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Differential Subordination and Superordination Results Associated with Mittag–Leffler Function

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Abstract: In this paper, we derive a number of interesting results concerning subordination and superordination relations for certain analytic functions associated with an extension of the Mittag–Leffler function.

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1. Definitions and Preliminaries

Let $\mathbb{H}$ be the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. Also, let $\mathbb{H}[a, n]$ denote the subclass of the functions $f \in \mathbb{H}$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C}). \tag{1}$$

Furthermore, let

$$A_m = \left\{ f \in \mathbb{H} \mid f(z) = z + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \dots \right\}.$$

Moreover, assume that $A = A_1$ which is the subclass of the functions $f \in \mathbb{H}$ of the form

$$f(z) = z + a_2 z^2 + \dots \quad . \tag{2}$$

For $f, g \in \mathbb{H}$, we say that the function f is subordinate to g, written symbolically as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$. In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [1,2]; see also [3]):

$$f(z) \prec g\,(z) \Leftrightarrow f(0) \prec g(0) \;\; and \;\; f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let λ and h be two analytic functions in $\mathbb{U}$, suppose

$$\Phi(r,s,t;z):\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}.$$

If λ and $\Phi(\lambda(z),z\lambda'(z),z^2\lambda''(z);z)$ are univalent functions in $\mathbb{U}$ and if λ satisfies the second-order superordination

$$h(z)\prec\Phi(\lambda(z),z\lambda'(z),z^2\lambda''(z);z), \tag{3}$$

then λ is called to be a solution of the differential superordination (3). (If f is subordinate to F, then F is superordination to f). An analytic function μ is called a subordinant of (3), if $\mu\prec\lambda$ for all the functions λ satisfying (3). A univalent subordinant $\widetilde{\mu}$ that satisfies $\mu\prec\widetilde{\mu}$ for all of the subordinants μ of (3), is called the best subordinant (cf., e.g., [2], see also [3]).

Miller and Mocanu [4] obtained sufficient conditions on the functions h, μ and Φ for which the following statement holds:

$$h(z)\prec\Phi(\lambda(z),z\lambda'(z),z^2\lambda''(z);z)\Rightarrow\mu(z)\prec\lambda(z). \tag{4}$$

The results of Miller and Mocanu [4] and Bulboaca [5] considered certain families of first-order differential superordination whenever superordination preserves integral operators [6]. Moreover, Ali et al. [7], used Bulboaca's results [5] and obtained the sufficient conditions for normalized analytic functions f to satisfy

$$\mu_1(z)\prec\frac{zf'(z)}{f(z)}\prec\mu_2(z), \tag{5}$$

where μ_1 and μ_2 are given univalent functions in $\mathbb{U}$ with $\mu_1(0)=1$. Also, Shanmugam et al. [8] obtained sufficient conditions for normalized analytic functions f to satisfy

$$\mu_1(z)\prec\frac{f(z)}{zf'(z)}\prec\mu_2(z),$$

and

$$\mu_1(z)\prec\frac{z^2f'(z)}{(f(z))^2}\prec\mu_2(z),$$

where μ_1 and μ_2 are given univalent functions in $\mathbb{U}$ with $\mu_1(0)=1$ and $\mu_2(0)=1$, while Obradovic and Owa [9] obtained some results of subordinations associated with $\left(\frac{f(z)}{z}\right)^{\delta}$.

Let $f\in A$. Attiya [10] introduced the operator $H^{\gamma,k}_{\alpha,\beta}(f)$, where $H^{\gamma,k}_{\alpha,\beta}(f):A\to A$ is defined by

$$H^{\gamma,k}_{\alpha,\beta}(f)=\mu^{\gamma,k}_{\alpha,\beta}*f(z)\qquad(z\in\mathbb{U}),$$

with $\beta,\gamma\in\mathbb{C}$, $\mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(k)-1\}$ and $\mathrm{Re}(k)>0$. Also, $\mathrm{Re}(\alpha)=0$ when $\mathrm{Re}(k)=1$; $\beta\neq0$. Here, $\mu^{\gamma,k}_{\alpha,\beta}$ is the generalized Mittag–Leffler function defined by [11], see also [10] and the symbol $(*)$ denotes the Hadamard product or convolution.

Due to the importance of Mittag–Leffler function, it is involved in many problems in natural and applied science.

A detailed investigation of Mittag–Leffler function has been studied by many authors see e.g., [11–16].

Attiya [10] noted that

$$H^{\gamma,k}_{\alpha,\beta}(f)(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha n)n!}a_nz^n. \tag{6}$$

From (6) follows (see [10])

$$z(H_{\alpha,\beta}^{\gamma,k}(f)(z))' = (\frac{\gamma+k}{k})(H_{\alpha,\beta}^{\gamma+1,k}(f)(z)) - \frac{\gamma}{k}(H_{\alpha,\beta}^{\gamma,k}(f)(z)) \tag{7}$$

and

$$\alpha z(H_{\alpha,\beta+1}^{\gamma,k}(f)(z))' = (\alpha+\beta)(H_{\alpha,\beta}^{\gamma,k}(f)(z)) - \beta(H_{\alpha,\beta+1}^{\gamma,k}(f)(z)). \tag{8}$$

In order to derive our results, we will use the following known definitions and lemmas.

Definition 1. *Ref [4]. Denote by μ the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where*

$$E(f) = \{\zeta : \zeta \in \partial\mathbb{U} \text{ and } \lim_{z\to\zeta} f(z) = \infty\}, \tag{9}$$

with $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(f)$.

Lemma 1. *Ref [3]. Let the function μ be univalent in the unit disc $\mathbb{U}$, and let θ and φ be analytic in a domain D containing $\mu(\mathbb{U})$, with $\varphi(w) \neq 0$ when $w \in \mu(\mathbb{U})$. Set $\mu(z) = z\mu'(z)\varphi(\mu(z))$, $h(z) = \theta(\mu(z)) + \mu(z)$ and suppose that*

(i) μ is a starlike function in $\mathbb{U}$ (i.e, $\mathrm{Re}\left(\frac{z\mu'(z)}{\mu(z)}\right) > 0$ for $z \in \mathbb{U}$),

(ii) $\mathrm{Re}\left(\frac{zh'(z)}{\mu(z)}\right) > 0$ for $z \in \mathbb{U}$.

If λ is analytic in $\mathbb{U}$ with $\lambda(0) = \mu(0)$, $\lambda(\mathbb{U}) \subseteq D$ and

$$\theta(\lambda(z)) + z\lambda'(z)\varphi(\lambda(z)) \prec \theta(\mu(z)) + z\mu'(z)\varphi(\mu(z)), \tag{10}$$

then $\lambda(z) \prec \mu(z)$, and μ is the best dominant.

Lemma 2. *Ref [6]. Let μ be a convex univalent function in the unit disc $\mathbb{U}$ and let ϑ and φ be analytic in a domain D containing $\mu(\mathbb{U})$. Suppose that*

(i) $\mathrm{Re}\left\{\frac{\vartheta'(\mu(z))}{\varphi(\mu(z))}\right\} > 0$ for $z \in \mathbb{U}$;

(ii) $z\mu'(z)\varphi(\mu(z))$ is starlike in $\mathbb{U}$.

If $\lambda \in \mathbb{H}[\mu(0), 1] \cap \mu$ with $\lambda(\mathbb{U}) \subseteq D$, and $\vartheta(\lambda(z)) + z\lambda'(z)\varphi(\lambda(z))$ is univalent in $\mathbb{U}$, and

$$\vartheta(\mu(z)) + z\mu'(z)\varphi(\mu(z)) \prec \vartheta(\lambda(z)) + z\lambda'(z)\varphi(\lambda(z)),$$

then $\mu(z) \prec \lambda(z)$, and μ is the best subordinant.

Lemma 3. *Ref [4]. Let μ be a convex function in $\mathbb{U}$ and let $\psi \in C$ with $\varkappa \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with*

$$\mathrm{Re}\left(1 + \frac{z\mu''(z)}{\mu'(z)}\right) > \max\left\{0; -\mathrm{Re}\left(\frac{\psi}{\varkappa}\right)\right\} \ (z \in \mathbb{U}).$$

If λ is analytic in $\mathbb{U}$, and

$$\psi\lambda(z) + \delta z\lambda'(z) \prec \psi\mu(z) + \varkappa z\mu'(z), \tag{11}$$

then $\lambda(z) \prec \mu(z)$, and μ is the best dominant.

Lemma 4. *Ref [17] Let μ be convex univalent in $\mathbb{U}$ and let $\delta \in C$, with $\mathrm{Re}(\delta) > 0$. If $\lambda \in \mathbb{H}[\mu(0), 1] \cap \mu$ and $\lambda(z) + \delta z\lambda'(z)$ is univalent in $\mathbb{U}$, then*

$$\mu(z) + \delta z\mu'(z) \prec \lambda(z) + \delta z\lambda'(z), \tag{12}$$

implies

$$\mu(z) \prec \lambda(z) \quad (z \in \mathbb{U})$$

and μ is the best subordinant.

In this paper we drive a number of interesting results concerning subordination and superordination relations for the operator $H_{\alpha,\beta}^{\gamma,k}(f)(z)$. Also, some of interesting sandwich results of the operator $H_{\alpha,\beta}^{\gamma,k}(f)(z)$ have been obtained.

2. Subordination and Superordination Results with $H_{\alpha,\beta}^{\gamma,k}(f)(z)$

Theorem 1. *Let μ be convex univalent in $\mathbb{U}$, with $\mu(0) = 1$, $\rho \in \mathbb{C}^*$, $\delta > 0$. Suppose μ satisfies*

$$\operatorname{Re}\left(1 + \frac{z\mu''(z)}{\mu'(z)}\right) > \max\left\{0; -\operatorname{Re}\left(\frac{\delta}{\rho}\right)\right\}. \tag{13}$$

If $f \in A$ satisfies the following subordination relation

$$\left(\frac{H_{\alpha,\beta}^{\gamma,k}(f)(z)}{z}\right)^{\delta} + \frac{\rho(\gamma+k)}{k}\left(\frac{H_{\alpha,\beta}^{\gamma,k}(f)(z)}{z}\right)^{\delta}\left(\frac{H_{\alpha,\beta}^{\gamma+1,k}(f)(z)}{H_{\alpha,\beta}^{\gamma,k}(f)(z)} - 1\right). \prec \mu(z) + \frac{\rho}{\delta}z\mu'(z) \tag{14}$$

then

$$\left(\frac{H_{\alpha,\beta}^{\gamma,k}(f)(z)}{z}\right)^{\delta} \prec \mu(z) \tag{15}$$

and $\mu(z)$ is the best dominant of (14).

Proof. Define the function λ by

$$\lambda(z) = \left(\frac{H_{\alpha,\beta}^{\gamma,k}(f)(z)}{z}\right)^{\delta} \quad (z \in \mathbb{U}). \tag{16}$$

The function λ is analytic in $\mathbb{U}$ and $\lambda(0) = 1$. Differentiating the function λ with respect to z logarithmically, we have

$$\frac{z\lambda'(z)}{\lambda(z)} = \delta\left[\frac{z\left(H_{\alpha,\beta}^{\gamma,k}(f)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k}(f)(z)} - 1\right].$$

In the resulting equation by using the identity (7), we have

$$\frac{z\lambda'(z)}{\lambda(z)} = \delta\left(\frac{\gamma+k}{k}\right)\left[\frac{H_{\alpha,\beta}^{\gamma+1,k}(f)(z)}{H_{\alpha,\beta}^{\gamma,k}(f)(z)} - 1\right].$$

Therefore,

$$\frac{z\lambda'(z)}{\delta} = \frac{(\gamma+k)}{k}\left[\frac{H_{\alpha,\beta}^{\gamma+1,k}(f)(z)}{H_{\alpha,\beta}^{\gamma,k}(f)(z)} - 1\right]\left(\frac{H_{\alpha,\beta}^{\gamma,k}(f)(z)}{z}\right)^{\delta}.$$

It follows from (14) that

$$\lambda(z) + \frac{\rho}{\delta} z\lambda'(z) \prec \mu(z) + \frac{\rho}{\delta} z\mu'(z).$$

Thus, an application of Lemma 3 with $\psi = 1$ and $\varkappa = \frac{\rho}{\delta}$, we obtain (15). □

In view of (8), and by using the similar method of proof the Theorem 1, we get the proof of Theorem 2.

Theorem 2. *Let μ be convex univalent in $\mathbb{U}$, with $\mu(0) = 1$, $\rho \in \mathbb{C}^*$, $\delta > 0$. Suppose μ satisfies (13). If $f \in A$ satisfies the subordination*

$$\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta} + \frac{\rho(\alpha+\beta)}{\alpha}\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta}\left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)} - 1\right). \prec \mu(z) + \frac{\rho}{\delta} z\mu'(z) \tag{17}$$

then

$$\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta} \prec \mu(z) \tag{18}$$

and $\mu(z)$ is the best dominant of (18).

Theorem 3. *Let $\zeta_i \in \mathbb{C}$ $(i = 1,2,3,4)$, $\delta > 0$, $\xi > 0$ (ξ is a real number) and μ be convex univalent in $\mathbb{U}$, with $\mu(0) = 1$, $\mu(z) \neq 0$ $(z \in \mathbb{U})$ and assume that μ satisfies*

$$\Re\left\{1 + \frac{\zeta_2}{\xi}\mu(z) + \frac{2\zeta_3}{\xi}\mu^2(z) + \frac{3\zeta_4}{\xi}\mu^3(z) + \frac{z\mu''(z)}{\mu'(z)} - \frac{z\mu'(z)}{\mu(z)}\right\} > 0. \tag{19}$$

Suppose that $\frac{z\mu'(z)}{\mu(z)}$ is starlike univalent in $\mathbb{U}$. Also, if $f \in A$ satisfies the following subordination relation:

$$\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z) \prec \zeta_1 + \zeta_2\mu(z) + \zeta_3\mu^2(z) + \zeta_4\mu^3(z) + \xi\frac{z\mu'(z)}{\mu(z)}, \tag{20}$$

where

$$\begin{aligned}\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z) \prec \zeta_1 &+ \zeta_2\left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)}\right)^{\delta} + \zeta_3\left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)}\right)^{2\delta} \\ &+ \zeta_4\left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)}\right)^{3\delta} + \xi\delta\frac{(\gamma+k)}{k}\left[\frac{H^{\gamma+2,k}_{\alpha,\beta}(f)(z)}{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)} - \frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)}\right] \\ &+ \frac{\xi\delta}{k}\left[\frac{H^{\gamma+2,k}_{\alpha,\beta}(f)(z)}{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)} - 1\right],\end{aligned} \tag{21}$$

then

$$\left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)}\right)^{\delta} \prec \mu(z)$$

and $\mu(z)$ is the best dominant of (20).

Proof. Define the function λ by

$$\lambda(z) = \left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)} \right)^{\delta} \quad (z \in \mathbb{U}). \tag{22}$$

The function λ is analytic in $\mathbb{U}$ and we note that $\lambda(0) = 1$.

After some computation and using (7), we have

$$\zeta_1 + \zeta_2 \lambda(z) + \zeta_3 \lambda^2(z) + \zeta_4 \lambda^3(z) + \xi \frac{z\lambda'(z)}{\lambda(z)} = \Omega(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi, \delta, \gamma, k, \alpha, \beta; z), \tag{23}$$

where $\Omega(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi, \delta, \gamma, k, \alpha, \beta; z)$ is given by (21).

From (20) and (23) we obtain

$$\zeta_1 + \zeta_2 \lambda(z) + \zeta_3 \lambda^2(z) + \zeta_4 \lambda^3(z) + \xi \frac{z\lambda'(z)}{\lambda(z)} \prec \zeta_1 + \zeta_2 \mu(z) + \zeta_3 \mu^2(z) + \zeta_4 \mu^3(z) + \xi \frac{z\mu'(z)}{\mu(z)}.$$

By setting

$$\theta(w) = \zeta_1 + \zeta_2 w + \zeta_3 w^2 + \zeta_4 w^3 \;\; and \quad \phi(w) = \frac{\xi}{w}, \;\; w \neq 0,$$

we see that θ is analytic in the complex plane $\mathbb{C}$ and ϕ is analytic in $\mathbb{C}^*$, also, $\phi(w) \neq 0$, $w \in \mathbb{C}^*$. Moreover

$$\mu(z) = z\mu'(z)\phi(\mu(z)) = \xi \frac{z\mu'(z)}{\mu(z)}$$

and

$$h(z) = \theta(\mu(z)) + \mu(z) = \zeta_1 + \zeta_2 \mu(z) + \zeta_3 \mu^2(z) + \zeta_4 \mu^3(z) + \xi \frac{z\mu'(z)}{\mu(z)}.$$

It is clear that $\mu(z)$ is starlike univalent in $\mathbb{U}$,

$$\mathrm{Re}\left(\frac{zh'(z)}{\mu(z)} \right) = \mathrm{Re}\left\{ 1 + \frac{\zeta_2}{\xi}\mu(z) + \frac{2\zeta_3}{\xi}\mu^2(z) + \frac{3\zeta_4}{\xi}\mu^3(z) + \frac{z\mu''(z)}{\mu'(z)} - \frac{z\mu'(z)}{\mu(z)} \right\} > 0\,.$$

Thus, from Lemma 1, we have $\lambda(z) \prec \mu(z)$. By using (22), we obtain the required result. □

In view of (8), and by using the similar method of proof of Theorem 3, we get the proof of Theorem 4

Theorem 4. *Let $\zeta_i \in \mathbb{C}$ $(i = 1,2,3,4)$, $\delta > 0, \xi > 0$ (ξ is a real number) and μ be convex univalent function in $\mathbb{U}$, with $\mu(0) = 1$, $\mu(z) \neq 0 (z \in \mathbb{U})$ and assume that the function μ satisfies (19). Also, let $\frac{z\mu'(z)}{\mu(z)}$ be starlike univalent in $\mathbb{U}$. If $f \in A$ satisfies (20), where*

$$\begin{aligned} \Omega(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi, \delta, \gamma, k, \alpha, \beta; z) \prec \zeta_1 + \zeta_2 \left(\frac{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)} \right)^{\delta} + \zeta_3 \left(\frac{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)} \right)^{2\delta} \\ + \zeta_4 \left(\frac{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)} \right)^{3\delta} + \xi\delta \frac{\alpha+\beta}{\alpha} \left[\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)} - \frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)} \right] \\ + \frac{\xi\delta}{\alpha} \left[\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)} - 1 \right], \end{aligned} \tag{24}$$

then

$$\left(\frac{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}\right)^{\delta} \prec \mu(z)$$

and $\mu(z)$ *is the best dominant of (20).*

Theorem 5. *Let* $\zeta_i \in \mathbb{C}$ $(i = 1,2,3,4), \xi > 0$ *(*ξ *is a real number) and* μ *be convex univalent in* $\mathbb{U}$*, with* $\mu(0) = 1$, $\mu(z) \neq 0 (z \in \mathbb{U})$ *and assume that* μ *satisfies (19). Also, if* $\frac{z\mu'(z)}{\mu(z)}$ *is starlike univalent in* $\mathbb{U}$*. Moreover, if* $f \in A$ *satisfies (20), where*

$$\begin{aligned}\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z) \prec \zeta_1 + \zeta_2\frac{zH^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta}(f)(z)\right)^2} + \zeta_3\frac{z^2\left(H^{\gamma+1,k}_{\alpha,\beta}(f)(z)\right)^2}{\left(H^{\gamma,k}_{\alpha,\beta}(f)(z)\right)^4} \\ + \zeta_4\frac{z^3\left(H^{\gamma+1,k}_{\alpha,\beta}(f)(z)\right)^3}{\left(H^{\gamma,k}_{\alpha,\beta}(f)(z)\right)^6} + \xi\frac{\gamma+k}{k}\left[1 + \frac{H^{\gamma+2,k}_{\alpha,\beta}(f)(z)}{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)} - 2\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)}\right] \\ + \frac{\xi}{k}\left[\frac{H^{\gamma+2,k}_{\alpha,\beta}(f)(z)}{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)} - 1\right],\end{aligned} \tag{25}$$

then

$$\frac{zH^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta}(f)(z)\right)^2} \prec \mu(z)$$

and $\mu(z)$ *is the best dominant of (20).*

Proof. Define the function λ by

$$\lambda(z) = \frac{zH^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta}(f)(z)\right)^2} \quad (z \in \mathbb{U}). \tag{26}$$

Then the function λ is analytic in $\mathbb{U}$ and $\lambda(0) = 1$.

We note that

$$\zeta_1 + \zeta_2\lambda(z) + \zeta_3\lambda^2(z) + \zeta_4\lambda^3(z) + \xi\frac{z\lambda'(z)}{\lambda(z)} = \Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z), \tag{27}$$

where $\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z)$ is given by (25).

From (20) and (27) we obtain

$$\zeta_1 + \zeta_2\lambda(z) + \zeta_3\lambda^2(z) + \zeta_4\lambda^3(z) + \xi\frac{z\lambda'(z)}{\lambda(z)} \prec \zeta_1 + \zeta_2\mu(z) + \zeta_3\mu^2(z) + \zeta_4\mu^3(z) + \xi\frac{z\mu'(z)}{\mu(z)}.$$

The remaining part of the proof of Theorem 5 is similar to that of Theorem 3 and hence we omit it. □

In view of (8), and by using the similar method of proof of Theorem 5, we get the proof Theorem 6.

Theorem 6. *Let* $\zeta_i \in \mathbb{C}$ $(i = 1,2,3,4), \xi > 0;$ real *and* μ *be convex univalent function in* $\mathbb{U}$, *with* $\mu(0) = 1$, $\mu(z) \neq 0 (z \in \mathbb{U})$ *and assume that* μ *satisfies (19). Also, let* $\frac{z\mu'(z)}{\mu(z)}$ *be starlike univalent in* $\mathbb{U}$. *If* $f \in A$ *satisfies (20), where*

$$\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z) \prec \zeta_1 + \zeta_2 \frac{zH^{\gamma,k}_{\alpha,\beta+2}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta+1}(f)(z)\right)^2} + \zeta_3 \frac{z^2\left(H^{\gamma,k}_{\alpha,\beta+2}(f)(z)\right)^2}{\left(H^{\gamma,k}_{\alpha,\beta+1}(f)(z)\right)^4}$$

$$+ \zeta_4 \frac{z^3\left(H^{\gamma,k}_{\alpha,\beta+2}(f)(z)\right)^3}{\left(H^{\gamma,k}_{\alpha,\beta+1}(f)(z)\right)^6} + \xi \frac{\alpha+\beta}{\alpha}\left[\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)} - 2\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}\right]$$
$$+ \frac{\xi}{\alpha}\left[\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)} - 1\right], \tag{28}$$

then

$$\frac{zH^{\gamma,k}_{\alpha,\beta+2}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta+1}(f)(z)\right)^2} \prec \mu(z)$$

and $\mu(z)$ *is the best dominant of (20).*

Remark 1. *Superordination results associated with* $H^{\gamma,k}_{\alpha,\beta}(f)(z)$ *can be done analogously by using Lemmas 2 and 4.*

3. Sandwich Results

Combining results of differential subordinations and superordinations, we get the following sandwich theorem.

Theorem 7. *Let* μ_1 *and* μ_2 *be convex univalent in* $\mathbb{U}$, *with* $\mu_1(0) = \mu_2(0) = 1$. *Suppose* μ_2 *satisfies (13),* $\delta > 0$ *and* $\mathrm{Re}\{\rho\} > 0$. *Let* $f \in A$ *satisfies*

$$\left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{z}\right)^{\delta} \in \mathbb{H}[1,1] \cap \mu$$

and

$$\left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{z}\right)^{\delta} + \frac{\rho(\gamma+k)}{k}\left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{z}\right)^{\delta}\left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)} - 1\right)$$

be univalent in $\mathbb{U}$. *If*

$$\mu_1(z) + \frac{\rho}{\delta} z\mu_1'(z) \prec \left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{z}\right)^{\delta} + \frac{\rho(\gamma+k)}{k}\left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{z}\right)^{\delta}\left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)} - 1\right)$$
$$\prec \mu_2(z) + \frac{\rho}{\delta} z\mu_2'(z)$$

then

$$\mu_1(z) \prec \left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{z}\right)^{\delta} \prec \mu_2(z)$$

and μ_1 and μ_2 *are respectively the best subordinate and best dominant.*

Theorem 8. *Let μ_1 and μ_2 be convex univalent in $\mathbb{U}$, with $\mu_1(0) = \mu_2(0) = 1$. Suppose μ_2 satisfies (13), $\delta > 0$ and $\mathrm{Re}\{\rho\} > 0$. Let $f \in A$ satisfies*

$$\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta} \in \mathbb{H}[1,1] \cap \mu$$

and

$$(1-\rho\frac{\beta}{\alpha})\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta} + \frac{\rho(\beta+\alpha)}{\alpha}\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta}\left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}\right)$$

be univalent in $\mathbb{U}$. If

$$\mu_1(z) + \frac{\rho}{\delta}z\mu_1'(z) \prec (1-\rho\frac{\beta}{\alpha})\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta} + \frac{\rho(\beta+\alpha)}{\alpha}\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta}\left(\frac{H^{\gamma,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}\right)$$
$$\prec \mu_2(z) + \frac{\rho}{\delta}z\mu_2'(z),$$

then

$$\mu_1(z) \prec \left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{z}\right)^{\delta} \prec \mu_2(z)$$

and μ_1 and *μ_2 are respectively the best subordinate and best dominant.*

Theorem 9. *Let μ_1 and μ_2 be convex univalent functions in $\mathbb{U}$, with $\mu_1(0) = \mu_2(0) = 1$. Suppose μ_1 satisfies*

$$\Re\left\{\frac{\zeta_2}{\xi}\mu_1(z) + \frac{2\zeta_3}{\xi}\mu_1^2(z) + \frac{3\zeta_4}{\xi}\mu_1^3(z)\right\} > 0. \tag{29}$$

and μ_2 satisfies (19). Let $f \in A$ satisfies $\left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)}\right)^{\delta} \in \mathbb{H}[1,1] \cap \mu$, and $\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z)$ is univalent in $\mathbb{U}$, where $\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z)$ is given by (21). If

$$\zeta_1 + \zeta_2\mu_1(z) + \zeta_3\mu_1^2(z) + \zeta_4\mu_1^3(z) + \xi\frac{z\mu_1'(z)}{\mu_1(z)} \prec \Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z)$$
$$\prec \zeta_1 + \zeta_2\mu_2(z) + \zeta_3\mu_2^2(z) + \zeta_4\mu_2^3(z) + \xi\frac{z\mu_2'(z)}{\mu_2(z)}, \tag{30}$$

then

$$\mu_1(z) \prec \left(\frac{H^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{H^{\gamma,k}_{\alpha,\beta}(f)(z)}\right)^{\delta} \prec \mu_2(z)$$

and μ_1 and *μ_2 are respectively the best subordinate and best dominant.*

Theorem 10. *Let μ_1 and μ_2 be convex univalent in $\mathbb{U}$, with $\mu_1(0) = \mu_2(0) = 1$. Suppose μ_1 satisfies (29), and μ_2 satisfies (19). Let $f \in A$ satisfies $\left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)}\right)^{\delta} \in \mathbb{H}[1,1] \cap \mu$ and $\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z)$ is univalent in $\mathbb{U}$, where $\Omega(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\xi,\delta,\gamma,k,\alpha,\beta;z)$ is given by (24). If (30) has been satisfied,* *then*

$$\mu_1(z) \prec \left(\frac{H^{\gamma,k}_{\alpha,\beta+1}(f)(z)}{H^{\gamma,k}_{\alpha,\beta+2}(f)(z)} \right)^{\delta} \prec \mu_2(z)$$

and μ_1 and μ_2 are respectively the best subordinate and best dominant.

Theorem 11. *Let μ_1 and μ_2 be convex univalent in $\mathbb{U}$, with $\mu_1(0) = \mu_2(0) = 1$. Suppose μ_1 satisfies (29), and μ_2 satisfies (19). Let $f \in A$ satisfies $\frac{zH^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta}(f)(z)\right)^2} \in \mathbb{H}[1,1] \cap \mu$ and $\Omega(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi, \delta, \gamma, k, \alpha, \beta; z)$ is univalent in $\mathbb{U}$, where $\Omega(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi, \delta, \gamma, k, \alpha, \beta; z)$ is given by (25). If (30) has been satisfied,*
then

$$\mu_1(z) \prec \frac{zH^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta}(f)(z)\right)^2} \prec \mu_2(z)$$

and μ_1 and μ_2 are respectively the best subordinate and best dominant.

Theorem 12. *Let μ_1 and μ_2 be convex univalent in $\mathbb{U}$, with $\mu_1(0) = \mu_2(0) = 1$. Suppose μ_1 satisfies (29), and μ_2 satisfies (19). Let $f \in A$ satisfies $\frac{zH^{\gamma,k}_{\alpha,\beta+2}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta+1}(f)(z)\right)^2} \in \mathbb{H}[1,1] \cap \mu$ and $\Omega(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi, \delta, \gamma, k, \alpha, \beta; z)$ is univalent in $\mathbb{U}$, where $\Omega(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi, \delta, \gamma, k, \alpha, \beta; z)$ is given by (28). If (30) has been satisfied,*
then

$$\mu_1(z) \prec \frac{zH^{\gamma,k}_{\alpha,\beta+2}(f)(z)}{\left(H^{\gamma,k}_{\alpha,\beta+1}(f)(z)\right)^2} \prec \mu_2(z)$$

and μ_1 and μ_2 are respectively the best subordinate and best dominant.

Remark 2. *By specifying the function Ω and selecting the particular values of α, β, γ and k we can derive a number of known results. Some of them are given below.*

(i) If we put $\gamma = k = 1$ and $\alpha = 0$ in Theorem 1, we obtain the results obtained by Murugusundaramoorthy and Magesh ([18], Corollary 3.3),

(ii) If we put $\gamma = k = 1$ and $\alpha = 0$ in Theorem 7 we obtain the results obtained by Raducanu and Nechita ([19], Corollary 3.10).

4. Conclusions

We obtained a number of interesting results concerning subordination and superordination relations for the operator $H^{\gamma,k}_{\alpha,\beta}(f)(z)$ of analytic functions associated with an extension of the Mittag–Leffler function in the open unit disk $\mathbb{U}$. Also, some of interesting sandwich results of the operator $H^{\gamma,k}_{\alpha,\beta}(f)(z)$ have been obtained.

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