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Article

On a Metric Affine Manifold with Several Orthogonal Complementary Distributions

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Abstract: A Riemannian manifold endowed with $k > 2$ orthogonal complementary distributions (called here an almost multi-product structure) appears in such topics as multiply twisted or warped products and the webs or nets composed of orthogonal foliations. In this article, we define the mixed scalar curvature of an almost multi-product structure endowed with a linear connection, and represent this kind of curvature using fundamental tensors of distributions and the divergence of a geometrically interesting vector field. Using this formula, we prove decomposition and non-existence theorems and integral formulas that generalize results (for $k = 2$) on almost product manifolds with the Levi-Civita connection. Some of our results are illustrated by examples with statistical and semi-symmetric connections.

Keywords: almost multi-product structure; mixed scalar curvature; integral formula; statistical connection; semi-symmetric connection; multiply twisted product; splitting

MSC: 53C15; 57R25

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1. Introduction

Distributions on a manifold (that is subbundles of the tangent bundle) appear in various situations and are used to build up notions of integrability, and specifically of a foliated manifold, e.g., [1,2]. In this article, we consider a connected m-dimensional Riemannian manifold (M, g) endowed with $k \geq 2$ pairwise orthogonal n_i-dimensional distributions $\mathcal{D}_i$ with dimension $\sum n_i = m$; thus, there exists an orthogonal splitting

$$TM = \mathcal{D}_1 \oplus \ldots \oplus \mathcal{D}_k.$$

This geometric structure, denoted here by $(M, g, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ and called a Riemannian almost multi-product structure (a Riemannian almost product structure when $k = 2$, e.g., [3]), appears in the theory of webs or nets (families of orthogonal foliations), see [4,5], and in recent studies of the curvature on multiply twisted and multiply warped products, e.g., [6,7].

A natural question is when $(M, g, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is decomposed (or splits locally) into the product of k manifolds. The best known result in this direction is the Decomposition theorem of de Rham, which states that "if each distribution $\mathcal{D}_i$ is parallel with respect to the Levi-Civita connection of M, then any point $p \in M$ has a neighborhood U, which is isometric to a product $M_1 \times \ldots \times M_k$ of Riemannian manifolds such that the submanifolds, which are parallel to the factor M_i, correspond to integral manifolds of the distribution $D_i|_U$. In the case that M is simply connected and complete the assertion is true with $U = M$". This theorem was generalized to multiply warped and twisted products, to pseudo-Riemannian and affinely connected manifolds, [8,9], and to more generally foliated manifolds and submanifolds.

On the other hand, many results of global Riemannian geometry (including splitting or decomposition of manifolds and integral formulas) are carried out using restrictions on the sign of curvature and the Stokes' (or divergence) theorem or its modifications for certain vector fields.

The mixed scalar curvature is the simplest curvature invariant of the almost multi-product structure, its research even for $k = 2$ led to many results, for example, integral formulas, splitting theorems and prescribing the curvature, e.g., [10–14].

The metric-affine geometry, founded by E. Cartan, generalizes Riemannian geometry: it uses a metric g and a linear connection $\bar\nabla$ instead of the Levi-Civita connection ∇ (of g), e.g., [1,15]. The following distinguished classes of metric-affine manifolds are considered important.

- Statistical manifolds, where the tensor $\bar\nabla g$ is symmetric in all its entries and connection $\bar\nabla$ is torsion-free, constitute an important class of metric-affine manifolds with applications in probability and statistics as well as in information geometry, e.g., [16,17].
- Riemann-Cartan manifolds, where the $\bar\nabla$-parallel transport preserves the metric, $\bar\nabla g = 0$, e.g., [11,18], with applications in physics; semi-symmetric connections constitute their special class, see [7,19].

In the article, we generalize results for $k = 2$ in [11–14] on almost product manifolds and twisted products. We introduce the mixed scalar curvature of $(M, g, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with respect to a non-Levi-Civita linear connection and represent this kind of curvature using fundamental tensors of the distributions and the divergence of a geometrically interesting vector field. Using this formula, we prove decomposition and non-existence theorems (sometimes called Liouville type theorems, e.g., [12,13]) and integral formulas (when M is compact or a certain vector field is compactly supported on M) for some classes of almost multi-product manifolds.

Section 2 contains definition and preliminary results. In Section 3 we prove new integral formulas for multi-product manifolds. In Section 4 we obtain splitting results for such manifolds (including multiply twisted products). Some of our results are illustrated by examples with statistical and semi-symmetric connections. We suggest that the concept of the mixed scalar curvature can be useful for differential geometry of multiply twisted and warped products as well as in the theory of webs and nets of foliations.

2. Preliminaries

Let (M, g) be a Riemannian manifold, and let ∇ denote its Levi-Civita connection. For any linear connection $\bar\nabla$ we consider the difference $\mathfrak{T} = \bar\nabla - \nabla$ (the *contorsion tensor*) and define auxiliary (1,2)-tensors $\mathfrak{T}^*$ and $\mathfrak{T}^\wedge$ by

$$\langle \mathfrak{T}^*_X Y, Z\rangle = \langle \mathfrak{T}_X Z, Y\rangle, \quad \mathfrak{T}^\wedge_X Y = \mathfrak{T}_Y X, \quad X, Y, Z \in \mathfrak{X}_M.$$

For the case of a statistical connection $\bar\nabla$ we have $\mathfrak{T}^\wedge = \mathfrak{T}$ and $\mathfrak{T}^* = \mathfrak{T}$. For Riemann-Cartan spaces we have $\mathfrak{T}^* = -\mathfrak{T}$, and $\bar\nabla$ is said to be a metric compatible connection.

For the curvature tensor $\bar R_{X,Y} = [\bar\nabla_Y, \bar\nabla_X] + \bar\nabla_{[X,Y]}$ of a linear connection $\bar\nabla$, we have

$$\bar R_{X,Y} - R_{X,Y} = (\nabla_Y \mathfrak{T})_X - (\nabla_X \mathfrak{T})_Y + [\mathfrak{T}_Y, \mathfrak{T}_X],$$

where $R_{X,Y} = [\nabla_Y, \nabla_X] + \nabla_{[X,Y]}$ is the curvature tensor of ∇. The scalar curvature $\bar{\rm S} = {\rm Tr}_g \overline{\rm Ric}$ is the function on M, where $\overline{\rm Ric}_{X,Y} = \frac{1}{2}\,{\rm Tr}(Z \to \bar R_{X,Z}\, Y + \bar R_{X,Z}\, Y)$ is the symmetric Ricci tensor of $\bar\nabla$.

Let $\{e_1, \ldots, e_m\}$ be a local adapted orthonormal frame on M, i.e., $e_a \in \mathcal{D}$ for $1 \le a \le n = \dim \mathcal{D}$. The *mixed scalar curvature* for two orthogonal complementary distributions $(\mathcal{D}, \mathcal{D}^\perp)$ on a Riemannian manifold (M^m, g) with a linear connection $\bar\nabla$ is defined in [11] by

$$\bar{\rm S}_{\mathcal{D},\mathcal{D}^\perp} = \frac{1}{2}\sum_{1\le a\le n,\ n<b\le m} \big(\langle \bar R_{e_a,e_b} e_a, e_b\rangle + \langle \bar R_{e_b,e_a} e_b, e_a\rangle\big). \tag{1}$$

If $\mathcal{D}$ is spanned by a unit vector field N, then $\bar{\rm S}_{\mathcal{D},\mathcal{D}^\perp} = \overline{\rm Ric}_{N,N}$. When $\mathfrak{T}=0$, the *mixed scalar curvature* for $(\mathcal{D},\mathcal{D}^\perp)$ is the function [14],

$${\rm S}_{\mathcal{D},\mathcal{D}^\perp} = \sum_{1\le a\le n,\ n<b\le m} \langle R_{e_a,e_b} e_a, e_b\rangle. \tag{2}$$

The mixed scalar curvature of $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$ is defined in [20] similarly to (2) as an averaged mixed sectional curvature. A plane in TM spanned by two vectors belonging to different distributions $\mathcal{D}_i$ and $\mathcal{D}_j$ will be called *mixed*, and its sectional curvature will be called mixed.

Given $(M,g;\mathcal{D}_1,\ldots\mathcal{D}_k)$, there exists a local adapted orthonormal frame $\{e_1,\ldots,e_m\}$ on M, i.e., $\{e_1,\ldots,e_{n_1}\}\subset\mathcal{D}_1$ and $\{e_{n_{i-1}+1},\ldots,e_{n_i}\}\subset\mathcal{D}_i$ for $i\ge 2$. All quantities defined below using such frame do not depend on the choice of this frame.

In the following definition we extend (1), see also as Definition 1.1 in [20].

Definition 1. *Given $(M,g,\bar\nabla;\mathcal{D}_1,\ldots\mathcal{D}_k)$, the following function on M will be called the mixed scalar curvature with respect to $\bar\nabla$:*

$$\bar{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \frac{1}{2}\sum_{i<j}\ \sum_{n_{i-1}<a\le n_i,\ n_{j-1}<b\le n_j} \big(\langle \bar R_{e_a,e_b}\, e_a,\, e_b\rangle + \langle \bar R_{e_b,e_a}\, e_b,\, e_a\rangle\big). \tag{3}$$

In particular, when $\mathfrak{T}=0$, the function on M

$${\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum_{i<j}\ \sum_{n_{i-1}<a\le n_i,\ n_{j-1}<b\le n_j} \langle R_{e_a,e_b}\, e_a,\, e_b\rangle$$

is the mixed scalar curvature of $(M,g;\mathcal{D}_1,\ldots\mathcal{D}_k)$ with respect to the Levi-Civita connection ∇.

Observe that the scalar curvature $\bar{\rm S}$ is decomposed as

$$\bar{\rm S} = 2\,\bar{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \textstyle\sum_i \bar{\rm S}\,|_{\mathcal{D}_i},$$

where $\bar{\rm S}\,|_{\mathcal{D}_i}$ is the scalar curvature of (M,g) along the plane field $\mathcal{D}_i$.

Proposition 1 (see [20] for $\mathfrak{T}=0$)**.** *For any $(M,g,\bar\nabla;\mathcal{D}_1,\ldots\mathcal{D}_k)$ we have the following decomposition of the mixed scalar curvature:*

$$2\,\bar{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \textstyle\sum_i \bar{\rm S}_{\mathcal{D}_i,\mathcal{D}_i^\perp}. \tag{4}$$

Proof. For any pair of complementary distributions $(\mathcal{D}_i,\mathcal{D}_i^\perp)$ on (M,g) we have

$$\bar{\rm S}_{\mathcal{D}_i,\mathcal{D}_i^\perp} = \sum_{n_{i-1}<a\le n_i,\ b\ne(n_{i-1},n_i]} \langle \bar R_{e_a,e_b} e_a, e_b\rangle.$$

Thus (4) follows directly from $\bar{\rm S}_{\mathcal{D}_i,\mathcal{D}_i^\perp} = \sum_{j\ne i}\bar{\rm S}_{\mathcal{D}_i,\mathcal{D}_j^\perp}$ and the definition (3). □

The symmetric second fundamental form $h_i:\mathcal{D}_i\times\mathcal{D}_i\to\mathcal{D}_i^\perp$ and the skew-symmetric integrability tensor $T_i:\mathcal{D}_i\times\mathcal{D}_i\to\mathcal{D}_i^\perp$ of $\mathcal{D}_i$ are defined by

$$2\,h_i(X,Y) = P_i^\perp(\nabla_X Y + \nabla_Y X),\quad 2\,T_i(X,Y) = P_i^\perp(\nabla_X Y - \nabla_Y X) = P_i^\perp\,[X,Y],$$

where $P_i: TM\to\mathcal{D}_i$ and $P_i^\perp: TM\to\mathcal{D}_i^\perp$ are orthoprojectors. The mean curvature vector field of $\mathcal{D}_i$ is $H_i = {\rm Tr}_g\, h_i$. Similarly, $h_i^\perp$, $H_i^\perp = {\rm Tr}_g\, h_i^\perp$, $T_i^\perp$ are defined for $\mathcal{D}_i^\perp$.

A distribution $\mathcal{D}_i$ is integrable if $T_i = 0$, and $\mathcal{D}_i$ is totally umbilical, harmonic, or totally geodesic, if $h_i = (H_i/n_i)\, g,\ H_i = 0$, or $h_i = 0$, respectively, e.g., [1].

Example 1. *Totally umbilical and totally geodesic integrable distributions appear on multiply twisted products. A multiply twisted product $F_1 \times_{u_2} F_2 \times \ldots \times_{u_k} F_k$ of Riemannian manifolds $(F_1, g_{F_1}), \ldots, (F_k, g_{F_k})$ is the product $M = F_1 \times \ldots \times F_k$ with the metric $g = g_{F_1} \oplus u_2^2\, g_{F_2} \oplus \ldots \oplus u_k^2\, g_{F_k}$, where $u_i : F_1 \times F_i \to (0, \infty)$ for $i \geq 2$ are smooth functions, see [7]. The twisted products (i.e., $k = 2$) and multiply warped products (i.e., $u_i : F_1 \to (0, \infty)$, see [6]) are special cases of multiply twisted products. Let contorsion tensors $\mathfrak{T}_{F_i}$ correspond to linear connections on (F_i, g_{F_i}). Then the contorsion tensor $\mathfrak{T} = \mathfrak{T}_{F_1} \oplus \ldots \oplus \mathfrak{T}_{F_k}$ corresponds to an adapted connection $\bar\nabla$ on M.*

Let $\mathcal{D}_i$ be the distribution on M obtained from the vectors tangent to horizontal lifts of F_i. The leaves tangent to $\mathcal{D}_i$ $(i \geq 2)$, are totally umbilical, with the mean curvature vector fields

$$H_i = -n_i P_1 \nabla(\log u_i)$$

tangent to $\mathcal{D}_1$, and the fibers (tangent to $\mathcal{D}_1$) are totally geodesic: $h_1 = 0$. On a multiply twisted product with $k > 2$ each pair of distributions is mixed totally geodesic: such (M, g) is diffeomorphic to the direct product, and the Lie bracket does not depend on metric. Since

$$\operatorname{div} H_i = -n_i\, (\Delta_1\, u_i)/u_i - (n_i^2 - n_i)\, \|P_1 \nabla u_i\|^2/u_i^2,$$

where Δ_1 is the Laplacian on $C^2(F_1)$, and we have

$$\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_{i\geq 2} n_i\, (\Delta_1\, u_i)/u_i\,.$$

The "musical" isomorphisms $\sharp$ and $\flat$ will be used for rank one and symmetric rank 2 tensors. For example, if $\omega \in \Lambda^1(M)$ is a 1-form and $X, Y \in \mathfrak{X}_M$ then $\omega(Y) = \langle \omega^\sharp, Y\rangle$ and $X^\flat(Y) = \langle X, Y\rangle$. For arbitrary (0,2)-tensors B and C we also have $\langle B, C\rangle = \operatorname{Tr}_g(B^\sharp C^\sharp) = \langle B^\sharp, C^\sharp\rangle$. The symmetric shape operator $(A_i)_Z$ of $\mathcal{D}_i$ with $Z \in \mathcal{D}_i^\perp$ and the skew-symmetric operator $(T_i^\sharp)_Z$ are defined by

$$\langle (A_i)_Z(X), Y\rangle = \, h_i(X, Y), Z\rangle, \quad \langle (T_i^\sharp)_Z(X), Y\rangle = \langle T_i(X, Y), Z\rangle, \quad X, Y \in \mathcal{D}_i.$$

Similarly, we define $(A_i^\perp)_Z$ and $(T_i^{\perp\sharp})_Z$ with $Z \in \mathcal{D}_i$. The squares of norms of tensors are given by

$$\langle h_i, h_i\rangle = \sum_{n_{i-1} < a,b \leq n_i} \langle h_i(e_a, e_b), h_i(e_a, e_b)\rangle, \quad \langle T_i, T_i\rangle = \sum_{n_{i-1} < a,b \leq n_i} \langle T_i(e_a, e_b), T_i(e_a, e_b)\rangle, \quad \text{etc.}$$

3. Integral Formulas

Integral formulas (usually obtained by applying the Divergence Theorem to appropriate vector fields) provide a powerful tool for proving global results in analysis and geometry, e.g., [10]. The first known integral formula for a closed Riemannian manifold endowed with a codimension one foliation tells us that the total (i.e., integral) mean curvature of the leaves vanishes, see [21]. The second formula in the series of total σ_k's—elementary symmetric functions of principal curvatures of the leaves—says that for a codimension one foliation with a unit normal N to the leaves the total σ_2 is a half of the total Ricci curvature in the N-direction, e.g., [10]:

$$\int_M \big(\,\sigma_2 - \frac{1}{2}\operatorname{Ric}_{N,N}\big)\, \mathrm{d}\operatorname{vol} = 0. \tag{5}$$

We immediately have two consequences of (5):

(a) if the Ricci curvature is nonpositive and not identically zero then $\mathcal{F}$ cannot be totally umbilical;

(b) if the Ricci curvature is nonnegative and not identically zero then $\mathcal{F}$ cannot be harmonic (i.e., with zero mean curvature of the leaves).

An integral formula in [14], containing the mixed scalar curvature of a Riemannian manifold endowed with two complementary orthogonal distributions, generalizes (5) and has many applications, e.g., survey [10]. In [11], this formula was extended for a metric affine almost product manifold (with a linear connection instead of the Levi-Civita connection). On the other hand, Walczak's result [14] was generalized in [20] for a Riemannian manifold with an almost multi-product structure, and here we continue this study for the case of arbitrary linear connection.

For the divergence of a vector field $X \in \mathfrak{X}_M$ we have

$$\operatorname{div} X = \operatorname{Tr}(\nabla X).$$

The following two lemmas on the mixed scalar curvature of (M, g) endowed with two complementary orthogonal distributions play a key role in this section.

Lemma 1 (see [14])**.** *For the mixed scalar curvature* $\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp}$ *of* $(M, g; \mathcal{D}, \mathcal{D}^\perp)$*, we have*

$$\operatorname{div}(H + H^\perp) = \mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} + \langle h, h\rangle + \langle h^\perp, h^\perp\rangle - \langle H^\perp, H^\perp\rangle - \langle H, H\rangle - \langle T, T\rangle - \langle T^\perp, T^\perp\rangle \quad (6)$$

Set $V(\mathcal{D}) = (\mathcal{D}\times\mathcal{D}^\perp)\cup(\mathcal{D}^\perp\times\mathcal{D})$. Define the partial traces of a contorsion tensor $\mathfrak{T}$ by

$$\operatorname{Tr}_{\mathcal{D}_i^\perp}\mathfrak{T} = \sum_{b\ne(n_{i-1},n_i]}\mathfrak{T}_{e_b}e_b, \quad \operatorname{Tr}_{\mathcal{D}_i}\mathfrak{T} = \sum_{n_{i-1}<a\le n_i}\mathfrak{T}_{e_a}e_a.$$

Lemma 2 (see Lemma 2 in [11])**.** *For* $(M, g, \bar\nabla = \nabla + \mathfrak{T}; \mathcal{D}, \mathcal{D}^\perp)$ *we get*

$$\begin{aligned}&\operatorname{div}\big(P\operatorname{Tr}_{\mathcal{D}^\perp}(\mathfrak{T}-\mathfrak{T}^*) + P^\perp\operatorname{Tr}_{\mathcal{D}}(\mathfrak{T}-\mathfrak{T}^*)\big) = 2(\bar{\mathrm{S}}_{\mathcal{D},\mathcal{D}^\perp} - \mathrm{S}_{\mathcal{D},\mathcal{D}^\perp})\\ &-\langle\operatorname{Tr}_{\mathcal{D}}\mathfrak{T}, \operatorname{Tr}_{\mathcal{D}^\perp}\mathfrak{T}^*\rangle - \langle\operatorname{Tr}_{\mathcal{D}^\perp}\mathfrak{T}, \operatorname{Tr}_{\mathcal{D}}\mathfrak{T}^*\rangle - \langle\operatorname{Tr}_{\mathcal{D}}(\mathfrak{T}-\mathfrak{T}^*) - \operatorname{Tr}_{\mathcal{D}^\perp}(\mathfrak{T}-\mathfrak{T}^*), H - H^\perp\rangle \quad (7)\\ &-\langle\mathfrak{T}-\mathfrak{T}^*+\mathfrak{T}^\wedge-\mathfrak{T}^{*\wedge},\ A^\perp - T^{\perp\sharp} + A - T^\sharp\rangle + \langle\mathfrak{T}^*,\ \mathfrak{T}^\wedge\rangle_{\,|\,V(\mathcal{D})}.\end{aligned}$$

Remark 1. *Using the auxiliary functions* $Q(\mathcal{D}, g)$ *and* $\bar Q(\mathcal{D}, g, \mathfrak{T})$*, given by*

$$\begin{aligned}Q(\mathcal{D}, g) &= \langle H^\perp, H^\perp\rangle + \langle H, H\rangle - \langle h, h\rangle - \langle h^\perp, h^\perp\rangle + \langle T, T\rangle + \langle T^\perp, T^\perp\rangle, &(8)\\ 2\,\bar Q(\mathcal{D}, g, \mathfrak{T}) &= \langle\operatorname{Tr}_{\mathcal{D}}\mathfrak{T}, \operatorname{Tr}_{\mathcal{D}^\perp}\mathfrak{T}^*\rangle + \langle\operatorname{Tr}_{\mathcal{D}^\perp}\mathfrak{T}, \operatorname{Tr}_{\mathcal{D}}\mathfrak{T}^*\rangle + \langle\operatorname{Tr}_{\mathcal{D}}(\mathfrak{T}-\mathfrak{T}^*) - \operatorname{Tr}_{\mathcal{D}^\perp}(\mathfrak{T}-\mathfrak{T}^*), H - H^\perp\rangle\\ &- \langle\mathfrak{T}^*,\ \mathfrak{T}^\wedge\rangle_{\,|\,V(\mathcal{D})} + \langle\mathfrak{T}-\mathfrak{T}^*+\mathfrak{T}^\wedge-\mathfrak{T}^{*\wedge},\ A^\perp - T^{\perp\sharp} + A - T^\sharp\rangle, &(9)\end{aligned}$$

Formulas (6) *and* (7) *can be written shortly as*

$$\begin{aligned}\operatorname{div}(H + H^\perp) &= \mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} - Q(\mathcal{D}, g), &(10)\\ \operatorname{div}\big(P\operatorname{Tr}_{\mathcal{D}^\perp}(\mathfrak{T}-\mathfrak{T}^*) + P^\perp\operatorname{Tr}_{\mathcal{D}}(\mathfrak{T}-\mathfrak{T}^*)\big) &= 2(\bar{\mathrm{S}}_{\mathcal{D},\mathcal{D}^\perp} - \mathrm{S}_{\mathcal{D},\mathcal{D}^\perp}) - 2\,\bar Q(\mathcal{D}, g, \mathfrak{T}). &(11)\end{aligned}$$

In a local adapted frame, the last term in (9) *and* $\langle\mathfrak{T}^*, \mathfrak{T}^\wedge\rangle_{\,|\,V(\mathcal{D})}$ *have the form*

$$\begin{aligned}&\langle\mathfrak{T}-\mathfrak{T}^*+\mathfrak{T}^\wedge-\mathfrak{T}^{*\wedge},\ A^\perp - T^{\perp\sharp} + A - T^\sharp\rangle\\ &= \sum_{a\le n_1,\, b>n_1}\big(\langle(\mathfrak{T}_{e_b} - \mathfrak{T}^*_{e_b})e_a + (\mathfrak{T}_{e_a} - \mathfrak{T}^*_{e_a})e_b,\ (A^\perp_{e_a} - T^{\perp\sharp}_{e_a})e_b + (A_{e_b} - T^\sharp_{e_b})e_a\rangle\big),\\ &\langle\mathfrak{T}^*,\ \mathfrak{T}^\wedge\rangle_{\,|\,V(\mathcal{D})} = \sum_{a\le n_1,\, b>n_1}\big(\langle\mathfrak{T}_{e_a}e_b,\ \mathfrak{T}^*_{e_b}e_a\rangle + \langle\mathfrak{T}^*_{e_a}e_b,\ \mathfrak{T}_{e_b}e_a\rangle\big).\end{aligned}$$

The following result generalizes (10) for $k > 2$ and a linear connection $\bar\nabla$ instead of ∇.

Proposition 2. *For an almost multi-product manifold $(M, g, \bar\nabla; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with a linear connection $\bar\nabla = \nabla + \mathfrak{T}$ we have*

$$\begin{aligned}&\operatorname{div}\sum\nolimits_i \Big(\frac{1}{2}\,\big(P_i \operatorname{Tr}_{\mathcal{D}_i^\perp}(\mathfrak{T}-\mathfrak{T}^*) + P_i^\perp \operatorname{Tr}_{\mathcal{D}_i}(\mathfrak{T}-\mathfrak{T}^*)\big) + H_i + H_i^\perp\Big)\\ &= 2\,\bar{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i \big(\bar Q(\mathcal{D}_i, g, \mathfrak{T}) + Q(\mathcal{D}_i, g)\big),\end{aligned} \tag{12}$$

where $Q(\mathcal{D}_i, g)$ and $\bar Q(\mathcal{D}_i, g, \mathfrak{T})$ are given in (8) and (9) with $\mathcal{D} = \mathcal{D}_i$.

Proof. Summing k copies of (10) with $\mathcal{D} = \mathcal{D}_i$ for $i = 1, \ldots, k$, and using (4) gives the equality (see also [20])

$$\operatorname{div}\sum\nolimits_i (H_i + H_i^\perp) = 2\,{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i Q(\mathcal{D}_i, g). \tag{13}$$

Summing k copies of (11) with $\mathcal{D} = \mathcal{D}_i$ for $i = 1, \ldots, k$ and using (4) gives the equality

$$\begin{aligned}&\frac{1}{2}\operatorname{div}\sum\nolimits_i \big(P_i \operatorname{Tr}_{\mathcal{D}_i^\perp}(\mathfrak{T}-\mathfrak{T}^*) + P_i^\perp \operatorname{Tr}_{\mathcal{D}_i}(\mathfrak{T}-\mathfrak{T}^*)\big)\\ &= 2\,\bar{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - 2\,{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i \bar Q(\mathcal{D}_i, g, \mathfrak{T}).\end{aligned} \tag{14}$$

Finally, the sum of (13) and (14) is (12). □

Theorem 1. *For a closed manifold M with an almost multi-product structure $(g, \bar\nabla; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ the following integral formula holds:*

$$\int_M \big(2\,\bar{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i (Q(\mathcal{D}_i, g) + \bar Q(\mathcal{D}_i, g, \mathfrak{T}))\big)\,{\rm d}\,{\rm vol}_g = 0\,. \tag{15}$$

Proof. Using the Divergence Theorem for (12), gives (15). □

Remark 2. *In Theorem 1 and in results below, instead of compactness of M, one may assume that certain vector fields under the divergence operator are compactly supported on M. For $\mathfrak{T} = 0$, the integral formula (15) reduces to the following result in [20]:*

$$\int_M \big(\,2\,{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i Q(\mathcal{D}_i, g)\,\big)\,{\rm d}\,{\rm vol}_g = 0\,. \tag{16}$$

Using the Divergence Theorem for (10) on a closed Riemannian manifold (M, g), gives the integral formula (16) for $k = 2$

$$\int_M \big({\rm S}_{\mathcal{D},\mathcal{D}^\perp} - Q(\mathcal{D}, g)\big){\rm d}\,{\rm vol}_g = 0,$$

and (11) and (10) give the following integral formula (15) for $k = 2$:

$$\int_M \big(\bar{\rm S}_{\mathcal{D},\mathcal{D}^\perp} - Q(\mathcal{D}, g) - \bar Q(\mathcal{D}, g, \mathfrak{T})\big){\rm d}\,{\rm vol}_g = 0.$$

Corollary 1. *For a closed manifold M endowed with an almost multi-product structure and a statistical connection $\bar\nabla = \nabla + \mathfrak{T}$, we have the following integral formula:*

$$\int_M \Big(2\,\bar{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i \big(Q(\mathcal{D}_i, g) + \langle \operatorname{Tr}_{\mathcal{D}_i}\mathfrak{T}, \operatorname{Tr}_{\mathcal{D}_i^\perp}\mathfrak{T}\rangle - \frac{1}{2}\langle \mathfrak{T}, \mathfrak{T}\rangle_{\,|V(\mathcal{D}_i)}\big)\Big)\,{\rm d}\,{\rm vol}_g = 0. \tag{17}$$

Proof. For $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with a statistical connection $\bar\nabla = \nabla + \mathfrak{T}$, we have for each i,

$$2\,\bar Q(\mathcal{D}_i, g, \mathfrak{T}) = 2\,\langle \operatorname{Tr}_{\mathcal{D}_i}\mathfrak{T},\ \operatorname{Tr}_{\mathcal{D}_i^\perp}\mathfrak{T}\rangle - \langle \mathfrak{T}, \mathfrak{T}\rangle_{\,|\,V(\mathcal{D}_i)},$$

see (19). Thus, (14) reduces to

$$2\,\bar{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - 2\,{\rm S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i \big(\langle {\rm Tr}_{\mathcal{D}_i^\perp} \mathfrak{T},\, {\rm Tr}_{\mathcal{D}_i} \mathfrak{T}\rangle - \frac{1}{2}\,\langle \mathfrak{T},\, \mathfrak{T}\rangle_{\,|V(\mathcal{D}_i)}\big) = 0. \tag{18}$$

Applying the Divergence Theorem gives (17) that also follows from (15). □

In the rest of this section we give examples with integral formulas for statistical and semi-symmetric connections.

Example 2. *(a) For the case of a statistical connection $\bar\nabla = \nabla + \mathfrak{T}$, the equality (9) simplifies as*

$$2\,\bar{Q}(\mathcal{D}, g, \mathfrak{T}) = 2\,\langle {\rm Tr}_{\mathcal{D}}\, \mathfrak{T},\ {\rm Tr}_{\mathcal{D}^\perp} \mathfrak{T}\rangle - \langle \mathfrak{T},\, \mathfrak{T}\rangle_{\,|\,V(\mathcal{D})}\,; \tag{19}$$

thus, (7) reduces to the equality

$$2\,\bar{\rm S}_{\mathcal{D},\mathcal{D}^\perp} - 2\,{\rm S}_{\mathcal{D},\mathcal{D}^\perp} - \langle {\rm Tr}_{\mathcal{D}^\perp} \mathfrak{T},\, {\rm Tr}_{\mathcal{D}}\, \mathfrak{T}\rangle + \frac{1}{2}\,\langle \mathfrak{T},\, \mathfrak{T}\rangle_{\,|V(\mathcal{D})} = 0. \tag{20}$$

Using (10) and (20) for a closed manifold M gives the following integral formula:

$$\int_M \big(2\,\bar{\rm S}_{\mathcal{D},\mathcal{D}^\perp} - Q(\mathcal{D}, g) - \langle {\rm Tr}_{\mathcal{D}}\, \mathfrak{T}, {\rm Tr}_{\mathcal{D}^\perp} \mathfrak{T}\rangle + \frac{1}{2}\langle \mathfrak{T},\, \mathfrak{T}\rangle_{\,|V(\mathcal{D})}\big)\,{\rm d\,vol}_g = 0.$$

(b) Let a Riemannian manifold (M^m, g) with a statistical connection $\bar\nabla$ admit a codimension-one foliation $\mathcal{F}$, and $\sigma_k(\mathcal{F})$ be elementary symmetric functions of principal curvatures of the leaves of $\mathcal{F}$. Let there exist unit normal vector field N to $\mathcal{F}$. Put $\mathcal{D} = span(N)$ and integrate the sum of (6) and (7) over a closed M. We get the integral formula generalizing (5):

$$\int_M \big(\,2\sigma_2(\mathcal{F}) - \overline{\rm Ric}_{N,N} - 2\,\langle \mathfrak{T}_N N,\ {\rm Tr}_{\mathcal{D}^\perp} \mathfrak{T}\rangle + \langle \mathfrak{T}_N,\, \mathfrak{T}_N\rangle_{\,|\,\mathcal{D}^\perp}\,\big)\,{\rm d\,vol}_g = 0. \tag{21}$$

(c) Let a Riemannian manifold (M^m, g) with a statistical connection $\bar\nabla$ admit m pairwise orthogonal codimension-one foliations $\mathcal{F}_i$, and $\sigma_k(\mathcal{F}_i)$ be elementary symmetric functions of principal curvatures of the leaves of $\mathcal{F}_i$. Let there exist unit vector fields N_i orthogonal to $\mathcal{F}_i$ and $\mathcal{D}_i = span(N_i)$. Writing down (21) for each N_i on a closed manifold M, and using

$$\langle \mathfrak{T},\, \mathfrak{T}\rangle_{\,|\,V(\mathcal{D}_i)} = \sum\nolimits_{j\ne i}\langle \mathfrak{T}_{N_i} N_j,\, \mathfrak{T}_{N_i} N_j\rangle,\quad \langle {\rm Tr}_{\mathcal{D}_i} \mathfrak{T},\ {\rm Tr}_{\mathcal{D}_i^\perp} \mathfrak{T}\rangle = \langle \mathfrak{T}_{N_i} N_i,\, \sum\nolimits_{j\ne i} \mathfrak{T}_{N_j} N_j,\rangle,$$

we obtain the following integral formulas for $1 \le i \le m$:

$$\begin{aligned}&\int_M \Big(2\sigma_2(\mathcal{F}_i) - \overline{\rm Ric}_{N_i,N_i}\\ &- \langle \mathfrak{T}_{N_i} N_i,\, \sum\nolimits_{j\ne i} \mathfrak{T}_{N_j} N_j,\rangle - \frac{1}{2}\sum\nolimits_{j\ne i}\langle \mathfrak{T}_{N_i} N_j,\, \mathfrak{T}_{N_i} N_j\rangle\Big)\,{\rm d\,vol}_g = 0.\end{aligned} \tag{22}$$

Summing m copies of (22) for $i = 1, \ldots, m$ and using $\bar{\rm S} = \sum_i \overline{\rm Ric}_{N_i,N_i}$, gives the integral formula with the scalar curvature $\bar{\rm S}$ of (M, g) (which also follows from (15) when $n_i = 1$),

$$\int_M \Big(\sum\nolimits_i \big(2\,\sigma_2(\mathcal{F}_i) - \langle \mathfrak{T}_{N_i} N_i,\, \sum_{j\ne i} \mathfrak{T}_{N_j} N_j,\rangle + \frac{1}{2}\sum_{j\ne i}\langle \mathfrak{T}_{N_i} N_j,\, \mathfrak{T}_{N_i} N_j\rangle\,\big) - \bar{\rm S}\Big)\,{\rm d\,vol}_g = 0.$$

For $\mathfrak{T} = 0$, the above formula simplifies to the following integral formula (see also [20]):

$$\int_M \big(\,2\sum\nolimits_i \sigma_2(\mathcal{F}_i) - {\rm S}\,\big)\,{\rm d\,vol}_g = 0. \tag{23}$$

We immediately have the following consequences of (23):
(a) if ${\rm S} < 0$, then each foliation $\mathcal{F}_i$ cannot be totally umbilical;
(b) if ${\rm S} > 0$, then each foliation $\mathcal{F}_i$ cannot be harmonic.

Example 3. *(a) Assume that $\bar\nabla$ is a semi-symmetric connection on (M^m, g) with complementary orthogonal distributions $(\mathcal{D}, \mathcal{D}^\perp)$. We have $\mathfrak{T}^* = -\mathfrak{T}$ (metric compatible connection) and*

$$\langle \mathfrak{T} - \mathfrak{T}^* + \mathfrak{T}^\wedge - \mathfrak{T}^{*\wedge},\ A^\perp - T^{\perp\sharp} + A - T^\sharp\rangle = 2\,\mathrm{Tr}(A_{P^\perp U} - A^\perp_{P^\top U}),$$
$$\langle \mathfrak{T}^*,\ \mathfrak{T}^\wedge\rangle_{\,|\,V(\mathcal{D})} = 0,$$
$$\mathrm{Tr}_{\mathcal{D}}\,\mathfrak{T} = PU - nU, \quad \mathrm{Tr}_{\mathcal{D}^\perp}\,\mathfrak{T} = P^\perp U - n^\perp U,$$

where $n = \dim\mathcal{D}$ and $n^\perp = \dim\mathcal{D}^\perp$. Thus, (7) takes the form

$$\begin{aligned} -\operatorname{div}(nP^\perp U + n^\perp PU) \ &= \bar{\mathrm{S}}_{\mathcal{D},\mathcal{D}^\perp} - \mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} + n\,n^\perp U - n\langle P^\perp U, P^\perp U\rangle - n^\perp\langle PU, PU\rangle \\ &- (n^\perp - n)\langle U,\ H - H^\perp\rangle - \mathrm{Tr}(A_{U^\perp} - A^\perp_{U^\top}). \end{aligned} \tag{24}$$

Using the Divergence Theorem for (24) and (10) on a closed Riemannian manifold (M, g), gives the following integral formula:

$$\int_M \big(\bar{\mathrm{S}}_{\mathcal{D},\mathcal{D}^\perp} - Q(\mathcal{D}, g) + n\,n^\perp U - n\langle P^\perp U, P^\perp U\rangle - n^\perp\langle PU, PU\rangle$$
$$- (n^\perp - n)\langle U,\ H - H^\perp\rangle - \mathrm{Tr}(A_{P^\perp U} - A^\perp_{P^\top U})\,\big)\,\mathrm{d\,vol}_g = 0.$$

(b) Next, consider an almost multi-product manifold $(M, g, \bar\nabla; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with a semi-symmetric connection $\bar\nabla = \nabla + \mathfrak{T}$. By (24) and (4), we have the equality

$$\begin{aligned} -\operatorname{div}\textstyle\sum_i(n_i P_i^\perp U + n_i^\perp P_i U) &= 2(\bar{\mathrm{S}}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}) + \textstyle\sum_i\big(n_i n_i^\perp\, U - n_i\langle P_i^\perp U, P_i^\perp U\rangle \\ - n_i^\perp\langle P_i U, P_i U\rangle &- (n_i^\perp - n_i)\langle U,\ H_i - H_i^\perp\rangle - \mathrm{Tr}(A_{i,P_i^\perp U} - A^\perp_{i,P_i^\top U})\big). \end{aligned} \tag{25}$$

Using the Divergence Theorem for (25) and (16) on a closed Riemannian manifold (M, g), gives the following integral formula:

$$\int_M \Big(2\,\bar{\mathrm{S}}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \textstyle\sum_i\big(n_i\, n_i^\perp\, U - n_i\langle P_i^\perp U, P_i^\perp U\rangle - n_i^\perp\langle P_i U, P_i U\rangle$$
$$- (n_i^\perp - n_i)\langle U,\ H_i - H_i^\perp\rangle - \mathrm{Tr}(A_{i,P_i^\perp U} - A^\perp_{i,P_i^\top U})) - Q(\mathcal{D}_i, g)\big)\Big)\,\mathrm{d\,vol}_g = 0.$$

4. Splitting and Nonexistence Theorems

Here, we apply Propositions 1 and 2 to obtain splitting results for almost multi-product manifolds and multiply twisted products.

We say that an almost multi-product manifold $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ *splits* if all distributions $\mathcal{D}_i$ are integrable and M is locally the direct product $M_1 \times \ldots \times M_k$ with canonical foliations tangent to $\mathcal{D}_i$. It is well known that if a simply connected manifold splits then it is the direct product.

We apply the submanifolds theory to almost multi-product manifolds.

Definition 2. *A pair $(\mathcal{D}_i, \mathcal{D}_j)$ with $i \ne j$ of distributions on $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ (with $k > 2$) is*
(a) mixed totally geodesic, if $h_{ij}(X, Y) = 0$ for all $X \in \mathcal{D}_i$ and $Y \in \mathcal{D}_j$.
(b) mixed integrable, if $T_{ij}(X, Y) = 0$ for all $X \in \mathcal{D}_i$ and $Y \in \mathcal{D}_j$.

Lemma 3 (see [20])**.** *If each pair $(\mathcal{D}_i, \mathcal{D}_j)$ with $i \ne j$ on $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is*
(a) mixed totally geodesic, then $h_{q_1,\ldots,q_r}(X, Y) = 0$,
(b) mixed integrable, then $T_{q_1,\ldots,q_r}(X, Y) = 0$,
where $q_1, \ldots, q_r$ is any subset of r distinct elements of $\{1, \ldots, k\}$ and $X \in \mathcal{D}_{q(1)}$, $Y \in \mathcal{D}_{q(2)}$.

The next definition is introduced to simplify the presentation of results. A linear connection $\bar\nabla = \nabla + \mathfrak{T}$ on $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ will be called *adapted* if $\mathfrak{T}$ is decomposed into $\mathcal{D}_i$-components,

$$\mathfrak{T}_X Y = 0 \quad (X \in \mathcal{D}_i,\ \ Y \in \mathcal{D}_j,\ \ i \ne j).$$

Lemma 4. *For an almost multi-product structure on M with an adapted statistical connection we have* $\bar{\mathrm{S}}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}$.

Proof. For statistical connection, the equality (18) is true. For adapted connection, we have $\bar{Q}(\mathcal{D}_i, g, \mathfrak{T}) = 0$ for $i \ge 1$, see Corollary 1. Thus, (18) implies the claim. □

The following splitting result generalizes Theorem 6 (with $k = 2$) in [12] and Theorem 2.1 in [20].

Theorem 2. *Suppose that an almost multi-product manifold* $(M, g, \bar\nabla; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ *with a statistical adapted connection* $\bar\nabla = \nabla + \mathfrak{T}$ *has integrable harmonic distributions* $\mathcal{D}_1, \ldots, \mathcal{D}_k$ *and each pair* $(\mathcal{D}_i, \mathcal{D}_j)$ *is mixed integrable. If* $\bar{\mathrm{S}}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} \ge 0$, *then* (M, g) *splits.*

Proof. From the equality $H_{1\ldots r} = P_{r+1\ldots k}(H_1 + \ldots + H_r)$ it follows that $H_i = 0$ for all $i \ge 1$, then $H_i^\perp = 0$ for all $i \ge 1$. Similarly (by Lemma 3), if $T_{ij} = 0$ for all $i \ge 1$, then $T_i^\perp = 0$ for all $i \ge 1$. By conditions, (8) with $\mathcal{D} = \mathcal{D}_i$, (12) and Corollary 1,

$$2\,\bar{\mathrm{S}}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \sum\nolimits_i (\|h_i\|^2 + \|h_i^\perp\|^2) = 0.$$

By the above, $h_i = 0$ $(i \ge 1)$. By well-known Decomposition theorem of de Rham, (M, g) splits. □

Observe that for $X \in \mathcal{D}_i$ and $Y \in \mathcal{D}_i^\perp$ we have

$$\operatorname{div}_i^\perp X = \operatorname{div} X + \langle X,\, H_i^\perp\rangle, \quad \operatorname{div}_i Y = \operatorname{div} Y + \langle Y,\, H_i\rangle, \tag{26}$$

where

$$\operatorname{div}_i X = \sum_{n_{i-1} < a \le n_i} \langle \nabla_{e_a} X, e_a\rangle, \quad \operatorname{div}_i^\perp X = \sum_{b \ne (n_{i-1}, n_i]} \langle \nabla_{e_b} X, e_b\rangle.$$

The following splitting result generalizes Theorem 2 in [14], see also Corollary 14 (where $k = 2$) in [11].

Theorem 3. *Suppose that an almost multi-product manifold* $(M, g, \bar\nabla; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ *with a statistical adapted connection* $\bar\nabla = \nabla + \mathfrak{T}$ *has integrable distributions* $\mathcal{D}_1, \ldots, \mathcal{D}_k$ *and each pair* $(\mathcal{D}_i, \mathcal{D}_j)$ *is mixed integrable. Suppose that* $\mathcal{D}_j$ *is harmonic (i.e.,* $H_j = 0$*) for some index* j *and* $H_i \in \mathcal{D}_j$ *and all* $i \ne j$. *If* $\bar{\mathrm{S}}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} > 0$, *then a foliation tangent to* $\mathcal{D}_j$ *has no compact leaves.*

Proof. By conditions, we have $H_i^\perp \in \mathcal{D}_j$ or all i. Assume that $\mathcal{D}_j$ has a compact leaf L. By (26), we have $\operatorname{div}_L H_i = \operatorname{div} H_i + \|H_i\|^2$ for $i \ne j$ and $\operatorname{div}_L H_i^\perp = \operatorname{div} H_i^\perp + \|H_i^\perp\|^2$ for all i. Thus,

$$\begin{aligned}\operatorname{div}_L \big(\sum\nolimits_{i\ne j} H_i + \sum\nolimits_i H_i^\perp\big) &= \operatorname{div} \big(\sum\nolimits_{i\ne j} H_i + \sum\nolimits_i H_i^\perp\big)\\ &\quad + \sum\nolimits_{i\ne j} \|H_i\|^2 + \sum\nolimits_i \|H_i^\perp\|^2.\end{aligned}$$

Therefore, integrating (12) along L and using Lemma 4, gives

$$\begin{aligned} 0 &= \int_L \operatorname{div}_L \big(\sum\nolimits_{i\ne j} H_i + \sum\nolimits_i H_i^\perp\big)\, d\operatorname{vol}_L \\ &= \int_L \big(2\,\bar{\mathrm{S}}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \sum\nolimits_i (\|h_i\|^2 + \|h_i^\perp\|^2)\big)\, d\operatorname{vol}_L > 0 \end{aligned}$$

– a contradiction. □

The following splitting result generalizes ([20], Theorem 2.2).

Theorem 4. *Suppose that an almost multi-product manifold $(M, g, \bar\nabla; \mathcal{D}_1, \dots, \mathcal{D}_k)$ with a statistical adapted connection $\bar\nabla = \nabla + \mathfrak{T}$ has totally umbilical distributions such that each pair $(\mathcal{D}_i, \mathcal{D}_j)$ is mixed totally geodesic, $\langle H_i, H_j\rangle = 0$ for all $i \neq j$. If (M, g) is complete open, $\|\xi\| \in \mathrm{L}^1(M, g)$ for $\xi = \sum_i (H_i + H_i^\perp)$ and $\bar{\mathrm{S}}_{\mathcal{D}_1,\dots,\mathcal{D}_k} \le 0$, then (M, g) splits.*

Proof. By assumptions and Lemma 4, from (12) we get

$$\operatorname{div}\xi = 2\,\bar{\mathrm{S}}_{\mathcal{D}_1,\dots,\mathcal{D}_k} - \sum\nolimits_i Q(\mathcal{D}_i, g), \tag{27}$$

where $Q(\mathcal{D}_i, g)$ is given in (8) with $\mathcal{D} = \mathcal{D}_i$. By conditions, for any $i \ge 1$ we have

$$\|H_i^\perp\|^2 - \|h_i^\perp\|^2 = \sum\nolimits_{j\neq i} \frac{n_j - 1}{n_j}\, \|P_i^\perp H_j\|^2 \ge 0,$$

where $P_i^\perp$ is the orthoprojector onto $\mathcal{D}_i^\perp$. Hence, $Q(\mathcal{D}_i, g) \ge 0$, and from $\bar{\mathrm{S}}_{\mathcal{D}_1,\dots,\mathcal{D}_k} \le 0$ and (27) we get $\operatorname{div}\xi \le 0$. By conditions and Lemma 5 below, $\operatorname{div}\xi = 0$. Thus, see (27), $\bar{\mathrm{S}}_{\mathcal{D}_1,\dots,\mathcal{D}_k} = 0$ and T_i and h_i vanish. By the Decomposition theorem of de Rham, (M, g) splits. □

Modifying Divergence theorem on a complete open manifold (M, g) gives the following.

Lemma 5 (see Proposition 1 in [22]). *Let (M^m, g) be a complete open Riemannian manifold endowed with a vector field ξ such that $\operatorname{div}\xi \ge 0$. If the norm $\|\xi\|_g \in \mathrm{L}^1(M, g)$ then $\operatorname{div}\xi \equiv 0$.*

The following corollary of Theorem 4 generalizes ([20], Corollary 4) with $\mathfrak{T} = 0$ and ([20], Corollary 22) with $k = 2$.

Corollary 2. *Let a multiply twisted product manifold (M, g) of k Riemannian manifolds be complete open and endowed with a statistical adapted connection and let $\langle H_i, H_j\rangle = 0$ for $i \neq j$. If $\bar{\mathrm{S}}_{\mathcal{D}_1,\dots,\mathcal{D}_k} \le 0$ and $\|\sum_i (H_i + H_i^\perp)\| \in L^1(M, g)$, then (M, g) is the direct product.*

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