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The Topological Entropy Conjecture

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Abstract: For a compact Hausdorff space X, let J be the ordered set associated with the set of all finite open covers of X such that there exists n_J, where n_J is the dimension of X associated with ∂. Therefore, we have $\check{H}_p(X;\mathbb{Z})$, where $0 \leq p \leq n = n_J$. For a continuous self-map f on X, let $\alpha \in J$ be an open cover of X and $L_f(\alpha) = \{L_f(U)|U \in \alpha\}$. Then, there exists an open fiber cover $\dot{L}_f(\alpha)$ of X^f induced by $L_f(\alpha)$. In this paper, we define a topological fiber entropy $ent_L(f)$ as the supremum of $ent(f, \dot{L}_f(\alpha))$ through all finite open covers of $X^f = \{L_f(U); U \subset X\}$, where $L_f(U)$ is the f-fiber of U, that is the set of images $f^n(U)$ and preimages $f^{-n}(U)$ for $n \in \mathbb{N}$. Then, we prove the conjecture $\log \rho \leq ent_L(f)$ for f being a continuous self-map on a given compact Hausdorff space X, where ρ is the maximum absolute eigenvalue of f_*, which is the linear transformation associated with f on the Čech homology group $\check{H}_*(X;\mathbb{Z}) = \bigoplus_{i=0}^{n} \check{H}_i(X;\mathbb{Z})$.

Keywords: algebra equation; Čech homology group; Čech homology germ; eigenvalue; topological fiber entropy

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1. Introduction

Recall that the pair (X, f) is called a topological dynamical system, which is induced by the iteration:

$$f^n = \underbrace{f \circ \cdots \circ f}_{n}, \qquad n \in \mathbb{N}$$

and f^0 is denoted the identity self-map on X, where X is a compact Hausdorff space and f is a continuous self-map on X. The preimage of a subset $A \subseteq X$ is denoted by $f^{-1}(A)$. If the preimage of $f^{-(n-1)}(A)$ is defined, then by induction, the preimage of $f^{-(n-1)}(A)$ is denoted by $f^{-n}(A)$, where $n \in \mathbb{Z}^+$.

1.1. Brief History

For a topological dynamical system (X, f), let α and β be the collections of the finite open cover of X, and let:

$$\begin{cases} \alpha \vee \beta = \{A \cap B; A \in \alpha, B \in \beta\}; \\ f^{-1}(\alpha) = \{f^{-1}(A); A \in \alpha\}, \\ f^{-1}(\alpha \vee \beta) = f^{-1}(\alpha) \vee f^{-1}(\beta); \\ \bigvee_{i=0}^{n-1} f^{-i}(\alpha) = \alpha \vee f^{-1}(\alpha) \vee \cdots \vee f^{-(n-1)}(\alpha), \quad n \in \mathbb{Z}^+. \end{cases} \tag{1}$$

For a finite open cover α of X, let $N(\alpha)$ be the infimum number of the subcover of α. Because X is compact, we get that $N(\alpha)$ is a positive integer. Hence, we define:

$$H(\alpha) = \log N(\alpha) \geq 0.$$

Following [1] (p. 81), if α, β are finite open covers of X, then we see:

$$\alpha < \beta \Longrightarrow H(\alpha) \leq H(\beta).$$

Definition 1 ([1], p. 89)**.** *For any given finite open cover α of X, define:*

$$ent(f, \alpha) = \lim_{n \to +\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-i}(\alpha)),$$

and define the topological entropy of f such that:

$$ent(f) = \sup_{\alpha} \{ent(f, \alpha)\},$$

where $\sup\limits_{\alpha}$ is through the all finite open cover of X.

For a compact manifold M, let $H_i(M; \mathbb{Z})$ be the i-th homology group of integer coefficients, where $0 \leq i \leq \dim M$. In 1974, M. Shub stated the topological entropy conjecture [2], which usually has been called the entropy conjecture [3], that is,

Conjecture 1. *The inequality:*

$$\log \rho \leq ent(f)$$

is valid or not for any C^1 self-map f on a compact manifold M, where $ent(f)$ is the topological entropy of f and ρ is the maximum absolute eigenvalue of f_*, which is the linear transformation associated with f on the homology group:

$$H_*(M; \mathbb{Z}) = \bigoplus_{i=0}^{\dim M} H_i(M; \mathbb{Z}).$$

In the first place, the inequality of Conjecture 1 is connected to the work of S. Smale [4–7], M. Shub [8,9], and D. P. Sullivan [10–12].

In 1975, Manning [13] proved that Conjecture 1 holds for any homeomorphism of manifolds X for which $\dim X \leq 3$, Shub and Williams [14] proved Conjecture 1 on manifolds M for no cycle diffeomorphisms, which are Axiom A; also, Ruelle and Sullivan [15] proved Conjecture 1 on manifolds M, which have an oriented expanding attractor $X \subset M$. In the same year, Pugh [16] proved that there is a homeomorphism f of some smooth M^8 such that Conjecture 1 is invalid.

In 1977, Misiurewicz et al. [17,18] proved that Conjecture 1 holds for any smooth maps on $X = S^n$ and for any continuous maps on $\mathbb{T}^n$ with $n \in \mathbb{Z}^+$.

In 1980, Katok [19] proved that if a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism f of a compact manifold has a Borel probability continuous (non-atomic) invariant ergodic measure with non-zero Lyapunov exponents, then it has positive topological entropy. In 1986, Katok [20] proved that if the universal covering space of X is homeomorphic to the Euclidean space, then Conjecture 1 holds for any $f \in C^\infty(X)$; also, he gave a counterexample explaining that the inequality of Conjecture 1 is invalid for a continuous map, that is on two-dimensional sphere S^2, there is $f \in C^0(S^2)$ such that:

$$0 = ent(f) < \log \rho.$$

For a C^∞ mapping, Yomdin [21] in 1987 and Newhouse [22] in 1989 proved Conjecture 1, respectively.

In 1992, for n-dimensional compact Riemannian manifolds with $n \in \mathbb{Z}^+$, Paternain made a relation between the geodesic entropy and topological entropy of the geodesic flow on the unit tangent bundle [23], which is an improvement of Manning's inequality [24].

In 1994, Ye [25] showed that homeomorphisms of Suslinian chainable continua and homeomorphisms of hereditarily decomposable chainable continua induced by square commuting diagrams on inverse systems of intervals have zero topological entropy.

In 1997, for a closed connected C^∞ manifold M^n with $n \in \mathbb{Z}^+$, Mañé [26] provided an equality to relate the exponential growth rate of geodesic entropy, as a function of T, which is parametrized by the arc length, with the topological entropy of the geodesic flow on the unit tangent bundle.

In 2000, Cogswell gave that μ-a.e. $x \in X$ is contained in an open disk $D_x \subset W^u(x)$, which exhibits an exponential volume growth rate greater than or equal to the measure-theoretic entropy of f with respect to μ, where $f \in C^{1+1}(X)$ and f is a measure-preserving transformation [27].

In 2002, Knieper et al. [28] showed that every orientable compact surface has a C^∞ open and dense set of Riemannian metrics whose geodesic flow has positive topological entropy.

In 2005, Bobok et al. [29] proved the inequality of Conjecture 1 for a compact manifold X and for any continuously differentiable map $f : X \to X$, which is m-fold at all regular values.

In 2006, Zhu [30] showed that for C^k-smooth random systems, the volume growth is bounded from above by the topological entropy on compact Riemannian manifolds.

In 2008, Marzantowicz et al. [3] proved the inequality of Conjecture 1 for all continuous mappings of compact nilmanifolds.

In 2010, Saghin et al. [31] proved the inequality of Conjecture 1 for partially hyperbolic diffeomorphism with a one-dimensional center bundle.

In 2013, Liao et al. [32] proved the inequality of Conjecture 1 for diffeomorphism away from ones with homoclinic tangencies.

In 2015, Liu et al. [33] proved the inequality of Conjecture 1 for diffeomorphism that are partially hyperbolic attractors.

In 2016, Cao et al. [34] proved the inequality of Conjecture 1 for dominated splittings without mixed behavior.

In 2017, Zang et al. [35] proved the inequality of Conjecture 1 for controllable dominated splitting.

In 2019, Lima et al. [36] developed symbolic dynamics for smooth flows with positive topological entropy on three-dimensional closed (compact and boundaryless) Riemannian manifolds.

In 2020, Hayashi [37] proved the inequality of Conjecture 1 for nonsingular C^1 endomorphisms away from homoclinic tangencies, extending the result of [32].

Lately, for results about random entropy expansiveness and dominated splittings, see [38], and for results about the relations of topological entropy and Lefschetz numbers, see [39–41]. Furthermore, for a variational principle for subadditive preimage topological pressure for continuous bundle random dynamical systems, see [42].

1.2. Motivation and Main Results

Conjecture 1 is not proven completely. For a compact Hausdorff space X, let J be the ordered set associated with the set of all finite open covers of X such that there exists n_J, where n_J is the dimension of X associated with ∂, which will become clear in Definition 3. Therefore, we have $\check{H}_p(X;\mathbb{Z})$, where $0 \le p \le n = n_J$. For a continuous self-map f on X, let $\alpha \in J$ be an open cover of X and $L_f(\alpha) = \{L_f(U)|U \in \alpha\}$. Then, there exists an open fiber cover $\dot{L}_f(\alpha)$ of X^f induced by $L_f(\alpha)$. In this paper, we define a topological fiber entropy $ent_L(f)$ as the supremum of $ent(f, \dot{L}_f(\alpha))$ through all finite open covers of

$X^f = \{L_f(U); U \subset X\}$, where $L_f(U)$ is the f-fiber of U, that is the set of images $f^n(U)$ and preimages $f^{-n}(U)$ for $n \in \mathbb{N}$. Then, we prove the inequality

$$\log \rho \leq ent_L(f).$$

where ρ is the maximum absolute eigenvalue of f_*, which is the linear transformation associated with f on the Čech homology group:

$$\check{H}_*(X;\mathbb{Z}) = \bigoplus_{i=0}^{n} \check{H}_i(X;\mathbb{Z}).$$

Specifically, in triangulable compact n-dimensional manifold M, we get:

$$H_*(M;\mathbb{Z}) = \check{H}_*(M;\mathbb{Z}).$$

Hence, Conjecture 1 is valid for topological fiber entropy.

In this paper, we always let $\Psi \in J$ be good enough and have enough refinement, i.e., satisfy all the necessary requirements of this paper. Define:

$$\alpha^c = \{A; \quad A^c \in \alpha, \quad \text{where} \quad A^c \cup A = X \quad \text{and} \quad A^c \cap A = \emptyset\},$$

and define:

$$\begin{cases} a_0 \cdots a_{i-1}\hat{a}_i, a_{i+1} \cdots a_p = a_0 \cdots a_{i-1}a_{i+1}, \cdots a_p; \\ a_0 \cdots a_{i-1}\underline{b}^{(i)} a_i \cdots a_p = a_0 \cdots a_{i-1} b a_i \cdots a_p; \\ a_0 \cdots a_{i-1}\underline{b}^{(i)}_{(k)} a_i \cdots a_p = \sum\limits_{m \in (k)} a_0 \cdots a_{i-1} b_m a_i \cdots a_p; \\ a_0 \cdots a_{i-1}\underline{b}^{(i)}_{\emptyset} \cdots a_p = \sum\limits_{m \in \emptyset} a_0 \cdots a_{i-1} b_m a_i \cdots a_p = a_0 \cdots a_{i-1} a_i \cdots a_p; \\ (k) = \{k_1, k_2, k_3, \cdots, k_n; n = \|\{a_0, \cdots, a_{i-1}, b_m, a_i, \cdots, a_p\}\| \geq 1, m \in \mathbb{Z}\}; \\ (a_0 \cdots \hat{b_{(k)}} \cdots a_p)^d = b_{k_1} \cdots b_{k_i} \cdots b_{k_n}, k_i \in (k). \end{cases}$$

2. Algebra Equation for the Boundary Operator

In this paper, let X be a compact Hausdorff space, $C^0(X)$ be the set of all continuous self-maps on X, and id be the identity map on X. Let α, β be finite open covers of X, if for any $B \in \beta$, there is $A \in \alpha$ such that $B \subseteq A$, then we define $\alpha \leq \beta$ and say that β is larger than α or β is a refinement of α. For $A \in \alpha$, let $\overline{A}$ be the closure of A and $\|A\|$ be the number of elements of A.

Definition 2 ([43], p. 541)**.** *Let X be a Hausdorff space, Ψ be a cover of X, and $U_0, U_1, U_2, \ldots, U_p \in \Psi$ with $p \in \mathbb{N}$. If $U_0 \bigcap U_1 \bigcap \cdots \bigcap U_p \neq \emptyset$, then we define a p-simplex σ_p. Hence, we get the p-th chain group C_p, the p-th homology group $H_p(\Psi;\mathbb{Z})$, and the p-th cohomology group $H^p(\Psi;\mathbb{Z})$, where:*

$$\cdots \to C_{p+1}(\Psi;\mathbb{Z}) \xrightarrow{\partial_{p+1}} C_p(\Psi;\mathbb{Z}) \xrightarrow{\partial_p} C_{p-1}(\Psi;\mathbb{Z}) \to \cdots$$

$$\partial_p(U_0 \cap \cdots \cap U_p) = \sum_{i}^{p} (-)^i (U_0 \cap \cdots \cap \hat{U}_i \cdots \cap U_p),$$

$$\partial_{p-1} \circ \partial_p = 0,\ B_p(\Psi;\mathbb{Z}) = im\ \partial_{p+1},\ Z_p(\Psi;\mathbb{Z}) = \ker \partial_p \text{ and } H_p(\Psi;\mathbb{Z}) = Z_p/B_p.$$

Let $C^p(\Psi;\mathbb{Z}) = \hom(C_p(\Psi;\mathbb{Z}),\mathbb{Z})$. Then, ∂_p induces a homomorphism $C^{p-1}(\Psi;\mathbb{Z}) \xrightarrow{\delta^p} C^p(\Psi;\mathbb{Z})$. We obtain that:

$$\cdots \leftarrow C^{p+1}(\Psi;\mathbb{Z}) \xleftarrow{\delta^{p+1}} C^p(\Psi;\mathbb{Z}) \xleftarrow{\delta^p} C^{p-1}(\Psi;\mathbb{Z}) \leftarrow \cdots,$$

$$\delta^{p+1} \circ \delta^p = 0,\ B^p(\Psi;\mathbb{Z}) = im\ \delta^p,\ Z^p(\Psi;\mathbb{Z}) = \ker \delta^{p+1} \text{ and } H^p(\Psi;\mathbb{Z}) = Z^p/B^p.$$

Lemma 1. *Let X be a Hausdorff space and* Ψ *be a finite open cover of X. Then, we get* $C^p(\Psi;\mathbb{Z}) \cong C_p(\Psi;\mathbb{Z})$ *with* $p \in \mathbb{N}$*. Moreover, let* $U^i = U_i^c$*.*

If:

$$c_p = U_0 \cap \cdots \cap U_p \in C_p(\Psi;\mathbb{Z}),$$

then:

$$c^p = U^0 \cup \cdots \cup U^p \neq X$$

is an isomorphic representation of the p-simplex of $C^p(\Psi;\mathbb{Z})$*.*

Proof. Because $\mathbb{Z}$ can be treated as a finite generated free ring [44], $C_p(\Psi;\mathbb{Z})$ can be treated as a finite-dimensional $\mathbb{Z}$-module space [45], and $C^p(\Psi;\mathbb{Z})$ can be treated as the dual $\mathbb{Z}$-module space of $C_p(\Psi;\mathbb{Z})$. With the property of the finite-dimensional $\mathbb{Z}$-module space, we get $C^p(\Psi;\mathbb{Z}) \cong C_p(\Psi;\mathbb{Z})$.

Because:

$$c_p = U_0 \cap \cdots \cap U_p \neq \varnothing \Longleftrightarrow c^p = U^0 \cup \cdots \cup U^p \neq X,$$

we get:

$$c_p \in C_p(\Psi;\mathbb{Z}) \Longleftrightarrow c^p \in C^p(\Psi;\mathbb{Z}).$$

That is, $U^0 \cup \cdots \cup U^p \neq X$ is an isomorphic representation of the p-simplex of $C^p(\Psi;\mathbb{Z})$. □

Definition 3. *Let X be a Hausdorff space,* Ψ *be a finite open cover of X, and J be the ordered set associated with the refinement of the finite open cover of X. Then, we define the function* $n_\Psi = \max\{n; n \in S\}$ *on J. Obviously, if* $\alpha, \beta \in J$ *and* $\alpha \leq \beta$*, then* $n_\alpha \leq n_\beta$*. If there exists* $n_J = \varinjlim_{\Psi \in J} n_\Psi$*, then we say that* n_J *is the dimension of X associated with* ∂*, where:*

$$S = \{n; \partial(U_0 \cdots \cap U_i \cdots U_n) \neq \partial(U_0 \cdots \cap U_i \cdots U_n \cap U_{n+1}), U_0, \cdots, U_{n+1} \in \Psi\}.$$

Definition 4. *Let X be a Hausdorff space,* Ψ *be a finite open cover of X, and* $0 \leq p \leq n = n_\Psi$*. If for any* $\sigma^p \in C^p(\Psi;\mathbb{Z})$*, there exists* $\sigma^n \in C^n(\Psi;\mathbb{Z})$ *such that* $\sigma^p = U^0 \cup \cdots \cup U^p$ *is the p-th surface of* σ^n *and:*

$$(U^0 \cup \cdots \cup \hat{U_{(k)}} \cdots \cup U^p)^d = U^{k_0} \cup \cdots \cup U^{k_{n-p+1}}.$$

Then, we say that X is a Poincaré space.

Lemma 2. *Let X be a Poincaré space and* Ψ *be its finite open cover. For* $0 \leq p \leq n = n_\Psi$*, we get that* $H^p(\Psi;\mathbb{Z}) \cong H_{n-p}(\Psi;\mathbb{Z})$*.*

Proof. By Lemma 1, we get the following chains of the mapping:

$$\begin{cases} \cdots \to C_{p+1}(\Psi;\mathbb{Z}) \xrightarrow{\partial_{p+1}} C_p(\Psi;\mathbb{Z}) \xrightarrow{\partial_p} C_{p-1}(\Psi;\mathbb{Z}) \to \cdots \\ \cdots \leftarrow C^{p+1}(\Psi;\mathbb{Z}) \xleftarrow{\delta^{p+1}} C^p(\Psi;\mathbb{Z}) \xleftarrow{\delta^p} C^{p-1}(\Psi;\mathbb{Z}) \leftarrow \cdots \end{cases} \quad (2)$$

For a fixed p-simplex in $C_p(\Psi;\mathbb{Z})$, we see the algebraic equation:

$$\begin{cases} <\partial c_p, c^{p-1}> = < c_p, \delta c^{p-1}>, \\ \partial_p(U_0 \cdots \cap U_i \cdots U_p) = \sum_{i=0}^{p} (-1)^i (U_0 \cdots \cap \hat{U}_i \cdots U_p), \\ \partial\emptyset = \delta\emptyset = 0, \\ < a, \emptyset > = < \emptyset, b > = 0; \end{cases} \tag{3}$$

and the algebraic equation:

$$\begin{cases} < \sum_{i=0}^{p} (-1)^i (U_0 \cdots \cap \hat{U}_i \cdots U_p), V_0 \cdots \cap \hat{V}_i \cdots V_p > = < c_p, \delta c^{p-1} >, \\ \sum_{i=0}^{p} (-)^i (V^0 \cdots \cup \underline{V_{(k)}^{(i)}} \cdots V^p) = \delta^p (V^0 \cdots \cup \hat{V_{(k)}} \cdots V^p). \end{cases} \tag{4}$$

If $(k) = \emptyset$, then we define:

$$(i) = \emptyset, \qquad (-1)^{\emptyset} = 0 \qquad \text{and} \qquad \delta^p(U^0 \cdots \cup \hat{U_{(k)}} \cdots U^p) = 0.$$

From (3) and (4), we obtain that:

$$\begin{cases} \partial_p(U_0 \cdots \cap U_i \cdots U_p) = \sum_{i=0}^{p} (-1)^i (U_0 \cdots \cap \hat{U}_i \cdots U_p), \\ \delta^p(U^0 \cdots \cup \hat{U_{(k)}} \cdots U^p) = \sum_{i=0}^{p} (-1)^i (U^0 \cdots \cup \underline{U_{(k)}^{(i)}} \cdots U^p). \end{cases} \tag{5}$$

That is,

$$\begin{cases} \partial_p(U_0 \cdots \cap U_i \cdots U_p) - \sum_{i=0}^{p} (-1)^i (U_0 \cdots \cap \hat{U}_i \cdots U_p) = 0, \\ \delta^p(U^0 \cdots \cup \hat{U_{(k)}} \cdots U^p) - \sum_{i=0}^{p} (-1)^i (U^0 \cdots \cup \underline{U_{(k)}^{(i)}} \cdots U^p) = 0, \\ \delta^{n-p+1}((U^0 \cdots \cup \hat{U_{(k)}} \cdots U^p)^d) = \delta^{n-p+1}(U^{k_1} \cdots \cup U^{k_m} \cdots U^{k_{n-p+1}}), \\ \delta^{n-p+1}(U^{k_1} \cdots \cup U^{k_m} \cdots U^{k_{n-p+1}}) = \sum_{i=0}^{p} (-1)^i (U^{k_1} \cdots \cup \underline{U_{(0,\cdots,p)}^{(i)}} \cdots U^{k_{n-p+1}}). \end{cases} \tag{6}$$

Let:

$$c_p = \sum z_m (U_0 \cdots \cap U_i \cdots U_p)_m.$$

Then, we see that:

$$c^{n-p} = \sum z_m ((U^0 \cdots \cup \hat{U_{(k)}} \cdots U^p)^d)_m, \quad \text{where} \quad z_m \in \mathbb{Z}.$$

Therefore, we obtain that:

$$\begin{cases} U_0 \cdots \cap U_i \cdots U_p \longleftrightarrow U^0 \cdots \cup \hat{U_{(k)}} \cdots U^p \longleftrightarrow (U^0 \cdots \cup \hat{U_{(k)}} \cdots U^p)^d, \\ c_p \in \ker \partial_p \Longleftrightarrow c^{n-p} \in \ker \delta^{n-p+1}, \\ c_p \in \ im\, \partial_{p+1} \Longleftrightarrow c^{n-p} \in \ im\, \delta^{n-p}. \end{cases} \tag{7}$$

Let:

$$\begin{cases} \partial_{\frac{ker}{im}}(C_p) = H_p(\Psi;\mathbb{Z}) = Z_p/B_p = \ker \partial_p / im\, \partial_{p+1}, \\ \partial^*_{\frac{ker}{im}}(C^p) = H^p(\Psi;\mathbb{Z}) = Z^p/B^p = \ker \delta^{p+1} / im\, \delta^p. \end{cases} \tag{8}$$

Then, ∂_p and δ^{n-p+1} are the dual solutions in the algebraic Equation (6). Similarly, $\partial_{\frac{ker}{im}}$ and $\partial^*_{\frac{ker}{im}}$ are the dual values in the algebraic Equation (8). All the processes of the dual maps are linear reversible, i.e., the same style as isomorphisms. Therefore, the p-th value of $\partial_{\frac{ker}{im}}$ on the C_p chain group is isomorphic to the $(n-p)$-th value of $\partial^*_{\frac{ker}{im}}$ on the C^{n-p} chain group, that is,

$$\partial_{\frac{ker}{im}}(C_p) \cong \partial^*_{\frac{ker}{im}}(C^{n-p}).$$

For this reason, we see that:

$$H^p(\Psi;\mathbb{Z}) \cong H_{n-p}(\Psi;\mathbb{Z}).$$

□

Like the linear equation in Euclidean space $\mathbb{R}^3$, let:

$$S_i : A_i x + B_i y + C_i z = 0$$

be a class of lines, or in other words, a class of planes:

$$S_i^* : A_i x + B_i y + C_i z = 0.$$

where $i \in \mathbb{Z}^+$ and $i \geq 2$.

The line and plane are a pair of duals. For a fixed space $\mathbb{R}^3$, the intrinsic relationships between lines or between planes are never changed. That is, f and g are two good maps such that they are linear, if:

$$f_i = f(S_i, S_{i-1}), \quad f_i^* = f^*(S_i^*, S_{i+1}^*), \quad g_i = g(f_i) \quad \text{and} \quad g_i^* = g(f_i^*).$$

then g_i and g_i^* is a pair of duals such that there is a natural relationship between g_i and g^*_{n-i}. For example, that natural relationship may be:

$$g_i = g^*_{n-i}, \quad \text{or} \quad g_i g^*_{n-i} = 1, \quad \text{or} \quad g_i + g^*_{n-i} = 0,$$

or:

$$g_i A_k + g_{n-i} B_k + C_k = 0 \text{ and } g^*_{n-i} A_k + g_i^* B_k + C_k = 0,$$

and so on. The dual outcomes and the representations of the natural relation between g_i and g^*_{n-i} only depend on the good maps f and g.

3. Germ and Dual of the Čech Homology

Definition 5 ([43], p. 542)**.** *Let X be a Hausdorff space and J be the ordered set associated with the set of all covers of X, $U_0, U_1, U_2, \ldots, U_p \in \Psi$ with $p \in \mathbb{N}$ and $\Psi \in J$. If $U_0 \bigcap U_1 \cap \cdots \cap U_p \neq \emptyset$, then we define a p-simplex σ_p. Hence, we get the p-th chain group C_p, the p-th homology group $H_p(\Psi;\mathbb{Z})$, and the p-th cohomology group $H^p(\Psi;\mathbb{Z})$. If $\Omega, \Psi \in J$ and $\Omega \leq \Psi$, then we get the homomorphisms:*

$$f_{\Psi\Omega} : H_p(\Psi;\mathbb{Z}) \to H_p(\Omega;\mathbb{Z}), \quad \textit{and} \quad f_{\Omega\Psi} : H^p(\Omega;\mathbb{Z}) \to H^p(\Psi;\mathbb{Z}).$$

Hence, we define the p-th Čech cohomology group:

$$\check{H}^p(X;\mathbb{Z}) = \varinjlim_{\Omega \in J} H^p(\Omega;\mathbb{Z}).$$

Following Definition 5, we have the following definition.

Definition 6. *Let X be a Hausdorff space and J be the ordered set associated with the set of all finite open covers of X such that there exist n_J. For $0 \le p \le n = n_J$, there exists the p-th Čech homology group:*

$$\check{H}_p(X;\mathbb{Z}) = \varprojlim_{\Omega \in J} H_p(\Omega;\mathbb{Z}).$$

Definition 7. *Let X be a Poincaré space and J be the ordered set associated with the set of all finite open covers of X such that there exists n_J. For $\Omega, \Psi \in J$, let $\Theta = \Psi \vee \Omega = \{\alpha \cap \beta; \alpha \in \Psi, \beta \in \Omega\}$. Then, we get homomorphisms*

$$f_{\Theta\Omega} : H_p(\Theta;\mathbb{Z}) \to H_p(\Omega;\mathbb{Z}) \quad \textit{and} \quad f_{\Theta\Psi} : H_p(\Theta;\mathbb{Z}) \to H_p(\Psi;\mathbb{Z}).$$

Following this, we can define the Čech homology germ $H_p(J;\mathbb{Z})$. Similarly, we define the Čech cohomology germ $H^p(J;\mathbb{Z})$. If there exists $\Gamma \in J$ such that, we get $H_p(\Psi;\mathbb{Z}) \cong H^{n-p}(\Psi;\mathbb{Z})$ for any $\Psi \in J$ whenever $\Gamma \le \Psi$, then we define:

$$H^p(J;\mathbb{Z}) \cong H_{n-p}(J;\mathbb{Z}),$$

where $n = n_J$.

By Lemma 2, Definitions 5–7, we get the following result.

Lemma 3. *Let X be a Poincaré space and J be the ordered set associated with the set of all finite open covers of X such that there exists n_J. For $0 \le p \le n = n_J$, we get that:*

$$\check{H}_p(X;\mathbb{Z}) \sim H_p(J;\mathbb{Z}) \quad \textit{and} \quad \check{H}^p(X;\mathbb{Z}) \sim H^p(J;\mathbb{Z}),$$

where $\sim$ means the different expressions for the same thing.

Definition 8. *Let X be a Poincaré space and J be the ordered set associated with the set of all finite open covers of X such that there exists n_J. For $n = n_J$, if:*

$$H_p(J;\mathbb{Z}) \cong H^{n-p}(J;\mathbb{Z}),$$

then we define:

$$\check{H}_p(X;\mathbb{Z}) \cong \check{H}^{n-p}(X;\mathbb{Z}).$$

4. f-Čech Homology

Definition 9. *Let X be a Hausdorff space, $U_i, V, W \subseteq X$ and $f \in C^0(X)$, where $0 \le i \le k$ and $k \in \mathbb{Z}$. Then, we define:*

$$\begin{cases} L_f(U) = (\cdots, f^{-n}(U), \cdots, f^{-1}(U), f^0(U), f^1(U), \cdots, f^n(U), \cdots), \\ f \circ L_f = L_f \circ f, \\ L_f(U) \cap L_f(V) = L_f(W), \textit{where } W = U \cap V, \\ L_f(U_0) \cdots \cap L_f(U_i) \cdots L_f(U_k) = L_f(U_0) \cap (L_f(U_1) \cdots \cap L_f(U_i) \cap L_f(U_k)), \\ L_{g+h}(U) = (\cdots, g^{-n}(U) \bigcup h^{-n}(U), \cdots, g^0(U) \bigcup h^0(U), \cdots, g^n(U) \bigcup h^n(U), \cdots), \\ L_f(\emptyset) = \emptyset, \\ L_{g \oplus h}(U) = L_{g+h}(U), \textit{when } g^{-1}(U) \cap h^{-1}(U) = \emptyset, \end{cases}$$

where $f^{-1}(U)$ is the preimage of U. We say that $L_f(U)$ is the f-fiber of U and let $X^f = \{L_f(U); U \subset X\}$.

If X is a compact space, then $X^{+\infty} = \prod_{i=-1}^{-\infty} X \times \underline{X} \times \prod_{i=1}^{+\infty} X$ is compact as well by the Tychonoff theorem. In fact, in Definition 9, $L_f(U)$ glues the preimage orbit and image orbit of U.

If X is a discrete Hausdorff space, then we get that $X^f\Big|_{\underline{X}\times\prod_{i=1}^{+\infty} X}$ is the direct limit space of (X, f) following [46], but $X^f\Big|_{\prod_{i=-1}^{-\infty} X\times\underline{X}}$ is not the inverse limit space of (X, f).

Definition 10. *Let X be a Hausdorff space, and let J be the ordered set associated with the set of all finite open covers of X. Let $f \in C^0(X), \Psi \in J$ and $U_0, \cdots, U_p \in \Psi$ with $p \in \mathbb{N}$. If:*

$$\sigma_p^f = L_f(U_0) \cap \cdots \cap L_f(U_p) \neq \emptyset,$$

then we define an f-Čech p-simplex σ_p^f. Hence, we get the f-Čech p-chain group $C_p(\Psi, f; \mathbb{Z})$, and we get the f-Čech p-th homology group $H_p(\Psi, f; \mathbb{Z})$, where:

$$\cdots \to C_{p+1}(\Psi, f; \mathbb{Z}) \xrightarrow{\partial_{p+1}^f} C_p(\Psi, f; \mathbb{Z}) \xrightarrow{\partial_p^f} C_{p-1}(\Psi, f; \mathbb{Z}) \to \cdots,$$
$$\partial_p^f(L_f(U_0) \cdots \cap L_f(U_i) \cdots L_f(U_p)) = \sum_{i=0}^{p} (-1)^i (L_f(U_0) \cdots \cap \hat{L}_f(U_i) \cdots L_f(U_p)).$$

It is easy to get that $\partial_{p-1}^f \circ \partial_p^f = 0$, that is,

$$\begin{aligned}
&\partial_{p-1}^f \circ \partial_p^f(L_f(U_0) \cdots \cap L_f(U_i) \cdots L_f(U_p)) \\
&= \sum_i^p (-1)^i \partial^f(L_f(U_0) \cdots \cap \hat{L}_f(U_i) \cdots L_f(U_p)) \\
&= \sum_i^p \sum_{j<i} (-1)^{i+j} (L_f(U_0) \cdots \cap \hat{L}_f(U_j) \cdots \cap \hat{L}_f(U_i) \cdots L_f(U_p)) \\
&+ \sum_i^p \sum_{j>i} (-1)^{i+j-1} (L_f(U_0) \cdots \cap \hat{L}_f(U_i) \cdots \cap \hat{L}_f(U_j) \cdots \cap L_f(U_p)) \\
&= 0.
\end{aligned}$$

Therefore, we see that:

$$B_p(\Psi, f; \mathbb{Z}) = im\, \partial_{p+1}^f, \quad Z_p(\Psi, f; \mathbb{Z}) = \ker \partial_p^f \quad \text{and}$$
$$H_p(\Psi, f; \mathbb{Z}) = Z_p(\Psi, f; G)/B_p(\Psi, f; \mathbb{Z}).$$

By Lemma 3 and Definition 9, we easily have the following lemma.

Lemma 4. *A Čech p-chain c_p is associated with an f-Čech p-chain c_p^f, that is $U_0 \cap U_1 \cap \cdots \cap U_p \neq \emptyset$ if and only if $L_f(U_0) \cap L_f(U_1) \cap \cdots \cap L_f(U_p) \neq \emptyset$. Therefore, the Čech p-chain group is isomorphic to the f-Čech p-chain group.*

Definition 11. *Let X be a Hausdorff space, $f \in C^0(X)$, and Ψ be a finite open cover of X. Let J be the ordered set associated with the refinement of the finite open cover of X. Then, we define the function $n_{\Psi,f} = \max\{n; n \in S\}$ on J. Obviously, if $\alpha, \beta \in J$ and $\alpha \leq \beta$, then $n_{\alpha,f} \leq n_{\beta,f}$. If there exists:*

$$n_{J,f} = \varinjlim_{\Psi \in J} n_{\Psi,f},$$

then we say that $n_{J,f}$ is the dimension of (X, f) associated with ∂^f, where:

$$S = \{n; \partial^f(L_f(U_0) \cdots \cap L_f(U_n)) \neq \partial(L_f(U_0) \cdots \cap L_f(U_n) \cap L_f(U_{n+1})), U_0, \cdots, U_{n+1} \in \Psi\}.$$

Similarly, with Definitions 5–7 and the following Definition 11, we obtain the following definition.

Definition 12. *Let X be a Hausdorff space, $f \in C^0(X)$, and J be the ordered set associated with the refinement of the finite open cover of X such that there exists $n_{J,f}$. Let $\Theta = \Psi \vee \Omega = \{\alpha \cap \beta; \alpha \in \Psi, \beta \in \Omega\}$ with $\Omega, \Psi \in J$. For $0 \le p \le n = n_{J,f}$, we get homomorphisms:*

$$f_{\Theta\Omega} : H_p(\Theta, f; \mathbb{Z}) \to H_p(\Omega, f; \mathbb{Z}) \quad \text{and} \quad f_{\Theta\Psi} : H_p(\Theta, f; \mathbb{Z}) \to H_p(\Psi, f; \mathbb{Z}).$$

Therefore, we get the pth f-Čech homology germ $H_p(J, f; \mathbb{Z})$ and the pth f-Čech homology group:

$$\check{H}_p(X, f; \mathbb{Z}) = \varprojlim_{\Omega \in J} H_p(\Omega, f; \mathbb{Z}).$$

Lemma 5. *Let X be a Hausdorff space, $f \in C^0(X)$, and J be the ordered set associated with the set of all finite open covers of X such that there exist n_J and $n_{J,f}$. Then, we have $n_J = n_{J,f}$, and we get $\check{H}_p(X, f; \mathbb{Z})$ and $\check{H}_p(X; \mathbb{Z})$, where $0 \le p \le n = n_J$. Moreover, for $\Psi \in J$, we get that:*

$$\begin{gathered} im\, \partial_{p+1} = B_p(\Psi; \mathbb{Z}) = B_p(\Psi, f; \mathbb{Z}) = im\, \partial^f_{p+1}, \\ \ker \partial_p = Z_p(\Psi; \mathbb{Z}) = Z_p(\Psi, f; \mathbb{Z}) = \ker \partial^f_p \quad \text{and} \\ Z_p(\Psi; G)/B_p(\Psi; \mathbb{Z}) = H_p(\Psi; \mathbb{Z}) = H_p(\Psi, f; \mathbb{Z}) = Z_p(\Psi, f; G)/B_p(\Psi, f; \mathbb{Z}). \end{gathered}$$

Using Lemmas 3, 5 and Definition 12, we see the following result.

Lemma 6. *Let X be a Hausdorff space, $f \in C^0(X)$, and J be the ordered set associated with the set of all finite open covers of X such that there exist $n_{J,f}$. For $0 \le p \le n = n_{J,f}$, we obtain:*

$$H_p(J, f; \mathbb{Z}) \sim \check{H}_p(X, f; \mathbb{Z}),$$

where $\sim$ means the different expressions for the same thing.

Furthermore, we can define the f-Čech cohomology germ $H^p(J, f; \mathbb{Z})$, the f-Čech cohomology group $\check{H}^p(X, f; \mathbb{Z})$, and the f-Poincaré space. Obviously, we get that $C_p(X; \mathbb{Z}) = C_p(X, id; \mathbb{Z})$. For convenience, let:

$$\begin{gathered} \check{H}_*(X; \mathbb{Z}) = \bigoplus_{i=0}^{n} \check{H}_i(X; \mathbb{Z}), \quad C_*(X; \mathbb{Z}) = \bigoplus_{i=0}^{n} C_i(X; \mathbb{Z}), \quad B_*(X; \mathbb{Z}) = \bigoplus_{i=0}^{n} B_i(X; \mathbb{Z}), \\ \check{H}_*(X, f; \mathbb{Z}) = \bigoplus_{i=0}^{n} \check{H}_i(X, f; \mathbb{Z}), \quad C_*(X, f; \mathbb{Z}) = \bigoplus_{i=0}^{n} C_i(X, f; \mathbb{Z}) \quad \text{and} \\ B_*(X, f; \mathbb{Z}) = \bigoplus_{i=0}^{n} B_i(X, f; \mathbb{Z}). \end{gathered}$$

By Lemmas 4 and 6, we have the following lemma.

Lemma 7. *Let X be a Hausdorff space, $f \in C^0(X)$, and J be the ordered set associated with the set of all finite open covers of X such that there exist n_J and $n_{J,f}$. Then, $n_J = n_{J,f}$ and for $n = n_J = n_{J,f}$. We have $\check{H}_p(X; \mathbb{Z})$ and $\check{H}_p(X, f; \mathbb{Z})$, where $0 \le p \le n$. Moreover, there are linear transformations f_* associated with f on $\check{H}_*(X; \mathbb{Z})$, on $C_*(X; \mathbb{Z})$, and on $\check{H}_*(X, f; \mathbb{Z})$, respectively. If E_{f_*} is the set of all eigenvalues of f_* and:*

$$\|E_{f_*}\| = \sup\{|a|; a \in E_{f_*}\},$$

then we obtain the inequalities:

$$\|E_{f_*}|_{\check{H}_*(X,f;\mathbb{Z})}\| \le \|E_{f_*}|_{Z_*(X,f;\mathbb{Z})}\| \le \|E_{f_*}|_{C_*(X,f;\mathbb{Z})}\|,$$
$$\|E_{f_*}|_{H_*(X;\mathbb{Z})}\| \le \|E_{f_*}|_{Z_*(X;\mathbb{Z})}\| \le \|E_{f_*}|_{C_*(X;\mathbb{Z})}\| \quad and \tag{9}$$
$$\|E_{f_*}|_{C_*(X;\mathbb{Z})}\| \le \|E_{f_*}|_{C_*(X,f;\mathbb{Z})}\|.$$

What is more, we can define the L_{C^0} category that its objects are X^f and its morphisms are continuous maps, where X is a Hausdorff space and f is a continuous self-map on X. Similarly, we can define the $\tilde{L}_{C^0}$ category for which its objects are $\check{H}_*(X,f;\mathbb{Z})$ and its morphisms are F_*, where X, Y are Hausdorff spaces, $f \in C^0(X), g \in C^0(Y)$, and F_* is associated with the continuous map $F : X^f \to Y^g$. Furthermore, we can define the homotopy and homeomorphism from X^f to X^g and research the relations between the elements of L_{C^0} and $\tilde{L}_{C^0}$.

Definition 13. *Let X, Y be compact Hausdorff spaces, $f \in C^0(X)$ and $g \in C^0(Y)$.*

(a) *If there exist continuous maps $F : X^f \to Y^g$ and $D : Y^g \to X^f$ such that $F \circ D = id_{Y^g}$ and $D \circ F = id_{X^f}$, then we say that X^f and Y^g are L_1-homotopy equivalent.*

(b) *If there exists a continuous map $F : X^f \times [0,1] \to Y^g$ such that $F(X^f, 0) = h(X^f)$ and $F(X^f, 1) = r(X^f)$, then we say that $h, r : X^f \to Y^g$ are the L_2-homotopy. Hence, h induces a homomorphism:*

$$h_* : \check{H}_*(X,f;\mathbb{Z}) \to \check{H}_*(Y,g;\mathbb{Z}),$$

and r_* induced by r.

Let L be the class of objects:

$$\{X^f; X \text{ is a compact Hausdorff space}, f \in C^0(X)\}.$$

For each pair $X^f, Y^g \in L$, let $mor_s(X^f, Y^g) = L_1(X^f, Y^g)$. By the definition of the L_1-homotopy and the composition function $\circ$, we get the category $(L, mor_s, \circ)$.

Let $\tilde{L}$ be the class of objects $\{\check{H}_*(X,f;\mathbb{Z}); X^f \in L\}$. Let:

$$mor_H(\check{H}_*(X,f;\mathbb{Z}), \check{H}_*(Y,g;\mathbb{Z}))$$

be the group homomorphism from $\check{H}_*(X,f;\mathbb{Z})$ to $\check{H}_*(Y,g;\mathbb{Z})$, where $\check{H}_*(X,f;\mathbb{Z})$, $\check{H}_*(Y,g;\mathbb{Z}) \in \tilde{L}$.

By the induced $*$ homomorphism of the L_1-homotopy and the composition function $\circ$, we get the category $(\tilde{L}, mor_H, \circ)$. Easily, we get a functor from $(L, mor_s, \circ)$ to $(\tilde{L}, mor_H, \circ)$.

Then, by diagram chasing, we see the following:

Theorem 2. *Let $f \in C^0(X), g \in C^0(Y)$, and let X and Y be compact Hausdorff spaces.*

(a) If X^f and Y^g are L_1-homotopy equivalent, then:

$$C_p(X,f;\mathbb{Z}) = C_p(X,g;\mathbb{Z}) \quad and \quad \check{H}_p(X,f;\mathbb{Z}) = \check{H}_p(X,g;\mathbb{Z}).$$

(b) If $h, r : X^f \to Y^g$ are the L_2-homotopy, then $h_* = r_*$.

Example 1. *Let $f \in C^0(X), g \in C^0(Y)$, and let X and Y be compact Hausdorff spaces. If there exists a homeomorphism F from X to Y such that $Ff = gF$, then:*

$$\check{H}_p(X,f;\mathbb{Z}) = \check{H}_p(X,g;\mathbb{Z}).$$

Example 2. *Let $f \in C^0(X), g \in C^0(Y)$, and let X and Y be compact Hausdorff spaces. If there exists a homeomorphism F from X to Y, then:*

$$\check{H}_p(X, f; \mathbb{Z}) = \check{H}_p(X, g; \mathbb{Z}).$$

Example 3. *Let $f \in C^0(X), g \in C^0(Y)$, and let X and Y be compact Hausdorff spaces. If there exists a continuous map $F : X \times [0,1] \to Y$ such that:*

$$\begin{cases} F(X,0) = h(X) \\ F(X,1) = r(X) \end{cases}$$

that is h and r are homotopies. Then, $h_* = r_*$, where:

$$h_* : \check{H}_*(X, f; \mathbb{Z}) \to \check{H}_*(Y, g; \mathbb{Z}) \quad \text{and} \quad r_* : \check{H}_*(X, f; \mathbb{Z}) \to \check{H}_*(Y, g; \mathbb{Z}).$$

5. Topological Fiber Entropy

In this section, X is a compact Hausdorff space and J is the set of all finite open covers of X such that there exists n_J. For $n = n_J$, we have $\check{H}_p(X; \mathbb{Z})$, where $0 \leq p \leq n$.

Let α be an open cover of X and $L_f(\alpha) = \{L_f(U) | U \in \alpha\}$. Then, there exists an open fiber cover $\dot{L}_f(\alpha)$ of X^f induced by $L_f(\alpha)$.

Definition 14. *For a fixed open fiber cover $\dot{L}_f(\alpha)$ of X^f, define:*

$$\begin{cases} \frac{f^{-1}(\dot{L}_f(\alpha))}{\dot{L}_f(\alpha)} = \max\limits_{U \in \alpha} \|\{f^{-1}\dot{L}_f(U) \cap \dot{L}_f(U)\}\|; \\ \frac{f(\dot{L}_f(\alpha))}{\dot{L}_f(\alpha)} = \max\limits_{U \in \alpha} \|\{f\dot{L}_f(U) \cap \dot{L}_f(U)\}\|; \\ L_d = \max\{\frac{f^{-1}(\dot{L}_f(\alpha))}{\dot{L}_f(\alpha)}, \frac{f(\dot{L}_f(\alpha))}{\dot{L}_f(\alpha)}\}; \\ ent(f, \dot{L}_f(\alpha)) = ent(f, \alpha) + \log L_d. \end{cases}$$

and define the topological fiber entropy of f by:

$$ent_L(f) = \sup_{\dot{L}_f(\alpha)} \{ent(f, \dot{L}_f(\alpha))\},$$

where $\sup\limits_{\dot{L}_f(\alpha)}$ is through all finite open covers of X^f.

Lemma 8 ([1], p. 102)**.** *If f is the shift operator on a k-symbolic space, then $ent(f) = \log k$.*

Corollary 1. *If f is the shift operator on a k-symbolic space, then:*

$$ent_L(f) = ent(f) + \log k = 2 \log k.$$

Example 4. *Let $\{1, 2, \cdots, k\} = X$ and $f : \begin{cases} \{1\} \to \{1, 2, \cdots, k\}, \\ \{2\} \to \{1, 2, \cdots, k\}, \\ \ \vdots \quad \vdots \qquad \vdots \\ \{k\} \to \{1, 2, \cdots, k\} \end{cases}$. Then:*

$$ent(f) = 0, \; ent_L(f) = 0.$$

Example 5. *Let $\{1, 2, \cdots, k\} = X$ and $f : \{1, 2, \cdots, k\} \to \{1\}$. Then:*

$$ent(f) = 0, \; ent_L(f) = 0.$$

Example 6. *Let* $[0,1] = X$ *and* $f(x) = kx, 0 < k < 1$. *Then:*

$$ent(f) = 0,\ ent_L(f) = 0.$$

Lemma 9. *For* $m \in \mathbb{Z}$ *and* $m > 2$, *there are* $p, q \in \mathbb{Z}$ *such that* $p \neq q$ *and* $m = p + q$, *where* $1 \leq p, 1 \leq q$.

Let $f \in C^0(X)$ and f_* be the linear transformation on $\check{H}_*(X, f; \mathbb{Z})$ associated with f. We say that a Čech eigenvalue chain is the chain belonging to an eigenvalue of f_*. Then, any Čech eigenchain can be associated with an open cover of X^f.

Lemma 10. *Let X be a compact Hausdorff space and J be the ordered set associated with the set of all finite open covers of X such that there exist* n_J *and* $n_{J,f}$. *Then,* $n_J = n_{J,f}$, *and for* $n = n_J = n_{J,f}$, *we have* $\check{H}_p(X; \mathbb{Z})$ *and* $\check{H}_p(X, f; \mathbb{Z})$, *where* $0 \leq p \leq n$. *Let* $\alpha \in J$ *be an open cover of X. If* $L_f(\alpha)$ *is a Čech eigenchain belonging to the eigenvalue m, then* $L_f(\alpha)$ *has a factor conjugating with a shift operator on m-symbolic space or* $L_d = m$, *where* $m \in \mathbb{N}$.

Proof. By Lemma 6, for an eigenchain $L_f = \sum_{i=0}^{k} a_i \check{\sigma}_i$ belonging to the eigenvalue m, there exists the f-Čech homology germ $H_p(J, f; \mathbb{Z})$ such that:

$$H_p(J, f; \mathbb{Z}) \sim \check{H}_p(X, f; \mathbb{Z}), \quad 0 \leq p \leq n_J.$$

where $\check{\sigma}_i \in \check{H}_*(X, f; \mathbb{Z})$ and $m, a_i \in \mathbb{Z}$.

Hence, there exists $\Phi \in J$ such that $L_f \in H_*(\Phi, f; G)$ and:

$$f_*(L_f) = m(L_f).$$

That can be extended to an equation on $C_*(\Phi, f; G)$, and we get the equation:

$$f_\sharp(\check{\sigma}_i) = m(\check{\sigma}_i), i \in \{0, \cdots, k\},$$

where $\check{\sigma}_i \in C_*(\Phi, f; G)$ and $m \in \mathbb{Z}$.

Just thinking of $f_\sharp$ on $C_*(\Phi, f; G)$, let $U_0, \cdots, U_j$ be open subsets of X and:

$$\check{\sigma}_i = L_f(U_0) \cap \cdots \cap L_f(U_j).$$

Then, we see:

$$L_f(U_\eta) = (\cdots, f^{-n}(U_\eta), \cdots, f^{-1}(U_\eta), f^0(U_\eta), f^1(U_\eta) \cdots, f^n(U_\eta), \cdots),$$

where $\eta \in \{0, \cdots, j\}$.

Therefore,

$$\begin{aligned} f_\sharp(\check{\sigma}_i) = &\quad f_\sharp(L_f(U_0) \bigcap \cdots \bigcap L_f(U_j)) = L_f(f(U_0)) \bigcap \cdots \bigcap L_f(f(U_j)) \\ = &\quad m(L_f(U_0) \bigcap \cdots \bigcap L_f(U_j)). \end{aligned}$$

That is,

$$\begin{aligned} & m(\bigcap_{\eta=0}^{j}(\cdots, f^{-n}(U_\eta), \cdots, f^{-1}(U_\eta), \underline{f^0(U_\eta)}, \cdots, f^n(U_\eta), \cdots)) \\ = & \bigcap_{\eta=0}^{j}(\cdots, f^{-n}(f(U_\eta)), \cdots, f^{-1}(f(U_\eta)), \underline{f^0(f(U_\eta))}, \cdots, f^n(f(U_\eta)), \cdots) \\ = & \bigcap_{\eta=0}^{j}(\cdots, f^{-(n-1)}(f(U_\eta)), \cdots, f^{-1}(f(U_\eta)), \underline{f(U_\eta)}, f^2(U_\eta), \cdots, f^{n+1}(U_\eta), \cdots). \end{aligned}$$

Therefore, we see that:

$$m(\bigcap_{\eta=0}^{j} L_f(U_\eta)) = (\bigcap_{\eta=0}^{j} L_f(f(U_\eta))).$$

Without loss of generality, let $j = 0$. Then:

$$L_f(f(U_0)) = m(L_f(U_0)).$$

If $L_f(U_0)$ is torsion, then the conclusion is trivial. Next, we only prove the conclusion for $L_f(U_0)$, which is torsion free. Now, let $L_f(U_0)$ be a torsion free element.

(i) $m = 0, 1$; the conclusion is trivial.

(ii) If $m = 2$, then there exists $U \subseteq f^{-1}(f(U_0))$ such that $U \nsubseteq U_0$ and $U_0 \nsubseteq U$, where U_0, U are non-empty open subsets of X.
If $f^{-1}(f(U_0)) = U_0$, then:

$$L_f(f(U_0)) = (L_f(U_0)) = 2(L_f(U_0));$$

this is a contradiction for the property that $\mathbb{Z}$ is a free group.
Because of $U \nsubseteq U_0$ and $U_0 \nsubseteq U$, with the property of the Hausdorff space, there exist points x, y such that $x \in U_0$, but $x \notin U$, and $y \in U$, but $y \notin U_0$. Then, there exist open neighborhoods $O(x)$ of x and $O(y)$ of y, respectively, such that:

$$x \in O(x) \subseteq U_0 \text{ but } O(x) \nsubseteq U \quad \text{and} \quad y \in O(y) \subseteq U \text{ but } O(y) \nsubseteq U_0.$$

That is, $O(x), O(y) \subseteq f^{-1}(f(U_0))$ and $O(x) \bigcap O(y) = \emptyset$.
Hence, $L_d = 2$, and for $m = 2$, the conclusion is true.

(iii) $m \geq 3$; from the mathematical induction, let the conclusion be right for $m = n - 1$. Then, we see the conclusion for $m = n$.
Using Lemma 9, we get $m = p + q, p \neq q$ and:

$$L_f(f(U_0)) = p(L_f(U_0)) + q(L_f(U_0)).$$

Therefore, there exists $f|_{U_0} = h + g$ such that:

$$L_h(f(U_0)) = p(L_f(U_0)) \quad \text{and} \quad L_g(f(U_0)) = q(L_f(U_0)).$$

(1) If $L_f \neq L_{h \oplus g}$, then using (ii) with the same computing, we get:

$$L_d = m.$$

(2) If $L_f = L_{h \oplus g}$, then we get:

$$L_f(f(U_0)) = L_h(f(U_0)) \oplus L_g(f(U_0));$$

else, we get:

$$h^{-1}(f(U_0)) \bigcap g^{-1}(f(U_0)) = W \neq \emptyset.$$

That is, we get $p(L_f(W)) = q(L_f(W))$, and it is a contradiction of the property that $\mathbb{Z}$ is a free group.

For $m = p + q$, we get that $p, q \leq n - 1$, and by mathematical induction, we obtain:

$$\begin{cases} h^{-1}(f(U_0)) \supseteq U_{0i}, U_{0j}, & U_{0i} \bigcap U_{0j} = \emptyset, 1 \leq i, j \leq p \\ g^{-1}(f(U_0)) \supseteq U_{1k}, U_{1l} & U_{1k} \bigcap U_{1l} = \emptyset, 1 \leq k, l \leq q \end{cases}$$

where U_{0i}, U_{0j}, U_{1k}, and U_{1l} are non-empty open subsets.

With the decomposition:

$$L_f(f(U_0)) = L_h(f(U_0)) \oplus L_g(f(U_0)),$$

we get that $U_i, U_j \subseteq f^{-1}(f(U_0))$, $U_i \bigcap U_j = \emptyset$, and U_i, U_j are non-empty open subsets of X, where $1 \leq i, j \leq m$.

Therefore, $L_d = m$ or there exists an m-symbolic space S_m conjugating with a shift operator on S_m, that is $L_f(U_0)$ has a factor conjugating with a shift operator on S_m.

Therefore, for $m = n$, the conclusion is right, and by mathematical induction, the conclusion is right for any eigenvalue m, where $m \in \mathbb{N}$. □

Now, we give the following definition.

Definition 15. *For two topological dynamic systems (X_1, f) and (X_2, g), if there exists a homeomorphism H from X_1 to X_2 such that $H \circ f = g \circ H$, then we say that H is a topological conjugacy from (X_1, f) to (X_2, g) or just say that (X_1, f) is topologically conjugate to (X_2, g); moreover, if $X = X_1 = X_2$, then we say that f is topologically conjugate to g on X.*

From the proof of Lemma 10, it is easy to see that $L_d(\cdot)$ is invariant for topological conjugacy. Furthermore, we know that the topological entropy $ent(\cdot)$ is invariant for topological conjugacy. Hence, we obtain that:

Proposition 1. *The topological fiber entropy is invariant for topological conjugacy.*

Theorem 3. *Let X be a compact Hausdorff space and J be the ordered set associated with the set of all finite open covers of X such that there exists n_J. For $n = n_J$, we have $\check{H}_p(X;\mathbb{Z})$, where $0 \leq p \leq n$. For $f \in C^0(X)$, we get:*

$$\log \|E_{f_* \check{H}_*(X;\mathbb{Z})}\| \leq ent_L(f),$$

Moreover, for $0 \leq p \leq n$, we get:

$$\log \|E_{f_* \check{H}_*(X,f;\mathbb{Z})}\| \leq ent_L(f).$$

Proof. It is easy to obtain that

$$ent_L(f) \geq ent(f, \dot{L}_f(\alpha)) \geq \log \|E_{f_*}|_{C_*(X,f;\mathbb{Z})}\| \geq \log \|E_{f_*}|_{\check{H}_*(X;\mathbb{Z})}\|$$

and:

$$ent_L(f) \geq ent(f, \dot{L}_f(\alpha)) \geq \log \|E_{f_*}|_{C_*(X,f;\mathbb{Z})}\| \geq \log \|E_{f_*}|_{\check{H}_*(X,f;\mathbb{Z})}\|.$$

□

By simple computing, we get the following results.

Proposition 2. *$ent_L(f) \geq ent(f)$; the inequality can be strict.*

Proposition 3. *$ent_L(id) = ent(id) = 0$, where id is the identical map.*

Corollary 2. *Let X be a compact Poincaré space and J be the ordered set associated with the set of all finite open covers of X such that there exists n_J. For $n = n_J$, we have $\check{H}_p(X;\mathbb{Z})$, where $0 \leq p \leq n$. The topological entropy conjecture is valid for the topological fiber entropy and Čech cohomology. Moreover, the topological entropy conjecture is valid for the topological fiber entropy and the f-Čech homology.*

Corollary 3. *In triangulable compact n-dimensional manifold M, the topological entropy conjecture is valid for the topological fiber entropy and homology group:*

$$H_*(M;\mathbb{Z}) = \bigoplus_{i=0}^{n} H_i(M;\mathbb{Z}),$$

where $H_i(M;\mathbb{Z})$ is the i-th integer coefficients' homology group of M.

6. Conclusions

If we replace $\mathbb{Z}$ with any free abelian group G that is finite generated, then the conclusion is also valid. Because the counterexample of A. B. Katok [20] is on a two-dimension sphere S^2 and $f \in C^0(S^2)$, with Corollary 3, we get that the inequality of the topological entropy conjecture is valid again with our definition, that is,

$$\log \rho \le ent_L(f).$$

Others may be more interested in what the topological fiber entropy $ent_L(f)$ measures. From the definition:

$$ent_L(f) = \sup_{\dot{L}_f(\alpha)} \{ent(f,\alpha) + \log L_d\},$$

we get that the topological fiber entropy $ent(f_L)$ is $\sup\limits_{\dot{L}_f(\alpha)}$ on the sum:

$$ent(f,\alpha) + \log L_d.$$

The first part $ent(f,\alpha)$ is the usually one. The second part $\log L_d$ is likely some fiber ratio or fiber degree of the dynamics (X,f); it is likely the "reference system" or "initial value" of the first part $ent(f,\alpha)$.

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