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Nonlinear Spectrum and Fixed Point Index for a Class of Decomposable Operators

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Abstract: We study a class of nonlinear operators that can be written as the composition of a linear operator and a nonlinear map. We obtain results on fixed point index based on parameters that are related to the definitions of nonlinear spectra. As a particular case, existence of positive solutions for a second-order differential equation with separated boundary conditions is proved. The result also provides a spectral interval for the corresponding Hammerstein integral operator.

Keywords: boundary value problem; cone; fixed point index; nonlinear spectrum; stably-solvable map

MSC: 47H10; 34B10

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1. Introduction

Nonlinear spectral theory has been shown to have applications in the study of existence of solutions for operator equations, particularly in integral equations [1,2]. On the other hand, fixed point index is well known as a popular technique to prove existence and multiplicity of positive solutions for Boundary Value Problems (BVPs). For example, a common method in studying differential equations with various boundary conditions is to convert the problem to an integral equation using the Green's function, then apply a fixed point theorem. Usually, the integral equation can be written as composition of a bounded linear operator and a nonlinear map.

In this paper, we are interested in operators in the form $LF : P \to P \subset E$, where L is a linear operator, F is a nonlinear map, and P is an order cone of the Banach space E. We obtain results on fixed point index of the nonlinear operator LF based on parameters that are related to the nonlinear spectra. We also extend the continuation principle for stably-solvable maps to the operator LF on a cone. The stably-solvable property is a key concept in the definition of nonlinear spectra [3,4]. As a particular case, we prove existence of positive solutions for a second-order differential equation with separated boundary conditions [5] and thus obtain a spectral interval for the Hammerstein integral operator.

Let E, F be Banach spaces and $f : E \to F$ be a continuous nonlinear map. The Furi–Martelli–Vignoli-spectrum (fmv-spectrum) [3,4] is defined by two parameters $d(f), \omega(f)$ and the stably-solvable property. Later, the Feng-spectrum [1,6] was introduced with the parameters $\omega(f), \nu(f)$ and $m(f)$. It is shown that the Feng-spectrum ($\sigma_F(f)$) contains all eigenvalues of the operator f.

We briefly review definitions of the related parameters. Let $\alpha(\Omega)$ denote the Measure of Noncompactness of $\Omega \subset E$ [1]. Then,

$$\alpha(f) = \inf\{k \geq 0 : \alpha(f(\Omega)) \leq k\alpha(\Omega) \ \text{ for every bounded } \ \Omega \subset E\},$$

$$\omega(f) = \sup\{k \geq 0; \alpha(f(\Omega)) \geq k\alpha(\Omega) \ \text{ for every bounded } \ \Omega \subset E\},$$

$$m(f) = \sup\{k \geq 0 : \|f(x)\| \geq k\|x\| \text{ for all } x \in E\},$$

$$d(f) = \liminf_{\|x\| \to \infty} \frac{\|f(x)\|}{\|x\|}, \quad |f| = \limsup_{\|x\| \to \infty} \frac{\|f(x)\|}{\|x\|},$$

where $|f|$ is called the quasinorm of f.

Definition 1. *The nonlinear map $f : E \to F$ is stably-solvable if and only if given any compact map $h : E \to F$ with $|h| = 0$, the equation*

$$f(x) = h(x)$$

has a solution in E.

Next, an order cone of Banach space introduces a partial order for the space so that positive solutions can be studied.

Definition 2. *Let E be a Banach space, P is a subset of E. P is called an order cone iff:*

(i) *P is closed, nonempty, and $P \neq \{0\}$;*
(ii) *$a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;*
(iii) *$x \in P$ and $-x \in P \Rightarrow x = 0$.*

Let P be an order cone of the Banach space E. For $r > 0$, denote $P_r = \{u \in P, \|u\| < r\}$, and $\partial P_r = \{u \in P, \|u\| = r\}$.

The following two lemmas on fixed point index [7] have been applied to prove existence of solutions for boundary value problems [8] and many other applications [7,9].

Lemma 1. *Let $N : P \to P$ be a completely continuous mapping. If*

$$Nu \neq \mu u, \text{ for all } u \in \partial P_r, \text{ and all } \mu \geq 1,$$

then the fixed point index $i(N, P_r, P) = 1$.

Lemma 2. *let $N : P \to P$ be a completely continuous mapping and satisfy $Nu \neq u$ for $u \in \partial P_r$. If $\|Nu\| \geq \|u\|$, for $u \in \partial P_r$, then the fixed point index $i(N, P_r, P) = 0$.*

2. Stably-Solvable Maps and Fixed Point Index

Let E be a Banach space and $P \subset E$ be an order cone. We consider the linear homeomorphism $L : E \to E$. It is known that [4,6]

$$m(L) \geq \frac{1}{\|L^{-1}\|}, \quad \omega(L) \geq \frac{1}{\|L^{-1}\|}, \quad d(L) = \frac{1}{\|L^{-1}\|}.$$

Let $F : P \to P$ be a nonlinear map. We use the following notations,

$$d(F)_P = \liminf_{x\in P, x\to\infty} \frac{\|F(x)\|}{\|x\|}, \quad |F|_P = \limsup_{x\in P, x\to\infty} \frac{\|F(x)\|}{\|x\|},$$

$$d(F)_0 = \liminf_{x\in P, x\to 0} \frac{\|F(x)\|}{\|x\|}, \quad |F|_0 = \limsup_{x\in P, x\to 0} \frac{\|F(x)\|}{\|x\|}.$$

The stably-solvable maps on a cone $P \subset E$ are defined below.

Definition 3. *The nonlinear map $F : P \to P$ is stably-solvable on the cone P if and only if given any compact map $h : P \to P$ with $|h|_P = 0$, the equation*

$$f(x) = h(x)$$

has a solution $x \in P$.

The following theorem is an extension of the continuation principle for stably-solvable maps to the class of decomposable operators $LF : P \to P$.

Theorem 1. *If $F : P \to P$ is stably-solvable on the cone P and $L : P \to P$ is bijective.*

(1) LF is also stably-solvable on P.

(2) Assume that $h : P \times [0,1] \to P$ is compact such that $h(x,0) = 0$ for all $x \in P$. Let

$$S = \{x \in P : LF(x) = h(x,t) \text{ for some } t \in [0,1]\}.$$

If $F(S)$ is bounded, then the equation

$$LF(x) = h(x,1)$$

has a solution $x \in P$.

Proof. (1) If $h : P \to P$ is a compact operator with $|h|_P = 0$. Then, $L^{-1}h : P \to P$ is compact and $|L^{-1}h|_P \leq \|L\||h|_P = 0$. Therefore, the equation

$$F(x) = L^{-1}h(x)$$

has a solution $x \in P$. Thus $LF(x) = h(x)$ has a solution. By definition, LF is stably-solvable on P.

(2) Consider the operator $L^{-1}h : P \times [0,1] \to P$. $L^{-1}h$ is compact and $L^{-1}h(x,0) = 0$. Let

$$S = \{x \in P : F(x) = L^{-1}h(x,t) \text{ for some } t \in [0,1]\}.$$

As F is stably-solvable on P, $S = \{x \in P : LF(x) = h(x,t)$ for some $t \in [0,1]\}$, and $F(S)$ is bounded by assumption (2), the equation $F(x) = L^{-1}h(x,1)$ has a solution $x \in P$. Thus $LF(x) = h(x,1)$ has a solution. □

Our next result is on the fixed point index of the nonlinear operator LF based on the parameters such as $|F|_P$ and $d(F)_P$ that are related to the definition of the fmv-spectrum [4].

Theorem 2. *Assume that $L : E \to E$ is a linear homeomorphism and $F : P \to P$ is a nonlinear map such that the composition $LF : P \to P$ is completely continuous.*

(1) *If* $|F|_P < d(L^{-1})$, *then there exists* $R_1 > 0$ *such that for all* $R > R_1$, $i(LF, P_R, P) = 1$.
(2) *If* $|F|_0 < d(L^{-1})$, *then there exists* $r_1 > 0$ *such that for all* $r < r_1$, $i(LF, P_r, P) = 1$.
(3) *If* $d(F)_P d(L) > 1$, *then there exists* $R_2 > 0$ *such that for all* $R > R_2$, $i(LF, P_R, P) = 0$.
(4) *If* $d(F)_0 d(L) > 1$, *then there exists* $r_2 > 0$ *such that for all* $r < r_2$, $i(LF, P_r, P) = 0$.

Proof. Define

$$O_1 = \{x \in P : LF(x) = \mu x,\ \mu \geq 1\},$$

and

$$O_2 = \{x \in P : \|F(x)\| \leq \|L^{-1}\|\|x\|\}.$$

We prove that under condition (1), O_1 is bounded. Condition (2) ensures that O_1 is bounded below. Thus, there exists $\delta > 0$ such that for $u \in E$, $\|u\| < \delta$, then $u \notin O_1$. Similarly, under condition (3), O_2 is bounded. Condition (4) implies O_2 is bounded below.

We only prove (1) and (4). (2) and (3) can be proved following the similar ideas.

Under condition (1), assume O_1 is unbounded. Then, there exist $x_n \in O_1$ such that $\|x_n\| \to \infty$ as $n \to \infty$.

$$\|L\|\|F(x_n)\| \geq \|LF(x_n)\| = \|\mu_n x_n\| \geq \|x_n\|. \tag{1}$$

$$\frac{\|F(x_n)\|}{\|x_n\|} \geq \frac{1}{\|L\|}. \tag{2}$$

Therefore, $|F|_P = \limsup_{x \in P, \|x\| \to \infty} \dfrac{\|F(x)\|}{\|x\|} \geq \dfrac{1}{\|L\|}$. This contradicts the condition $|F|_P < d(L^{-1}) = \frac{1}{\|L\|}$.

On the other hand, if condition (4) holds, assume there exists $x_n \in O_2$ such that $\|x_n\| \to 0$ as $n \to \infty$. We have

$$\frac{\|F(x_n)\|}{\|x_n\|} \leq \|L^{-1}\|.$$

Thus,

$$d(F)_0 = \liminf_{x \in P, \|x\| \to 0} \frac{\|F(x)\|}{\|x\|} \leq \|L^{-1}\| = \frac{1}{d(L)}.$$

This contradicts the assumption $d(F)_0 d(L) > 1$.

Next, if O_1 is bounded, we can select R large enough such that

$$LFx \neq \mu x, \text{ for all } x \in \partial P_R, \text{ and all } \mu \geq 1.$$

By Lemma 1, we have $i(LF, P_R, P) = 1$.

On the other hand, if O_1 is bounded below, we can select r small enough such that

$$LFx \neq \mu x, \text{ for all } x \in \partial P_r, \text{ and all } \mu \geq 1.$$

Again by Lemma 1, we have $i(LF, P_r, P) = 1$.

If O_2 is bounded, we can select R large enough such that $\|F(x)\| > \|L^{-1}\|\|x\|$ for $x \in \partial P_R$. Then, $LF(x) \neq x$ for all $x \in \partial P_R$. Otherwise, if there exists $x_0 \in \partial P_R$ such that $LF(x_0) = x_0$, we would get the contradiction $F(x_0) = L^{-1}(x_0)$ and $\|F(x_0)\| = \|L^{-1}(x_0)\| \leq \|L^{-1}\|\|x_0\|$. Next,

$$\|LFx\| \geq \frac{1}{\|L^{-1}\|}\|F(x)\| \geq \|x\|, \text{ for all } x \in \partial P_R.$$

By Lemma 2, we have $i(LF, P_R, P) = 0$

Similarly, if O_2 is bounded below, we can select r small enough such that

$$\|LFx\| \geq \|x\|, \text{ for all } x \in \partial P_r.$$

By Lemma 2, we have $i(LF, P_r, P) = 0$
The proof is complete. □

Theorem 2 can be used to prove existence of positive solutions for nonlinear operator equations involving a parameter.

Theorem 3. *Let L and F be defined as Theorem 2. Assume that*

$$d(F)_0 > \|L^{-1}\| \text{ and } |F|_P < \frac{1}{\|L\|}.$$

Then, the operator equation $\lambda LF(x) = x$ has a positive solution $x \in P$ for $1 \leq \lambda < \frac{d(L^{-1})}{|F|_P}$.

Proof. The condition $d(F)_0 > \|L^{-1}\|$ implies $d(F)_0 d(L) > 1$ and $|F|_P < \frac{1}{\|L\|}$ ensures $\frac{d(L^{-1})}{|F|_P} > 1$. For $\lambda \geq 1$, we have $d(F)_0 d(\lambda L) = \lambda d(F)_0 d(L) \geq d(F)_0 d(L) > 1$. By Theorem 2 (4), there exists $r > 0$ small enough such that $i(\lambda LF, P_r, P) = 0$. On the other side, if $\lambda < \frac{d(L^{-1})}{|F|_P}$, then $|F|_P < \frac{d(L^{-1})}{\lambda} = d((\lambda L)^{-1})$. By Theorem 2 (1), there exists $R > 0$ large enough such that $i(\lambda LF, P_R, P) = 1$. Therefore, there exists a fixed point $\lambda LF(x) = x$, $x \in \Omega_R \setminus \overline{\Omega_r}$, where $\Omega_R = \{x : x \in P, \|x\| < R\}$. □

As the Feng-spectrum contains all eigenvalues and it is closed [6], the following result on spectral interval follows from Theorem 3.

Corollary 1. *Under the conditions of Theorem 3, the nonlinear operator LF has the spectral interval*

$$[\|L\||F|_P, 1] \subset \sigma_F(LF).$$

3. Positive Solutions and Spectral Interval for BVPs

In this section, we study the following second-order differential equation with separated boundary conditions:

$$u''(t) + \lambda f(t, u(t)) = 0, \ t \in [0,1], \tag{3}$$

$$\theta u(0) - \alpha u'(0) = 0, \tag{4}$$

$$\gamma u(1) + \beta u'(1) = 0, \tag{5}$$

where $\theta, \alpha, \beta > 0, \gamma \geq 0, \lambda > 0$, and $f : [0,1] \times (0, \infty) \to \mathbb{R}^+$ is continuous and non-negative. When $\lambda = 1$, problem (3)–(5) was studied in [9] under the conditions that $\alpha > 0$, $\beta > 0$ and $\theta\gamma + \theta\beta + \alpha\gamma > 0$. Conditions (4) and (5) are an extension of the boundary conditions $\alpha u(0) - \beta u'(0) = 0, u'(1) = 0$ studied in [10], and a special case of the non-local boundary value problem involving linear functionals $au(0) - bu'(0) = \alpha[u], u'(1) = \beta[u]$ [5,11,12]. Equation (5) can also been seen as the limiting case of the basic three-point boundary value problem [13], $\sigma u'(1) + u(\eta) = 0$, as $\eta \to 1^-$. It is known that the three-point boundary value problem can be explained as a model of a thermostat with a temperature controller [13–15].

In the following, we prove existence of positive solutions of BVP (3)–(5) using Lemmas 1 and 2 and obtain a spectral interval for the corresponding Hammerstein integral operator that can be written as the composition of a linear operator L and a nonlinear map F.

Notice that existence of a solution for (3)–(5) is equivalent to the existence of a fixed point for the following Hammerstein operator [5]:

$$N(\lambda, u)(t) = \lambda \int_0^1 G(t,s) f(s, u(s)) ds, \tag{6}$$

where the Green's function

$$G(t,s) = \begin{cases} \dfrac{(\alpha+\theta s)(\gamma+\beta-\gamma t)}{\theta(\gamma+\beta)+\alpha\gamma} & 0 \le s \le t \le 1, \\ \dfrac{(\alpha+\theta t)(\gamma+\beta-\gamma s)}{\theta(\gamma+\beta)+\alpha\gamma} & 0 \le t \le s \le 1. \end{cases} \tag{7}$$

Let $C[0,1]$ denote a Banach space of continuous functions with the norm

$$\|u\| = \max\{|u(t)| : t \in [0,1]\}.$$

We use the cone P with parameter $0 < c_0 < 1$:

$$P = \{u \in C[0,1] : u(t) \ge c_0\|u\|, \text{ for } t \in [0,1]\},$$

$$c_0 = \begin{cases} \dfrac{\alpha}{\alpha+\theta}, & if\ \gamma = 0, \\ \dfrac{\alpha}{\alpha+\theta}, & if\ \gamma \ne 0,\ \dfrac{\beta}{\gamma} - \dfrac{\alpha}{\theta} \ge 1, \\ \dfrac{\beta}{\beta+\gamma}, & if\ \gamma \ne 0,\ \dfrac{\beta}{\gamma} - \dfrac{\alpha}{\theta} \le -1, \\ \dfrac{\alpha\beta}{(\alpha+\theta)(\gamma+\beta)}, & if\ \gamma \ne 0,\ -1 < \dfrac{\beta}{\gamma} - \dfrac{\alpha}{\theta} < 1. \end{cases} \tag{8}$$

Define the operators L and F: $C[0,1] \to C[0,1]$:

$$(Lu)(t) = \int_0^1 G(t,s)u(s)ds, \quad (Fu)(t) = f(t, u(t)), \ u \in C[0,1]. \tag{9}$$

Then, $N(\lambda, u) = \lambda(LF)(u)$. Note that the linear operator L is not a homeomorphism on the space $C[0,1]$. However, we will show that $L : P \to P$ and is injective on P. Following Lemma 2.1 of [5], we know that the Green's function G satisfies the strong positivity condition [9]:

$$c_0 G(s,s) \le G(t,s) \le G(s,s), \quad \text{for } 0 \le t, s \le 1. \tag{10}$$

For $\forall u \in P$, (10) ensures that

$$\begin{aligned} c_0\|N(\lambda,u)\| &\le c_0 \int_0^1 \lambda G(s,s) f(s,u(s)) ds \\ &\le \int_0^1 \lambda G(t,s) f(s,u(s)) ds = N(\lambda,u). \end{aligned} \tag{11}$$

Therefore, $N(\lambda, P) \subset P$.

We first prove a property of the linear operator L that is related to the so-called u_0-positive linear operator on a cone [16], that later was generalized to u_0-positive linear operator relative to a pair of cones [9,17]. The following lemma shows that L actually satisfies stronger conditions than the requirements of u_0-positive linear operators.

Lemma 3. *Let L be defined by (9). Then $L : P \to P$ is completely continuous and satisfies*

$$k_1 u(1) \le Lu \le k_2 u(1), \;\; \textit{for any } u \in P, \tag{12}$$

for some $k_1, k_2 > 0$.

Proof. For $\forall u \in P$, by property (10), we have

$$c_0 \|Lu\| \le c_0 \int_0^1 G(s,s)u(s)ds \le \int_0^1 G(t,s)u(s)ds = Lu,$$

So $L(P) \subset P$. Moreover,

$$c_0 u(1) \le c_0 \|u\| \le u(t) \le \|u\| \le \frac{u(1)}{c_0}, \;\; t \in [0,1].$$

Thus

$$\begin{aligned} \left(c_0^2 \int_0^1 G(s,s)ds\right)u(1) &= \int_0^1 c_0 G(s,s) c_0 u(1) ds \\ &\le \int_0^1 G(t,s) c_0 \|u\| ds \le \int_0^1 G(t,s)u(s)ds \end{aligned}$$

and

$$\begin{aligned} \int_0^1 G(t,s)u(s)ds &\le \int_0^1 G(t,s)\|u\|ds \\ &\le \int_0^1 G(t,s)\frac{u(1)}{c_0}ds \le \left(\frac{1}{c_0}\int_0^1 G(s,s)ds\right)u(1). \end{aligned}$$

Let $k_1 = c_0^2 \int_0^1 G(s,s)ds$, $k_2 = \frac{1}{c_0}\int_0^1 G(s,s)ds$, then

$$k_1 u(1) \le Lu \le k_2 u(1).$$

Applying the Ascoli-Arzela theorem, we can prove that L is completely continuous. □

Remark 1. *The constants k_1 and k_2 (12) can be calculated using (7) and (8).*

$$\begin{aligned} \int_0^1 G(s,s)ds &= \frac{\frac{1}{6}\theta\gamma + \frac{1}{2}\theta\beta + \frac{1}{2}\alpha\gamma + \alpha\beta}{(\theta\gamma + \theta\beta + \alpha\gamma)} \\ &= \frac{1}{2} + \frac{3\alpha\beta - \theta\gamma}{3(\theta\gamma + \theta\beta + \alpha\gamma)} \\ & \begin{cases} = \frac{1}{2} & \textit{if } \theta\gamma = 3\alpha\beta, \\ < \frac{1}{2} & \textit{if } \theta\gamma > 3\alpha\beta, \\ > \frac{1}{2} & \textit{if } \theta\gamma < 3\alpha\beta. \end{cases} \end{aligned}$$

As

$$k_1 = c_0^2 \int_0^1 G(s,s)ds, \;\; k_2 = \frac{1}{c_0}\int_0^1 G(s,s)ds,$$

$k_1 > \frac{c_0{}^2}{2}$ if $\theta\gamma < 3\alpha\beta$ and $k_2 < \frac{1}{2c_0}$ if $\theta\gamma > 3\alpha\beta$. If $\theta\gamma = 3\alpha\beta$, then $k_1 = \frac{c_0{}^2}{2}$ and $k_2 = \frac{1}{2c_0}$. In the special case, $\alpha = \theta$ and $\gamma = \beta$, we can calculate that $k_1 = \frac{13}{288}$ and $k_2 = \frac{26}{9}$ for the boundary conditions $u(0) - u'(0) = 0$, $u(1) + u'(1) = 0$.

Next, property (12) ensures that

$$c_0k_1\|u\| \leq Lu \leq k_2\|u\|, \text{ for any } u \in P. \tag{13}$$

For $u \in P$, if $L(u) = 0$, then $u = 0$. Therefore, L is injective on P. The spectral radius of L, $r(L) > 0$ [9]. We now prove existence of a positive solution for problem (3)–(5) which implies a spectral interval for the operator LF. The proof follows similar ideas as that of [8].

Theorem 4. *Assume that $f(t,x) > 0$ for $x > 0$. Denote*

$$d(f) = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x}, \ |f|_0 = \limsup_{x\to 0} \max_{t\in[0,1]} \frac{f(t,x)}{x}.$$

If $d(f) = \infty, 0 < |f|_0 < \infty$, then BVP (3) has at least one positive solution for $\lambda \in \left(0, \frac{1}{|f|_0 r(L)}\right)$.

Proof. Let $\lambda < \frac{1}{|f|_0 r(L)}$. Select $\epsilon > 0$ small enough such that $\lambda(|f|_0 + \epsilon)r(L) < 1$. Assume $\delta > 0$ such that $\frac{f(t,x)}{x} < |f|_0 + \epsilon$ for $x \in (0, 2\delta)$. Therefore, we have $N(\lambda, u) \neq \mu u$ for $u \in \partial P_\delta$, and $\mu \geq 1$. Otherwise, there exist $u_0 \in \partial P_\delta$ and $\mu_0 \geq 1$ such that $N(\lambda, u_0) = \mu_0 u_0$. Then

$$\mu_0 u_0(t) = N(\lambda, u_0)(t) \leq \lambda(|f|_0 + \epsilon) \int_0^1 G(t,s)u_0(s)ds = \lambda(|f|_0 + \epsilon)Lu_0(t).$$

Thus $Lu_0(t) \geq \frac{\mu_0}{\lambda(|f|_0+\epsilon)} u_0(t)$, this implies $r(L) \geq \frac{\mu_0}{\lambda(|f|_0+\epsilon)}$. As $\lambda(|f|_0 + \epsilon)r(L) < 1$, we have a contradiction. By Lemma 1, $i(N, P_\delta, P) = 1$.

On the other hand, select M large enough such that

$$\lambda M c_0 \int_0^1 G(1,s)ds > 1.$$

As $d(f) = \infty$, there exists $M_1 > 0$, such that $\frac{f(t,x)}{x} > M$ for $x > M_1$. We take $M_1 > \max\{c_0, 2\delta\}$ and let $R = \frac{M_1}{c_0}$. For $u \in \partial P_R$, we have

$$u(t) \geq c_0\|u\| = M_1 \text{ for } t \in [0,1].$$

Therefore,

$$\|N(\lambda, u)\| \geq \lambda \int_0^1 G(1,s)f(s,u(s))ds \geq \lambda M c_0\|u\| \int_0^1 G(1,s)ds > \|u\|.$$

By Lemma 2, $i(N, P_R, P) = 0$. From the property of fixed point index,

$$i(N, P_R \setminus \bar{P}_\delta, P) = i(N, P_R, P) - i(N, P_\delta, P) = -1$$

Therefore, N has a fixed point in $P_R \setminus P_\delta$. □

Remark 2. *Theorem 4 implies that the decomposable nonlinear operator LF has a spectral interval $[|f|_0 r(L), \infty) \subset \sigma_F(LF)$ and the spectral radius $r(LF) = \infty$ [6].*

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