

Article

Methods for Constructing Complex Solutions of Nonlinear PDEs Using Simpler Solutions

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Abstract: This paper describes a number of simple but quite effective methods for constructing exact solutions of nonlinear partial differential equations that involve a relatively small amount of intermediate calculations. The methods employ two main ideas: (i) simple exact solutions can serve to construct more complex solutions of the equations under consideration and (ii) exact solutions of some equations can serve to construct solutions of other, more complex equations. In particular, we propose a method for constructing complex solutions from simple solutions using translation and scaling. We show that in some cases, rather complex solutions can be obtained by adding one or more terms to simpler solutions. There are situations where nonlinear superposition allows us to construct a complex composite solution using similar simple solutions. We also propose a few methods for constructing complex exact solutions to linear and nonlinear PDEs by introducing complex-valued parameters into simpler solutions. The effectiveness of the methods is illustrated by a large number of specific examples (over 30 in total). These include nonlinear heat equations, reaction–diffusion equations, wave type equations, Klein–Gordon type equations, equations of motion through porous media, hydrodynamic boundary layer equations, equations of motion of a liquid film, equations of gas dynamics, Navier–Stokes equations, and some other PDEs. Apart from exact solutions to ‘ordinary’ partial differential equations, we also describe some exact solutions to more complex nonlinear delay PDEs. Along with the unknown function at the current time, $u = u(x, t)$, these equations contain the same function at a past time, $w = u(x, t - \tau)$, where $\tau > 0$ is the delay time. Furthermore, we look at nonlinear partial functional-differential equations of the pantograph type, which, in addition to the unknown $u = u(x, t)$, also contain the same functions with dilated or contracted arguments, $w = u(px, qt)$, where p and q are scaling parameters. We propose an efficient approach to construct exact solutions to such functional-differential equations. Some new exact solutions of nonlinear pantograph-type PDEs are presented. The methods and examples in this paper are presented according to the principle “from simple to complex”.

Keywords: exact solutions; nonlinear PDEs; reaction–diffusion equations; wave type equations; PDEs with constant and variable delay; pantograph-type PDEs; functional-differential equations



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1. Introduction

1.1. Preliminary Remarks

Exact solutions of nonlinear partial differential equations and methods for their construction are necessary for the development, analysis, and verification of various mathematical models used in natural and engineering sciences, as well as for testing approximate analytical and numerical methods. There are several basic methods for finding exact solutions and constructing reductions of nonlinear partial differential equations: the method of group analysis of differential equations (the method of searching for classical symmetries) [1–6], methods for finding for nonclassical symmetries [7–10], the direct Clarkson–Kruskal method [11–14], methods for generalized separation of variables [13–15], methods for functional separation of variables [14,16–18], the method of differential constraints [13,14,19], the method of truncated Painlevé expansions [13,20,21], and use of conservation laws to obtain exact solutions [22–24]. The application of these methods requires considerable special training and, as a rule, is accompanied by time-consuming analysis and a large volume of analytical transformations and intermediate calculations.

This paper describes a number of simple, but quite effective, methods for constructing exact solutions of nonlinear partial differential equations, which do not require much special training and lead to a relatively small amount of intermediate calculations. These methods are based on the following two simple, but very important, ideas:

- simple exact solutions can serve as a basis for constructing more complex solutions of the equations under consideration,
- exact solutions to some equations can serve as the basis for constructing solutions to other more complex equations.

The effectiveness of the proposed methods is illustrated by a large number of specific examples. Nonlinear heat equations, reaction–diffusion equations, wave type equations, Klein–Gordon type equations, equations of motion in porous media, hydrodynamic boundary layer equations, equations of motion of a liquid film, equations of gas dynamics, Navier–Stokes equations, and some other PDEs are considered. In addition to exact solutions of ‘ordinary’ partial differential equations, some exact solutions of more complex nonlinear delay PDEs with constant and variable delay and pantograph-type functional-differential equations with partial derivatives are also described.

The methods and examples in the article are presented according to the principle “from simple to complex”. For the convenience of a wide audience with different mathematical backgrounds, the authors tried to do their best, wherever possible, to avoid special terminology.

1.2. Concept of ‘Exact Solution’ for Nonlinear PDEs

In this article, the term ‘exact solution’ for nonlinear partial differential equations will be used in cases where the solution is expressed:

- (i) in terms of elementary functions, functions included in the equation (this is necessary when the equation contains arbitrary functions), and indefinite or/and definite integrals;
- (ii) through solutions of ordinary differential equations or systems of such equations.

Combinations of cases (i) and (ii) are also allowed. In case (i), the exact solution can be presented in explicit, implicit, or parametric form.

Remark 1. Exact solutions of nonlinear diffusion and wave type PDEs can be found, for example, in [4,5,9,10,13–18,25–44].

2. Construction of Complex Solutions from Simple Solutions by Translation and Scale Transformations

2.1. Some Definitions. Simplest Transformations

We say that a partial differential equation,

$$F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{1}$$

is invariant with respect to a one-parameter invertible transformation,

$$x = X(\bar{x}, \bar{t}, \bar{u}, a), \quad t = T(\bar{x}, \bar{t}, \bar{u}, a), \quad u = U(\bar{x}, \bar{t}, \bar{u}, a), \tag{2}$$

if, after substituting expressions (2) into (1), we obtain exactly the same equation

$$F(\bar{x}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{t}}, \bar{u}_{\bar{t}\bar{t}}, \dots) = 0.$$

It is important to note that the free parameter a , which can take values in a certain interval (a_1, a_2) , is not included in Equation (1).

Transformations that preserve the form of Equation (1) transform a solution of the considered equation into a solution of the same equation.

A function $I(x, t, u)$ (different from a constant and independent of a) is called an invariant of transformation (2) if it is preserved under this transformation, i.e.,

$$I(x, t, u) = I(\bar{x}, \bar{t}, \bar{u})$$

for all admissible values of the parameter a .

A solution $u = \Phi(x, t)$ of Equation (1) is called invariant if, under the transformation (2), it transforms into exactly the same solution $\bar{u} = \Phi(\bar{x}, \bar{t})$.

Further, we will consider only one-parameter transformations of the form

$$\begin{aligned} x &= \bar{x} + b_1, & t &= \bar{t} + b_2, & u &= \bar{u} + b_3 & (\text{translation}); \\ x &= c_1 \bar{x}, & t &= c_2 \bar{t}, & u &= c_3 \bar{u} & (\text{scaling}), \end{aligned}$$

and the composition of these transformations. Here b_n and c_n ($n = 1, 2, 3$) are constants depending on the free parameter a . Such transformations will be called the simplest transformations.

The following example shows how to determine the invariants of the simplest transformations and the form of the corresponding invariant solutions.

Example 1. Consider a transformation consisting of the translation in x and the scaling in t and u :

$$x = \bar{x} - m \ln a, \quad t = a\bar{t}, \quad u = a^k \bar{u}, \tag{3}$$

where k and m are some constants. Excluding the parameter a , we find two functionally independent invariants:

$$I_1 = x + m \ln t, \quad I_2 = ut^{-k}. \tag{4}$$

If the considered equation is invariant under transformation (3), then it admits an invariant solution, which can be represented as $I_2 = \varphi(I_1)$ [1,4] or $u = t^k \varphi(z)$, where $z = x + m \ln t$. Substituting the resulting expression into the original equation we arrive at an ordinary differential equation for the function $\varphi = \varphi(z)$.

2.2. Construction of Complex Solutions from Simpler Solutions. Examples

Simple one-term solutions in the form of a product of functions of different variables are most easily found by the method of separation of variables (the simplest solutions of this type $u = Ax^\alpha t^\beta$ are easily determined from the equations under consideration by the method of undefined coefficients). The methods for constructing more complex solutions based on such solutions are described below.

First, we will consider a simple multiplicative separable solution of the special form

$$u = t^k \varphi_1(x), \tag{5}$$

where k is some constant and $\varphi_1(x)$ is some function. Such solutions do not change (are invariant) under the scaling transformation

$$t = a\bar{t}, \quad u = a^k \bar{u}. \tag{6}$$

Below, in the form of a proposition, we describe a method that allows us to construct more complex solutions based on one-term solutions of the form (5).

Proposition 1. *Let the equation*

$$F(t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{7}$$

which does not explicitly depend on the x , has a simple solution of the form (5), and does not change under the scaling transformation (6) (i.e., Equation (7) has the same property as the original solution (5)). Then this equation also has a more complex solution of the form

$$u = t^k \varphi_2(z), \quad z = x + m \ln t, \tag{8}$$

where m is an arbitrary constant.

Proof. Equation (7) does not explicitly depend on the spatial variable and is invariant under the translation in x . Consider transformation (3), which is a composition of the translation in x and scaling in t and u (see (6)). Transformation (3) has two functionally independent invariants (4). Therefore, the solution invariant under transformation (3) has the form (8) (see Example 1). □

Remark 2. *In general, the form of functions $\varphi_1(x)$ and $\varphi_2(x)$, which are included in the original solution (5) and the more complex solution (8), respectively, may differ and $\psi_2(z)|_{m=0} \neq \varphi_1(x)$.*

Consider now a simple multiplicative separable solution of the special form

$$u = x^n \psi_1(t), \tag{9}$$

where n is some constant and $\psi_1(t)$ is some function. Such solutions do not change (are invariant) under the scaling transformation

$$x = a\bar{x}, \quad u = a^n \bar{u}. \tag{10}$$

A more complex solution than (9) can be obtained by using Proposition 1, redefining the constants and variables in (5)–(8) accordingly. A different, but equivalent method for constructing a more complex solution is described below, which is sometimes more convenient to use in practice.

Proposition 2. *Let the equation*

$$F(x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{11}$$

which does not explicitly depend on t , has a simple solution of the form (9), and does not change under the scaling transformation (10) (i.e., Equation (11) has the same property as the original solution (9)). Then this equation also has a more complex solution of the form

$$u = e^{-npt} \psi_2(y), \quad y = xe^{pt}, \tag{12}$$

where p is an arbitrary constant.

Proof. Equation (11) is invariant under the translation in t . Consider a transformation that is a composition of the translation in t and the scaling in x and u (see (10)):

$$x = a\bar{x}, \quad t = \bar{t} - \frac{1}{p} \ln a, \quad u = a^n \bar{u}, \tag{13}$$

where p is an arbitrary constant ($p \neq 0$). Transformation (13) has two functionally independent invariants $I_1 = y = xe^{pt}$ and $I_2 = e^{npt}u$. Therefore, a solution invariant with respect to transformation (13) has the form (12). \square

Obviously, Equation (11) admits the degenerate solution (12) with $p = 0$.

Remark 3. *In (12), the function argument y is linear in x . Therefore, solution (12) is easy to differentiate with respect to x . This solution representation should be used for equations which contain partial derivatives with respect to x of a higher order than with respect to t .*

Example 2. Consider the Boussinesq equation

$$u_t = a(uu_x)_x, \tag{14}$$

which describes the unsteady flow of groundwater in a porous medium with a free surface [45].

Equation (14) has a simple exact solution,

$$u = -\frac{x^2}{6at}, \tag{15}$$

which is simultaneously a solution of two types (5) and (9). Let us consider in order both possibilities of constructing more complex solutions based on solution (15).

1°. Solution (15) and Equation (14) retain their form under the scaling transformation $t = c\bar{t}, u = \bar{u}/c$. Therefore, by virtue of Proposition 1 Equation (14) admits a more complex exact solution,

$$u = \frac{\varphi(z)}{t}, \quad z = x + k \ln t,$$

where the function $\varphi = \varphi(z)$ satisfies the ordinary differential equation (hereinafter ODE):

$$k\varphi'_z - \varphi = a(\varphi\varphi'_z)'_z. \tag{16}$$

Note that Equation (16) for $k = 0$ admits a one-parameter family of solutions in the form of a quadratic polynomial,

$$\varphi = -\frac{x^2}{6a} + Cx - \frac{3aC^2}{2},$$

where C is an arbitrary constant. For $C = 0$ this solution coincides with the original solution (15).

2°. Solution (15) and Equation (14) retain their form also under the scaling transformation $x = c\bar{x}, u = c^2\bar{u}$. Therefore, by virtue of Proposition 2, Equation (14) admits another exact solution

$$u = e^{-2pt}\psi(y), \quad y = xe^{pt},$$

where p is an arbitrary constant and the function $\psi = \psi(y)$ is described by the ODE:

$$p\psi\psi'_y - 2p\psi = a(\psi\psi'_y)'_y.$$

Example 3. Consider now the Guderley equation

$$u_{xx} = au_yu_{yy}, \tag{17}$$

which is used to describe transonic gas flows [46].

Equation (17) admits a simple exact solution,

$$u = \frac{y^3}{3ax^2}, \tag{18}$$

which is a special case of two types of solutions (5) and (9). Let us consider in order both possibilities of constructing more complex solutions based on solution (18).

1°. Solution (18) and Equation (17) retain their form under the scaling transformation $x = c\bar{x}, u = c^{-2}\bar{u}$. Therefore, by virtue of Proposition 1, Equation (17) has a more complex exact solution of the form

$$u = x^{-2}\varphi(z), \quad z = y + m \ln x,$$

where the function $\varphi = \varphi(z)$ is described by the second-order ODE:

$$m^2\varphi''_{zz} - 5m\varphi'_z + 6\varphi = a\varphi'_z\varphi''_{zz}.$$

For $m = 0$, this equation admits a one-parameter family of solutions in the form of a cubic polynomial,

$$\varphi(z) = \frac{z^3}{3a} + Cz^2 + aC^2z + \frac{a^2C^3}{3},$$

where C is an arbitrary constant. For $C = 0$ this solution coincides with the original solution (18).

2°. Solution (18) and Equation (17) retain their form also under the scaling transformation $y = c\bar{y}, u = c^3\bar{u}$. Therefore, by virtue of Proposition 2, one can also obtain another more complex exact solution,

$$u = e^{-3px}\psi(z), \quad z = ye^{px},$$

where p is an arbitrary constant and the function $\psi = \psi(z)$ is described by the ODE:

$$p^2z^2\psi''_{zz} - 5p^2z\psi'_z + 9p^2\psi = a\psi'_z\psi''_{zz}.$$

Example 4. In gas dynamics, there is a nonlinear wave equation,

$$u_{tt} = a(u^b u_x)_x, \quad b \neq 0, \tag{19}$$

which admits a simple exact solution of the form

$$u = a^{-1/b} x^{2/b} t^{-2b}. \tag{20}$$

This solution belongs to both classes of solutions (5) and (9). Therefore, based on solution (20), we can construct two more complex solutions described below.

1°. Solution (20) and Equation (19) are invariant under the scaling transformation $t = c\bar{t}, u = c^{-2/b}\bar{u}$. By virtue of Proposition 1, Equation (19) has a more complex solution of the form

$$u = t^{-2/b} \varphi(z), \quad z = x + m \ln t,$$

where the function $\varphi = \varphi(z)$ is described by the ODE:

$$m^2 \varphi''_{zz} - \frac{m(b+4)}{b} \varphi'_z + \frac{2(b+2)}{b^2} \varphi = a(\varphi^b \varphi'_z)'_z.$$

2°. Solution (20) and Equation (19) are also invariant under the scaling transformation $x = c\bar{x}, u = c^{2/b}\bar{u}$. Therefore, by virtue of Proposition 2, Equation (19) admits another solution

$$u = e^{-2pt/b} \psi(y), \quad y = xe^{pt},$$

where p is an arbitrary constant and the function $\psi = \psi(y)$ satisfies the second-order nonlinear ODE:

$$p^2 y^2 \psi''_{yy} + \frac{p^2(b-4)}{b} y \psi'_y + \frac{4p^2}{b^2} \psi = a(\psi^b \psi'_y)'_y.$$

Example 5. The system of boundary layer equations on a flat plate by introducing a stream function is reduced to one nonlinear third-order PDE:

$$u_y u_{xy} - u_x u_{yy} = \nu u_{yyy}, \tag{21}$$

where ν is the kinematic viscosity of the fluid [47].

Equation (21) has a simple solution of the form

$$u = \frac{6\nu x}{y}, \tag{22}$$

which generates two more complex solutions.

1°. Solution (22) and Equation (21) do not change under the scaling transformation $x = a\bar{x}, u = a\bar{u}$. Therefore, by virtue of Proposition 1, Equation (21) has a more complex solution of the form

$$u = x\varphi(z), \quad z = y + k \ln x,$$

where k is an arbitrary constant and the function $\varphi = \varphi(z)$ satisfies the ODE:

$$-\varphi \varphi''_{zz} + (\varphi'_z)^2 = \nu \varphi'''_{zzz}.$$

2°. Solution (22) and Equation (21) do not change also under the scaling transformation $y = a\bar{y}, u = \bar{u}/a$. Therefore, by virtue of Proposition 2, Equation (21) admits another solution

$$u = e^{px} \psi(z), \quad z = ye^{px},$$

where p is an arbitrary constant and the function $\psi = \psi(z)$ is described by the ODE:

$$p\psi \psi''_{zz} - 2p(\psi'_z)^2 = \nu \psi'''_{zzz}.$$

Example 6. Consider a fourth-order nonlinear evolution equation describing the change in the film thickness of a heavy viscous liquid moving along a horizontal superhydrophobic surface with a variable surface tension coefficient,

$$u_t = [(au^3 + bx^{2/3}u^2)(u_x - c(x^2u_{xx})_x)]_x, \tag{23}$$

where $a, b,$ and c are some constants [48,49].

Equation (23) has a simple solution of the form

$$u = x^{2/3}f(t), \tag{24}$$

where the function $f = f(t)$ is described by the first-order ODE with separable variables

$$f'_t = \frac{10}{81}(2c + 9)f^3(af + b).$$

Solution (24) and Equation (23) are invariant under the scaling transformation $x = k\bar{x}, u = k^{2/3}\bar{u}$. Therefore, by virtue of Proposition 2, Equation (23) also has a more complex solution of the form

$$u = e^{-2pt/3}\psi(y), \quad y = xe^{pt},$$

where the function $\psi = \psi(y)$ satisfies the fourth-order nonlinear ODE [48,49]:

$$py\psi'_y - \frac{2}{3}p\psi = [(a\psi^3 + by^{2/3}\psi^2)(\psi'_y - (cy^2\psi''_{yy})'_y)]'_y,$$

where p is an arbitrary constant.

Example 7. Consider a n th-order nonlinear PDE of the form

$$u_t = u^s F(u_x/u, u_{xx}/u, \dots, u_x^{(n)}/u). \quad s \neq 1. \tag{25}$$

Equation (25) has a simple solution,

$$u = t^{1/(1-s)}\varphi(x), \tag{26}$$

where the function $\varphi = \varphi(x)$ is described by the ODE:

$$\frac{\varphi}{1-s} = \varphi^s F(\varphi'_x/\varphi, \varphi''_{xx}/\varphi, \dots, \varphi_x^{(n)}/\varphi).$$

Solution (26) and Equation (25) are invariant under the scaling transformation $t = a\bar{t}, u = a^{1/(1-s)}\bar{u}$. Therefore, by virtue of Proposition 1, Equation (25) also has a more complex solution of the form

$$u = t^{1/(1-s)}\varphi(z), \quad z = x + m \ln t,$$

where m is an arbitrary constant and the function $\theta = \theta(z)$ satisfies the ODE:

$$m\varphi'_z + \frac{\varphi}{1-s} = \varphi^s F(\varphi'_z/\varphi, \varphi''_{zz}/\varphi, \dots, \varphi_z^{(n)}/\varphi).$$

Proposition 3. Let the equation

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{27}$$

which does not explicitly depend on x and t (and therefore admits the traveling-wave solution [13]) does not change under scaling of the unknown function

$$u = c\bar{u}, \tag{28}$$

where $c > 0$ is an arbitrary constant. Then this equation admits an exact solution (more complicated than the traveling-wave solution) of the form

$$u = e^{kt} \varphi(z), \quad z = px + qt, \tag{29}$$

where $k, p,$ and q are arbitrary constants ($pq \neq 0$).

Proof. Consider a transformation that is a composition of translations in x and t and scaling of the unknown function (28):

$$x = \bar{x} + \frac{1}{p} \ln a, \quad t = \bar{t} - \frac{1}{q} \ln a, \quad u = a^{-k/q} \bar{u}, \tag{30}$$

where $a > 0$ is an arbitrary constant ($c = a^{-k/q}$), p and q are some constants ($pq \neq 0$). The transformation (30) preserves the form of Equation (27) and has two functionally independent invariants $I_1 = z = px + qt$ and $I_2 = e^{-kt}u$. Therefore, a solution that is invariant with respect to transformation (30), can be represented as (29). Solution of the form (29) is obtained from the invariant solution by applying the scaling transformation (28). □

Example 8. Consider the nonlinear heat-type equation

$$u_t = au_{xx} + uf(u_x/u), \tag{31}$$

where $f = f(\xi)$ is an arbitrary function.

Equation (31) is invariant under the scaling transformation (28). Therefore, by virtue of Proposition 3, this equation has a solution of the form (29), where the function $\varphi = \varphi(z)$ satisfies the nonlinear ODE:

$$k\varphi + q\varphi'_z = ap^2\varphi''_{zz} + \varphi f(p\varphi'_z/\varphi).$$

Example 9. Consider a more complex nonlinear PDE of order n ,

$$u_t = uF(u_x/u, u_{xx}/u, \dots, u_x^{(n)}/u). \tag{32}$$

Equation (32) is invariant under the scaling transformation (28). Therefore, by virtue of Proposition 3, this equation has a solution of the form (29), where the function $\varphi = \varphi(z)$ satisfies the nonlinear ODE:

$$k\varphi + q\varphi'_z = \varphi F(p\varphi'_z/\varphi, p^2\varphi''_{zz}/\varphi, \dots, p^n\varphi_z^{(n)}/\varphi),$$

where $F(w_1, w_2, \dots, w_n)$ is an arbitrary function.

2.3. Generalization to Nonlinear Multidimensional Equations

The above Propositions 1–3 allow obvious generalizations to the case of an arbitrary number of spatial variables.

Example 10. Consider the nonlinear heat equation with n spatial variables

$$u_t = a \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(u^k \frac{\partial u}{\partial x_i} \right), \quad k \neq 0. \tag{33}$$

Equation (33) admits a simple multiplicative separable solution,

$$u = t^{-1/k} \varphi(x_1, \dots, x_n), \tag{34}$$

where the function $\varphi = \varphi(x_1, \dots, x_n)$ satisfies the stationary equation

$$-\frac{1}{k} \varphi = a \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\varphi^k \frac{\partial \varphi}{\partial x_i} \right).$$

Solution (34) and Equation (33) are invariant under the scaling transformation $t = c\bar{t}$, $u = c^{-k}\bar{u}$. Therefore, by virtue of Proposition 1, Equation (33) also has a more complex solution of the form

$$u = t^{-1/k} \theta(z_1, \dots, z_n), \quad z_i = x_i + m_i \ln t,$$

where m_i are arbitrary constants, and the function $\theta = \theta(z_1, \dots, z_n)$ satisfies the stationary equation

$$-\frac{1}{k} \theta + \sum_{i=1}^n m_i \frac{\partial \theta}{\partial z_i} = a \sum_{i=1}^n \frac{\partial}{\partial z_i} \left(\theta^k \frac{\partial \theta}{\partial z_i} \right).$$

2.4. Generalization to Nonlinear Systems of Coupled Equations

The above Propositions 1–3 can also be used to find exact solutions of systems of coupled PDEs.

Example 11. Consider the nonlinear system consisting of two coupled reaction–diffusion equations

$$\begin{aligned} u_t &= a(u^b u_x)_x + u f(u/v), \\ v_t &= a(v^b v_x)_x + v g(u/v), \end{aligned} \tag{35}$$

where a and b are some constants ($b \neq 0$), and $f(z)$ and $g(z)$ are arbitrary functions.

System of Equation (35) has a simple solution of the form

$$u = x^{2/b} \varphi(t), \quad v = x^{2/b} \psi(t), \tag{36}$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are described by the system of first-order ODEs:

$$\begin{aligned} \varphi'_t &= \frac{2a(b+2)}{b^2} \varphi^{b+1} + \varphi f(\varphi/\psi), \\ \psi'_t &= \frac{2a(b+2)}{b^2} \psi^{b+1} + \psi g(\varphi/\psi). \end{aligned}$$

Solution (36) and system of Equation (35) are invariant under the scaling transformation $x = c\bar{x}$, $u = c^{2/b}\bar{u}$, $v = c^{2/b}\bar{v}$. Therefore, by virtue of Proposition 2, the system of Equation (35) also has a more complex solution of the form

$$u = e^{-2mt/b} \Phi(z), \quad v = e^{-2mt/b} \Psi(z), \quad z = xe^{mt},$$

where the functions $\Phi = \Phi(z)$ and $\Psi = \Psi(z)$ are described by the ODE system:

$$\begin{aligned} mz\Phi'_z - \frac{2m}{b} \Phi &= a(\Phi^b \Phi'_z)'_z + \Phi f(\Phi/\Psi), \\ mz\Psi'_z - \frac{2m}{b} \Psi &= a(\Psi^b \Psi'_z)'_z + \Psi f(\Phi/\Psi), \end{aligned}$$

where m is an arbitrary constant.

Example 12. Consider another nonlinear system consisting of two coupled reaction–diffusion equations

$$\begin{aligned} u_t &= a(u^b u_x)_x + u^{b+1} f(u/v), \\ v_t &= a(v^b v_x)_x + v^{b+1} g(u/v), \end{aligned} \tag{37}$$

where a and b are some constants ($b \neq 0$), and $f(z)$ and $g(z)$ are arbitrary functions. System of Equation (37) has a simple solution of the form

$$u = t^{-1/b}\varphi(x), \quad v = t^{-1/b}\psi(x), \tag{38}$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are described by the second-order ODE system

$$\begin{aligned} -\frac{\varphi}{b} &= a(\varphi^b \varphi'_x)'_x + \varphi^{b+1}f(\varphi/\psi), \\ -\frac{\psi}{b} &= a(\psi^b \psi'_x)'_x + \psi^{b+1}g(\varphi/\psi). \end{aligned}$$

Solution (38) and system of Equation (37) are invariant under the scaling transformation $t = c\bar{t}$, $u = c^{-1/b}\bar{u}$, $v = c^{-1/b}\bar{v}$. By virtue of Proposition 1, the system of Equation (37) also has a more complex solution of the form

$$u = t^{-1/b}\Phi(z), \quad v = t^{-1/b}\Psi(z), \quad z = x + m \ln t,$$

where m is an arbitrary constant, and the functions $\Phi = \Phi(z)$ and $\Psi = \Psi(z)$ satisfy the system ODE:

$$\begin{aligned} -\frac{\Phi}{b} + m\Phi'_z &= a(\Phi^b \Phi'_z)'_z + \Phi^{b+1}f(\Phi/\Psi), \\ -\frac{\Psi}{b} + m\Psi'_z &= a(\Psi^b \Psi'_z)'_z + \Psi^{b+1}g(\Phi/\Psi). \end{aligned}$$

3. Construction of Complex Solutions by Adding Terms or Combining Two Solutions

3.1. Construction of Complex Solutions by Adding Terms to Simpler Solutions

In some cases, simple solutions can be generalized by adding one or more additional terms to them, which leads to more complex solutions with generalized separation of variables [13–15]. We demonstrate the possible course of reasoning in such cases using the examples of the Boussinesq Equation (14) and the Guderley Equation (17).

Example 13. As mentioned earlier, the Boussinesq Equation (14) has a solution with a simple separation of variables (quadratic in x , see (15)), which we write as

$$u = \varphi(t)x^2, \quad \varphi(t) = -1/(6at). \tag{39}$$

Let us try to find a more complex solution in the form of the sum

$$u(x, t) = \varphi(t)x^2 + \psi(t)x^k, \quad k \neq 2, \tag{40}$$

whose first term coincides with the solution (39). The second term of formula (40) includes the function $\psi(t)$ and the coefficient k , which must be found.

Substituting (40) in (14), after elementary transformations we get

$$(\varphi'_t - 6a\varphi^2)x^2 + [\psi'_t - a(k+1)(k+2)\varphi\psi]x^k - ak(2k-1)\psi^2x^{2k-2} = 0. \tag{41}$$

Since this equality must hold identically for any x , the functional coefficients for various powers of x in (41) must be zero. Thus, there are two possible cases $k = 0$ and $k = 1/2$ (both correspond to the vanishing of the coefficient at x^{2k-2}), which must be considered separately.

1°. *The first case.* Substituting $k = 0$ into (41), to define the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$, we have the system of ODEs:

$$\varphi'_t - 6a\varphi^2 = 0, \quad \psi'_t - 2a\varphi\psi = 0,$$

the general solution of which is determined by the formulas

$$\varphi(t) = -\frac{1}{6a(t + C_1)}, \quad \psi(t) = \frac{C_2}{|t + C_1|^{1/3}}, \tag{42}$$

where C_1 and C_2 are arbitrary constants.

2°. *The second case* (the Barenblatt–Zeldovich dipole solution [50]). Substituting $k = 1/2$ into (41), we obtain a system of ODEs for determining the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$:

$$\varphi'_t - 6a\varphi^2 = 0, \quad \psi'_t - \frac{15}{4}a\varphi\psi = 0.$$

The general solution of this system is

$$\varphi(t) = -\frac{1}{6a(t + C_1)}, \quad \psi(t) = \frac{C_2}{|t + C_1|^{5/8}}. \tag{43}$$

Given the formulas (40), (42), (43), as a result, we obtain two three-parameter generalized separable solutions of Equation (14):

$$u = -\frac{1}{6a(t + C_1)}(x + C_3)^2 + \frac{C_2}{|t + C_1|^{1/3}},$$

$$u = -\frac{1}{6a(t + C_1)}(x + C_3)^2 + \frac{C_2}{|t + C_1|^{5/8}}(x + C_3)^{1/2},$$

where for the sake of greater generality, an arbitrary translation in x is additionally added.

Remark 4. *The wave type equation with quadratic nonlinearity*

$$u_{tt} = a(uu_x)_x,$$

also admits solutions of the form (40) with $k = 0$ and $k = 1/2$.

Example 14. Let us now return to the Goderley Equation (17). This equation admits the simple exact solution (18), which we write in the form

$$u = f(x)y^3, \quad f(x) = 1/(3ax^2).$$

We will look for more complex solutions (with generalized separation of variables) Equation (17) in the form

$$u(x, y) = \varphi(x)y^k + \psi(x), \tag{44}$$

where the functions $\varphi(x)$ and $\psi(x)$ and the constant $k \neq 0$ are determined in the subsequent analysis (solution (18) is a particular case of solution (44) for $k = 3$ and $\psi = 0$).

It is important to note that binomial solutions of the form (44) are quite often encountered in practice and are the simplest generalized separable solutions of nonlinear PDEs.

Substituting (44) in (17), after rearranging the terms, we come to the relation

$$\varphi''_{xx}y^k - ak^2(k - 1)\varphi^2y^{2k-3} + \psi''_{xx} = 0, \tag{45}$$

which contains the power functions y^k and y^{2k-3} and must be satisfied identically for any y .

Consider two cases: $\psi''_{xx} = 0$ and $\psi''_{xx} \neq 0$.

1°. *The first case.* When $\psi''_{xx} = 0$ we get a binomial equation with separable variables, which can be satisfied if we set

$$k = 3, \quad \varphi''_{xx} - 18a\varphi^2 = 0. \tag{46}$$

The general solution of the autonomous ODE (46) can be represented in the implicit form

$$x = \pm \int (12a\varphi^3 + C_1)^{-1/2} d\varphi + C_2.$$

Moreover, this equation admits a particular solution of the power form $\varphi = \frac{1}{3a}(x + C_1)^{-2}$, which leads to a three-parameter exact solution of Equation (17):

$$u = \frac{1}{3a}(x + C_1)^{-2}y^3 + C_2x + C_3, \tag{47}$$

where $C_1, C_2,$ and C_3 are arbitrary constants.

2°. *Second case.* To balance the function $\psi''_{xx} \neq 0$ with second term in equality (45), we must set $k = 3/2$. As a result, we obtain a binomial equation, which can be satisfied by setting

$$\varphi''_{xx} = 0, \quad \psi''_{xx} = \frac{9}{8}a\varphi^2.$$

These equations are easily integrated and lead to a four-parameter exact solution of Equation (17):

$$u = (C_1x + C_2)y^{3/2} + \frac{3a}{32C_1^2}(C_1x + C_2)^4 + C_3x + C_4, \tag{48}$$

where $C_1, C_2, C_3,$ and C_4 are arbitrary constants.

Example 15. Let us return to the hydrodynamic boundary layer Equation (21). It is easy to verify that this equation admits the self-similar solution [51]:

$$u = F(\xi), \quad \xi = y/x, \tag{49}$$

where the function $F = F(\xi)$ satisfies the third-order ODE: $-(F'_z)^2 = \nu F'''_{zzz}$.

We look for a more general solution of Equation (21) by adding the function $\varphi(x)$ to (49):

$$u = F(\xi) + \varphi(x), \quad \xi = y/x.$$

Simple calculations show that $\varphi(x) = a \ln x$, where a is an arbitrary constant. As a result, we obtain a non-self-similar solution of the boundary layer Equation (21) of the form [13]:

$$u = F(\xi) + a \ln x, \quad \xi = y/x,$$

where the function $F = F(\xi)$ is described by the third-order ODE: $-(F'_z)^2 - aF''_{zz} = \nu F'''_{zzz}$.

3.2. Construction of Compound Solutions (Nonlinear Superposition of Solutions)

In some cases, two similar but different solutions of the considered nonlinear PDE can be combined to obtain a more general composite solution. We demonstrate the possible course of reasoning in such cases by examples of the Guderley equation and the nonlinear diffusion equation with a second-order volume reaction.

Example 16. From expressions (47) and (48), it follows that the Guderley Equation (17) has two solutions of the same type $u_1 = \varphi y^{3/2} + \psi$ and $u_2 = \varphi y^3 + \psi$, which differ from each other by the exponent y . This circumstance suggests an attempt to construct a more general solution of Equation (17), that includes both terms with different exponents at once. In other words, we are looking for a composite solution of the form

$$u(x, y) = \varphi_1(x)y^3 + \varphi_2(x)y^{3/2} + \psi(x). \tag{50}$$

Substitute it in Equation (17). After combining the functional factors for power-functions $y^{3n/2}$ ($n = 0, 1, 2$), we get

$$(\varphi''_1 - 18a\varphi_1^2)y^3 + (\varphi''_2 - \frac{45}{4}a\varphi_1\varphi_2)y^{3/2} + \psi'' - \frac{9}{8}a\varphi_2^2 = 0.$$

For this equality to hold for any y , it is necessary to equate the functional factors of $y^{3n/2}$ to zero. As a result, we arrive at the system of ODEs:

$$\begin{aligned} \varphi_1'' - 18a\varphi_1^2 &= 0, \\ \varphi_2'' - \frac{45}{4}a\varphi_1\varphi_2 &= 0, \\ \psi'' - \frac{9}{8}a\varphi_2^2 &= 0. \end{aligned} \tag{51}$$

Thus, it is constructively proved that Equation (17) admits a solution of the form (50) (this solution was obtained in [52]).

It can be shown that the system (51) admits the exact solution

$$\begin{aligned} \varphi_1 &= \frac{1}{3a}(x + C_1)^{-2}, \\ \varphi_2 &= C_2(x + C_1)^{5/2} + C_3(x + C_1)^{-3/2}, \\ \psi &= \frac{3a}{112}C_2^2(x + C_1)^7 + \frac{3}{8}aC_2C_3(x + C_1)^3 + \frac{9}{16}aC_3^2(x + C_1)^{-1} + C_4x + C_5. \end{aligned}$$

Example 17. Let us now consider a nonlinear diffusion equation with the second-order volume reaction

$$u_t = a(uu_x) - bu^2. \tag{52}$$

The procedure for constructing a composite solution of this equation will be carried out in two stages: first, we will find two fairly simple solutions, and then, using these solutions, we will construct a composite solution.

1°. *Solutions of exponential form in x .* Exact generalized separable solutions of Equation (52) are sought in the form

$$u(x, t) = \varphi(t)e^{\lambda x} + \psi(t), \tag{53}$$

where functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ and the constant λ are to be determined in the subsequent analysis. Substituting (53) in (52) and collecting similar terms at exponents $e^{n\lambda x}$ ($n = 0, 1, 2$), we get

$$(b - 2a\lambda^2)\varphi^2e^{2\lambda x} + [\varphi'_t + (2b - a\lambda^2)\varphi\psi]e^{\lambda x} + \psi'_t + b\psi^2 = 0.$$

Since this equality must be satisfied identically for any x , the functional factors of $e^{n\lambda x}$ must be equated to zero. As a result, we come to the differential–algebraic system

$$\begin{aligned} b - 2a\lambda^2 &= 0, \\ \varphi'_t + (2b - a\lambda^2)\varphi\psi &= 0, \\ \psi'_t + b\psi^2 &= 0, \end{aligned}$$

which allows two solutions

$$\lambda = \pm \left(\frac{b}{2a}\right)^{1/2}, \quad \varphi = \frac{C_1}{|t + C_2|^{3/2}}, \quad \psi = \frac{1}{b(t + C_2)}, \tag{54}$$

where C_1 and C_2 are arbitrary constants.

2°. *Composite solution of exponential form in x .* From relations (53) and (54), it follows that Equation (52) has two solutions $u_{1,2} = \varphi e^{\pm\lambda x} + \psi$. They differ in structure from each other only by the sign of the exponent λ .

This circumstance suggests trying to construct a more general solution of Equation (52), which includes both exponential terms at once. In other words, we are looking for a composite solution of the form

$$u(x, t) = \varphi_1(t)e^{-\lambda x} + \varphi_2(t)e^{\lambda x} + \psi(t), \quad \lambda = \left(\frac{b}{2a}\right)^{1/2}. \tag{55}$$

Substituting (55) in (52), after elementary transformations we have

$$[(\varphi_1)'_t + \frac{3}{2}b\varphi_1\psi]e^{-\lambda x} + [(\varphi_2)'_t + \frac{3}{2}b\varphi_2\psi]e^{\lambda x} + \psi'_t + b(2\varphi_1\varphi_2 + \psi^2) = 0.$$

Equating the functional factors of $e^{n\lambda x}$ ($n = 0, \pm 1$) to zero, we arrive at the first-order ODE system

$$\begin{aligned} (\varphi_1)'_t + \frac{3}{2}b\varphi_1\psi &= 0, \\ (\varphi_2)'_t + \frac{3}{2}b\varphi_2\psi &= 0, \\ \psi'_t + b(2\varphi_1\varphi_2 + \psi^2) &= 0. \end{aligned} \tag{56}$$

Thus, it has been proved that Equation (52) admits the solution of the form (55).

By excluding ψ from the first two equations in (56), we obtain the equality $(\varphi_1)'_t/\varphi_1 = (\varphi_2)'_t/\varphi_2$. This implies that $\varphi_1 = A\varphi(t)$, $\varphi_2 = B\varphi(t)$, where A and B are arbitrary constants. Therefore, the generalized separable solution (55) is reduced to the form

$$u(x, t) = \varphi(t)(Ae^{-\lambda x} + Be^{\lambda x}) + \psi(t), \quad \lambda = \left(\frac{b}{2a}\right)^{1/2}, \tag{57}$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are described by the nonlinear system of two ODEs:

$$\begin{aligned} \varphi'_t + \frac{3}{2}b\varphi\psi &= 0, \\ \psi'_t + b(2AB\varphi^2 + \psi^2) &= 0. \end{aligned} \tag{58}$$

By excluding t , this autonomous system is reduced to one ODE, which is homogeneous and therefore can be integrated [53]. Note that the system of Equation (58) for $AB > 0$ admits two simple solutions

$$\varphi = \pm \frac{1}{3b\sqrt{AB}(t + C)}, \quad \psi = \frac{2}{3b(t + C)},$$

which define the solution (57) in the form of a product of functions of different arguments.

3°. *Solution of trigonometric type in x .* When writing formulas (55) and (57) implicitly, it was assumed that $ab > 0$. For $ab < 0$, we have

$$\lambda = i\beta, \quad \beta = \left(-\frac{b}{2a}\right)^{1/2}, \quad i^2 = -1.$$

In this case, in solution (57) instead of exponential functions, trigonometric functions appear, i.e., it can be represented in the form

$$u(x, t) = \varphi(t)[A_1 \cos(\beta x) + B_1 \sin(\beta x)] + \psi(t), \quad \beta = \left(-\frac{b}{2a}\right)^{1/2}, \tag{59}$$

where A_1 and B_1 are arbitrary constants. Substituting (59) into Equation (52) and performing calculations similar to those in Item 2°, we obtain the following nonlinear system of ODEs for the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$:

$$\begin{aligned} \varphi'_t + \frac{3}{2}b\varphi\psi &= 0, \\ \psi'_t + b[\frac{1}{2}(A_1^2 + B_1^2)\varphi^2 + \psi^2] &= 0. \end{aligned} \tag{60}$$

This system allows for two simple solutions

$$\varphi = \pm \frac{2}{3b\sqrt{A_1^2 + B_1^2}(t + C)}, \quad \psi = \frac{2}{3b(t + C)},$$

which determine the solution (59) in the form of a product of functions of different arguments.

4. The Use of Complex-Valued Parameters for Constructing Exact Solutions

4.1. Linear Partial Differential Equations

In the case of linear partial differential equations, the following proposition can be used to construct more complex solutions from simpler solutions.

Proposition 4. *Let a linear homogeneous PDE with two independent variables x and t have a one-parameter solution of the form $u = \varphi(x, t, c)$, where c is a parameter that is not included in the original equation. Then the considered equation also has two two-parameter solutions*

$$u_1 = \operatorname{Re} \varphi(x, t, a + ib), \quad u_2 = \operatorname{Im} \varphi(x, t, a + ib), \tag{61}$$

where a and b are arbitrary real constants, $\operatorname{Re} z$ and $\operatorname{Im} z$ are real and imaginary parts of complex number z .

Proof. The validity of the proposition follows from the linearity of the equation and from the fact that the solution $u = \varphi(x, t, c)$ is also a solution for $c = a + ib$. \square

Proposition 4 implies the validity of the following two consequences:

Corollary 1. *Let a linear homogeneous PDE not depend explicitly on the independent variable t and have a solution $u = \varphi(x, t)$. Then this equation also has two one-parameter families of solutions*

$$u_1 = \operatorname{Re} \varphi(x, t + ia), \quad u_2 = \operatorname{Im} \varphi(x, t + ia),$$

where a is an arbitrary real constant.

Corollary 2. *Let a linear homogeneous PDE not depend explicitly on the independent variable x and have a solution $u = \varphi(x, t)$. Then this equation also has two one-parameter families of solutions*

$$u_1 = \operatorname{Re} \varphi(x + ia, t), \quad u_2 = \operatorname{Im} \varphi(x + ia, t),$$

where a is an arbitrary real constant.

Example 18. Consider the linear heat equation

$$u_t - u_{xx} = 0. \tag{62}$$

It is easy to verify that this equation admits an exact solution of the exponential form

$$u = \exp(c^2t + cx),$$

where c is an arbitrary parameter.

Using Proposition 4, we obtain two more complicated two-parameter families exact solutions of Equation (62):

$$\begin{aligned} u_1 &= \operatorname{Re} \exp(c^2t + cx)|_{c=a+ib} = \exp[(a^2 - b^2)t + ax] \cos[b(2at + x)], \\ u_2 &= \operatorname{Im} \exp(c^2t + cx)|_{c=a+ib} = \exp[(a^2 - b^2)t + ax] \sin[b(2at + x)]. \end{aligned}$$

Example 19. Consider the linear wave equation

$$u_{tt} - u_{xx} = 0. \tag{63}$$

It is easy to verify that Equation (63) admits translation transformations for both independent variables and has the particular solution

$$u = \frac{x}{x^2 - t^2}. \tag{64}$$

Making a translation in solution (64) with an imaginary parameter in t and using Corollary 1, we find two more complicated one-parameter families of solutions to Equation (63):

$$u_1 = \operatorname{Re} \frac{x}{x^2 - (t + ia)^2} = \frac{x(x^2 - t^2 + a^2)}{(x^2 - t^2 + a^2)^2 + 4a^2t^2},$$

$$u_2 = \operatorname{Im} \frac{x}{x^2 - (t + ia)^2} = \frac{2axt}{(x^2 - t^2 + a^2)^2 + 4a^2t^2}.$$

Making a translation in solution (64) with an imaginary parameter in x and using Corollary 2, we find two other one-parameter families of solutions:

$$u_3 = \operatorname{Re} \frac{x + ia}{(x + ia)^2 - t^2} = \frac{x(x^2 - t^2 + a^2)}{(x^2 - t^2 - a^2)^2 + 4a^2x^2},$$

$$u_4 = \operatorname{Im} \frac{x + ia}{(x + ia)^2 - t^2} = \frac{a(x^2 + t^2 + a^2)}{(x^2 - t^2 - a^2)^2 + 4a^2x^2}.$$

Example 20. Consider the linear heat equation

$$u_t = u_{xx} + \frac{1}{x}u_x, \tag{65}$$

which describes two-dimensional processes with axial symmetry, where x is the radial coordinate. It is easy to verify that Equation (65) admits a translation transformation with respect to the variable t and has the particular solution

$$u = \frac{1}{t} \exp\left(-\frac{x^2}{4t}\right). \tag{66}$$

Making a translation in solution (66) with an imaginary parameter in the variable t and using Corollary 1, we find two more complicated one-parameter families of solutions:

$$u_3 = \operatorname{Re} \frac{1}{t + ia} \exp\left(-\frac{x^2}{4(t + ia)}\right) =$$

$$= \frac{1}{t^2 + a^2} \exp\left(-\frac{x^2t}{4(t^2 + a^2)}\right) \left(t \cos \frac{ax^2}{4(t^2 + a^2)} + a \sin \frac{ax^2}{4(t^2 + a^2)}\right),$$

$$u_4 = \operatorname{Im} \frac{1}{t + ia} \exp\left(-\frac{x^2}{4(t + ia)}\right) =$$

$$= \frac{1}{t^2 + a^2} \exp\left(-\frac{x^2t}{4(t^2 + a^2)}\right) \left(a \cos \frac{ax^2}{4(t^2 + a^2)} - t \sin \frac{ax^2}{4(t^2 + a^2)}\right).$$

Example 21. Consider the linear wave equation with variable coefficients

$$u_{tt} - (xu_x)_x = 0. \tag{67}$$

This equation admits a translation transformation with respect to t and has the exact solution

$$u = \frac{Ct}{(4x - t^2)^{3/2}}, \tag{68}$$

where C is an arbitrary constant.

By making a translation in solution (68) an imaginary parameter in t and using the Corollary 1, we can find two more complicated one-parameter families of solutions by formulas

$$u_1 = \operatorname{Re} \frac{C(t + ia)}{(4x - (t + ia)^2)^{3/2}}, \quad u_2 = \operatorname{Im} \frac{C(t + ia)}{(4x - (t + ia)^2)^{3/2}}. \tag{69}$$

The final form of these solutions is not presented here, due to the cumbersomeness of their recording. The solution u_1 was obtained in [54] and was used to describe the propagation of localized disturbances in one-dimensional shallow water over an inclined bottom. Note, that in [55], another exact solution of the Equation (67) was obtained by integrating the parameter a .

Proposition 5. *Let a linear homogeneous PDE have a one-parameter solution of the form $u = \varphi(x, t, c)$, where c is a real parameter that is not included in the equation. Then, by n -fold differentiation or integration of this solution, one can obtain other exact solutions of the considered equation [56,57].*

Corollary 3. *Exact solutions of linear PDEs, which do not explicitly depend on the independent variable t , can be constructed by differentiating or/and integrating with respect to parameters a and b in solutions (61) that are obtained by introducing the complex parameter $c = a + ib$.*

Remark 5. *In [55], by integrating formulas (69) with parameter a , it was obtained a new solution of Equation (67).*

4.2. Nonlinear Partial Differential Equations

1°. In some cases, it is possible to obtain another solution from one solution, passing from real parameters to complex ones in such a way that the transformed equation and the solution remain real. Let us explain this with a few examples.

Example 22. Let us return again to Equation (52). It is easy to verify that its trigonometric solution (59) and the system of Equation (60) can be obtained from the solution exponential form (57) and system of Equation (58), if in the latter we formally set

$$e^{\lambda x} = e^{i\beta x} = \cos(\beta x) + i \sin(\beta x), \quad e^{-\lambda x} = e^{-i\beta x} = \cos(\beta x) - i \sin(\beta x), \tag{70}$$

$$A = \frac{1}{2}(A_1 + iB_1), \quad B = \frac{1}{2}(A_1 - iB), \quad A_1 = A + B, \quad B_1 = i(B - A).$$

Example 23. Consider the equation

$$u_t = au_{xx} + uf(u_x^2 - bu^2), \tag{71}$$

where $f(w)$ is an arbitrary function.

It is easy to verify that Equation (71) has the simple multiplicative separable solution (exponential in x):

$$u = \psi(t)e^{\lambda x}, \tag{72}$$

where the parameter λ and the function $\psi(t)$ are to be determined in the subsequent analysis. Substituting (72) in (71), we obtain two solutions of the form (72), where

$$\lambda = \pm\sqrt{b}, \quad \psi'_t = [ab + f(0)]\psi. \tag{73}$$

The presence of two solutions of the same type corresponding to $\pm\lambda$ suggests trying them ‘combined’ and looking for a more general composite solution of the form

$$u = \psi(t)(Ae^{\lambda x} + Be^{-\lambda x}), \tag{74}$$

where A and B are some constants. Substituting (74) in Equation (71), we obtain a solution of the form (74), where A and B are arbitrary constants, and the function $\psi = \psi(t)$ satisfies the nonlinear ODE:

$$\psi'_t = ab\psi + \psi f(-4ABb\psi^2). \tag{75}$$

Substituting (70) into (74) and (75), we arrive at a new solution containing already trigonometric functions in x ,

$$u = \varphi(t)[A_1 \cos(\beta x) + B_1 \sin(\beta x)], \quad \beta = \sqrt{-b},$$

where A_1 and B_1 are arbitrary constants, and the function $\psi = \psi(t)$ is described by a nonlinear ODE:

$$\psi'_t = ab\psi + \psi f(-(A_1^2 + B_1^2)b\psi^2).$$

Remark 6. Exact solutions of the nonlinear hyperbolic equation

$$u_{tt} = au_{xx} + u f(u_x^2 - bu^2)$$

are constructed in the same way.

2°. Exact solutions of some nonlinear PDEs can be obtained using the proposition below.

Proposition 6. Let a nonlinear PDE have an exact solution involving trigonometric functions of the form

$$u = F(x, t, A \cos(\beta x) + B \sin(\beta x), \beta^2), \tag{76}$$

where A , B , and β are free real parameters that are not included in the considered equation. Then this equation also has the exact solution involving hyperbolic functions:

$$u = F(x, t, \bar{A} \cosh(\lambda x) + \bar{B} \sinh(\lambda x), -\lambda^2), \tag{77}$$

where \bar{A} , \bar{B} , λ are free real parameters. The converse is also true: if an equation has the exact solution (77), then it also has the exact solution (76).

Solution (77) is obtained from (76) by renaming the parameters $\beta = i\lambda$, $A = \bar{A}$, $B = -i\bar{B}$, $i^2 = -1$.

Example 24. Consider the fourth-order nonlinear equation

$$u_y(\Delta u)_x - u_x(\Delta u)_y = \nu \Delta \Delta u, \quad \Delta u = u_{xx} + u_{yy}, \tag{78}$$

to which the stationary Navier–Stokes equations are reduced in the planar case [58].

Equation (78) has the exact solution

$$u(x, y) = [\bar{A} \sinh(\lambda x) + \bar{B} \cosh(\lambda x)]e^{-\gamma y} + \frac{\nu}{\gamma}(\gamma^2 + \lambda^2)x.$$

Therefore, this equation also has the exact solution

$$u(x, y) = [A \sin(\beta x) + B \cos(\beta x)]e^{-\gamma y} + \frac{\nu}{\gamma}(\gamma^2 - \beta^2)x.$$

These solutions and other examples of this kind can be found in [13].

5. Using Solutions of Simpler Equations for Construct Solutions to Complex Equations

Preliminary remarks. It is often possible to use solutions of simpler equations to construct exact solutions to complex differential equations. In this section, we will illustrate the reasoning in such cases for nonlinear PDEs (see Section 5.1), as well as for more complex nonlinear partial functional differential equations (see Sections 5.2–5.4).

5.1. Nonlinear Partial Differential Equations

The following example shows how precise solutions of nonlinear reaction–diffusion equations can be used to generate exact solutions to wave type equations.

Example 25. Consider the reaction–diffusion equation with quadratic nonlinearity

$$u_t = a(uu_x)_x + bu, \quad (79)$$

which admits several simple exact solutions, which are given below and expressed in elementary functions (see, for example [13]).

1°. The additive separable solution:

$$u = -\frac{b}{6a}x^2 + \psi(t), \quad (80)$$

where $\psi(t) = C \exp(\frac{2}{3}bt)$ and C is an arbitrary constant.

2°. The multiplicative separable solution:

$$u = \psi(t)x^2, \quad (81)$$

where $\psi(t) = -be^{bt}(6ae^{bt} + C)^{-1}$ and C is an arbitrary constant.

3°. The generalized separable solution:

$$u = \psi_1(t)x^2 + \psi_2(t), \quad (82)$$

where $\psi_1(t) = -be^{bt}(6ae^{bt} + C_1)^{-1}$ and $\psi_2(t) = C_2e^{bt}(6ae^{bt} + C_1)^{-1/3}$, and C_1 and C_2 are arbitrary constants.

4°. The generalized separable solution:

$$u = \psi_1(t)x^2 + \psi_2(t)\sqrt{x}, \quad (83)$$

where $\psi_1(t) = -be^{bt}(6ae^{bt} + C_1)^{-1}$ and $\psi_2(t) = C_2e^{bt}(6ae^{bt} + C_1)^{-5/8}$, and C_1 and C_2 are arbitrary constants.

Let us now consider a nonlinear wave type equation with a quadratic nonlinearity of the form

$$u_{tt} = a(uu_x)_x + bu. \quad (84)$$

Equations (79) and (84) differ only in the order of the derivative with respect to t in the left parts of the equations. Since the right-hand sides of these equations involving derivatives with respect to x are the same, it is natural to assume that the power structure of solutions with respect to x of both equations will also be the same, and only the functional factors that depend on t will change for different powers of x .

In other words, we look for exact solutions of wave type PDE (84) in the same form as solutions of reaction–diffusion PDE (79). As a result, we get the following four exact solutions of PDE (84):

1°. The additive separable solution of the form (80), where the function $\psi = \psi(t)$ is described by the ODE:

$$\psi''_{tt} = -\frac{1}{3}b\psi + b\psi.$$

2°. The multiplicative separable solution of the form (81), where the function $\psi = \psi(t)$ is described by the ODE:

$$\psi''_{tt} = 6a\psi^2 + b\psi.$$

3°. The generalized separable solution of the form (82), where the functions $\psi_1 = \psi_1(t)$ and $\psi_2 = \psi_2(t)$ are described by the ODEs:

$$\begin{aligned} \psi''_1 &= 6a\psi_1^2 + b\psi_1, \\ \psi''_2 &= 2a\psi_1\psi_2 + b\psi_2. \end{aligned}$$

4°. The generalized separable solution of the form (83), where the functions $\psi_1 = \psi_1(t)$ and $\psi_2 = \psi_2(t)$ are described by the ODEs:

$$\begin{aligned} \psi''_1 &= 6a\psi_1^2 + b\psi_1, \\ \psi''_2 &= \frac{15}{4}a\psi_1\psi_2 + b\psi_2. \end{aligned}$$

The considered example is a good illustration of a rather general fact, which is a consequence of the results of [15] (see also [13,14]) and can be formulated as the following proposition.

Proposition 7. *Let the evolution partial differential equation*

$$u_t = F[u], \tag{85}$$

where $F[u] \equiv F(u, u_x, \dots, u_x^{(n)})$ is the nonlinear differential operator in x , has a generalized separable solution of the form

$$u = \sum_{k=1}^m \psi_k(t) \varphi_k(x). \tag{86}$$

Then, the more complex partial differential equation

$$L_1[u] = L_2[w], \quad w = F[u], \tag{87}$$

where L_1 and L_2 are any linear differential operators in t ,

$$L_1[u] = \sum_{i=0}^k a_i(t)u_t^{(i)}, \quad L_2[w] = \sum_{j=0}^m b_j(t)w_t^{(j)},$$

also has the generalized separable solution of the form (86) with the same functions $\varphi_k(x)$ (but with other functions $\psi(t)$).

Remark 7. *In the Equations (85) and (87), the nonlinear operator F can explicitly depend on the variables x and t .*

Let us now give an example of constructing an exact solution that cannot be obtained by using Proposition 7.

Example 26. Consider the n th-order nonlinear PDE:

$$u_{tt} = uF(u_x/u, u_{xx}/u, \dots, u_x^{(n)}/u), \tag{88}$$

which differs from (32) only in the order of the derivative with respect to t on the left part of the equation.

We look for the solution of Equation (88) in the same form as the solution of Equation (32). Substituting (29) in (88), for the function $\varphi = \varphi(z)$ we obtain a nonlinear ODE:

$$k^2\varphi + 2kq\varphi'_z + q^2\varphi''_{zz} = \varphi F(p\varphi'_z/\varphi, p^2\varphi''_{zz}/\varphi, \dots, p^n\varphi_z^{(n)}/\varphi).$$

5.2. Partial Differential Equations with Delay

In biology, biophysics, biochemistry, chemistry, medicine, control theory, climate model theory, ecology, economics, and many other areas there are nonlinear systems, the rate of change of parameters of which depends not only on the current state of the system at a given time, but also on the state system at some previous time [59]. The differential equations that describe such processes, in addition to the unknown function $u = u(x, t)$ also include the function $w = u(x, t - \tau)$, where t is time, $\tau > 0$ is the constant delay. In some cases, we consider situations where the delay depends on the time, $\tau = \tau(t)$.

The presence of a delay significantly complicates the analysis of such equations. Although nonlinear PDEs with constant delay allow solutions of the traveling wave type $u = u(z)$, where $z = x + \lambda t$ (see, for example, [59–62]), they do not allow self-similar solutions of the form $u = t^\beta \varphi(xt^\lambda)$, which often have simpler PDEs without delay.

More complex than traveling wave solutions, exact solutions of nonlinear reaction–diffusion type equations with delay were obtained in [63–72]. Exact solutions of nonlinear Klein–Gordon type equations with delay and related nonlinear hyperbolic equations are given in [71–76]. Below, with specific examples, we will show how exact solutions of nonlinear delay PDEs can be found by using solutions of simpler PDEs without delay.

Example 27. Let us consider a nonlinear reaction–diffusion equation with a constant delay,

$$u_t = a(uu_x)_x + bw, \quad w = u(x, t - \tau). \tag{89}$$

Equation (89) is more complicated than the ODE without delay (79) and goes into it at $\tau = 0$. The presence of the delay in (89) does not affect the nonlinear term containing derivatives in x . Therefore, we can assume that the power structure of solutions in x of both equations will be the same, and only the functional factors that depend on t will change.

In other words, we look for exact solutions PDE with delay (89) in the same form, as solutions simpler PDE without delay (79). As a result, we get the following four exact solutions of the nonlinear delay PDE (89):

1°. The additive separable solution of the form (80), where the function $\psi = \psi(t)$ is described by the linear delay ODE:

$$\psi'_t = -\frac{1}{3}b\psi + b\bar{\psi}, \quad \bar{\psi} = \psi(t - \tau).$$

2°. The multiplicative separable solution of the form (81), where the function $\psi = \psi(t)$ is described by the nonlinear delay ODE:

$$\psi'_t = 6a\psi^2 + b\bar{\psi}, \quad \bar{\psi} = \psi(t - \tau).$$

3°. The generalized separable solution of the form (82), where the functions $\psi_1 = \psi_1(t)$ and $\psi_2 = \psi_2(t)$ are described by the delay ODEs:

$$\begin{aligned} \psi'_1 &= 6a\psi_1^2 + b\bar{\psi}_1, & \bar{\psi}_1 &= \psi_1(t - \tau), \\ \psi'_2 &= 2a\psi_1\psi_2 + b\bar{\psi}_2, & \bar{\psi}_2 &= \psi_2(t - \tau). \end{aligned}$$

4°. The generalized separable solution of the form (83), where the functions $\psi_1 = \psi_1(t)$ and $\psi_2 = \psi_2(t)$ are described by the delay ODEs:

$$\begin{aligned} \psi_1' &= 6a\psi_1^2 + b\bar{\psi}_1, & \bar{\psi}_1 &= \psi_1(t - \tau), \\ \psi_2' &= \frac{15}{4}a\psi_1\psi_2 + b\bar{\psi}_2, & \bar{\psi}_2 &= \psi_2(t - \tau). \end{aligned}$$

Example 28. More complex than (89), nonlinear PDE with variable delay

$$u_t = a(uu_x)_x + bw, \quad w = u(x, t - \tau(t)),$$

where $\tau(t)$ is an arbitrary function, also admits four exact solutions of the form (80)–(83).

Example 29. The reaction–diffusion equation with logarithmic nonlinearity

$$u_t = au_{xx} + u(b \ln u + c), \tag{90}$$

admits the exact functional separable solution [15]:

$$u(x, t) = \exp[\psi_2(t)x^2 + \psi_1(t)x + \psi_0(t)], \tag{91}$$

where the functions $\psi_n = \psi_n(t)$ are described by the nonlinear system of ODEs:

$$\begin{aligned} \psi_2' &= 4a\psi_2^2 + b\psi_2, \\ \psi_1' &= 4a\psi_1\psi_2 + b\psi_1, \\ \psi_0' &= a(\psi_1^2 + 2\psi_2) + b\psi_0 + c. \end{aligned}$$

Let us now consider a more complex nonlinear reaction–diffusion equation with a constant delay,

$$u_t = au_{xx} + u(b \ln w + c), \quad w = u(x, t - \tau). \tag{92}$$

PDE with delay (92) in the special case $\tau = 0$ passes into the simpler PDE without delay (90). For $\tau = 0$ the solution of delay PDE (92), as for Equation (90), is sought in the form (91). As a result, for the functions $\psi_n = \psi_n(t)$, we obtain the nonlinear system delay ODEs:

$$\begin{aligned} \psi_2' &= 4a\psi_2^2 + b\bar{\psi}_2, & \bar{\psi}_2 &= \psi_2(t - \tau), \\ \psi_1' &= 4a\psi_1\psi_2 + b\bar{\psi}_1, & \bar{\psi}_1 &= \psi_1(t - \tau), \\ \psi_0' &= a(\psi_1^2 + 2\psi_2) + b\bar{\psi}_0 + c, & \bar{\psi}_0 &= \psi_0(t - \tau). \end{aligned}$$

Example 30. More complex than (92), nonlinear PDE with variable delay

$$u_t = au_{xx} + u(b \ln w + c), \quad w = u(x, t - \tau(t)),$$

where $\tau(t)$ is an arbitrary function, also admits a solution with functional separation of variables of the form (91) [64].

Remark 8. In [71,72], methods for constructing generalized traveling-wave solutions for nonlinear PDEs with delay by using exact solutions more simple PDEs without delay were proposed. This method is suitable for constructing exact solutions in both explicit and implicit form.

5.3. Pantograph-Type Partial Differential Equations

In this section, we will consider functional-differential equations with partial derivatives of the pantograph type, which in addition to the unknown $u = u(x, t)$, also contain the same functions with dilated or contracted arguments, $w = u(px, qt)$, where p and q

are scaling parameters (for equations with variable delay we have $0 < p < 1, 0 < q < 1$). Pantograph-type ODEs and PDEs are used for mathematical modeling of various processes in engineering [77], biology [78–82], astrophysics [83], electrodynamics [84], the theory of populations [85], number theory [86], stochastic games [87], graph theory [88], risk and queuing theories [89], the theory of neural networks [90].

Below, with specific examples, it is shown that the exact solutions of nonlinear pantograph-type PDEs can be found by using simpler ‘ordinary’ PDEs, which do not contain the unknown function with dilated or contracted arguments.

Example 31. Let us first consider a pantograph-type reaction–diffusion equation with quadratic nonlinearity,

$$u_t = a(uu_x)_x + bw, \quad w = u(px, qt). \tag{93}$$

Equation (93) is more complicated than the ODE without argument scaling (79) and passes into it at $p = q = 1$. The presence in (93) of dilation in w does not affect the nonlinear term containing derivatives with respect to x . Therefore, we can assume that the power structure of solutions in x of both equations will be the same, and only the functional factors that depend on t will change (and for the additive separate solution, a factor in x^2 will change).

In other words, we look for exact solutions of pantograph-type PDE (93) in the same form as solutions of ‘ordinary’ simpler PDE (79). As a result, we get the following four exact pantograph-type solutions (93):

1°. The additive separable solution is

$$u = -\frac{bp^2}{6a}x^2 + \psi(t); \quad \psi'_t = -\frac{1}{3}bp^2\psi + b\bar{\psi}, \quad \bar{\psi} = \psi(qt).$$

2°. The multiplicative separable solution has the form (81), where the function $\psi = \psi(t)$ is described by the pantograph-type ODE:

$$\psi'_t = 6a\psi^2 + bp^2\bar{\psi}, \quad \bar{\psi} = \psi(qt).$$

3°. The generalized separable solution has the form (82), where the functions $\psi_1 = \psi_1(t)$ and $\psi_2 = \psi_2(t)$ are described by the pantograph-type ODEs:

$$\begin{aligned} \psi'_1 &= 6a\psi_1^2 + bp^2\bar{\psi}_1, & \bar{\psi}_1 &= \psi_1(qt), \\ \psi'_2 &= 2a\psi_1\psi_2 + b\bar{\psi}_2, & \bar{\psi}_2 &= \psi_2(qt). \end{aligned}$$

4°. The generalized separable solution has the form (83), where the functions $\psi_1 = \psi_1(t)$ and $\psi_2 = \psi_2(t)$ are described by the pantograph-type ODEs:

$$\begin{aligned} \psi'_1 &= 6a\psi_1^2 + bp^2\bar{\psi}_1, & \bar{\psi}_1 &= \psi_1(qt), \\ \psi'_2 &= \frac{15}{4}a\psi_1\psi_2 + b\sqrt{p}\bar{\psi}_2, & \bar{\psi}_2 &= \psi_2(qt). \end{aligned}$$

Example 32. The reaction–diffusion equation with power-law nonlinearity

$$u_t = au_{xx} + bu^k, \tag{94}$$

for $k \neq 1$ admits a self-similar solution [25]:

$$u(x, t) = t^{\frac{1}{1-k}} U(z), \quad z = xt^{-1/2}, \tag{95}$$

where the function $U = U(z)$ is described by the nonlinear ODE:

$$\frac{1}{1-k}U - \frac{1}{2}zU'_z = aU''_{zz} + U^k,$$

Let us now consider a much more complex nonlinear partial functional-differential equation of the pantograph-type

$$u_t = au_{xx} + bw^k, \quad w = u(px, qt), \tag{96}$$

where p and q are free parameters ($p > 0, q > 0$). Parameter values $0 < p < 1$ and $0 < q < 1$ correspond to equations with proportional delay in two arguments.

The functional-differential Equation (96) in the special case $p = q = 1$ passes into the ‘ordinary’ partial differential Equation (94). For $k \neq 1$ the solution of the pantograph-type PDE (96), as for Equation (94), is sought in the form (95). As a result, for the function $U = U(z)$, we obtain a nonlinear ODE of the pantograph-type [91]:

$$\frac{1}{1-k}U - \frac{1}{2}zU'_z = aU''_{zz} + bq^{\frac{k}{1-k}}W^k, \quad W = U(sz), \quad s = pq^{-1/2}. \tag{97}$$

Remark 9. Equation (96) with proportional delays for $0 < p, q < 1$ in the special case $p = q^{1/2}$ has an exact solution expressed in terms of the solution of the ODE without delay (97) with $s = 1$; for $p < q^{1/2}$, Equation (96) reduces to the delay ODE with $s < 1$; and for $p > q^{1/2}$, to the ODE with contracted argument for $s > 1$. Moreover, a solution of the ODE (96) for $p, q > 1$ for appropriate values of the parameters p and q can also be expressed in terms of the solution of the ODE with delay ($s < 1$), without delay ($s = 1$), and with contracted argument ($s > 1$).

Example 33. Let us now consider the reaction–diffusion equation with exponential nonlinearity

$$u_t = au_{xx} + be^{\lambda u}, \tag{98}$$

which for $\lambda \neq 0$ admits the invariant solution [25]:

$$u(x, t) = U(z) - \frac{1}{\lambda} \ln t, \quad z = xt^{-1/2}, \tag{99}$$

where the function $U = U(z)$ is described by the nonlinear ODE:

$$-\frac{1}{\lambda} - \frac{1}{2}zU'_z = aU''_{zz} + be^{\lambda U}.$$

Let us now consider a much more complex nonlinear functional-differential equation of the pantograph-type

$$u_t = au_{xx} + be^{\lambda w}, \quad w = u(px, qt), \tag{100}$$

where p and q are free parameters ($p > 0, q > 0$).

Partial functional-differential Equation (100) for $p = q = 1$ passes into the ‘ordinary’ partial differential Equation (98). For $\lambda \neq 0$ the solution of the pantograph-type Equation (100), as for Equation (98), is sought in the form (99). As a result, for the function $U = U(z)$, we obtain a nonlinear ODE of the pantograph-type [91]:

$$-\frac{1}{\lambda} - \frac{1}{2}zU'_z = aU''_{zz} + \frac{b}{q}e^{\lambda W}, \quad W = U(sz), \quad s = pq^{-1/2}.$$

In the special case for $p = q^{1/2}$ this equation is a standard ODE (without dilated or contracted arguments).

Example 34. It is easy to show that the nonlinear Klein–Gordon type equation

$$u_{tt} = au_{xx} + u(b \ln u + c) \tag{101}$$

allows the multiplicative separable solution

$$u(x, t) = \varphi(x)\psi(t). \tag{102}$$

More complicated than (101), the nonlinear pantograph-type PDE:

$$u_{tt} = au_{xx} + u(b \ln w + c), \quad w = u(px, qt), \tag{103}$$

also has the multiplicative separable solution (102), where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are described by the nonlinear pantograph-type ODEs:

$$\begin{aligned} a\varphi''_{xx} + \varphi(b\bar{\varphi} + c) &= 0, & \bar{\varphi} &= \varphi(px); \\ \psi''_{tt} = b\psi \ln \bar{\psi}, & & \bar{\psi} &= \psi(qt). \end{aligned}$$

Example 35. More complex than (103), the partial functional-differential equation

$$u_{tt} = au_{xx} + u(b \ln w + c), \quad w = u(\zeta(x), \eta(t)), \tag{104}$$

where $\zeta(x)$ and $\eta(t)$ are arbitrary functions, also allows a solution with the separation of variables of the form (102).

In particular, for $\zeta(x) = x - \tau_1$ and $\eta(t) = t - \tau_2$, where τ_1 and τ_2 are some positive constants, Equation (104) is a partial differential equation with two constant delays.

5.4. Approach for Constructing Exact Solutions of Functional Partial Differential Equations

Below, a rather general approach for constructing exact solutions of functional partial differential equations of the pantograph-type is formulated in the form following principle.

The principle of analogy of solutions. *Structure of exact solutions to partial functional-differential equations of the form*

$$F(u, w, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \quad w = u(px, qt) \tag{105}$$

often (but not always) is determined by the structure of solutions to simpler partial differential equations:

$$F(u, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0. \tag{106}$$

Equation (106) does not contain the unknown functions with dilated or contracted arguments; it is obtained from (105) by formally replacing w by u .

The solutions discussed in Examples 29–32 were constructed by using the principle of analogy of solutions. Below are two more complex examples.

Example 36. Consider the pantograph-type reaction–diffusion equation with power-law nonlinearities

$$u_t = au_{xx} + bu^m w^k, \quad w = u(px, qt), \tag{107}$$

that is more complex than (96).

Following the principle of analogy of solutions, we set $w = u$ in Equation (107). As a result, we arrive at the equation

$$u_t = au_{xx} + bu^{m+k},$$

which, after renaming $m + k$ to k coincides with Equation (94). Taking into account that the solution of Equation (94) is determined by formula (95), the solution of Equation (107) (by renaming k to $m + k$ in Equation (95)) is sought in the form [91]:

$$u(x, t) = t^{\frac{1}{1-m-k}} U(z), \quad z = xt^{-1/2}, \quad k \neq 1 - m.$$

As a result, for the function $U = U(z)$ we get the nonlinear pantograph-type ODE:

$$aU''_{zz} + \frac{1}{2}zU'_z - \frac{1}{1-m-k}U + bq^{\frac{k}{1-m-k}}U^mW^k = 0,$$

$$W = U(sz), \quad s = pq^{-1/2}.$$

Example 37. Let us now consider the pantograph-type reaction–diffusion equation with exponential nonlinearities

$$u_t = au_{xx} + be^{\mu u + \lambda w}, \quad w = u(px, qt), \tag{108}$$

that is more complex than (100). Following the principle of analogy of solutions, we set $w = u$ in Equation (108). As a result, we arrive at the equation

$$u_t = au_{xx} + be^{(\mu + \lambda)u},$$

which, after renaming $\mu + \lambda$ by λ coincides with Equation (97). Taking into account that the solution of Equation (100) is determined by formula (99), the solution of Equation (108) (by renaming λ to $\mu + \lambda$ in Equation (99)) is sought in the form

$$u(x, t) = U(z) - \frac{1}{\mu + \lambda} \ln t, \quad z = xt^{-1/2}, \quad \mu \neq -\lambda.$$

As a result, for the function $U = U(z)$ we obtain the nonlinear pantograph-type ODE:

$$aU''_{zz} + \frac{1}{2}zU'_z + \frac{1}{\mu + \lambda} + bq^{-\frac{\lambda}{\mu + \lambda}}e^{\mu U + \lambda W} = 0,$$

$$W = U(sz), \quad s = pq^{-1/2}.$$

Remark 10. In [91], a number of exact solutions of nonlinear pantograph PDEs of diffusion and wave types are obtained, which well confirm the principle of analogy of solutions for pantograph-type PDEs.

6. Brief Conclusions

A number of simple, but quite effective, methods for constructing exact solutions of nonlinear partial differential equations that require a relatively small amount of intermediate calculations are described. These methods are based on the following two main ideas: (i) simple exact solutions can serve as the basis for constructing more complex solutions of the considered equations, (ii) exact solutions of some equations can serve as a basis for constructing solutions of more complex equations. The effectiveness of the proposed methods is illustrated by a large number of specific examples of constructing exact solutions of nonlinear heat equations, reaction–diffusion equations, wave type equations, hydrodynamics equations and some other PDEs. In addition to exact solutions to partial differential equations, some exact solutions to nonlinear delay PDEs and pantograph-type PDEs are also described. The principle of analogy of solutions is formulated, which allows us to constructively find exact solutions to such partial functional-differential equations.

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