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On the Canonical Foliation of an Indefinite Locally Conformal Kähler Manifold with a Parallel Lee Form

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Abstract: We study the semi-Riemannian geometry of the foliation $\mathcal{F}$ of an indefinite locally conformal Kähler (l.c.K.) manifold M, given by the Pfaffian equation $\omega = 0$, provided that $\nabla\omega = 0$ and $c = \|\omega\| \neq 0$ (ω is the Lee form of M). If M is conformally flat then every leaf of $\mathcal{F}$ is shown to be a totally geodesic semi-Riemannian hypersurface in M, and a semi-Riemannian space form of sectional curvature $c/4$, carrying an indefinite c-Sasakian structure. As a corollary of the result together with a semi-Riemannian version of the de Rham decomposition theorem any geodesically complete, conformally flat, indefinite Vaisman manifold of index $2s$, $0 < s < n$, is locally biholomorphically homothetic to an indefinite complex Hopf manifold $\mathbb{C}H^n_s(\lambda)$, $0 < \lambda < 1$, equipped with the indefinite Boothby metric $g_{s,n}$.

Keywords: indefinite locally conformal Kähler manifold; indefinite Hopf manifold; indefinite Boothby metric; indefinite Vaisman manifold; Lee vector field; Lee form; canonical foliation; indefinite Sasakian structure

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1. Reminder of l.c.K. Geometry and Statement of Main Results

Let M be a complex n-dimensional indefinite Hermitian manifold, of index $0 \leq \nu < 2n$, with the complex structure J and the semi-Riemannian metric g (ν is necessarily even, i.e., $\nu = 2s$). (M, J, g) is an *indefinite Kähler manifold* if $\nabla J = 0$, where ∇ is the Levi-Civita connection of (M, g). Indefinite Kähler manifolds were studied by M. Barros and A. Romero [1]. An indefinite Hermitian manifold M is an *indefinite locally conformal Kähler (l.c.K.)* manifold if there is an open cover $\{U_i\}_{i\in I}$ of M and a family $\{f_i\}_{i\in I}$ of C^∞ functions $f_i : U_i \to \mathbb{R}$ such that $\left(U_i\,,\, e^{-f_i}\, g\right)$ is an indefinite Kähler manifold for any $i \in I$. In the positive definite case ($\nu = 0$) l.c.K. structures were introduced by P. Libermann (cf. [2]) more than sixty years ago. The "result" by T. Aubin (that any compact l.c.K. manifold is actually Kähler, cf. [3]) slowed down scientific investigation in this area until I. Vaisman's 1976 work pointed out (cf. [4]) the counterexample of a complex Hopf manifold $\mathbb{C}H^n_0(\lambda)$ with the Boothby metric $g_{0,n}$ and disproved (as Boothby's metric $g_{0,n}$ is l.c.K. yet the first Betti number of the (compact, complex) manifold $\mathbb{C}H^n_0(\lambda)$ is 1, so that $\mathbb{C}H^n_0(\lambda)$ admits no globally defined Kähler metrics) Aubin's "finding". Despite the impressive advancement of science (an account of which up to the year 1998 is provided by the monograph [5]) regarding the geometry of l.c.K. structures, it was not until the work by K.L. Duggal et al. that indefinite l.c.K. manifolds were introduced (cf. [6]), and the startling differences between the definite and indefinite cases were emphasized (cf. also [7]).

If we set $g_i = e^{-f_i} g$ then $g_j = e^{f_i - f_j}\, g_i$ i.e., the indefinite Kähler metrics g_j and g_i are conformally related. By a result in [6] any two conformally related indefinite Kähler metrics are actually homothetic. Hence, for any $i, j \in I$ there is $c_{ji} \in \mathbb{R}$ such that $f_i - f_j = c_{ji}$ on $U_i \cap U_j$. In particular, the (locally defined) differential forms $\{df_i\}_{i\in I}$ glue up to a (globally defined) closed differential 1-form ω (the *Lee form* of M) such that $\omega\big|_{U_i} = df_i$

for any $i \in I$]. The Lee form was discovered by H.C. Lee (cf. [8]) in the positive definite case. Let Ω (respectively Ω_j) be the 2-form associated to (g, J) (respectively (g_j, J)) e.g., $\Omega(X,Y) = g(X, JY)$ for any $X, Y \in \mathfrak{X}(M)$. Then $\Omega_j = e^{-f_j}\,\Omega$ hence $d\Omega = \omega \wedge \Omega$.

The *Lee field* is the tangent vector field $B \in \mathfrak{X}(M)$ determined by $g(B, X) = \omega(X)$ for any $X \in \mathfrak{X}(M)$. Let us set $c = g(B,B) \in C^\infty(M)$ and $\mathrm{Sing}(B) = \{x \in M : B_x = 0\}$. Unlike the positive definite case, it may be that $c = 0$ yet $\mathrm{Sing}(B) = \emptyset$ (provided that B is lightlike).

Let D be the *Weyl connection* i.e.,

$$D_X Y = \nabla_X Y - \frac{1}{2}\Big\{\omega(X)\,Y + \omega(Y)\,X - g(X,Y)\,B\Big\} \tag{1}$$

for any $X, Y \in \mathfrak{X}(M)$. The pointwise restriction of D to U_i is the Levi–Civita connection of (U_i, g_i), hence $DJ = 0$. In addition, $Dg = \omega \otimes g$. An indefinite l.c.K. manifold is *conformally flat* if $R^D = 0$ (R^D is the curvature tensor field of the Weyl connection).

Indefinite l.c.K. manifolds with $\nabla\omega = 0$ (the indefinite counterpart of generalized Hopf manifolds, cf. I. Vaisman [9]) were studied in [6]. Any such manifold carries a natural foliation $\mathcal{F}$, tangent to the distribution defined by the Pfaffian equation $\omega = 0$. Additionally, $c \in \mathbb{R}$ so that B is spacelike (respectively timelike, or lightlike) if $c > 0$ (respectively if $c < 0$, or $c = 0$). Adapting the terminology in the monograph [5] to the indefinite case, any indefinite l.c.K. manifold with a parallel Lee form ω will be referred to as an *indefinite Vaisman manifold*. By Theorem 1 in ([6], p. 9), for every indefinite Vaisman manifold (M, J, g) of index $2s$, $0 < s < n$, with $\mathrm{Sing}(B) = \emptyset$, either (i) $c \neq 0$, and then every leaf $L \in M/\mathcal{F}$ is a totally geodesic semi-Riemannian hypersurface of (M, g) of index

$$\mathrm{ind}(L) = \begin{cases} 2s & \text{if } c > 0 \\ 2s-1 & \text{if } c < 0, \end{cases}$$

or (ii) $c = 0$ and then every leaf of $\mathcal{F}$ is a totally geodesic lightlike hypersurface of (M, g). The notion of a totally geodesic submanifold in statement (i) is the ordinary notion in semi-Riemannian geometry (cf. e.g., Definition 12 in [10], p. 104) i.e., the shape tensor of each semi-Riemannian leaf L vanishes. As to statement (ii) the adopted notion is typical of lightlike geometry (cf. e.g., A. Bejancu & K.L. Duggal [11]) and is perhaps less familiar to the scientific community devoted to the study of (semi) Riemannian geometry. In the present paper we focus on the semi-Riemannian case ($c \neq 0$) and relegate the study of the lightlike case to further work. Therefore, we recall but briefly the constructions most relevant (to statement (ii) above) in Appendix A. Our main result is as follows.

Theorem 1. *Let M be a conformally flat indefinite Vaisman manifold with $c \neq 0$. Then every leaf of the canonical foliation $\mathcal{F}$ is a semi-Riemannian space form of sectional curvature $c/4$, carrying an indefinite c-Sasakian structure. Vice versa, for every indefinite c-Sasakian manifold $\big(N, (\varphi, \xi, \eta, \gamma)\big)$ the product manifold $M = N \times \mathbb{R}$ together with the complex structure $J = f + c^{-1}\,(\omega \otimes A - \theta \otimes B)$ and the indefinite Hermitian metric $g = p^*\gamma + c^{-1}\,\omega \otimes \omega$ is an indefinite Vaisman manifold whose Lee vector and Lee form are*

$$B = \sqrt{|c|}\,\frac{\partial}{\partial t} \in \mathfrak{X}(M), \quad \omega = \epsilon(c)\,\sqrt{|c|}\,dt,$$

and $g(B,B) = c$. Moreover if (N, γ) has constant sectional curvature $c/4$ then the Weyl connection of M is flat.

Here $A = JB$ and $\theta = -\omega \circ J$ are respectively the *anti-Lee field* and *anti-Lee form*. In addition, f is the $(1,1)$ tensor field on M given by $f = \varphi$ on $T(N)$ and $f(B) = 0$, $p : M \to N$ is the projection, and $\epsilon(c) = \mathrm{sign}\,(c)$. Applying Theorem 1 together with the semi-Riemannian version (due to H. Wu [12]) of the de Rham decomposition theorem (cf. Theorem 6.1 in [13], p. 187) yields the following.

Corollary 1. *Let M be a connected, geodesically complete, conformally flat, complex n-dimensional ($n \geq 3$) indefinite Vaisman manifold of index $2s$ ($0 < s < n$) with $c = \|\omega\| \neq 0$. Then the universal semi-Riemanian covering manifold of M is*

$$\left(\mathbb{C}^n \setminus \Lambda_{\epsilon(c)}\,,\, 4\,c^{-1}\,|\zeta|_{s,n}^{-2}\,\epsilon_j\,d\zeta^j \otimes d\overline{\zeta}^j\right)$$

where $\Lambda_{\epsilon(c)} = \left\{\zeta \in \mathbb{C}^n \,:\, \epsilon(c)\,\epsilon_j\,\left|\zeta^j\right|^2 \geq 0\right\}$.

Here $\epsilon_1 = \cdots = \epsilon_s = -1 = -\epsilon_{s+1} = \cdots = -\epsilon_n$.

The paper is organized as follows. Section 2 is devoted to the construction and main properties of indefinite complex Hopf manifolds $\mathbb{C}H^n_s(\lambda)$, which stay to ordinary complex Hopf manifolds (equipped with the positive definite Boothby metric) as M. Barros and A. Romero's indefinite complex projective spaces $\mathbb{C}P^{n-1}_s(4)$ stay to ordinary complex projective spaces (equipped with the Fubini-Study metric). Unlike ordinary Hopf manifolds $\mathbb{C}H^N(\lambda)$, indefinite Hopf manifolds are noncompact. Topologically each $\mathbb{C}H^n_s(\lambda)$ consists of two connected components $\Omega_\pm$ (such that $\mathbb{C}H^{n-s}(\lambda)$ (respectively $\mathbb{C}H^s(\lambda)$) is a strong deformation retract of Ω_+ (respectively of Ω_-)). Section 3 discusses the local structure of conformally flat indefinite Vaisman manifolds (indefinite l.c.K. manifolds with a parallel Lee form), in the spirit of nowadays classical work by I. Vaisman (cf. [9,14] in the positive definite case) and provides proofs to the main results (Theorem 1 and Corollary 1). The assumption of geodesic completeness in Theorem 1 is only needed in order to apply H. Wu's semi-Riemannian de Rham decomposition theorem (cf. [12]). Section 4 states a few open problems. Appendix A presents the construction of the lightlike transversal vector bundle $\mathrm{tr}(T\mathcal{F}) \to M$ and the derivation of the lightlike analogs to Gauss–Weingarten formulas (paving the road towards the study of the case $c = 0$).

2. Indefinite Hopf Manifolds

Indefinite complex Hopf manifolds (introduced in [6], p. 11) are our main examples of indefinite Vaisman manifolds. We recall their construction, for further use. Let $\mathbb{C}^n_s$ be $\mathbb{C}^n$ with the Hermitian form

$$b_{s,n}(z,w) = -\sum_{j=1}^{s} z_j\overline{w}_j + \sum_{j=s+1}^{n} z_j\overline{w}_j\,, \quad z,w \in \mathbb{C}^n\,.$$

Let $\Lambda = \left\{z \in \mathbb{C}^n \setminus \{0\} \,:\, -\sum_{j=1}^{s}\left|z_j\right|^2 + \sum_{j=s+1}^{n}\left|z_j\right|^2 = 0\right\}$ be the null cone in $\mathbb{C}^n_s$ and let $\Lambda_0 = \Lambda \cup \{0\}$. For every $\lambda \in \mathbb{C} \setminus \{0\}$

$$F_\lambda(z) = \lambda\,z, \quad z \in \mathbb{C}^n \setminus \Lambda_0\,,$$

is a holomorphic transformation of $\mathbb{C}^n \setminus \Lambda_0$. Let

$$G_\lambda = \{F^m_\lambda \,:\, m \in \mathbb{Z}\}$$

be the discrete group generated by F_λ. By Theorems 2 and 3 in ([6], p. 11–13), if $n > 1$, $0 < s < n$, and $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$, then G_λ acts freely on $\mathbb{C}^n \setminus \Lambda_0$ as a properly discontinuous group of holomorphic transformations, hence the quotient space

$$\mathbb{C}H^n_s(\lambda) = \left(\mathbb{C}^n \setminus \Lambda_0\right)/G_\lambda$$

is a complex manifold. In addition,

$$g_{s,n} = |z|_{s,n}^{-2}\Big(-\sum_{j=1}^{s} dz^j \odot d\overline{z}^j + \sum_{j=s+1}^{n} dz^j \odot d\overline{z}^j\Big), \quad |z|_{s,n} := \left|b_{s,n}(z,z)\right|^{\frac{1}{2}}\,,$$

is a semi-Riemannian metric on $\mathbb{C}H^n_s(\lambda)$ organizing it as an indefinite l.c.K. manifold. If $0 < \lambda < 1$ then
$$\mathbb{C}H^n_s(\lambda) \approx \Sigma^{2n-1} \times S^1$$
(a diffeomorphism) where $\Sigma^{2n-1} = \{z \in \mathbb{C}^n \,:\, |z|_{s,n} = 1\}$. In particular $\mathbb{C}H^n_s(\lambda)$ is noncompact. If $\Lambda_\pm = \{z \in \mathbb{C}^n \,:\, \pm b_{s,n}(z,z) \geq 0\}$ (so that $\partial\,\Lambda_\pm = \Lambda_0$) then $\mathbb{C}H^n_s(\lambda)$ has two connected components $\Omega_\pm = \big(\Lambda_\pm \setminus \Lambda_0\big)/G_\lambda$
$$\Omega_+ \approx S^{2n-1}_{2s} \times S^1\,, \quad \Omega_- \approx H^{2n-1}_{2s-1} \times S^1\,.$$
The indefinite l.c.K. manifold $\big(\mathbb{C}H^n_s(\lambda),\, g_{s,n}\big)$ has a parallel Lee form locally given by $\omega = -d\log|z|^2_{s,n}$. The corresponding Lee field is
$$B = -2\,a(z)\left(z^j\,\frac{\partial}{\partial z^j} + \overline{z}^j\,\frac{\partial}{\partial \overline{z}^j}\right),$$
$$a(z) := \text{sign}\,\big(b_{s,n}(z,z)\big) = \pm 1, \quad z \in \Lambda_\pm \setminus \Lambda_0\,.$$
In particular, if $B_\pm = B\big|_{\Omega_\pm}$ then B_+ is spacelike and B_- is timelike. Let
$$D = \{2\pi i a + (\log\lambda)\, b \,:\, a, b \in \mathbb{Z}\} \;\; (0 < \lambda < 1)$$
and let us consider the torus $T^1_{\mathbb{C}} = \mathbb{C}/D$. Then $T^1_{\mathbb{C}}$ acts freely on $\mathbb{C}H^n_s(\lambda)$ and
$$p : \Omega_+ \to \mathbb{C}P^{n-1}_s(4), \quad p\big(\Pi(z)\big) = z \cdot \mathbb{C}^*\,,$$
is a principal $T^1_{\mathbb{C}}$-bundle and a semi-Riemannian submersion of Ω_+ (equipped with the semi-Riemannian metric $g_{s,n}$) onto the indefinite complex projective space $\mathbb{C}P^{n-1}_s(4)$, where $\Pi : S^{2n-1}_{2s} \to \mathbb{C}P^{n-1}_s(4)$ is the indefinite Hopf fibration. The semi-Riemannian metric $g_{s,n}$ is referred to as the *indefinite Boothby metric*. Its positive definite counterpart $g_{0,n}$ was discovered by W.M. Boothby [15]. As to the notations above, we set $\mathbb{R}^N_\nu = \big(\mathbb{R}^N,\, h_{\nu,N}\big)$ with
$$h_{\nu,N}(x,y) = -\sum_{j=1}^{\nu} x_j y_j + \sum_{j=\nu+1}^{N} x_j y_j\,,$$
so that $S^N_\nu \subset \mathbb{R}^{N+1}_\nu$ and $H^N_\nu \subset \mathbb{R}^{N+1}_{\nu+1}$ are respectively the pseudo-sphere and pseudo-hyperbolic space
$$S^N_\nu(r) = \big\{x \in \mathbb{R}^{N+1} \,:\, h_{\nu,\,N+1}(x,x) = r^2\big\}, \quad S^N_\nu = S^N_\nu(1),$$
$$H^N_\nu(r) = \big\{x \in \mathbb{R}^{N+1} \,:\, h_{\nu+1,\,N+1}(x,x) = -r^2\big\}, \quad H^N_\nu = H^N_\nu(1).$$
The indefinite complex projective space is (cf. [1])
$$\mathbb{C}P^{n-1}_s(k) = \big(\Lambda_+ \setminus \Lambda_0\big)/\mathbb{C}^*\,, \quad \mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$
Let $k > 0$ and let
$$\Pi : S^{2n-1}_{2s}\left(\frac{2}{\sqrt{k}}\right) \to \mathbb{C}P^{n-1}_s(k), \quad \Pi(z) = z \cdot \mathbb{C}^*\,,$$
be the indefinite Hopf fibration (a principal S^1-bundle). S^1 acts on $S^{2n-1}_{2s}\big(2/\sqrt{k}\big)$ as a group of isometries; hence (by adapting Proposition E.3 in [16], p. 7, to the semi-Riemannian context) there is a unique semi-Riemannian metric of index $2s$ on $\mathbb{C}P^{n-1}_s(k)$ such that Π is a semi-Riemannian submersion and $\mathbb{C}P^{n-1}_s(k)$ is an indefinite complex space form of (constant) holomorphic sectional curvature k. Let $\mathbb{C}P^N = \big(\mathbb{C}^{N+1} \setminus \{0\}\big)/\mathbb{C}^*$ be the ordinary

complex projective space. By a result in [1], p. 57, $\mathbb{C}P_s^{n-1}(k)$ and $\mathbb{C}P^{n-1-s}$ are homotopy equivalent; hence, $\mathbb{C}P_s^{n-1}(k)$ is simply connected.

3. Conformally Flat Indefinite l.c.K. Manifolds

Since $\mathbb{C}_s^n$ is flat, the indefinite Hopf manifold $\mathbb{C}H_s^n(\lambda)$ equipped with the indefinite Boothby metric $g_{s,n}$ is (by the very definition of $g_{s,n}$) conformally flat. By a result of I. Vaisman [14], any conformally flat complete Vaisman manifold is locally biholomorphically homothetic to a Hopf manifold with the Boothby metric. The purpose of the present paper is to recover Vaisman's result (cf. *op. cit.*) to the semi-Riemannian setting. Let M be an indefinite l.c.K. manifold, and let B, A, ω, and θ be its Lee and anti-Lee vector fields (respectively differential forms). Then

$$\theta(X) = g(X, A), \quad \theta(B) = \omega(A) = 0, \quad \theta(A) = \omega(B) = c.$$

Moreover (by $DJ = 0$)

$$(\nabla_X J)Y = -\frac{1}{2}\{\theta(Y)X + \omega(Y)JX + \Omega(X,Y)B - g(X,Y)A\}. \tag{2}$$

Indefinite l.c.K. manifolds with $\nabla\omega = 0$ may be characterized as follows.

Lemma 1. *Let M be an indefinite l.c.K. manifold. Then M has a parallel Lee form if and only if one of the following relations holds*

$$\begin{gathered}\nabla_X B = 0, \quad \nabla_B X = [B, X], \\ (D_X\omega)Y = \omega(X)\omega(Y) - (c/2)\, g(X,Y), \\ D_X B = -(c/2)\, X, \quad D_B X = [B, X] - (c/2)\, X,\end{gathered} \tag{3}$$

for any $X \in \mathfrak{X}(M)$.

The proof of Lemma 1 is straightforward and hence omitted. We collect a few properties of the Lee and anti-Lee forms and vector fields in the following

Lemma 2. *Let M be an indefinite Vaisman manifold. Then*

$$\mathcal{L}_B J = 0, \quad \mathcal{L}_A J = 0, \quad \mathcal{L}_B g = 0, \quad \mathcal{L}_A g = 0, \tag{4}$$

$$d\theta = \omega \wedge \theta + (c/2)\,\Omega\,. \tag{5}$$

Here $\mathcal{L}_X$ is the Lie derivative at the direction X.

Proof of Lemma 2. The first relation in (4) follows from the fact that D is torsion free and $DJ = 0$. The second relation in (4) is a consequence of

$$\nabla_X A = \frac{1}{2}\{\omega(X)A - \theta(X)B - c\, JX\}, \tag{6}$$

$$\nabla_A J = 0, \quad \nabla_B J = 0.$$

Next, the fact that both the Lee and anti-Lee fields are Killing vector fields follows from $\nabla g = 0$. Finally, to prove (5) one performs the following calculation

$$2\,(d\theta)(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]) =$$

$$= -X(\omega(JY)) + Y(\omega(JX)) + \omega(J[X,Y]) =$$

(as $\nabla\omega = 0$ and ∇ is torsion free)

$$= \omega((\nabla_Y J)X) - \omega((\nabla_X J)Y) =$$

(by (2))

$$= \frac{1}{2}\{\theta(Y)\omega(X) + \omega(Y)\omega(JX) + c\,\Omega(X,Y)$$
$$-\theta(X)\omega(Y) - \omega(X)\omega(JY) - c\,\Omega(Y,X)\} =$$

(by $\theta = -\omega \circ J$)

$$= \omega(X)\theta(Y) - \omega(Y)\theta(X) + c\,\Omega(X,Y).$$

□

Let M be an indefinite Vaisman manifold. For the remainder of the section we assume that $\mathrm{Sing}(B) = \emptyset$ and $c \neq 0$. The leafwise bundle metric γ induced by g on $T(\mathcal{F})$ is semi-Riemannian of index

$$\mathrm{ind}(T(\mathcal{F}),\gamma) = \begin{cases} 2s, & c > 0, \\ 2s-1, & c < 0. \end{cases}$$

Then (cf. also (2) in [6], p. 9)

$$T(M) = T(\mathcal{F}) \oplus \mathbb{R}B \tag{7}$$

and every $X \in \mathfrak{X}(M)$ may be decomposed as

$$X = X_{\mathcal{F}} + c^{-1}\omega(X)B$$

for some $X_{\mathcal{F}} \in T(\mathcal{F})$. Let $\tan : T(M) \to T(\mathcal{F})$ be the natural projection associated to the decomposition (7). Let us set $\xi = \tan(A)$ so that (again by (7)) $A = \xi + \lambda\, B$ for some $\lambda \in C^\infty(M)$. Then $0 = \omega(A) = \lambda\, c$; hence $A = \xi \in T(\mathcal{F})$ i.e., on any indefinite l.c.K. manifold, the anti-Lee field is tangent to the leafs of the canonical foliation $\mathcal{F}$. Next let us consider the endomorphism

$$\varphi : T(\mathcal{F}) \to T(\mathcal{F}), \quad \varphi X = \tan(JX), \quad X \in T(\mathcal{F}).$$

Then $\varphi(\xi) = \tan(JA) = -\tan(B)$; hence

$$\varphi(\xi) = 0. \tag{8}$$

Next we set

$$\eta(X) = \gamma(X\,,\,\xi), \quad X \in T(\mathcal{F}). \tag{9}$$

Then $\eta(\xi) = g(A,A)$ i.e.,

$$\eta(\xi) = c. \tag{10}$$

Note that η is the pullback of θ to $T(\mathcal{F})$. It may be easily shown that

$$\varphi X = JX + c^{-1}\,\eta(X)\,B, \tag{11}$$

$$\eta \circ \varphi = 0, \tag{12}$$

$$\gamma(\varphi X, \varphi Y) = \gamma(X,Y) - c^{-1}\eta(X)\eta(Y), \tag{13}$$

for any $X, Y \in T(\mathcal{F})$. In addition,

$$\varphi^2 = -I + c^{-1}\eta \otimes \xi. \tag{14}$$

An inspection of (8)–(10) and (12)–(14) shows that the restriction of $(\varphi, \xi, \eta, \gamma)$ to a leaf $L \in M/\mathcal{F}$ obeys to all axioms defining an almost contact metric structure on L (in the sense of D.E. Blair [17], p. 19–21), except for the occurrence of the constant $1/c$. Let us set

$$\Phi(X, Y) = \gamma(X, \varphi Y), \quad X, Y \in T(\mathcal{F}),$$

so that (by (5))

$$d\eta = (c/2)\,\Phi. \tag{15}$$

on $T(\mathcal{F}) \otimes T(\mathcal{F})$. Next we consider

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi\{[\varphi X, Y] + [X, \varphi Y]\}$$

for any $X, Y \in T(\mathcal{F})$. A calculation (relying on (11)) yields

$$N_\varphi + 2c^{-1}\,(d\eta) \otimes \xi = 0 \tag{16}$$

on $T(\mathcal{F}) \otimes T(\mathcal{F})$. The property (16) is an obvious semi-Riemannian analog to *normality* (cf. e.g., [17], p. 49). Let L be a leaf of $\mathcal{F}$. Then (by (6))

$$(\nabla^L_X \eta)Y = (d\eta)(X, Y) = (c/2)\,\Phi(X, Y), \tag{17}$$

for any $X, Y \in T(L)$. Here ∇^L is the Levi–Civita connection of (L, γ_L) (and γ_L is the pointwise restriction of γ to L). By (2)

$$(\nabla_X J)Y = \frac{1}{2}\{\gamma(X, Y)\xi - \eta(Y)X - \Phi(X, Y)B\},$$

for any $X, Y \in T(\mathcal{F})$. Therefore

$$\nabla^L \varphi = \frac{1}{2}\{\gamma \otimes \xi - I \otimes \eta\} \tag{18}$$

on $T(L) \otimes T(L)$ (I is the identical transformation). Let R^∇, R^D, and R^L be respectively the curvature tensor fields of ∇, D, and ∇^L. As B is parallel

$$R^\nabla(X, Y)B = 0, \quad X, Y \in T(M). \tag{19}$$

Similarly (as $DB = -(c/2)\,I$ and D is torsion-free)

$$R^D(X, Y)B = 0, \quad X, Y \in T(M). \tag{20}$$

By (1)

$$R^D(X, Y)Z = R^\nabla(X, Y)Z - \frac{c}{4}(X \wedge Y)Z - \tag{21}$$

$$-\frac{1}{4}\{\omega(X)Y - \omega(Y)X\}\omega(Z) + \frac{1}{2}\{\omega(X)g(Y, Z) - \omega(Y)g(X, Z)\},$$

for any $X, Y, Z \in T(M)$. Here

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

In particular for any $X, Y, Z \in T(\mathcal{F})$

$$R^D(X, Y)Z = R^\nabla(X, Y)Z - (c/4)\,(X \wedge Y)Z. \tag{22}$$

Every leaf $L \in M/\mathcal{F}$ is totally geodesic in (M, g) hence (by the Gauss–Codazzi equations)

$$R^\nabla(X, Y)Z = R^L(X, Y)Z$$

for any $X, Y, Z \in T(L)$. This ends the proof.

Lemma 3. *Let M be a conformally flat (i.e., $R^D = 0$) indefinite Vaisman manifold with $c \neq 0$. Then, every leaf of the canonical foliation $\mathcal{F}$ is a semi-Riemannian space form of sectional curvature $c/4$.*

At this point we may clarify the role played by the geometric structure of the leafs of $\mathcal{F}$ in the classification of indefinite Vaisman manifolds. We need to recall a few notions of indefinite Sasakian geometry (following mainly K.L. Duggal [18] and T. Takahashi [19]). Let $c \in \mathbb{R} \setminus \{0\}$. Let N be a real $(2n+1)$-dimensional C^∞ manifold, and let φ, ξ and η be respectively a $(1,1)$-tensor field, a tangent vector field, and a differential 1-form on N such that

$$\varphi^2 = -I + c^{-1}\,\eta \otimes \xi, \quad \eta(\xi) = c, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0.$$

The synthetic object (φ, ξ, η) is referred to as an *almost c-contact structure* on N. A semi-Riemannian metric γ on N is *compatible* to (φ, ξ, η) if

$$\gamma(\varphi X, \varphi Y) = \gamma(X, Y) - c^{-1}\eta(X)\eta(Y),$$

for any $X, Y \in T(N)$. In addition, an almost c-contact structure is *normal* if

$$N_\varphi + 2c^{-1}(d\eta) \otimes \xi = 0.$$

Let (φ, ξ, η) be an almost c-contact structure and γ a compatible semi-Riemannian metric. Let us set $\Phi(X, Y) = \gamma(X, \varphi Y)$ for any $X, Y \in \mathfrak{X}(N)$. If

$$d\eta = (c/2)\,\Phi$$

then $(\varphi, \xi, \eta, \gamma)$ is said to satisfy the *c-contact condition*, and $(\varphi, \xi, \eta, \gamma)$ is a *c-contact metric structure*. An *indefinite c-Sasakian manifold* is a manifold N endowed with a normal c-contact metric structure. The underlying semi-Riemannian metric γ is referred to as an *indefinite c-Sasakian metric*. The properties in (8), (10), (12)–(14) and (16) may then be rephrased as follows.

Lemma 4. *Every leaf of the canonical foliation of an indefinite Vaisman manifold with $c \neq 0$ is an indefinite c-Sasakian manifold.*

Viceversa let $c \in \mathbb{R} \setminus \{0\}$ and let $(N, (\varphi, \xi, \eta, \gamma))$ be an indefinite c-Sasakian manifold. Let us set $M = N \times \mathbb{R}$ and

$$B = \sqrt{|c|}\,\frac{\partial}{\partial t}, \quad \omega = \epsilon(c)\,\sqrt{|c|}\,dt,$$

where t is the natural coordinate function on $\mathbb{R}$ and $\epsilon(c) \in \{\pm 1\}$ is the sign of c. Then $\omega(B) = c$ and $\omega(X) = 0$ for any $X \in \mathfrak{X}(N)$. Let $p : M \to N$ be the canonical projection. Then

$$g = p^*\gamma + c^{-1}\,\omega \otimes \omega \tag{23}$$

is a semi-Riemannian metric on M and

$$g(X, Y) = \gamma(X, Y), \quad g(X, B) = 0, \quad g(B, B) = c,$$

for any $X, Y \in T(N)$. Next we set

$$\theta = p^*\eta, \quad A = \theta^\sharp,$$

where $\sharp$ denotes raising of indices with respect to g i.e., $g(A, V) = \theta(V)$ for any $V \in \mathfrak{X}(M)$. Moreover we extend φ to a $(1,1)$ tensor field f on M by declaring that $f(B) = 0$. Then

$$J = f + c^{-1}\{\omega \otimes A - \theta \otimes B\} \tag{24}$$

is an almost complex structure on M compatible to g i.e., (M, J, g) is an indefinite almost Hermitian manifold. A calculation shows that the normality property of (φ, ξ, η) implies $N_J = 0$ i.e., J is an actual complex structure. We set as customary $\Omega(V, W) = g(V, JW)$ for any $V, W \in \mathfrak{X}(M)$. Then

$$\Omega = 2c^{-1}\,\theta \wedge \omega + p^*\Phi. \tag{25}$$

The proof is straightforward and hence omitted. By the c-contact condition $p^*\Phi = 2c^{-1}d\theta$, hence $d\Omega = \omega \wedge \Omega$ i.e., (M, J, g) is an indefinite l.c.K. manifold whose Lee form is ω. It may be easily checked that $\nabla\omega = 0$. We obtain the following.

Lemma 5. *Let $c \in \mathbb{R} \setminus \{0\}$ and $(N, (\varphi, \xi, \eta, \gamma))$ an indefinite c-Sasakian manifold. Then $M = N \times \mathbb{R}$ together with the complex structure (24) and the indefinite Hermitian metric (23) is an indefinite l.c.K. manifold with a parallel Lee form whose Lee vector $B \in \mathfrak{X}(M)$ satisfies $g(B, B) = c$. Moreover if (N, γ) has constant sectional curvature $c/4$ then the Weyl connection of M is flat ($R^D = 0$).*

The last statement in Lemma 5 may be proved by observing that the canonical foliation $\mathcal{F}$ of the indefinite l.c.K. manifold $M = N \times \mathbb{R}$ is given by $M/\mathcal{F} = \{N \times \{t\} : t \in \mathbb{R}\}$ followed by applying the identity (22). Let $0 < \lambda < 1$ and let $\mathbb{C}H^n_s(\lambda)$ be the indefinite Hopf manifold. Let $\pi : \mathbb{C}^n \setminus \Lambda_0 \to \mathbb{C}H^n_s(\lambda)$ be the canonical projection. Let us consider the C^∞ diffeomorphism

$$F : \mathbb{C}H^n_s(\lambda) \to \Sigma^{2n-1} \times S^1,$$
$$F(\pi(z)) = \left(\frac{z}{|z|_{s,n}}, \exp\left(\frac{2\pi i \log |z|_{s,n}}{\log \lambda} \right) \right), \quad z \in \mathbb{C}^n \setminus \Lambda_0\,,$$

with the obvious inverse

$$F^{-1}(\zeta, w) = \pi\left(\lambda^{(\arg w)/(2\pi)}\zeta\right), \quad \zeta \in \Sigma^{2n-1}, \quad w \in S^1,$$

where $\arg : \mathbb{C} \to [0, 2\pi)$. Note that

$$\Sigma^{2n-1} \cap \Lambda_+ = S^{2n-1}_{2s}\,, \quad \Sigma^{2n-1} \cap \Lambda_- = H^{2n-1}_{2s-1}\,.$$

Let $g_{s,n}$ be the indefinite Boothby metric and let us set

$$\tilde{g} = G^* g_{s,n}\,, \quad G =: F^{-1}\,.$$

We need to compute the explicit local expression of $\tilde{g}$. To this end we set $I = (0, 1) \subset \mathbb{R}$ and $U = \{\exp(2\pi i t) : t \in I\} \subset S^1$. Then $G : \Sigma^{2n-1} \times U \to \mathbb{C}H^n_s(\lambda)$ is given by

$$G(\zeta, \exp(2\pi i t)) = \pi(\lambda^t \zeta), \quad \zeta \in \Sigma^{2n-1}, \quad t \in I.$$

Let $D \subset \mathbb{C}^n$ be an open set such that $\pi : D \to \pi(D) \subset \mathbb{C}H^n_s(\lambda)$ is a local diffeomorphism and let $(z^1, \ldots, z^n)$ be the corresponding complex coordinates on $\pi(D)$. Then

$$G^* dz^j = \lambda^t\left[d\zeta^j + (\log \lambda)\,\zeta^j\,dt\right], \quad G^* d\overline{z}^j = \lambda^t\left[d\overline{\zeta}^j + (\log \lambda)\,\overline{\zeta}^j\,dt\right],$$

hence

$$\tilde{g} = \epsilon_j\,d\zeta^j \odot d\overline{\zeta}^j + (\log \lambda)^2\,dt \otimes dt = \tag{26}$$
$$= -\sum_{j=1}^{s} d\zeta^j \odot d\overline{\zeta}^j + \sum_{j=s+1}^{n} d\zeta^j \odot d\overline{\zeta}^j + (\log \lambda)^2\,dt \otimes dt$$

as $\overline{\zeta}_j d\zeta^j + \zeta^j d\overline{\zeta}_j = 0$ along Σ^{2n-1}. Here $\zeta_j = \zeta^j$. Also $\epsilon_j = -1$ for $1 \leq j \leq s$ and $\epsilon_j = 1$ for $s+1 \leq j \leq n$. Let $g_{j\bar{k}} = g_{s,n}(Z_j, \overline{Z}_k)$ where $Z_j = \partial/\partial z^j$. Then

$$g_{j\bar{k}} = \frac{1}{2}\,|z|^{-2}_{s,n}\,\epsilon_j\,\delta_{jk}$$

hence the 2-form $\Omega_{j\bar{k}} = -i g_{j\bar{k}}$ may be written as

$$\Omega = -i|z|^{-2}_{s,n}\left(-\sum_{j=1}^{s} dz^j \wedge d\overline{z}^j + \sum_{j=s+1}^{n} dz^j \wedge d\overline{z}^j\right)$$

and taking the exterior derivative we obtain the familiar formula $d\Omega = \omega \wedge \Omega$ (accounting for the fact that $g_{s,n}$ is an indefinite l.c.K. metric) where

$$\omega = |z|^{-2}_{s,n}\left[\sum_{j=1}^{s}\left(\overline{z}^j\,dz^j + z^j\,d\overline{z}^j\right) - \sum_{j=s+1}^{n}\left(\overline{z}^j\,dz^j + z^j\,d\overline{z}^j\right)\right].$$

A straightforward calculation shows that

$$F^* d\zeta^j = |z|^{-1}_{s,n}\left[dz^j - \frac{1}{2}\,\epsilon_k |z|^{-2}_{s,n} z^j\left(\overline{z}^k\,dz^k + z^k\,d\overline{z}^k\right)\right],$$

$$F^* dt = |z|^{-2}_{s,n}\,\pi\,i\,(\log\lambda)^{-1}\,h\,\epsilon_j\left(\overline{z}^j\,dz^j + z^j\,d\overline{z}^j\right),$$

where

$$h = \exp\left(\frac{2\pi i \log|z|_{s,n}}{\log\lambda}\right).$$

In particular

$$\omega = -\frac{\log\lambda}{\pi\,i\,h}\,F^* dt. \tag{27}$$

Let us assume from now on that M is a conformally flat indefinite Vaisman manifold with $c \neq 0$. Let $\tilde{M}$ be the universal covering space of M. By a semi-Riemannian version of the de Rham decomposition theorem due to H. Wu (cf. [12]) it follows that $\tilde{M}$ with the lifted metric is isometric to $\tilde{L} \times \mathbb{R}$, where $\tilde{L}$ is the universal covering space of an arbitrary fixed leaf $L \in M/\mathcal{F}$. Yet (by Lemma 4) $\tilde{L}$ carries an indefinite Sasakian structure (obtained as the lift of the structure on L). Then (by Lemma 5) $\tilde{L} \times \mathbb{R}$ carries the indefinite l.c.K. structure associated to the indefinite Sasakian structure on $\tilde{L}$. Moreover, as M is conformally flat, each leaf L is a totally geodesic submanifold and a space form of (constant) sectional curvature $k = c/4$. Then (by a classical result in the theory of space forms, cf. J.A. Wolf [20], p. 68) we have

$$\tilde{L} \approx \begin{cases} S^{2n-1}_{2s}\left(\dfrac{2}{\sqrt{c}}\right), & c > 0, \\[2ex] H^{2n-1}_{2s}\left(\dfrac{2}{\sqrt{|c|}}\right), & c < 0, \end{cases}$$

(a global isometry). Let $(\varphi, \xi, \eta, \gamma)$ be the lift to $\tilde{L}$ of the indefinite Sasakian structure on L (induced by the indefinite l.c.K. structure on M). Then $\tilde{L} \times \mathbb{R}$ carries an indefinite l.c.K. structure (J, g) with the Lee form $\omega = \epsilon(c)\sqrt{|c|}dt$. Let $\hat{g}$ be a local indefinite Kähler metric of $\tilde{L} \times \mathbb{R}$. As M is conformally flat so does $\tilde{M} \approx \tilde{L} \times \mathbb{R}$; hence, $\hat{g}$ is flat, so that $\hat{g} = \epsilon_j dz^j \odot d\overline{z}^j$ for some local complex coordinate system (z^j) on $\tilde{M}$. Thus

$$g = \exp\left(\epsilon(c)\,t\,\sqrt{|c|}\right)\epsilon_j\,dz^j \odot d\overline{z}^j. \tag{28}$$

It should be observed that a priori $\tilde{L} \times \mathbb{R}$ carries yet another complex structure $\bar{J}$ obtained as the lift to $\tilde{M}$ of the complex structure of M. To see that these complex structures actually coincide let us consider the commutative diagram

$$\begin{tikzcd} \tilde{M} \approx \tilde{L} \times \mathbb{R} \arrow[r, "\tilde{p}"] \arrow[d, "\pi_M"'] & \tilde{L} \arrow[d, "\pi_L"] \\ M \approx L \times \mathbb{R} \arrow[r, "p"] & L \end{tikzcd}$$

where π_M, π_L, p, $\tilde{p}$ are projections. Let $(\varphi, \xi, \eta, \gamma)$ be the indefinite Sasakian structure of L and $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{\gamma})$ its lift to $\tilde{L}$. Then

$$\tilde{\gamma} = \pi_L^* \gamma, \quad \tilde{\eta} = \pi_L^* \eta,$$

and $\tilde{\xi} \in \mathfrak{X}(\tilde{L})$ is given by $\tilde{\gamma}(X, \tilde{\xi}) = \tilde{\eta}(X)$ for any $X \in \mathfrak{X}(\tilde{L})$. By the proof of Lemma 5 if $c \in \mathbb{R} \setminus \{0\}$

$$\tilde{B} = \sqrt{|c|}\, \frac{\partial}{\partial t}, \quad \tilde{\omega} = \epsilon(c) \sqrt{|c|}\, dt,$$

are respectively the Lee field and the Lee form on $(\tilde{L} \times \mathbb{R}, \tilde{J}, \tilde{g})$. Here

$$\tilde{g} = \tilde{p}^* \tilde{\gamma} + c^{-1} \tilde{\omega} \otimes \tilde{\omega} = (\pi_L \circ \tilde{p})^* \gamma + \epsilon(c)\, dt \otimes dt,$$

while $\tilde{J}$ is the complex structure. Next if $\tilde{\theta} = \tilde{p}^* \tilde{\eta} = (\pi_L \circ \tilde{p})^* \eta$ then

$$\tilde{\Omega} = \tilde{p}^* \tilde{\Phi} + 2c^{-1} \tilde{\theta} \wedge \tilde{\omega} = (\pi_L \circ \tilde{p})^* \Phi + 2c^{-1} \tilde{\theta} \wedge \tilde{\omega}$$

is the Kähler 2-form of $\tilde{L} \times \mathbb{R}$. On the other hand let $\bar{\omega}$ be the lift of the Lee form (of the indefinite l.c.K. manifold (M, g, J)) to $(\tilde{M}, \bar{J}, \bar{g})$. Precisely

$$\bar{\omega} = \pi_M^* \omega, \quad \bar{g} = \pi_M^* g.$$

Let $\bar{B} \in \mathfrak{X}(\tilde{M})$ be given by $\bar{\omega}(X) = \bar{g}(X, \bar{B})$ for any $X \in \mathfrak{X}(\tilde{M})$. Then $\bar{g}$ is an indefinite Hermitian metric on $\tilde{M}$, and a calculation shows that $d\bar{\Omega} = \bar{\omega} \wedge \bar{\Omega}$ so that $(\tilde{M}, \bar{J}, \bar{g})$ is an indefinite l.c.K. manifold with the Lee form $\bar{\omega}$. As $M \approx L \times \mathbb{R}$ and $g = p^* \gamma + \epsilon(c)\, dt \otimes dt$ and $\bar{g} = (p \circ \pi_M)^* \gamma + \epsilon(c)\, dt \otimes dt$, one has

$$\bar{B} = \epsilon(c) \bar{\omega}\left(\frac{\partial}{\partial t}\right) \frac{\partial}{\partial t}$$

hence

$$\bar{g}(\bar{B}, \bar{B}) = c, \quad \bar{\omega}\left(\frac{\partial}{\partial t}\right)^2 = |c|.$$

By replacing t with $-t$ if necessary we may assume that $\bar{\omega}(\partial/\partial t) > 0$ when $c > 0$ and $\bar{\omega}(\partial/\partial t) < 0$ when $c < 0$ i.e.,

$$\bar{\omega}\left(\frac{\partial}{\partial t}\right) = \epsilon(c) \sqrt{|c|}.$$

We may then conclude that

$$\bar{B} = \sqrt{|c|} \frac{\partial}{\partial t} = \tilde{B}, \quad \bar{\omega} = \tilde{\omega}.$$

Finally

$$\bar{\Omega} = (p \circ \pi_M)^* \Phi + 2c^{-1} (p \circ \pi_M)^* \eta \wedge \bar{\omega}.$$

Yet $p \circ \pi_M = \pi_L \circ \tilde{p}$ where from $\bar{g} = \tilde{g}$ and $\bar{\Omega} = \tilde{\Omega}$. We may conclude that $\bar{J} = \tilde{J}$.

Let $\phi : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n \setminus \Lambda_{\epsilon(c)}$ be given by

$$\phi(z,t) = \exp\left(-\frac{\epsilon(c)t\sqrt{|c|}}{2}\right) z, \quad z \in \mathbb{C}^n, \quad t \in \mathbb{R}.$$

Then

$$\phi^* d\zeta^j = \exp\left(-\frac{\epsilon(c)t\sqrt{|c|}}{2}\right)\left\{dz^j - \frac{\epsilon(c)\sqrt{|c|}}{2} z^j dt\right\}$$

hence (by setting $g_0 = \epsilon_j \, d\zeta^j \odot d\overline{\zeta}^j$)

$$\phi^* g_0 = \exp\left(-\epsilon(c)t\sqrt{|c|}\right)\left\{\epsilon_j \, dz^j \odot d\overline{z}^j - \right.$$

$$\left. -\epsilon(c)\sqrt{|c|}\epsilon_j \left[z^j d\overline{z}^j + \overline{z}^j dz^j\right] \odot dt + (|c|/4)\, |z|^2_{s,n} dt \otimes dt \right\}.$$

As $\tilde{L}$ is described by the equation $\epsilon_j z^j \overline{z}^j = \epsilon(c) 4/|c|$ it follows that

$$\mathbf{j}^* \left[\epsilon_j (z^j d\overline{z}^j + \overline{z}^j dz^j)\right] = 0$$

and $\hat{g} = \mathbf{j}^*(\epsilon_j dz^j \odot d\overline{z}^j)$, where $\mathbf{j} : \tilde{L} \hookrightarrow \mathbb{C}^n$ is the inclusion. Then

$$\mathbf{j}^* \phi^* g_0 = \exp(-\epsilon(c)t\sqrt{|c|})\{\hat{g} + \epsilon(c)\, dt \otimes dt\}.$$

Next let us observe that $\mathbf{h} = \phi \circ (\mathbf{j} \times 1_{\mathbb{R}})$ is a diffeomorphism

$$S^{2n-1}_{2s}(2/\sqrt{c}) \times \mathbb{R} \approx \mathbb{C}^n \setminus \Lambda_+, \quad c > 0,$$

$$H^{2n-1}_{2s}(2/\sqrt{-c}) \times \mathbb{R} \approx \mathbb{C}^n \setminus \Lambda_+, \quad c < 0.$$

Moreover

$$\mathbf{h}^*\left(\frac{4}{c}\, |\zeta|^{-1}_{s,n}\, g_0\right) = (\mathbf{j} \times 1_{\mathbb{R}})^* \left\{\frac{4}{c} \exp\left(\epsilon(c)t\sqrt{|c|}\right) |z|^{-1}_{s,n} \phi^* g_0\right\} =$$

$$= \hat{g} + \epsilon(c)\, dt \otimes dt.$$

On the other hand if $\mathbf{i} : \tilde{L} \hookrightarrow \tilde{L} \times \mathbb{R}$ is the map $z \mapsto (z, 0)$ then $\gamma = \mathbf{i}^* \hat{g}$. Next if $p : \tilde{L} \times \mathbb{R} \to \tilde{L}$ and $q : \tilde{L} \times \mathbb{R} \to \mathbb{R}$ are the natural projections, then

$$g = p^*\gamma + \epsilon(c) q^*(dt \otimes dt), \quad \hat{g} = (\mathbf{i} \circ p)^* \hat{g} = p^* \gamma.$$

Consequently

$$\mathbf{h}^*\left(\frac{4}{c}\, |\zeta|^{-1}_{s,n}\, g_0\right) = g.$$

Finally if J is the complex structure on $\tilde{L} \times \mathbb{R}$ induced by the canonical complex structure of $\mathbb{C}^n$ via h, then h is a biholomorphism of $(\tilde{L} \times \mathbb{R}, J)$ onto $\mathbb{C}^n \setminus \Lambda_{\epsilon(c)}$ and g is an indefinite Hermitian metric with respect to J. Corollary 1 is proved.

4. Conclusions and Open Problems

Any indefinite Vaisman manifold admits (cf. [6]) two canonical foliations $\mathcal{F}$ and $\mathcal{G}$, the first of which is given by the Pfaffian equation $\omega = 0$, while the second is tangent to the distribution spanned by the Lee and anti-Lee fields i.e., $T(\mathcal{G}) = \mathbb{R}B \oplus \mathbb{R}A$. The objective of the present paper was the investigation of the leafwise geometric structure of $\mathcal{F}$, which turned out to be indefinite c-Sasakian, for instance in the sense of K.L. Duggal [18].

Our study is confined to the semi-Riemannian case ($c \neq 0$), and in that case knowledge of the first-order geometric structure of $\mathcal{F}$ together with the semi-Riemannian de Rham decomposition theorem (cf. [12]) leads to the local description of the metric structure of any geodesically complete, conformally flat, indefinite Vaisman manifold (cf. Corollary 1 above). It is an open question whether these considerations carry over to the lightlike case. Besides from relying on a theory of the second fundamental form for the lightlike foliation $\mathcal{F}$ of the semi-Riemannian manifold M (whose starting ingredients are briefly described in Appendix A, by adapting the treatment of a single submanifold (such as in [11]) to that of a foliation (such as in [21])), an attempt to solve the posed problem will require a lightlike version of the de Rham decomposition theorem. Besides the metric structure, the first canonical foliation $\mathcal{F}$ inherits from M a tangential CR structure, i.e., each leaf of L is a CR manifold, of CR dimension $n-1$, nondegenerate if $c \neq 0$. The study of the pseudohermitian geometry (in the sense of [22]) of the leaves of $\mathcal{F}$ is a matter of a "work in progress". We anticipate that every leaf of $\mathcal{F}$ is a pseudo-Einstein manifold of pseudohermitian sectional curvature (in the sense of [23]) $H(\sigma) = 1$ and pseudohermitian scalar curvature (in the sense of [22]) $\rho = 2n(n-1)$ yet non-spherical (which may be seen, provided that $n \geq 3$, by computing the Chern-Moser tensor (in the sense of [24]) of a leaf). Once again the case $c = 0$ is open. To further recover (from the definite to the indefinite case) results on conformally flat l.c.K. manifolds, one should rely on the classical Gray–Hervella classification (cf. [25]) and consider the work by V.F. Kirichenko [26].

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Appendix A. Canonical Foliations with Lightlike Leaves

Let us assume that $c = 0$ so that $B \in T(\mathcal{F})$. We set (cf. e.g., [11], p. 140)

$$\mathrm{Rad}\, T(\mathcal{F})_x = T(\mathcal{F})_x \cap T(\mathcal{F})_x^{\perp}, \quad x \in M.$$

Then, $B \in \mathrm{Rad}\, T(\mathcal{F})$. By Proposition 2.2 in [11], p. 6, $\dim_{\mathbb{R}} T(\mathcal{F})_x = 2n-1$ yields $\dim_{\mathbb{R}} T(\mathcal{F})_x^{\perp} = 1$, for any $x \in M$. If $\mathrm{Sing}(B) = \emptyset$ then $T(\mathcal{F})^{\perp} = \mathbb{R}B$. Consequently (by Proposition 1.1 in [11], p. 78) every leaf $L \in M/\mathcal{F}$ is a lightlike hypersurface in (M, g). As another consequence every $Y \in T(\mathcal{F})^{\perp}$ is orthogonal to the Lee field, hence $Y \in T(\mathcal{F})$ implying $T(\mathcal{F})^{\perp} \subset T(\mathcal{F})$ i.e., $\mathrm{Rad}\, T(\mathcal{F}) = T(\mathcal{F})^{\perp}$. Let $S(T\mathcal{F})$ be a C^{∞} distribution on M such that

$$T(\mathcal{F}) = S(T\mathcal{F}) \oplus_{\mathrm{orth}} T(\mathcal{F})^{\perp}.$$

The portion of $S(T\mathcal{F})$ over a leaf of $\mathcal{F}$ is a *screen distribution* on that leaf (the adopted terminology is that of [11], p. 78). Then, one may apply Proposition 2.1 in [11], p. 5, at a point $x \in M$, to the unique leaf of $\mathcal{F}$ passing through x, to conclude that $S(T\mathcal{F})$ is nondegenerate, and then

$$T(M) = S(T\mathcal{F}) \oplus_{\mathrm{orth}} T(\mathcal{F})^{\perp}.$$

Note that $S(T\mathcal{F})^{\perp}$ has rank 2 and $T(\mathcal{F})^{\perp} \subset S(T\mathcal{F})^{\perp}$. One has the following:

Lemma A1. *Let $\pi : E \to M$ be a vector subbundle of $S(T\mathcal{F})^{\perp} \to M$ such that $S(T\mathcal{F})^{\perp} = T(\mathcal{F})^{\perp} \oplus E$. Let $V \in C^{\infty}(U, E)$ be a nowhere zero section, with $U \subset M$ open. Then* (i) *$\omega(V) \neq 0$ everywhere in U. Let $N_V \in C^{\infty}\big(U, S(T\mathcal{F})^{\perp}\big)$ be given by*

$$N_V = \frac{1}{\omega(V)} \left\{ V - \frac{g(V,V)}{2\,\omega(V)}\, B \right\}.$$

If $V' \in C^\infty(U', E)$ is another nonzero section, with $U' \subset M$ open and $U \cap U' \neq \emptyset$, then (ii) $N_V = N_{V'}$ *on* $U \cap U'$. *Let $x \in M$ and let $U \subset M$ be an open neighborhood of x such that $E_U = \pi^{-1}(U)$ is trivial. Let us set*

$$\mathrm{tr}(T\mathcal{F})_x = \mathbb{R}N_V(x).$$

Then (iii) $\mathrm{tr}(T\mathcal{F})_x$ *is well defined and the disjoint union* $\mathrm{tr}(T\mathcal{F}) = \bigcup_{x\in M} \mathrm{tr}(T\mathcal{F})_x$ *is the total space of a lightlike vector subbundle* $\mathrm{tr}(T\mathcal{F}) \to M$ *of* $S(T\mathcal{F})^\perp \to M$ *such that*

$$S(T\mathcal{F})^\perp = T(\mathcal{F})^\perp \oplus \mathrm{tr}(T\mathcal{F}).$$

Finally (iv) *the definition of* $\mathrm{tr}(T\mathcal{F})$ *does not depend upon the choice of complement E to $T(\mathcal{F})^\perp$ in $S(T\mathcal{F})^\perp$.*

Cf. Lemma 1 in [6], p. 10. Note that Lemma 1 is an adaptation of Theorem 1.1 in ([11], p. 79) (applying to the foliation $\mathcal{F}$ of M, rather than a single lightlike hypersurface of M). Again by adopting the terminology in [11], p. 79, the portion of $\mathrm{tr}(T\mathcal{F})$ over a leaf $L \in M/\mathcal{F}$ is the *lightlike transversal vector bundle* of L with respect to the screen distribution $S(T(\mathcal{F})\big|_L$. Gathering the decompositions above

$$T(M) = S(T\mathcal{F}) \oplus_{\mathrm{orth}} \left[T(\mathcal{F})^\perp \oplus \mathrm{tr}(T\mathcal{F})\right] = T(\mathcal{F}) \oplus \mathrm{tr}(T\mathcal{F}). \tag{A1}$$

Let $\tan : T(M) \to T(\mathcal{F})$ and $\mathrm{tra} : T(M) \to \mathrm{tr}(T\mathcal{F})$ be the canonical projections associated to the direct sum decomposition (A1). Let us set

$$\nabla^{\mathcal{F}}_X Y = \tan\left(\nabla_X Y\right), \quad h(X,Y) = \mathrm{tra}\left(\nabla_X Y\right),$$

$$A_V X = -\tan\left(\nabla_X V\right), \quad \nabla^{\mathrm{tr}}_X V = \mathrm{tra}\left(\nabla_X V\right),$$

for any $X, Y \in T(\mathcal{F})$, and any $V \in \mathrm{tr}(T\mathcal{F})$. Then $\nabla^{\mathcal{F}}$ is a connection in $T(\mathcal{F}) \to M$, h is a $\mathrm{tr}(T\mathcal{F})$-valued bilinear symmetric form on $T(\mathcal{F})$, A_V is an endomorphism of $T(\mathcal{F})$, and ∇^{tr} is a connection in $\mathrm{tr}(T\mathcal{F}) \to M$. In particular

$$\nabla_X Y = \nabla^{\mathcal{F}}_X Y + h(X,Y), \quad \nabla_X V = -A_V X + \nabla^{\mathrm{tr}}_X V,$$

(the Gauss and Weingarten formulas for $\mathcal{F}$ in (M, g)). Cf. also [21]. The pointwise restrictions of $\nabla^{\mathcal{F}}$, ∇^{tr}, h and A_V to a leaf of $\mathcal{F}$ are respectively the induced connections, the second fundamental form, and the shape operator of that leaf (cf. [11], p. 83), as a lightlike submanifold of (M, g). A leaf L of $\mathcal{F}$ is *totally geodesic* if every geodesic of $\nabla^{\mathcal{F}}$ lying on L is also a geodesic of (M, g). A lightlike version of the classical de Rham decomposition theorem (cf. [13], pp. 187–193) or of its semi-Riemannian analog (cf. [12,27]) is so far unknown.

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