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On the Reciprocal Sums of Products of Balancing and Lucas-Balancing Numbers

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Abstract: Recently Panda et al. obtained some identities for the reciprocal sums of balancing and Lucas-balancing numbers. In this paper, we derive general identities related to reciprocal sums of products of two balancing numbers, products of two Lucas-balancing numbers and products of balancing and Lucas-balancing numbers. The method of this paper can also be applied to even-indexed and odd-indexed Fibonacci, Lucas, Pell and Pell–Lucas numbers.

Keywords: balancing numbers; Lucas-balancing numbers; Fibonacci numbers; reciprocal; floor function

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1. Introduction

The classical Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ are generated from the recurrence relation $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. As is well known, the Fibonacci numbers possess many interesting properties and appear in a variety of application fields [1].

Recently Ohtsuka and Nakamura [2] reported an interesting property of the Fibonacci numbers and proved the following identities:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\ F_n - F_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \text{ is odd,} \end{cases} \tag{1}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\ F_{n-1}F_n, & \text{if } n \geq 1 \text{ and } n \text{ is odd,} \end{cases} \tag{2}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Following the work of Ohtsuka and Nakamura, diverse results in the same direction have been reported in the literature [3–15].

A positive integer n is called the balancing number if [16]

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some positive integer r. As shown in [16], the balancing numbers $\{B_n\}_{n=0}^{\infty}$ satisfy the recurrence relation $B_n = 6B_{n-1} - B_{n-2}$ $(n \geq 2)$ with the initial conditions $B_0 = 0$ and $B_1 = 1$. The balancing numbers are useful in studying the Diophantine equations [17,18]. The numbers $\{C_n\}_{n=0}^{\infty}$ with $C_n = \sqrt{8B_n^2 + 1}$ are called the Lucas-balancing numbers [19] and obtained from the recurrence relation $C_n = 6C_{n-1} - C_{n-2}$ $(n \geq 2)$ with the initial conditions $C_0 = 1$ and $C_1 = 3$.

Panda et al. [20] recently studied the reciprocal sums of balancing and Lucas-balancing numbers, and obtained various identities. For example, they showed that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k} \right)^{-1} \right\rfloor = B_n - B_{n-1} - 1 \ \ (n \geq 1), \tag{3}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k^2} \right)^{-1} \right\rfloor = B_n^2 - B_{n-1}^2 - 1 \ \ (n \geq 1), \tag{4}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k}} \right)^{-1} \right\rfloor = B_{2n} - B_{2n-2} - 1 \ \ (n \geq 1), \tag{5}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k}^2} \right)^{-1} \right\rfloor = B_{2n}^2 - B_{2n-2}^2 - 1 \ \ (n \geq 1), \tag{6}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k B_{k+1}} \right)^{-1} \right\rfloor = B_n B_{n+1} - B_{n-1} B_n - 1 \ \ (n \geq 1), \tag{7}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k} B_{2k+2}} \right)^{-1} \right\rfloor = B_{2n+1}^2 - B_{2n-1}^2 - 2 \ \ (n \geq 1), \tag{8}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_k} \right)^{-1} \right\rfloor = C_n - C_{n-1} \ \ (n \geq 2), \tag{9}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_k^2} \right)^{-1} \right\rfloor = C_n^2 - C_{n-1}^2 \ \ (n \geq 1), \tag{10}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k}} \right)^{-1} \right\rfloor = C_{2n} - C_{2n-2} \ \ (n \geq 1), \tag{11}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k}^2} \right)^{-1} \right\rfloor = C_{2n}^2 - C_{2n-2}^2 \ \ (n \geq 1), \tag{12}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_k C_{k+1}} \right)^{-1} \right\rfloor = C_n C_{n+1} - C_{n-1} C_n + 1 \ \ (n \geq 1), \tag{13}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k} C_{2k+2}} \right)^{-1} \right\rfloor = C_{2n+1}^2 - C_{2n-1}^2 + 8 \ \ (n \geq 1), \tag{14}$$

etc.

We note that (3), (4) and (9), (10) also can be obtained, respectively from ([12] [Theorem 2.1]) and ([12] [Theorem 2.2]).

In this paper, we derive general identities related to reciprocal sums of products of two balancing numbers, products of two Lucas-balancing numbers and products of balancing and Lucas-balancing numbers. The results obtained here not only include most identities in [20] as special cases but also can be used to derive similar identities for even-indexed and odd-indexed Fibonacci, Lucas, Pell and Pell–Lucas numbers.

2. Results

For the ease of presentation, we use the notation $G_n = S(G_0, G_1, a, b)$ to denote the numbers $\{G_n\}_{n=0}^{\infty}$ generated from the recurrence relation

$$G_n = aG_{n-1} + bG_{n-2} \ \ (n \geq 2),$$

with the initial conditions G_0 and G_1.

To deal with the balancing numbers $B_n = S(0, 1, 6, -1)$ and Lucas-balancing numbers $C_n = S(1, 3, 6, -1)$ in a unified manner, we consider the numbers $G_n = S(G_0, G_1, a, -1)$, where G_0 is a nonnegative integer, G_1 and a are positive integers. As in [12], we assume that

$$a \geq \max\{3, 1 + G_0/G_1\}.$$

Firstly we present two lemmas which will be used to prove our main results. For $G_n = S(G_0, G_1, a, -1)$ and $H_n = S(H_0, H_1, a, -1)$, define

$$\begin{aligned}
\Phi_G &:= aG_0G_1 - G_0^2 - G_1^2, \\
\Phi_H &:= aH_0H_1 - H_0^2 - H_1^2, \\
\Delta_n &:= \frac{G_{n-1}G_{n+1}H_{n+m-1}H_{n+m+1} - G_n^2H_{n+m}^2}{G_{n+1}H_{n+m+1} - G_{n-1}H_{n+m=1}}, \\
\Delta_m &:= \lim_{n\to\infty} \Delta_n.
\end{aligned}$$

Lemma 1 (See [21])**.** *For $G_n = S(G_0, G_1, a, -1)$, we have*

$$G_n^2 - G_{n-r}G_{n+r} = (G_1G_r - G_0G_{r+1})Q_r,$$

where $Q_n = S(0, 1, a, -1)$.

Lemma 2. *For $G_n = S(G_0, G_1, a, -1)$ and $H_n = S(H_0, H_1, a, -1)$, we have*

$$\Delta_m = \frac{(\Phi_H G_1^2 + \Phi_G H_{m+1}^2)\alpha^4 - 2(\Phi_H G_0G_1 + \Phi_G H_m H_{m+1})\alpha^3 + (\Phi_H G_0^2 + \Phi_G H_m^2)\alpha^2}{G_1H_{m+1}\alpha^6 - (G_1H_m + G_0H_{m+1})\alpha^5 + G_0H_0\alpha^4 - G_1H_{m+1}\alpha^2 + (G_1H_m + G_0H_{m+1})\alpha - G_0H_m},$$

where

$$\alpha = \frac{a + \sqrt{a^2 - 4}}{2}.$$

Proof. From Lemma 1, we have

$$G_{n-1}G_{n+1}H_{n+m-1}H_{n+m+1} - G_n^2H_{n+m}^2 = \Phi_H G_n^2 + \Phi_G H_{n+m}^2 + \Phi_G\Phi_H.$$

G_n and H_{n+m} can be expressed as [21]

$$\begin{aligned}
G_n &= \frac{G_1(\alpha^n - \beta^n) - G_0(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}, \\
H_{n+m} &= \frac{H_{m+1}(\alpha^n - \beta^n) - H_m(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta},
\end{aligned}$$

where $\alpha > 1$ and $0 < \beta < 1$ are solutions of the equation $x^2 - ax + 1 = 0$, i.e.,

$$\alpha = \frac{a + \sqrt{a^2 - 4}}{2} \quad \text{and} \quad \beta = \frac{a - \sqrt{a^2 - 4}}{2}.$$

Since

$$\begin{aligned}&(\alpha-\beta)^2(\Phi_H G_n^2+\Phi_G H_{n+m}^2+\Phi_G\Phi_H)\\ =\ &\alpha^{2n-2}\big[(\Phi_H G_1^2+\Phi_G H_{m+1}^2)\alpha^2-2(\Phi_H G_0 G_1+\Phi_G H_m H_{m+1})\alpha+(\Phi_H G_0^2+\Phi_G H_m^2)\big]+\gamma_n,\end{aligned}$$

and

$$\begin{aligned}&(\alpha-\beta)^2(G_{n+1}H_{n+m+1}-G_{n-1}H_{n+m-1})\\ =\ &\alpha^{2n-4}\big[G_1H_{m+1}\alpha^6-(G_1H_m+G_0H_{m+1})\alpha^5+G_0H_0\alpha^4-G_1H_{m+1}\alpha^2+(G_1H_m+G_0H_{m+1})\alpha-G_0H_m\big]+\delta_n,\end{aligned}$$

where $\lim_{n\to\infty}\gamma_n=0$ and $\lim_{n\to\infty}\delta_n=0$, then the proof is completed. □

Now we state our main results.

Theorem 1. *For $G_n=S(G_0,G_1,a,-1)$ and $H_n=S(H_0,H_1,a,-1)$, there exists a positive integer N such that*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{G_kH_{k+m}}\right)^{-1}\right\rfloor=G_nH_{n+m}-G_{n-1}H_{n+m-1}+g_m,\quad \text{if } n\geq N, \tag{15}$$

where

$$g_m=\lfloor\Delta_m\rfloor.$$

Proof. Consider

$$\begin{aligned}&\frac{1}{G_nH_{n+m}-G_{n-1}H_{n+m-1}+g_m}-\frac{1}{G_{n+1}H_{n+m+1}-G_nH_{n+m}+g_m}-\frac{1}{G_nH_{n+m}}\\ =\ &\frac{X_1}{(G_nH_{n+m}-G_{n-1}H_{n+m-1}+g_m)(G_{n+1}H_{n+m+1}-G_nH_{n+m}+g_m)G_nH_{n+m}},\end{aligned}$$

where

$$\begin{aligned}X_1&=G_{n-1}G_{n+1}H_{n+m-1}H_{n+m+1}-G_n^2H_{n+m}^2-g_m(G_{n+1}H_{n+m+1}-G_{n-1}H_{n+m-1})-g_m^2\\ &=(G_{n+1}H_{n+m+1}-G_{n-1}H_{n+m-1})(\Delta_n-g_m)-g_m^2.\end{aligned}$$

Since Δ_n converges to Δ_m and $\Delta_m-g_m>0$, there exists a positive integer n_0 such that $X_1>0$ if $n\geq n_0$ or

$$\frac{1}{G_nH_{n+m}}<\frac{1}{G_nH_{n+m}-G_{n-1}H_{n+m-1}+g_m}-\frac{1}{G_{n+1}H_{n+m+1}-G_nH_{n+m}+g_m},\quad \text{if } n\geq n_0.$$

Repeatedly applying the above inequality, we have

$$\sum_{k=n}^{\infty}\frac{1}{G_kH_{k+m}}<\frac{1}{G_nH_{n+m}-G_{n-1}H_{n+m-1}+g_m},\quad \text{if } n\geq n_0. \tag{16}$$

Similarly,

$$\begin{aligned}&\frac{1}{G_nH_{n+m}-G_{n-1}H_{n+m-1}+g_m+1}-\frac{1}{G_{n+1}H_{n+m+1}-G_nH_{n+m}+g_m+1}-\frac{1}{G_nH_{n+m}}\\ =\ &\frac{X_2}{(G_nH_{n+m}-G_{n-1}H_{n+m-1}+g_m+1)(G_{n+1}H_{n+m+1}-G_nH_{n+m}+g_m+1)G_nH_{n+m}},\end{aligned}$$

where

$$\begin{aligned}
X_2 &= G_{n-1}G_{n+1}H_{n+m-1}H_{n+m+1} - G_n^2H_{n+m}^2 - G_{n+1}H_{n+m+1} + G_{n-1}H_{n+m-1} \\
&\quad -g_m(G_{n+1}H_{n+m+1} - G_{n-1}H_{n+m-1}) - (g_m+1)^2 \\
&= (G_{n+1}H_{n+m+1} - G_{n-1}H_{n+m-1})(\Delta_n - g_m) - G_{n+1}H_{n+m+1} + G_{n-1}H_{n+m-1} - (g_m+1)^2.
\end{aligned}$$

Since Δ_n converges to Δ_m and $0 < \Delta_m - g_m < 1$, then there exists a positive integer n_1 such that $X_2 < 0$ if $n \geq n_1$ or

$$\frac{1}{G_nH_{n+m} - G_{n-1}H_{n+m-1} + g_m} - \frac{1}{G_{n+1}H_{n+m+1} - G_nH_{n+m} + g_m} < \frac{1}{G_nH_{n+m}}, \quad \text{if } n \geq n_1,$$

from which we obtain

$$\frac{1}{G_nH_{n+m} - G_{n-1}H_{n+m-1} + g_m + 1} < \sum_{k=n}^{\infty} \frac{1}{G_kH_{k+m}}, \quad \text{if } n \geq n_1. \tag{17}$$

Then (15) follows from (16) and (17). □

Setting $G_n = H_n = S(0,1,6,-1)$ in Theorem 1, we obtain Corollary 1 below.

Corollary 1. *For balancing numbers $B_n = S(0,1,6,-1)$, there exists a positive integer N such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_kB_{k+m}} \right)^{-1} \right\rfloor = B_nB_{n+m} - B_{n-1}B_{n+m-1} + g_m, \quad \textit{if } n \geq N, \tag{18}$$

where

$$g_m = \left\lfloor \frac{-(B_{m+1}^2+1)\alpha^3 + 2B_mB_{m+1}\alpha^2 - B_m^2\alpha}{B_{m+1}\alpha^5 - B_m\alpha^4 - B_{m+1}\alpha + B_m} \right\rfloor,$$

with $\alpha = 3 + 2\sqrt{2}$.

For balancing numbers, $g_0 = g_1 = -1$ and we obtain (4) and (7) from (18). In addition we have

$$\begin{aligned}
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_kB_{k+2}} \right)^{-1} \right\rfloor &= B_nB_{n+2} - B_{n-1}B_{n+1} - 2, \quad \text{if } n \geq 1, \\
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_kB_{k+3}} \right)^{-1} \right\rfloor &= B_nB_{n+3} - B_{n-1}B_{n+2} - 6, \quad \text{if } n \geq 1,
\end{aligned}$$

etc.,

Setting $G_n = H_n = S(1,3,6,-1)$ in Theorem 1, we obtain Corollary 2 below.

Corollary 2. *For Lucas-balancing numbers $C_n = S(0,1,6,-1)$, there exists a positive integer N such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_kC_{k+m}} \right)^{-1} \right\rfloor = C_nC_{n+m} - C_{n-1}C_{n+m-1} + g_m, \quad \textit{if } n \geq N, \tag{19}$$

where

$$g_m = \left\lfloor \frac{(8C_{m+1}^2+72)\alpha^4 - (16C_mC_{m+1}+48)\alpha^3 + (8C_m^2+8)\alpha^2}{3C_{m+1}\alpha^6 - (3C_m + C_{m+1})\alpha^5 + \alpha^4 - 3C_{m+1}\alpha^2 + (3C_m + C_{m+1})\alpha - C_m} \right\rfloor,$$

with $\alpha = 3 + 2\sqrt{2}$.

For Lucas-balancing numbers, $g_0 = 0$ and $g_1 = 1$ and we obtain (10) and (13) from (19). In addition we have

$$\begin{aligned}
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_kC_{k+2}} \right)^{-1} \right\rfloor &= C_nC_{n+2} - C_{n-1}C_{n+1} + 8, \quad \text{if } n \geq 1, \\
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_kC_{k+3}} \right)^{-1} \right\rfloor &= C_nC_{n+3} - C_{n-1}C_{n+2} + 46, \quad \text{if } n \geq 1,
\end{aligned}$$

etc.

Setting $G_n = S(0,1,6,-1)$ and $H_n = S(1,3,6,-1)$ in Theorem 1, we obtain Corollary 3 below.

Corollary 3. *For balancing numbers* $B_n = S(0,1,6,-1)$ *and Lucas-balancing numbers* $C_n = S(0,1,6,-1)$, *there exists a positive integer N such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_kC_{k+m}} \right)^{-1} \right\rfloor = B_nC_{n+m} - B_{n-1}C_{n+m-1} + g_m, \quad \text{if } n \geq N, \tag{20}$$

where

$$g_m = \left\lfloor \frac{(8 - C_{m+1}^2)\alpha^3 + 2C_mC_{m+1}\alpha^2 - C_m^2\alpha}{C_{m+1}\alpha^5 - C_m\alpha^4 - C_{m+1}\alpha + C_m} \right\rfloor,$$

with $\alpha = 3 + 2\sqrt{2}$.

From (20), we have

$$\begin{aligned}
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_kC_k} \right)^{-1} \right\rfloor &= B_nC_n - B_{n-1}C_{n-1} - 1, \quad \text{if } n \geq 1, \\
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_kC_{k+1}} \right)^{-1} \right\rfloor &= B_nC_{n+1} - B_{n-1}C_n - 1, \quad \text{if } n \geq 1, \\
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_kC_{k+2}} \right)^{-1} \right\rfloor &= B_nC_{n+2} - B_{n-1}C_{n+1} - 3, \quad \text{if } n \geq 1,
\end{aligned}$$

etc.

Setting $G_n = S(1,3,6,-1)$ and $H_n = S(0,1,6,-1)$ in Theorem 1, we obtain Corollary 4 below.

Corollary 4. *For balancing numbers* $B_n = S(0,1,6,-1)$ *and Lucas-balancing numbers* $C_n = S(0,1,6,-1)$, *there exists a positive integer N such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{k+m}C_k} \right)^{-1} \right\rfloor = B_{n+m}C_n - B_{n+m-1}C_{n-1} + g_m, \quad \text{if } n \geq N, \tag{21}$$

where

$$g_m = \left\lfloor \frac{(8B_{m+1}^2 - 9)\alpha^4 - (16B_m B_{m+1} - 6)\alpha^3 + (8B_m^2 - 1)\alpha^2}{3B_{m+1}\alpha^6 - (3B_m + B_{m+1})\alpha^5 - 3B_{m+1}\alpha^2 + (3B_m + B_{m+1})\alpha - B_m} \right\rfloor,$$

with $\alpha = 3 + 2\sqrt{2}$.

From (21), we have

$$\begin{aligned} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{k+1}C_k} \right)^{-1} \right\rfloor &= B_{n+1}C_n - B_n C_{n-1}, \quad \text{if } n \geq 1, \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{k+2}C_k} \right)^{-1} \right\rfloor &= B_{n+2}C_n - B_{n+1}C_{n-1} + 2, \quad \text{if } n \geq 1, \end{aligned}$$

etc.

We can obtain similar results for the even-indexed and odd-indexed numbers of $G_n = S(G_0, G_1, a, -1)$ and $H_n = S(H_0, H_1, a, -1)$. It is easily seen that

$$\begin{aligned} G_{2n} &= (a^2 - 2)G_{2n-2} - G_{2n-4}, \\ G_{2n+1} &= (a^2 - 2)G_{2n-1} - G_{2n-3}. \end{aligned}$$

Let $G_n^e = G_{2n}$ and $G_n^o = G_{2n+1}$. Then

$$\begin{aligned} G_n^e &= S(G_0^e, G_1^e, a^2 - 2, -1), \\ G_n^o &= S(G_0^o, G_1^o, a^2 - 2, -1), \end{aligned}$$

where $G_0^e = G_0$, $G_1^e = aG_1 - G_0$, $G_0^o = G_1$ and $G_1^o = (a^2 - 1)G_1 - aG_0$. Similarly

$$\begin{aligned} H_n^e &= S(H_0^e, H_1^e, a^2 - 2, -1), \\ H_n^o &= S(H_0^o, H_1^o, a^2 - 2, -1), \end{aligned}$$

where $H_0^e = H_0$, $H_1^e = aH_1 - H_0$, $H_0^o = H_1$ and $H_1^o = (a^2 - 1)H_1 - aH_0$.

As before, for $U_n \in \{G_n^e, G_n^o, H_n^e, H_n^o\}$ and $V_n \in \{G_n^e, G_n^o, H_n^e, H_n^o\}$, let

$$\begin{aligned} \Phi_U &:= (a^2 - 2)U_0U_1 - U_0^2 - U_1^2, \\ \Phi_V &:= (a^2 - 2)V_0V_1 - V_0^2 - V_1^2, \\ \hat{\Delta}_n &:= \frac{U_{n-1}U_{n+1}V_{n+m-1}V_{n+m+1} - U_n^2 V_{n+m}^2}{U_{n+1}V_{n+m+1} - U_{n-1}V_{n+m=1}}, \\ \hat{\Delta}_m &:= \lim_{n \to \infty} \hat{\Delta}_n. \end{aligned}$$

Then

$$\hat{\Delta}_m = \frac{(\Phi_V U_1^2 + \Phi_U V_{m+1}^2)\hat{\alpha}^4 - 2(\Phi_V U_0U_1 + \Phi_U V_m V_{m+1})\hat{\alpha}^3 + (\Phi_V U_0^2 + \Phi_U V_m^2)\hat{\alpha}^2}{U_1V_{m+1}\hat{\alpha}^6 - (U_1V_m + U_0V_{m+1})\hat{\alpha}^5 + U_0V_0\hat{\alpha}^4 - U_1V_{m+1}\hat{\alpha}^2 + (U_1V_m + U_0V_{m+1})\hat{\alpha} - U_0V_m},$$

where

$$\hat{\alpha} = \frac{a^2 - 2 + \sqrt{(a^2 - 2)^2 - 4}}{2},$$

and we obtain the following results.

Theorem 2. *For $G_n = S(G_0, G_1, a, -1)$ and $H_n = S(H_0, H_1, a, -1)$, let $U_n \in \{G_n^e, G_n^o, H_n^e, H_n^o\}$ and $V_n \in \{G_n^e, G_n^o, H_n^e, H_n^o\}$. Then, for each pair (U_n, V_n), there exists a positive integer N such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k V_{k+m}} \right)^{-1} \right\rfloor = U_n V_{n+m} - U_{n-1} V_{n+m-1} + \hat{g}_m, \quad \text{if } n \geq N, \tag{22}$$

where

$$\hat{g}_m = \lfloor \hat{\Delta}_m \rfloor.$$

For balancing numbers $B_n = S(0, 1, 6, -1)$, setting $U_n = V_n = B_n^e$ in Theorem 2, we obtain Corollary 5 below.

Corollary 5. *For balancing numbers $B_n = S(0, 1, 6, -1)$, there exists a positive integer N such that*

$$\begin{aligned} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k} B_{2k+2m}} \right)^{-1} \right\rfloor &= B_{2n} B_{2n+2m} - B_{2n-2} B_{2n+2m-2} + \hat{g}_m \\ &= B_{2n+m}^2 - B_{2n+m-2}^2 + \hat{g}_m, \quad \text{if } n \geq N, \end{aligned} \tag{23}$$

where

$$\hat{g}_m = \left\lfloor \frac{-(6B_{2m+2}^2 + 216)\hat{\alpha}^3 + 12 B_{2m} B_{2m+2} \hat{\alpha}^2 - 6 B_{2m}^2 \hat{\alpha}}{B_{2m+2}\hat{\alpha}^5 - B_{2m}\hat{\alpha}^4 - B_{2m+2}\hat{\alpha} + B_{2m}} \right\rfloor,$$

with $\hat{\alpha} = 17 + 12\sqrt{2}$.

For balancing numbers, $\hat{g}_0 = -1$ and $\hat{g}_1 = -2$, and (6) and (8) are obtained from (23). In addition we have

$$\begin{aligned} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k} B_{2k+4}} \right)^{-1} \right\rfloor &= B_{2n+2}^2 - B_{2n}^2 - 37, \quad \text{if } n \geq 1, \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_{2k} B_{2k+6}} \right)^{-1} \right\rfloor &= B_{2n+3}^2 - B_{2n+1}^2 - 1223, \quad \text{if } n \geq 1, \end{aligned}$$

etc.

For Lucas-balancing numbers $C_n = S(1, 3, 6, -1)$, setting $U_n = V_n = C_n^e$ in Theorem 2, we obtain Corollary 6 below.

Corollary 6. *For Lucas-balancing numbers $C_n = S(1, 3, 6, -1)$, there exists a positive integer N such that*

$$\begin{aligned} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k} C_{2k+2m}} \right)^{-1} \right\rfloor &= C_{2n} C_{2n+2m} - C_{2n-2} C_{2n+2m-2} + \hat{g}_m \\ &= C_{2n+m}^2 - C_{2n+m-2}^2 + \hat{g}_m, \quad \text{if } n \geq N, \end{aligned} \tag{24}$$

where

$$\hat{g}_m = \left\lfloor \frac{(288C_{2m+2}^2 + 83232)\hat{\alpha}^4 - (576 C_{2m} C_{2m+2} + 9792)\hat{\alpha}^3 + (288 C_{2m}^2 + 288)\hat{\alpha}^2}{17C_{2m+2}\hat{\alpha}^6 - (17C_{2m} + C_{2m+2})\hat{\alpha}^5 + \hat{\alpha}^4 - 17C_{2m+2}\hat{\alpha}^2 + (17C_{2m} + C_{2m+2})\hat{\alpha} - C_{2m}} \right\rfloor,$$

with $\hat{\alpha} = 17 + 12\sqrt{2}$.

For Lucas-balancing numbers, $\hat{g}_0 = 0$ and $\hat{g}_1 = 8$, and (12) and (14) are obtained from (24). In addition we have

$$\begin{aligned} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k}C_{2k+4}} \right)^{-1} \right\rfloor &= C_{2n+2}^2 - C_{2n}^2 + 288, \quad \text{if } n \geq 1, \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{C_{2k}C_{2k+6}} \right)^{-1} \right\rfloor &= C_{2n+3}^2 - C_{2n+1}^2 + 9783, \quad \text{if } n \geq 1, \end{aligned}$$

etc.

For $U_n \in \{Gn^e, G_n^o, H_n^e, H_n^o\}$ and $V_n \in \{G_n^e, G_n^o, H_n^e, H_n^o\}$, we have sixteen pairs of (U_n, V_n), and we can obtain more identities from Theorem 2. Other identities are left to the interested readers.

3. Discussion

In this paper, we derived general identities related to reciprocal sums of products of two balancing numbers, products of two Lucas-balancing numbers and products of balancing and Lucas-balancing numbers. Repeatedly applying Theorem 2, we can obtain similar results for (B_{4n}, C_{4n}), (B_{8n}, C_{8n}), etc.

The method of this paper can also be applied to even-indexed and odd-indexed numbers of $G_n = S(G_0, G_1, a, 1)$. In fact, for the numbers of the form $G_n = S(G_0, G_1, a, 1)$, we have

$$\begin{aligned} G_{2n} &= (a^2 + 2)G_{2n-2} - G_{2n-4}, \\ G_{2n+1} &= (a^2 + 2)G_{2n-1} - G_{2n-3}. \end{aligned}$$

Hence Theorem 2 can be used to obtain various identities for even-indexed and odd-indexed Fibonacci, Lucas, Pell and Pell–Lucas numbers.

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