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Level Operators over Intuitionistic Fuzzy Index Matrices

Krassimir Atanassov *,†, Peter Vassilev † and Olympia Roeva †

Department of Bioinformatics and Mathematical Modelling, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 105, 1113 Sofia, Bulgaria; peter.vassilev@gmail.com (P.V.); olympia@biomed.bas.bg (O.R.)

* Correspondence: krat@bas.bg

† These authors contributed equally to this work.

Abstract: The index matrix (IM) is an extension of the ordinary matrix with indexed rows and columns. Over IMs' standard matrix operations are defined and a lot of other ones that do not exist in the standard case. Intuitionistic fuzzy IMs (IFIMs) are modification of the IMs, when their elements are intuitionistic fuzzy pairs (IFPs). Extended IFIMs are IFIMs whose indices of the rows and columns are evaluated by IFPs. Different operations, relations and operators over IFIMs, and some specific ones, are defined for EIFIMs. In the paper, twelve new level operators are defined for EIFIMs and in the partial case, over IFIMs. The proposed level operators fall into two groups: operators that change the values of the EIFIM elements and operators that change the IFPs associated to the indices of the rows and columns. The basic properties of the operators are studied.

Keywords: index matrix; intuitionistic fuzzy pair; level operation

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1. Introduction

The matrix theory is one of the basic areas of algebra (see, e.g., [1,2]). Different operations are defined over matrices, but each one of these operations is possible only when some conditions hold. For example, we cannot sum two matrices with different dimensions.

The concept of an index matrix (IM) was introduced in 1987 in [3] to enable two matrices with different dimensions to be summed. Later, this concept was extended essentially and not only operations, but also relations and operators were defined over IM. Their properties were discussed in a series of papers and a comprehensive overview can be found in the book [4].

Initially, by analogy with the standard matrices, the elements of the IMs were real numbers. Later, these elements obtained the form of sentences and predicates, functions, etc. One of the basic IM-modifications are intuitionistic fuzzy IMs (IFIMs) whose elements are intuitionistic fuzzy pairs (IFPs) [5]. A further modification are the extended IFIMs (EIFIMs), i.e., IFIMs whose indices of the rows and columns are evaluated by IFPs.

The 3-dimensional intuitionistic fuzzy index matrices (3-D IFIMS) introduced in [4] were further studied and new definitions were proposed in [6]. Subsequently, some basic operations and modal operators over the 3-D IFIMS were investigated in [7,8]. Some applications of the apparatus of IM have been discussed in [9,10]. Other examples of applications to number theory are given in [11].

To widen the area of application of IM, it is necessary to enrich the theory with additional operators. Following this, in the present study we introduce new level operators for better treatment and aggregation of information. We also establish some of the level operators properties.

In [12], for the first time, level operators were defined over IMs for the cases when the elements are real numbers in a fixed (finite) interval. Here, we continue this research, expanding it to the case with IFIMs having as elements IFPs, keeping all notations previously used to make the differences more pronounced.

The paper is structured as follows. In Section 2, short definitions of the relevant concepts are given. In Section 3, the new operators are introduced and their basic properties are studied. In Section 4, an example is provided. Final remarks and conclusion is presented in Section 5.

2. Preliminaries

Initially, we give some remarks on IFPs. The IFP is an object with the form $\langle a, b\rangle$, where $a, b \in [0,1]$ and $a + b \leq 1$, that is used as a tool for evaluation. Its components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$.

The following relations have been defined in [5]:

$$\begin{array}{lll} x < y & \text{iff} & a < c \text{ and } b > d \\ x \leq y & \text{iff} & a \leq c \text{ and } b \geq d \\ x = y & \text{iff} & a = c \text{ and } b = d \\ x \geq y & \text{iff} & a \geq c \text{ and } b \leq d \\ x > y & \text{iff} & a > c \text{ and } b < d \end{array}$$

We define analogous of operations "conjunction" and "disjunction":

$$\begin{aligned} x \wedge y &= \langle \min(a, c), \max(b, d)\rangle, \\ x \vee y &= \langle \max(a, c)), \min(b, d)\rangle, \\ x + y &= \langle a + c - ac, bd\rangle, \\ x.y &= \langle ac, b + d - bd\rangle. \end{aligned}$$

When "$\circ$" is one of these operations, then we see that it has a pair of two sub-operations. For example, if "$\circ$" is "$\wedge$", then its sub-operations are "min" and "max", while if "$\circ$" is "$\vee$", then its sub-operations are "max" and "min". Let us denote these operations by "$\circ_1$" and "$\circ_2$", respectively.

In [5], definitions of 185 operations' "implication" are given, while in [13–15] for each one of these implications, three conjunctions and three disjunctions are juxtaposed.

Let $\mathcal{I}$ be a fixed set. By IFIM with index sets K and L ($K, L \subset \mathcal{I}$), we denote the object:

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \ldots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j}\rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1}\rangle & \ldots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle & \ldots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n}\rangle \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \ldots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j}\rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$.

For brevity, we can denote the above object by $[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$, where

$$K = \{k_1, k_2, ..., k_m\},$$

$$L = \{l_1, l_2, ..., l_n\},$$

for $1 \leq i \leq m$, and $1 \leq j \leq n$:

$$\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0,1].$$

Now, for the mentioned above sets K and L, the EIFIM is defined by:

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$:

$$\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0,1],$$

$$\alpha_i^k, \beta_i^k, \alpha_i^k + \beta_i^k \in [0,1],$$
$$\alpha_j^l, \beta_j^l, \alpha_j^l + \beta_j^l \in [0,1]$$

and here and below,

$$K^* = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | 1 \leq i \leq m\},$$
$$L^* = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | 1 \leq j \leq n\}.$$

Therefore,

$$K^* \subseteq K \times [0,1] \times [0,1],$$
$$L^* \subseteq L \times [0,1] \times [0,1].$$

Let for $P, Q \subset \mathcal{I}$:

$$P^* = \{\langle p_r, \alpha_r^p, \beta_r^p \rangle | p_r \in P\},$$
$$Q^* = \{\langle q_s, \alpha_s^q, \beta_s^q \rangle | q_s \in Q\}.$$

Then

$$K^* \subset P^* \text{ iff } (K \subset P) \ \& \ (\forall k_i = p_r \in K)((\alpha_i^k < \alpha_r^p) \ \& \ (\beta_i^k > \beta_r^p)),$$
$$K^* \subseteq P^* \text{ iff } (K \subseteq P) \ \& \ (\forall k_i = p_r \in K)((\alpha_i^k \leq \alpha_r^p) \ \& \ (\beta_i^k \geq \beta_r^p)),$$
$$L^* \subset Q^* \text{ iff } (L \subset Q) \ \& \ (\forall l_j = q_s \in L)((\alpha_j^l < \alpha_s^q) \ \& \ (\beta_j^l > \beta_s^q)),$$
$$L^* \subseteq Q^* \text{ iff } (L \subseteq Q) \ \& \ (\forall l_j = q_s \in L)((\alpha_j^l \leq \alpha_s^q) \ \& \ (\beta_j^l \geq \beta_s^q)).$$

For the EIFIMs $A = [K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}], B = [P^*, Q^*, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle\}]$, operations that are analogous to the standard matrix operations of addition and multiplication, as well as other specific ones (see [4,12]) are defined as given below. Some technical oversights existing in [4] were pointed out in [12].

Addition-$(\circ)$

$$A \oplus_{(\circ)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup P\},$$
$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cup Q\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u \in K - P \\ \alpha_r^p, & \text{if } t_u \in P - K \\ \circ_1(\alpha_i^k, \alpha_r^p), & \text{if } t_u = k_i = p_r \in K \cap P \end{cases},$$

$$\beta_u^t = \begin{cases} \beta_i^k, & \text{if } t_u = k_i \in K - P \\ \beta_r^p, & \text{if } t_u = p_r \in P - K \\ \circ_2(\beta_i^k, \beta_r^p) & \text{if } t_u = k_i = p_r \in K \cap P \end{cases},$$

$$\alpha_w^v = \begin{cases} \alpha_j^l, & \text{if } v_w = l_j \in L - Q \\ \alpha_s^q, & \text{if } v_u = q_s \in Q - L \\ \circ_1(\alpha_j^l, \alpha_s^q), & \text{if } v_w = l_j = q_s \in L \cap Q \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w = l_j \in L - Q \\ \beta_s^q, & \text{if } v_w = q_s \in Q - L \\ \circ_2(\beta_j^l, \beta_s^q), & \text{if } v_w = l_j = q_s \in L \cap Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \quad \text{and } v_w = l_j \in L - Q \\ & \quad \text{or } t_u = k_i \in K - P \\ & \quad \text{and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \\ & \quad \text{and } v_w = q_s \in Q - L \\ & \quad \text{or } t_u = p_r \in P - K \\ & \quad \text{and } v_w = q_s \in Q; \\ \langle \circ_1(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad \circ_2(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \quad \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

Termwise multiplication-(○)

$$A \otimes_{(\circ)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cap P\},$$
$$V^* = L^* \cap Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cap Q\},$$

for $t_u = k_i = p_r \in K \cap P$:

$$\alpha_u^t = \min(\alpha_i^k, \alpha_r^p),$$
$$\beta_u^t = \max(\beta_i^k, \beta_r^p),$$

for $v_w = l_j = q_s \in L \cap Q$:
$$\alpha^v_w = \min(\alpha^l_j, \alpha^q_s),$$
$$\beta^v_w = \max(\beta^l_j, \beta^q_s),$$
and
$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \circ_1(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \circ_2(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle.$$

Let "$*$" be another operation and let it have sub-operations $\langle *_1, *_2 \rangle$.

Multiplication-($\circ, *$)

$$A \odot_{(\circ,*)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = (K \cup (P - L))^* = \{\langle t_u, \alpha^t_u, \beta^t_u \rangle | t_u \in K \cup (P - L)\},$$
$$V^* = (Q \cup (L - P))^* = \{\langle v_w, \alpha^v_w, \beta^v_w \rangle | v_w \in Q \cup (L - P)\},$$
$$\alpha^t_u = \begin{cases} \alpha^k_i, & \text{if } t_u = k_i \in K \\ \alpha^p_r, & \text{if } t_u = p_r \in P - L \end{cases},$$
$$\beta^t_u = \begin{cases} \beta^k_i, & \text{if } t_u = k_i \in K \\ \beta^p_r, & \text{if } t_u = p_r \in P - L \end{cases},$$
$$\alpha^{v_w} = \begin{cases} \alpha^l_j, & \text{if } v_w = l_j \in L - P \\ \alpha^q_s, & \text{if } t_w = q_s \in Q \end{cases},$$
$$\beta^{v_w} = \begin{cases} \beta^l_j, & \text{if } v_w = l_j \in L - P \\ \beta^q_s, & \text{if } t_w = q_s \in Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =$$
$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q \\ \Big\langle \underset{l_j = p_r \in L \cap P}{\circ_1} (*_1(\mu_{k_i,l_j}, \rho_{p_r,q_s})), & \text{if } t_u = k_i \in K \\ \underset{l_j = p_r \in L \cap P}{\circ_2} (*_2(\nu_{k_i,l_j}, \sigma_{p_r,q_s})) \Big\rangle, & \text{and } v_w = q_s \in Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

For example, when $\circ = \langle \max, \min \rangle$ and $* = \langle \min, \max \rangle$, we obtain operation multiplication-($\max, \min$) from [4].

Let

$$A = \begin{array}{c|ccc} & c,\langle 0.1,0.6\rangle & d,\langle 0.7,0.1\rangle & e,\langle 1.0,0.0\rangle \\ \hline a,\langle 0.5,0.4\rangle & \langle 0.3,0.2\rangle & \langle 0.4,0.4\rangle & \langle 0.9,0.1\rangle \\ b,\langle 0.3,0.5\rangle & \langle 0.5,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0.8,0.0\rangle \end{array}$$

and

$$B = \begin{array}{c|cc} & h,\langle 0.7,0.2\rangle & c,\langle 0.2,0.7\rangle \\ \hline a,\langle 0.2,0.3\rangle & \langle 0.4,0.3\rangle & \langle 0.2,0.2\rangle \\ f,\langle 0.1,0.4\rangle & \langle 0.5,0.2\rangle & \langle 0.4,0.4\rangle \\ g,\langle 0.7,0.1\rangle & \langle 0.6,0.1\rangle & \langle 0.3,0.6\rangle \end{array}.$$

Then

$$A \oplus_{(\wedge)} B = \begin{array}{c|cccc} & c,\langle 0.1,0.7\rangle & d,\langle 0.7,0.1\rangle & e,\langle 1.0,0.0\rangle & h,\langle 0.7,0.2\rangle \\ \hline a,\langle 0.2,0.4\rangle & \langle 0.2,0.2\rangle & \langle 0.4,0.4\rangle & \langle 0.9,0.1\rangle & \langle 0.4,0.3\rangle \\ b,\langle 0.3,0.5\rangle & \langle 0.5,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0.8,0.0\rangle & \langle 0.0,1.0\rangle \\ f,\langle 0.1,0.4\rangle & \langle 0.4,0.4\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.5,0.2\rangle \\ g,\langle 0.7,0.1\rangle & \langle 0.3,0.6\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.6,0.1\rangle \end{array}$$

and

$$A \otimes_{(\vee)} B = \begin{array}{c|c} & c,\langle 0.2,0.6\rangle \\ \hline a,\langle 0.5,0.3\rangle & \langle 0.3,0.2\rangle \end{array}.$$

Let

$$C = \begin{array}{c|c} & h,\langle 0.7,0.2\rangle \\ \hline c,\langle 0.2,0.3\rangle & \langle 0.4,0.3\rangle \\ f,\langle 0.1,0.4\rangle & \langle 0.5,0.2\rangle \\ d,\langle 0.7,0.1\rangle & \langle 0.6,0.1\rangle \end{array},$$

then

$$A \odot_{(\wedge,\vee)} C = \begin{array}{c|cc} & h,\langle 0.7,0.2\rangle & e,\langle 1.0,0.0\rangle \\ \hline a,\langle 0.5,0.4\rangle & ((\langle 0.3,0.2\rangle \wedge \langle 0.4,0.3\rangle) \vee ((\langle 0.4,0.4\rangle \wedge \langle 0.6,0.1\rangle) & \langle 0.9,0.1\rangle \\ b,\langle 0.3,0.5\rangle & ((\langle 0.5,0.3\rangle \wedge \langle 0.4,0.3\rangle) \vee ((\langle 0.6,0.3\rangle \wedge \langle 0.6,0.1\rangle) & \langle 0.8,0.0\rangle \\ f,\langle 0.1,0.4\rangle & \langle 0.5,0.2\rangle & \langle 0.0,1.0\rangle \end{array}$$
$$= \begin{array}{c|cc} & h,\langle 0.7,0.2\rangle & e,\langle 1.0,0.0\rangle \\ \hline a,\langle 0.2,0.3\rangle & \langle 0.3,0.3\rangle \vee \langle 0.4,0.4\rangle & \langle 0.9,0.1\rangle \\ b,\langle 0.2,0.3\rangle & \langle 0.4,0.3\rangle \vee \langle 0.6,0.3\rangle & \langle 0.8,0.0\rangle \\ f,\langle 0.1,0.4\rangle & \langle 0.5,0.2\rangle & \langle 0.0,1.0\rangle \end{array} = \begin{array}{c|cc} & h,\langle 0.7,0.2\rangle & e,\langle 1.0,0.0\rangle \\ \hline a,\langle 0.2,0.3\rangle & \langle 0.4,0.3\rangle & \langle 0.9,0.1\rangle \\ b,\langle 0.2,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0.8,0.0\rangle \\ f,\langle 0.1,0.4\rangle & \langle 0.5,0.2\rangle & \langle 0.0,1.0\rangle \end{array}.$$

A lot of relations are defined over two EIFIMs. Here, we use only five of them:
The strict relation "inclusion about value":

$$A \subset_v B \text{ iff } B \supset_v A \text{ iff } (K^* = P^*) \;\&\; (L^* = Q^*) \;\&\; (\forall k \in K)(\forall l \in L)$$
$$(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

The non-strict relation "inclusion about value":

$$A \subseteq_v B \text{ iff } B \supseteq_v A \text{ iff } (K^* = P^*) \;\&\; (L^* = Q^*) \;\&\; (\forall k \in K)(\forall l \in L)$$
$$(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

The strict relation "inclusion about dimension":

$$A \subset_d B \text{ iff } B \supset_d A \text{ iff } (((K \subset P) \& (L \subset Q)) \vee ((K \subseteq P) \& (L \subset Q))$$

$$\vee((K \subset P)\&(L \subseteq Q)))\ \&\ (\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

The non-strict relation "inclusion about dimension":

$$A \subseteq_d B \text{ iff } B \supseteq_d A \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

The relation "equality about dimension":

$$A = B \text{ iff } A \subseteq_v B\&B \supseteq_v A \text{ iff } A \subseteq_d B\&B \supseteq_d A$$
$$\text{iff } (K^* = P^*)\ \&\ (L^* = Q^*)\ \&\ (\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle = \langle c_{k,l}, d_{k,l}\rangle).$$

3. Definition and Properties of the New Level Operators

The proposed new level operators fall into two groups. The operators from the first group are applicable as over EIFIMs, as well as over IFIMs, while the operators from the second group are specific only for EIFIMs.

3.1. Definition and Properties of the Operators from the First Group

Let us have the IM $A = [K^*, L^*, \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle]$, where $\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle$ is an IFP. Then, for two fixed numbers $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$, we define:

$$\mathcal{L}^{>}_{\alpha,\beta}(A) = [K^*, L^*, \{\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle\}],$$

where

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle > \langle \alpha, \beta\rangle \\ \langle 0, 1\rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\geq}_{\alpha,\beta}(A) = [K^*, L^*, \{\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle\}],$$

where

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle \geq \langle \alpha, \beta\rangle \\ \langle 0, 1\rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{<}_{\alpha,\beta}(A) = [K^*, L^*, \{\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle\}],$$

where

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} < \langle \alpha, \beta\rangle \\ \langle 1, 0\rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\leq}_{\alpha,\beta}(A) = [K^*, L^*, \{\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle\}],$$

where

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle \leq \langle \alpha, \beta\rangle \\ \langle 1, 0\rangle, & \text{otherwise} \end{cases}.$$

For the EIFIM A from the example in Section 2, we have:

$$\mathcal{L}^{>}_{0.5,0,3}(A) = \begin{array}{c|ccc} & c, \langle 0.1, 0.6\rangle & d, \langle 0.7, 0.1\rangle & e, \langle 1.0, 0.0\rangle \\ \hline a, \langle 0.5, 0.4\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.9, 0.1\rangle \\ b, \langle 0.3, 0.5\rangle & \langle 0.0, 1.0\rangle & \langle 0.6, 0.3\rangle & \langle 0.8, 0.0\rangle \end{array},$$

$$\mathcal{L}^{\geq}_{0.5,0,3}(A) = \begin{array}{c|ccc} & c, \langle 0.1, 0.6\rangle & d, \langle 0.7, 0.1\rangle & e, \langle 1.0, 0.0\rangle \\ \hline a, \langle 0.5, 0.4\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.9, 0.1\rangle \\ b, \langle 0.3, 0.5\rangle & \langle 0.5, 0.3\rangle & \langle 0.6, 0.3\rangle & \langle 0.8, 0.0\rangle \end{array},$$

$$\mathcal{L}^{<}_{0.6,0,3}(A) = \begin{array}{c|ccc} & c, \langle 0.1, 0.6\rangle & d, \langle 0.7, 0.1\rangle & e, \langle 1.0, 0.0\rangle \\ \hline a, \langle 0.5, 0.4\rangle & \langle 0.3, 0.2\rangle & \langle 0.4, 0.4\rangle & \langle 1.0, 0.0\rangle \\ b, \langle 0.3, 0.5\rangle & \langle 0.5, 0.3\rangle & \langle 1.0, 0.0\rangle & \langle 1.0, 0.0\rangle \end{array},$$

$$\mathcal{L}^{\leq}_{0.6,0,3}(A) = \begin{array}{c|ccc} & c, \langle 0.1, 0.6\rangle & d, \langle 0.7, 0.1\rangle & e, \langle 1.0, 0.0\rangle \\ \hline a, \langle 0.5, 0.4\rangle & \langle 0.3, 0.2\rangle & \langle 0.4, 0.4\rangle & \langle 1.0, 0.0\rangle \\ b, \langle 0.3, 0.5\rangle & \langle 0.5, 0.3\rangle & \langle 0.6, 0.3\rangle & \langle 1.0, 0.0\rangle \end{array}.$$

Let for $K, L \subset \mathcal{I}$:

$$O^*_{K^*,L^*} = [K^*, L^*, \{\langle 0, 1\rangle\}],$$
$$E^*_{K^*,L^*} = [K^*, L^*, \{\langle 1, 0\rangle\}],$$

denote the zero and unit EIFIMs with index sets K^* and L^*. Obviously, the following equalities are valid for every two fixed numbers $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$:

$$\mathcal{L}^{>}_{\alpha,\beta}(O^*_{K^*,L^*}) = O^*_{K^*,L^*},$$
$$\mathcal{L}^{\geq}_{\alpha,\beta}(O^*_{K^*,L^*}) = O^*_{K^*,L^*},$$
$$\mathcal{L}^{<}_{\alpha,\beta}(O^*_{K^*,L^*}) = E^*_{K^*,L^*},$$
$$\mathcal{L}^{\leq}_{\alpha,\beta}(O^*_{K^*,L^*}) = E^*_{K^*,L^*},$$
$$\mathcal{L}^{>}_{\alpha,\beta}(E^*_{K^*,L^*}) = E^*_{K^*,L^*},$$
$$\mathcal{L}^{\geq}_{\alpha,\beta}(E^*_{K^*,L^*}) = E^*_{K^*,L^*},$$
$$\mathcal{L}^{<}_{\alpha,\beta}(E^*_{K^*,L^*}) = O^*_{K^*,L^*},$$
$$\mathcal{L}^{\leq}_{\alpha,\beta}(E^*_{K^*,L^*}) = O^*_{K^*,L^*}.$$

Theorem 1. *For each EIFIM A and for every two IFPs* $\langle \alpha, \beta\rangle, \langle \gamma, \delta\rangle$*:*

$$\mathcal{L}^{>}_{\langle\alpha,\beta\rangle}(\mathcal{L}^{>}_{\langle\gamma,\delta\rangle}(A)) = \mathcal{L}^{>}_{\langle\max(\alpha,\gamma),\min(\beta,\delta)\rangle}(A),$$
$$\mathcal{L}^{\geq}_{\langle\alpha,\beta\rangle}(\mathcal{L}^{\geq}_{\langle\gamma,\delta\rangle}(A)) = \mathcal{L}^{\geq}_{\langle\max(\alpha,\gamma),\min(\beta,\delta)\rangle}(A),$$
$$\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(\mathcal{L}^{<}_{\langle\gamma,\delta\rangle}(A)) = \mathcal{L}^{<}_{\langle\min(\alpha,\gamma),\max(\beta,\delta)\rangle}(A),$$
$$\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(\mathcal{L}^{\leq}_{\langle\gamma,\delta\rangle}(A)) = \mathcal{L}^{\leq}_{\langle\min(\alpha,\gamma),\max(\beta,\delta)\rangle}(A).$$

Proof. Let the above EIFIM A be given. Then

$$B = \mathcal{L}^{>}_{\langle\alpha,\beta\rangle}(\mathcal{L}^{>}_{\langle\gamma,\delta\rangle}(A)) = \mathcal{L}^{>}_{\langle\alpha,\beta\rangle}([K^*, L^*, \{\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle\}]) = [K, L, \{\langle \tau_{k_i,l_j}, \upsilon_{k_i,l_j}\rangle\}],$$

where

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle > \langle \gamma, \delta\rangle \\ \langle 0, 1\rangle, & \text{otherwise} \end{cases}$$

and

$$\langle \tau_{k_i,l_j}, \upsilon_{k_i,l_j} \rangle = \begin{cases} \langle \rho_{k_i,l_j}, \sigma_{k_i,l_j} \rangle, & \text{if } \langle \rho_{k_i,l_j}, \sigma_{k_i,l_j} \rangle > \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}.$$

If $\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle > \langle \max(\alpha,\gamma), \min(\beta,\delta) \rangle$, then

$$\langle \tau_{k_i,l_j}, \upsilon_{k_i,l_j} \rangle = \langle \rho_{k_i,l_j}, \sigma_{k_i,l_j} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle;$$

if $\langle \alpha, \beta \rangle \geq \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle > \langle \gamma, \delta \rangle$, then

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle,$$

but $\langle \tau_{k_i,l_j}, \upsilon_{k_i,l_j} \rangle = \langle 0, 1 \rangle$; if $\langle \gamma, \delta \rangle \geq \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle > \langle \alpha, \beta \rangle$, then

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j} \rangle = \langle 0, 1 \rangle$$

and hence, $\langle \tau_{k_i,l_j}, \upsilon_{k_i,l_j} \rangle = \langle 0, 1 \rangle$; if $\langle \min(\alpha,\gamma), \max(\beta,\delta) \rangle \geq \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$, then

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j} \rangle = \langle 0, 1 \rangle$$

and hence, $\langle \tau_{k_i,l_j}, \upsilon_{k_i,l_j} \rangle = \langle 0, 1 \rangle$.

Therefore,

$$\langle \tau_{k_i,l_j}, \upsilon_{k_i,l_j} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle > \langle \max(\alpha,\gamma), \min(\beta,\delta) \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases},$$

i.e.,

$$B = \mathcal{L}^{>}_{\langle \max(\alpha,\gamma), \min(\beta,\delta) \rangle}(A).$$

The remaining equalities are proved in the same manner. □

Corollary 1. *For each EIFIM A and for every two IFPs $\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle$:*

$$\mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(\mathcal{L}^{>}_{\langle \gamma,\delta \rangle}(A)) = \mathcal{L}^{>}_{\langle \gamma,\delta \rangle}(\mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A)),$$

$$\mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(\mathcal{L}^{\geq}_{\langle \gamma,\delta \rangle}(A)) = \mathcal{L}^{\geq}_{\langle \gamma,\delta \rangle}(\mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(A)),$$

$$\mathcal{L}^{<}_{\langle \alpha,\beta \rangle}(\mathcal{L}^{<}_{\langle \gamma,\delta \rangle}(A)) = \mathcal{L}^{<}_{\langle \gamma,\delta \rangle}(\mathcal{L}^{<}_{\langle \alpha,\beta \rangle}(A)),$$

$$\mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(\mathcal{L}^{\leq}_{\langle \gamma,\delta \rangle}(A)) = \mathcal{L}^{\leq}_{\langle \gamma,\delta \rangle}(\mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(A)).$$

Theorem 2. *Let the two EIFIMs A and B be given and let $\langle \alpha, \beta \rangle$ be an arbitrary IFP. Then*

(a) $\mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A) \oplus_{(+)} \mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(B) \supseteq_v \mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A \oplus_{(+)} B)$,

(b) $\mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A) \oplus_{(.)} \mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(B) \subseteq_v \mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A \oplus_{(.)} B)$,

(c) $\mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A) \oplus_{(\vee)} \mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(B) = \mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A \oplus_{(\vee)} B)$,

(d) $\mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A) \oplus_{(\wedge)} \mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(B) = \mathcal{L}^{>}_{\langle \alpha,\beta \rangle}(A \oplus_{(\wedge)} B)$,

(e) $\mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(A) \oplus_{(+)} \mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(B) \supseteq_v \mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(A \oplus_{(+)} B)$,

(f) $\mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(A) \oplus_{(.)} \mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(B) \subseteq_v \mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(A \oplus_{(.)} B)$,

(g) $\mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(A) \oplus_{(\vee)} \mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(B) = \mathcal{L}^{\geq}_{\langle \alpha,\beta \rangle}(A \oplus_{(\vee)} B)$,

(h) $\mathcal{L}^{\geq}_{\langle\alpha,\beta\rangle}(A)\oplus_{(\wedge)}\mathcal{L}^{\geq}_{\langle\alpha,\beta\rangle}(B)=\mathcal{L}^{\geq}_{\langle\alpha,\beta\rangle}(A\oplus_{(\wedge)}B)$,

(i) $\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(A)\oplus_{(+)}\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(B)\subseteq_v\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(A\oplus_{(+)}B)$,

(j) $\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(A)\oplus_{(.)}\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(B)\supseteq_v\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(A\oplus_{(.)}B)$,

(k) $\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(A)\oplus_{(\vee)}\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(B)=\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(A\oplus_{(\vee)}B)$,

(l) $\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(A)\oplus_{(\wedge)}\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(B)=\mathcal{L}^{<}_{\langle\alpha,\beta\rangle}(A\oplus_{(\wedge)}B)$,

(m) $\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A)\oplus_{(+)}\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(B)\subseteq_v\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A\oplus_{(+)}B)$,

(n) $\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A)\oplus_{(.)}\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(B)\supseteq_v\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A\oplus_{(.)}B)$,

(o) $\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A)\oplus_{(\vee)}\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(B)=\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A\oplus_{(\vee)}B)$,

(p) $\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A)\oplus_{(\wedge)}\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(B)=\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A\oplus_{(\wedge)}B)$.

Proof. *(p)* Let EIFIM A have the above form and let

$$B=[P^*,Q^*\langle\rho_{p_r,q_s},\sigma_{p_r,q_s}\rangle],$$

where $\langle\rho_{p_r,q_s},\sigma_{p_r,q_s}\rangle$ is an IFP.

From definition of operation $\oplus_{(\wedge)}$, we have:

$$\begin{aligned}&\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}(A\oplus_{(\wedge)}B)\\ &=\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}([K^*\cup P^*,L^*\cup Q^*,\{\langle\min(\mu_{k_i,l_j},\rho_{p_r,q_s}),\max(\nu_{k_i,l_j},\sigma_{p_r,q_s})\rangle\}])\\ &=\mathcal{L}^{\leq}_{\langle\alpha,\beta\rangle}([K^*\cup P^*,L^*\cup Q^*,\langle\tau_{u_v,w_t},\upsilon_{u_v,w_t}\rangle]),\\ &=[K^*\cup P^*,L^*\cup Q^*,\langle\varphi_{u_v,w_t},\psi_{u_v,w_t}\rangle],\end{aligned}$$

where

$$\langle\tau_{u_v,w_t},\upsilon_{u_v,w_t}\rangle=\langle\min(\mu_{k_i,l_j},\rho_{p_r,q_s}),\max(\nu_{k_i,l_j},\sigma_{p_r,q_s})\rangle$$

and

$$\langle\varphi_{u_v,w_t},\psi_{u_v,w_t}\rangle=\begin{cases}\langle\tau_{u_v,w_t},\upsilon_{u_v,w_t}\rangle, & \text{if } \langle\tau_{u_v,w_t},\upsilon_{u_v,w_t}\rangle\leq\langle\alpha,\beta\rangle\\ \langle 1,0\rangle, & \text{otherwise}\end{cases}.$$

Sequentially, we will study the different cases for $\langle\tau_{u_v,w_t},\upsilon_{u_v,w_t}\rangle$.

1. If $\langle\tau_{u_v,w_t},\upsilon_{u_v,w_t}\rangle=\langle\mu_{k_i,l_j},\nu_{k_i,l_j}\rangle$, then

$$\langle\varphi_{u_v,w_t},\psi_{u_v,w_t}\rangle=\begin{cases}\langle\mu_{k_i,l_j},\nu_{k_i,l_j}\rangle, & \text{if } \langle\mu_{k_i,l_j},\nu_{k_i,l_j}\rangle\leq\langle\alpha,\beta\rangle\\ \langle 1,0\rangle, & \text{otherwise}\end{cases}$$

and therefore, the elements with indices $\langle k_i,l_j\rangle$ will coincide in the two IMs $\mathcal{L}^{\leq}_{\overline{\alpha}}(A\oplus_{(\wedge)}B)$ and $\mathcal{L}^{\leq}_{\overline{\alpha}}(A)\oplus_{(\wedge)}\mathcal{L}^{\leq}_{\overline{\alpha}}(B)$.

2. If $\langle\tau_{u_v,w_t},\upsilon_{u_v,w_t}\rangle=\langle\rho_{p_r,q_s},\sigma_{p_r,q_s}\rangle$, then

$$\langle\varphi_{u_v,w_t},\psi_{u_v,w_t}\rangle=\begin{cases}\langle\rho_{p_r,q_s},\sigma_{p_r,q_s}\rangle, & \text{if } \langle\rho_{p_r,q_s},\sigma_{p_r,q_s}\rangle\leq\langle\alpha,\beta\rangle\\ \langle 1,0\rangle, & \text{otherwise}\end{cases}$$

and therefore, the elements with indices $\langle p_r, q_s\rangle$ will coincide in the two IMs $\mathcal{L}^{\leq}_{\overline{\alpha}}(A \oplus_{(\wedge)} B)$ and $\mathcal{L}^{\leq}_{\overline{\alpha}}(A) \oplus_{(\wedge)} \mathcal{L}^{\leq}_{\overline{\alpha}}(B)$.

3. If $\langle \tau_{u_v,w_t}, \upsilon_{u_v,w_t}\rangle = \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle$, then

$$\langle \varphi_{u_v,w_t}, \psi_{u_v,w_t}\rangle$$

$$= \begin{cases} \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle, & \text{if } \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle \leq \langle \alpha, \beta\rangle \\ \langle 1, 0\rangle, & \text{otherwise.} \end{cases}$$

Now, there are the following four subcases:

3.1. if

$$\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle \leq \langle \alpha, \beta\rangle$$

and

$$\langle \rho_{k_i,l_j}, \sigma_{k_i,l_j}\rangle \leq \langle \alpha, \beta\rangle,$$

then these elements will keep their values in $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(A)$ and in $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(B)$, respectively and their conjunction will coincide with the value $\langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle$ in EIFIM $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(A \oplus_{(\wedge)} B)$.

3.2. if

$$\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle \leq \langle \alpha, \beta\rangle$$

and

$$\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle \not\leq \langle \alpha, \beta\rangle,$$

i.e.,

$$\rho_{p_r,q_s} > \alpha \text{ and } \sigma_{p_r,q_s} > \beta, \text{ or } \rho_{p_r,q_s} \leq \alpha \text{ and } \sigma_{p_r,q_s} > \beta, \text{ or } \rho_{p_r,q_s} > \alpha \text{ and } \sigma_{p_r,q_s} \leq \beta,$$

then only the a-element will keep its value in $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(A)$, while the b-element will obtain value $\langle 1, 0\rangle$ in $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(B)$. Therefore, the value of $\langle \varphi_{u_v,w_t}, \psi_{u_v,w_t}\rangle = \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle$ in EIFIM $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(A \oplus_{(\wedge)} B)$ will be equal to its corresponding value in the EIFIM $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(A) \oplus_{(\wedge)} \mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(B)$.

3.3. if

$$\mu_{k_i,l_j} > \alpha \text{ and } \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle \not\leq \langle \alpha, \beta\rangle$$

and

$$\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle \leq \langle \alpha, \beta\rangle,$$

i.e.,

$$\nu_{k_i,l_j} > \beta, \text{ or } \mu_{k_i,l_j} \leq \alpha \text{ and } \nu_{k_i,l_j} > \beta, \text{ or } \mu_{k_i,l_j} > \alpha \text{ and } \nu_{k_i,l_j} \leq \beta,$$

then only the b-element will keep its value in $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(B)$, while the a-element will obtain value $\langle 1, 0\rangle$ in $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(A)$. Therefore, the value of $\langle \varphi_{u_v,w_t}, \psi_{u_v,w_t}\rangle = \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle$ in EIFIM $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(A \oplus_{(\wedge)} B)$ will be equal to its corresponding value in IM $\mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(A) \oplus_{(\wedge)} \mathcal{L}^{\leq}_{\langle \alpha,\beta\rangle}(B)$.

3.4. if

$$\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle \not\leq \langle \alpha, \beta\rangle$$

and

$$\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle \not\leq \langle \alpha, \beta \rangle,$$

then the a- and b-elements will obvain value $\langle 1, 0 \rangle$ in IMs $\mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(A)$ and $\mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(B)$. Therefore, the value of $\langle \varphi_{u_v,w_t}, \psi_{u_v,w_t} \rangle = \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle$ in IM $\mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(A \oplus_{(\wedge)} B)$ will be equal to $\langle 1, 0 \rangle$ and to its corresponding value in IM $\mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(A) \oplus_{(\wedge)} \mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(B)$.

4. If $\langle \tau_{u_v,w_t}, \upsilon_{u_v,w_t} \rangle = \langle 1, 0 \rangle$, then $\langle \varphi_{u_v,w_t}, \psi_{u_v,w_t} \rangle = \langle 1, 0 \rangle$ and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(A \oplus_{(\wedge)} B)$ and $\mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(A) \oplus_{(\wedge)} \mathcal{L}^{\leq}_{\langle \alpha,\beta \rangle}(B)$.

Hence, relation $=$ between the two IMs exists.

The other cases are proved in the same manner. □

If in Theorem 2 we replace operation $\oplus_{(\circ)}$ by operation $\otimes_{(\circ)}$ or $\odot_{(\circ,*)}$, the inequalities will keep their forms and the proofs will be analogous.

Now, we will modify the operator **automatic reduction** from [4] to the form:

$$@(A) = [P^*, Q^*, \{b_{p_r,q_s}\}],$$

where $P \subseteq K, Q \subseteq L$ are index sets with the following property:

$$(\forall k \in K^* - P^*)(\forall l \in L^*)(a_{k_i,l_j} = \langle 0, 1 \rangle) \;\&\; (\forall k \in K^*)(\forall l \in L^* - Q^*)(a_{k_i,l_j} = \langle 0, 1 \rangle)$$

$$\&\; (\forall p_r = k_i \in P)(\forall q_s = l_j \in Q)(b_{p_r,q_s} = a_{k_i,l_j}).$$

Therefore, this operator removes the rows and columns of EIFIM A that contain only elements $\langle 0, 1 \rangle$.

3.2. Definition and Properties of the Operators from the Second Group

Let us have the EIFIM

$$A = [K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

and for the two numbers $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$:

$$\mathcal{L}^{>}_{R,\alpha,\beta}(A) = [\overline{K}^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where

$$\overline{K}^* = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | 1 \leq i \leq m\}$$

and

$$\langle \gamma_i^k, \delta_i^k \rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } \langle \alpha_i^k, \beta_i^k \rangle > \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\geq}_{R,\alpha,\beta}(A) = [\overline{K}^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where

$$\overline{K}^* = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | 1 \leq i \leq m\}$$

and

$$\langle \gamma_i^k, \delta_i^k \rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } \langle \alpha_i^k, \beta_i^k \rangle \geq \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{<}_{R,\alpha,\beta}(A) = [\overline{K}^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where

$$\overline{K}^* = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | 1 \leq i \leq m\}$$

and

$$\langle \gamma_i^k, \delta_i^k \rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } \langle \alpha_i^k, \beta_i^k \rangle < \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\leq}_{R,\alpha,\beta}(A) = [\overline{K}^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where

$$\overline{K}^* = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | 1 \leq i \leq m\}$$

and

$$\langle \gamma_i^k, \delta_i^k \rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } \langle \alpha_i^k, \beta_i^k \rangle \leq \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{>}_{C,\alpha,\beta}(A) = [K^*, \overline{L}^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where

$$\overline{L}^* = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | 1 \leq i \leq m\}$$

and

$$\langle \gamma_i^l, \delta_i^l \rangle = \begin{cases} \langle \alpha_i^l, \beta_i^l \rangle, & \text{if } \langle \alpha_i^l, \beta_i^l \rangle > \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\geq}_{C,\alpha,\beta}(A) = [K^*, \overline{L}^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where

$$\overline{L}^* = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | 1 \leq i \leq m\}$$

and

$$\langle \gamma_i^l, \delta_i^l \rangle = \begin{cases} \langle \alpha_i^l, \beta_i^l \rangle, & \text{if } \langle \alpha_i^l, \beta_i^l \rangle \geq \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{<}_{C,\alpha,\beta}(A) = [K^*, \overline{L}^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where

$$\overline{L}^* = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | 1 \leq i \leq m\}$$

and

$$\langle \gamma_i^l, \delta_i^l \rangle = \begin{cases} \langle \alpha_i^l, \beta_i^l \rangle, & \text{if } \langle \alpha_i^l, \beta_i^l \rangle < \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\leq}_{C,\alpha,\beta}(A) = [K^*, \overline{L}^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where

$$\overline{L}^* = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | 1 \leq i \leq m\}$$

and

$$\langle \gamma_i^l, \delta_i^l \rangle = \begin{cases} \langle \alpha_i^l, \beta_i^l \rangle, & \text{if } \langle \alpha_i^l, \beta_i^l \rangle \leq \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

Theorem 3. *For each EIFIM A and for every two IFPs* $\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle$*:*

$$\mathcal{L}^z_{R,\alpha,\beta}(\mathcal{L}^y_{C,\alpha,\beta}(A)) = \mathcal{L}^y_{C,\alpha,\beta}(\mathcal{L}^z_{R,\alpha,\beta}(A)),$$

where $y, z \in \{>, \geq, <, \leq\}$.

We omit the proof of Theorem 3 and the next theorem (Theorem 4) as they may be proved similarly to Theorem 2.

Finally, we modify the operator **automatic reduction** from Section 3.1 to the forms:

$$@_R(A) = [P^*, L^*, \{b_{p_r,q_s}\}],$$

where $P^* \subseteq K^*$ is the index sets with the following property:

$$(\forall k \in K - P)(\langle \alpha_i^k, \beta_i^k \rangle = \langle 0, 1 \rangle) \ \& \ (\forall p_r = k_i \in P)(\forall q_s = l_j \in L)(b_{p_r,q_s} = a_{k_i,l_j}),$$

$$@_C(A) = [K^*, Q^*, \{b_{p_r,q_s}\}],$$

where $Q^* \subseteq L^*$ is the index sets with the following property:

$$(\forall l \in L - Q)(\langle \alpha_i^l, \beta_i^l \rangle = \langle 0, 1 \rangle) \ \& \ (\forall q_s = l_j \in Q)(\forall p_r = k_i \in K)(b_{p_r,q_s} = a_{k_i,l_j}).$$

Therefore, this operator removes the rows and columns of EIFIM A in which indices have evaluation $\langle 0, 1 \rangle$.

Theorem 4. *For each EIFIM A and for every two IFPs* $\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle$*:*

$$@_C(@_R(A) = @_R(@_C(A).$$

Theorem 1 keeps its sense when we change operator $\mathcal{L}^z_{\alpha,\beta}$ with each one of the operators $\mathcal{L}^z_{R,\alpha,\beta}$ and $\mathcal{L}^z_{C,\alpha,\beta}$, where $z \in \{>, \geq, <, \leq\}$.

Theorem 2 keeps also its sense for each one of operations $\oplus_\circ, \otimes_\circ, \odot_{\circ,*}$ when we change operator $\mathcal{L}^z_{\alpha,\beta}$ with each one of the operators $\mathcal{L}^z_{R,\alpha,\beta}$ and $\mathcal{L}^z_{C,\alpha,\beta}$, where $z \in \{>, \geq, <, \leq\}$, and relations $\subseteq_v$ and $\supseteq_v$ with relations $\subseteq_d$ and $\supseteq_d$, respectively.

4. An Example

A possible application of the newly proposed level operators to assessment of the student's training is considered.

Let us have s students $S_1, \dots S_s$ that have (by the current moment) scores $\langle \mu_i^S, \nu_i^S \rangle$ for $i = 1, \dots s$. Let these students study d different disciplines $D_1, \dots, D_d$, that have scores received by the previous procedures $\langle \mu_j^D, \nu_j^D \rangle$ for $j = 1, \dots d$.

The student's scores (following n_i previous examinations) after each examination are calculated by the formula

$$\langle \mu_{i,new}^S, \nu_{i,new}^S \rangle = \begin{cases} \langle \frac{n_i \mu_{i,old}^S + 1}{n_i + 1}, \frac{n_i \nu_{i,old}^S}{n_i + 1} \rangle, & \text{the evaluation of the student's examination is high} \\ \langle \frac{n_i \mu_{i,old}^S}{n_i + 1}, \frac{n_i \nu_{i,old}^S + 1}{n_i + 1} \rangle, & \text{the evaluation of the student's examination is poor} \\ \langle \frac{n_i \mu_{i,old}^S}{n_i}, \frac{n_i \nu_{i,old}^S + 1}{n_i + 1} \rangle, & \text{the student had not attended the exam} \end{cases}.$$

Obviously, $\langle \mu_{i,new}^S, \nu_{i,new}^S \rangle$ is an IFP.

The discipline's score (after m previous training procedures) after each examination of the students $S_1, \dots S_s$, is calculated by the formula

$$\langle \mu_{j,new}^D, \nu_{j,new}^D \rangle = \left\langle \frac{m\mu_{j,old}^D + \frac{s_g}{s}}{m+1}, \frac{m\nu_{j,old}^D + \frac{s_b}{s}}{m+1} \right\rangle,$$

where s_g is the number of the students who received high score, and s_p is the number of the students who had poor score. Therefore, the number $s - s_g - s_p$ corresponds to the number of the students who had not attended the examination and obviously, $\langle \mu_{j,new}^D, \nu_{j,new}^D \rangle$ is an IFP.

The sets of triples

$$S^* = \{\langle S_i, \mu_{i,new}^S, \nu_{i,new}^S \rangle | 1 \leq i \leq s\}$$

and

$$D^* = \{\langle D_j, \mu_{j,new}^D, \nu_{j,new}^D \rangle | 1 \leq j \leq d\}$$

are the indices of the EIFIM

$$A = [S^*, D^*, \{\langle \mu_{S_i,D_j}, \nu_{S_i,D_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & D_1, \langle \mu_1^D, \nu_1^D \rangle & \dots & D_j, \langle \mu_j^D, \nu_j^D \rangle & \dots & D_n, \langle \mu_n^D, \nu_n^D \rangle \\ \hline S_1, \langle \mu_1^S, \nu_1^S \rangle & \langle \mu_{S_1,D_1}, \nu_{S_1,D_1} \rangle & \dots & \langle \mu_{S_1,D_j}, \nu_{S_1,D_j} \rangle & \dots & \langle \mu_{S_1,D_n}, \nu_{S_1,D_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ S_i, \langle \mu_i^S, \nu_i^S \rangle & \langle \mu_{S_i,D_1}, \nu_{S_i,D_1} \rangle & \dots & \langle \mu_{S_i,D_j}, \nu_{S_i,D_j} \rangle & \dots & \langle \mu_{S_i,D_n}, \nu_{S_i,D_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ S_m, \langle \mu_m^S, \nu_m^S \rangle & \langle \mu_{S_m,D_1}, \nu_{S_m,D_1} \rangle & \dots & \langle \mu_{S_m,D_j}, \nu_{S_m,D_j} \rangle & \dots & \langle \mu_{S_m,D_n}, \nu_{S_m,D_n} \rangle \end{array},$$

and IFP $\langle \mu_{S_i,D_j}, \nu_{S_i,D_j} \rangle$ corresponds to the evaluation of the ith student on the examination of jth discipline. The scores as IFPs can be calculated in the following manner: $\langle \frac{p}{r}, \frac{q}{r} \rangle$, where r is the number of all problems that the student had to solve, p is the number of correctly solved problems, q is the number of wrongly solved problems and $r - p - q$ is the number of the problems that were not solved, e.g., for the lack of time.

Using operators $\mathcal{L}^{>}$ and $\mathcal{L}^{\geq}$ over EIFIM A, we will determine the higher evaluations, using operators $\mathcal{L}^{<}$ and $\mathcal{L}^{\leq}$, the lower evaluations, using operators $\mathcal{L}_R^{>}$ and $\mathcal{L}_R^{\geq}$, the students with higher scores, using operators $\mathcal{L}_R^{<}$ and $\mathcal{L}_R^{\leq}$, the students with lower scores, using operators $\mathcal{L}_C^{>}$ and $\mathcal{L}_C^{\geq}$, the disciplines with higher scores, using operators $\mathcal{L}_C^{<}$ and $\mathcal{L}_C^{\leq}$, the disciplines with lower scores.

It is possible to describe the elements in the matrix A, i.e., the scores, as natural or real numbers, instead of IFPs, when this is convenient.

5. Conclusions

The IMs were introduced to resolve some particular problems related to the extensions of some modifications of Petri nets [3]. Over the years, they have been enriched with different relations, operations and operators. In addition to the standard two dimensional IMs, 3- and n- dimensional IMs were defined. With [12] and the current research, we initiate a new direction of research over IM—introduction and studying of group of new level operators. It is important to note that many of the operations, relations and operators do not exist in the ordinary matrices. Moreover, the new level operators are different in sense compared to the fuzzy and intuitionistic fuzzy types of level operators (cf. [16]).

In the near future, other properties of the introduced level operators will be studied. These operators can find applications in different areas, such as intuitionistic fuzzy graphs, intuitionistic fuzzy cognitive maps and others (see [16]), that will be deeply and thoroughly investigated and analyzed in sufficient detail in a series of future studies.

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