$R_{A,B,h_s,h_t} := \sum_{(j_1,j_2)\in[m_1]\times[m_2]} \sum_{(i_1,i_2)\in\mathbb{S}_{h_2}(c_{j_1 j_2})} \sum_{t=m_3 r_3+1}^{n_3-h_t} \frac{\mathbb{I}_{\left\{\frac{X_{t+h_t}(i_1,i_2)}{u_n}\in B,\ \frac{X_t(c_{j_1 j_2})}{u_n}\in A\right\}}}{\#\mathbb{S}_{h_s}(c_{j_1 j_2})}.$$

We can therefore write (22) in terms of empirical processes of cluster functionals (4) and use Lindeberg CLT for cluster functionals on random fields (Theorem 1) together with suitable conditions of joint distributions, in order to prove the convergence in distribution of the iso-extremogram estimator.

For this, first of all, we make some considerations: the normalized random variables are defined here by $X_{\mathbf{n},(i_1,i_2,t)} = u_n^{-1} X_t(i_1,i_2)$, where $\mathbf{n} = (n_1,n_2,n_3)$ and $n = n_1 n_2 n_3$; and the random blocks $(Y_{\mathbf{n},j_1 j_2 j_3})_{(j_1,j_2,j_3)\in D_{\mathbf{n},3}}$ as in (2). We define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathcal{F}_{A,B} :=$

$\{f_{A,B,h_s,h_t} : h_s, h_t \in \mathbb{N}_0\}$ as the family of cluster functionals defined in (21). Moreover, for the set A, bounded away from zero, let $v_n := \mathbb{P}(u_n^{-1} X_0(0,0) \in A)$.

Secondly, consider the following conditions:

(Cov′) For each $h_s, h'_s, h_t, h'_t \in \mathbb{N}_0$,

$$\sum_{\mathbf{i}\in\mathbb{S}_{h_s}(c)} \sum_{\mathbf{i}'\in\mathbb{S}_{h'_s}(c)} \sum_{t=1}^{r_3-h_t} \sum_{t'=1}^{r_3-h'_t} \frac{\mathbb{P}\Big(u_n^{-1}(X_t(c), X_{t'}(c)) \in A^2, (X_{t+h_t}(\mathbf{i}), X_{t'+h'_t}(\mathbf{i}')) \in B^2\Big)}{r v_n \cdot \#\mathbb{S}_{h_s}(c) \cdot \#\mathbb{S}_{h'_s}(c)}$$

and

$$\sum_{\mathbf{i}\in\mathbb{S}_{h_s}(c)} \sum_{t=1}^{r_3-h_t} \sum_{t'=1}^{r_3} \frac{\mathbb{P}(u_n^{-1}(X_t(c), X_{t'}(c)) \in A^2, X_{t+h_t}(\mathbf{i}) \in B)}{r v_n \cdot \#\mathbb{S}_{h_s}(c)}$$

converge to $\sigma_{A,B}((h_s,h_t),(h'_s,h'_t))$ and $\sigma'_{A,B}(h_s,h_t)$, respectively, where $r = r_1 r_2 r_3$ and $c = (\lceil (r_1+1)/2 \rceil, \lceil (r_2+1)/2 \rceil)$ (the "center" of the block $B_{11} = [r_1] \times [r_2]$).

(C) $$\sum_{(c,t),(c',t')\in C(r_1,r_2)\times[n_3]} \mathbb{P}\Big(u_n^{-1}(X_t(c), X_{t'}(c')) \in A \times A\Big) = \mathcal{O}(1),$$

where $C(r_1,r_2) := \{c_{ij} \in [n_1] \times [n_2] : (i,j) \in [m_1] \times [m_2]\}$ is set of the "centers" of the blocks $B_{ij} = [(i-1)r_1 + 1 : ir_1] \times [(j-1)r_2 + 1 : jr_2]$.

Proposition 1 (CLT for the iso-extremogram estimator). *Assume that the following conditions hold for the $\mathbb{R}^k$-valued space–time process*

$$X = \Big\{X_t(\mathbf{s}) : (\mathbf{s},t) \in \mathbb{Z}^2 \times [0,\infty)\Big\}.$$

1. *The process X is stationary in both space and time and satisfies Condition (I).*
2. *The sequence (u_n) is such that (18) holds. Moreover, $r \ll v_n^{-1} \ll n$ and $\sqrt{n v_n} \ll r \ll n v_n r_3$, where $n = n_1 n_2 n_3$, $r = r_1 r_2 r_3$, $r_i \ll n_i$ and $r_i = r_{n_i,i} \longrightarrow \infty$, for $i = 1,2,3$.*
3. *Conditions (Cov′) and (C) hold, and the Lindeberg condition (Lin) is satisfied for the normalized variables $X_{\mathbf{n},(\mathbf{s},t)} = u_n^{-1} X_t(\mathbf{s})$ together with the family of cluster functionals $\mathcal{F}_{A,B}$. Moreover, for each $k \in \mathbb{N}$, the coefficient $T^*_{\mathbf{n},\mathbf{t}}(\mathbf{f}_k)$ defined in (15) converges to zero as $\mathbf{n} \to \infty$, for all $k-$tuple of cluster functionals $(f_1,\ldots,f_k) \in \mathcal{F}^k_{A,B}$ and all $\mathbf{t} \in \mathbb{R}^k$. The same assumption holds together with the family $\mathcal{F}_A := \{f_{A,A,0,0}\}$, which contains a single functional.*

Then, for each $(L_s, L_t) \in \mathbb{N}_0 \times \mathbb{N}_0$,

$$\frac{\sqrt{n v_{\mathbf{n}}}}{r_1 r_2}\Big(\widehat{\rho^*}_{A,B}(h_s,h_t) - \rho^*_{A,B,n}(h_s,h_t)\Big)_{0\le h_s\le L_s,\ 0\le h_t\le L_t} \xrightarrow[\mathbf{n}\to\infty]{\mathcal{D}} \mathcal{N}(0, \Sigma_{A,B,L_s,L_t}), \tag{23}$$

where $\rho^*_{A,B,n}(h_s,h_t) := \mathbb{P}(u_n^{-1} X_{h_t}(h_s \vec{e}_1) \in B \,|\, u_n^{-1} X_0(\mathbf{0}) \in A)$ and Σ_{A,B,L_s,L_t} is the covariance matrix, defined by the coefficients

$$\sigma_{\mathbf{h},\mathbf{h}'} = \sigma_{A,B}(\mathbf{h},\mathbf{h}') - \rho^*_{A,B}(\mathbf{h}')\sigma'_{A,B}(\mathbf{h}) - \rho^*_{A,B}(\mathbf{h})\sigma'_{A,B}(\mathbf{h}') + \rho^*_{A,B}(\mathbf{h})\rho^*_{A,B}(\mathbf{h}')\sigma'_{A,A}(\mathbf{0}),$$

with $\mathbf{h}, \mathbf{h}' \in [0 : L_s] \times [0 : L_t]$.

Proof. Consider the expression (22) of the iso-extremogram estimator. Then, for $(h_s,h_t) \in [0 : L_s] \times [0, L_t]$, we obtain that

$$\frac{\sqrt{n v_n}}{r_1 r_2}\Big(\widehat{\rho^*}_{A,B}(h_s,h_t) - \rho^*_{A,B,n}(h_s,h_t)\Big)$$
$$= \frac{Z_{\mathbf{n}}(f_{A,B,h_s,h_t}) - \Big(\frac{m h_t v_n}{\sqrt{n v_n}} + Z_{\mathbf{n}}(f_{A,A,0,0})\Big)\rho^*_{A,B,n}(h_s,h_t) + \frac{\delta_{\mathbf{n}}}{\sqrt{n v_n}} + R}{\frac{r_1 r_2}{\sqrt{n v_n}} Z_{\mathbf{n}}(f_{A,A,0,0}) + 1 + \frac{r_1 r_2 R_{A,A,0,0}}{n v_n}}, \tag{24}$$

where $Z_{\mathbf{n}}(\cdot)$ denotes the empirical process of cluster functionals (4). Furthermore, here $R = (nv_n)^{-1}\Big(R_{A,B,h_s,h_t} - \rho^*_{A,B,n} R_{A,A,0,0}\Big)$ and $m = m_1 m_2 m_3$.

Now, notice that Chebyshev's inequality applied on the random variables R and $r_1 r_2 R_{A,A,0,0}/\ (nv_n)$ implies that they converge to zero in probability as $\mathbf{n} \to \infty$. Similarly, applying Chebyshev's inequality together with the condition $\sqrt{nv_n} = o(r)$, we prove that $(nv_n)^{-1/2}\delta_{\mathbf{n}} \xrightarrow{P} 0$, as $\mathbf{n} \to \infty$. This last condition ($\sqrt{nv_n} = o(r)$) also guarantees that $mh_t v_n (nv_n)^{-1/2} \underset{\mathbf{n}\to\infty}{\longrightarrow} 0$. Again, Chebyshev's inequality on the random variable $\frac{r_1 r_2}{\sqrt{nv_n}} Z_{\mathbf{n}}(f_{A,A,0,0})$, followed by Condition (C) and $r = o(nv_n r_3)$, implies that this converges to zero in probability as $\mathbf{n} \to \infty$. Thus,

$$\frac{\sqrt{nv_n}}{r_1 r_2}\Big(\widehat{\rho^*}_{A,B}(h_s,h_t) - \rho^*_{A,B,n}(h_s,h_t)\Big) = Z_{\mathbf{n}}(f_{A,B,h_s,h_t}) - \rho^*_{A,B,n}(h_s,h_t) Z_{\mathbf{n}}(f_{A,A,0,0}) + o(1).$$

From Theorem 1, the assumption 3 implies that $(Z_n(f_{A,B,h_s,h_t}))_{(h_s,h_t)\in[0:L_s]\times[0:L_t]}$ converges to a centered Gaussian random variable with covariance matrix

$$(\sigma_{A,B}(\mathbf{h},\mathbf{h}'))_{\mathbf{h},\mathbf{h}'\in[0:L_s]\times[0:L_t]},$$

for each $(L_s, L_t) \in \mathbb{N}_0^2$. Using the same argument, we prove that $Z_n(f_{A,A,0,0})$ converges to a centered Gaussian variable with variance $\sigma_{A,A}(\mathbf{0},\mathbf{0})$.

Finally, considering the existence of $\sigma'_{A,B}$ in (Cov'), we obtain the desired result. □

4. Conclusions and Perspectives

We have proved Lindeberg lemmas for cluster functionals on stationary random fields. This allowed us to obtain a CLT for the finite-dimensional marginal distributions of the empirical process (4) of cluster functionals of stationary random fields under the classical Lindeberg condition and the convergence to zero of a sequence $T_{\mathbf{n}}$ that summarizes the dependence between the blocks of values of the random field. Moreover, we have introduced a new spatio–temporal measure of serial extremal dependence: the iso-extremogram, a type of correlogram for extreme values of space–time processes. Under precise conditions, we have proved that the iso-extremogram estimator is asymptotically Gaussian.

In all our results, it can be noted that the sequence $T_{\mathbf{n}}$ converges to zero if the random field satisfies short range dependence conditions; either mixing or weak-dependence conditions. However, in this work we do not specify such conditions because it is not the aim of this paper, but of course it will be presented in a forthcoming applied statistics article including numerical simulations. To obtain a general idea of how to simplify the coefficient $T_{\mathbf{n}}$ using weak dependence coefficients, the reader is referred to Gómez-García [5] which deals with the time series framework.

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