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Some Fixed Point Results of Weak-Fuzzy Graphical Contraction Mappings with Application to Integral Equations

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Abstract: The present paper aims to introduce the concept of weak-fuzzy contraction mappings in the graph structure within the context of fuzzy cone metric spaces. We prove some fixed point results endowed with a graph using weak-fuzzy contractions. By relaxing the continuity condition of mappings involved, our results enrich and generalize some well-known results in fixed point theory. With the help of new lemmas, our proofs are straight forward. We furnish the validity of our findings with appropriate examples. This approach is completely new and will be beneficial for the future aspects of the related study. We provide an application of integral equations to illustrate the usability of our theory.

Keywords: fuzzy cone metric space; fixed point; weak-contraction; graph structure

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1. Introduction

The theory of fixed points centers on the process of solving the equation of the form $T(\mu) = \mu$. We discuss a new concept that overlaps between metric fixed point theory and graph theory. This new area yields interesting generalizations of the Banach contraction principle [1] in metric spaces endowed with a graph. The fixed point techniques have received considerable attention due to their broad applications in many applied sciences to solve diverse problems in engineering, game theory, physics, computer science, image recovery and signal processing, control theory, communications, and geophysics.

In 1965, Zadeh [2] introduced the fuzzy sets. Kramosil and Michálek [3] introduced the notion of fuzzy metric space. George and Veeramani [4] modified the description of fuzzy metric spaces due to Kramosil and Michálek. Gregori and Sapena [5] introduced the concept of fuzzy contractive mappings. On the other hand, the results were applied to metric spaces provided with a partial order by Ran and Reuring [6]. To find a solution to some special matrix equations was also one of the great charms of the fixed point theorists. To this end, the work of EL-Sayed and Andre' [7] was a pioneer one. Later on, Nieto and Rodriguez Lopez [8] extended the work of [6] and applied their results to solve some differential equations.

One natural question is whether contractive conditions may be found that indicate the presence of a fixed point in an entire metric space, but do not imply continuity?

A mapping $T : U \to U$, where (U, d) is a metric space is said to be a contraction map [1], if there exists $0 < \alpha < 1$, such that for all $\mu, \nu \in U$;

$$d(T\mu, T\nu) \leq \alpha d(\mu, \nu) \tag{1}$$

The following result was defined by Kannan [9], in which the above question was answered affirmatively. If $T : U \to U$ is a complete metric space where (U, d) satisfies inequality:

$$d(T\mu, T\nu) \leq \alpha[d(\mu, T\mu) + d(\nu, T\nu)], \quad \mu, \nu \in U, \tag{2}$$

where $0 < \alpha < \frac{1}{2}$, then a unique fixed point will be in T. The mapping T need not to be continuous, the references therein (see [9,10]).

In 2008, Jachymiski [11] initiated a novel idea in fixed point theory, where the author evoked graph structure on metric spaces instead of order structure. According to this concept, Banach's contraction condition will be satisfied only for the edges of the graph. If $\hat{\mathcal{G}}$ is a directed graph and $E(\hat{\mathcal{G}})$ is the set of its edges then the contraction condition is:

$$d(T\mu, T\nu) \leqslant \beta d(\mu, \nu), \quad \beta \in (0,1), \forall(\mu, \nu) \in E(\hat{\mathcal{G}})$$

Some noteworthy efforts done on this concept can be seen in [12–17]. Starting from these results, we aim to make a methodical study of fixed point theorem in fuzzy metric spaces endowed with a graph.

In 2016, Usman [18] generalized a new class of F-contractions in b-metric spaces and to obtain existence theorems for Volterra-type integral inclusion. In 2017, Kamran et al. [19] introduced a new class of comparison functions to present some fixed point theorems with an extended b-metric space. For various applications of fixed points in metric spaces (see [20–23]).

In this paper, we introduce the weak-fuzzy contractions conditions from fuzzy cone metric spaces and prove some fixed point results for such mappings in the sense of Grabiec [24]. Without taking the continuity of the mapping T into consideration, our results unify and enrich the results of Jachymski [11], Gregori and Sapena [5] in the framework of fuzzy metric spaces. Our proofs modify the findings in existing literature. We call this contraction weak-fuzzy graphical contraction (wfgc) and discuss some slip-ups in this context. Throughout our discussion, we shall write fuzzy cone metric space as FCM-space in short.

The article is organized into five sections. In Section 2, we define the preliminaries and some basic definitions which help readers to understand our results easily. In Section 3, we establish some novel results of complete FCM-space with a weak-contraction has a unique fixed point endowed with a graph. We define the related definitions before the main result. In Section 4, we validate the obtained results via the existence of solution of an integral equation in graphical mapping. Few interesting examples are provided to explain our results. Finally, in Section 5, we discuss the conclusion and future directions of our work.

2. Preliminaries

We start by recalling some definitions and properties of fuzzy metric spaces and contractive mappings.

Definition 1 ([25]). *An operation $* : [0,1]^2 \to [0,1]$ is called continuous t-norm, if it satisfying the following conditions:*

(i) $*$ *is commutative, associative and continuous.*
(ii) $1 * a = a$ *and* $a * b \leq c * d$, *whenever* $a \leq c$ *and* $b \leq d, \forall a, b, c, d \in [0,1]$.

The classical examples of continuous T-norm are T_L, T_P and T_M defined as;

(a) *The Minimum operator* $T_M(a,b) = \min\{a,b\}$;
(b) *The product operator* $T_P(a,b) = ab$;
(c) *The Lukasiewicz's norm* $T_L(a,b) = \max\{a+b-1, 0\}$.

Lemma 1 ([26]). *If $*$ is a continuous t-norm, and $\{\varphi_n\}, \{\phi_n\}$ and $\{\psi_n\}$ are sequences such that $\varphi_n \to \varphi$, $\phi_n \to \phi$ and $\psi_n \to \psi$. Then*

$$\overline{\lim_{n\to\infty}}(\varphi_n * \phi_n * \psi_n) = \varphi * \overline{\lim_{n\to\infty}}\, \phi * \psi$$
$$\underline{\lim_{n\to\infty}}(\varphi_n * \phi_n * \psi_n) = \varphi_* \underline{\lim_{n\to\infty}}\, \phi * \psi$$

where $\overline{\lim}_{n\to\infty}$ and $\underline{\lim}_{n\to\infty}$ stands for limits supremum and limit infimum for left continuous and right continuous, respectively.

Definition 2 ([25]). *Let E be a real Banach space, a subset P of E is called cone if;*

(i) $P \neq \emptyset$, *closed and* $P \neq \{\theta\}$, *where* θ *is the zero element of* E;
(ii) *If* $a, b \geq 0$ *and* $\mu, \nu \in P$, *then* $a\mu + b\nu \in P$;
(iii) *If both* $\mu, -\mu \in P$, *then* $\mu = \theta$.

The cone P is called normal if there is a number $k > 0$ such that for all $\mu, \nu \in E$

$$0 \leq \mu \leq \nu \to \|\mu\| \leq k\|\nu\|$$

Throughout in our discussion we suppose E *is a Banach space,* P *is a cone in* E *with* $int(\mathrm{P}) \neq \phi$ *and* $\leq$ *is a partial ordering with respect to* P.

The following definition of fuzzy metric space was introduced by George and Veeramani [4]. We are concerned with this concept of fuzzy metric space.

Definition 3 ([4]). *A* 3*-tuple* $(U, \mathcal{M}_r, *)$ *is called fuzzy cone metric space, if* P *is a cone of* E, *U is an arbitrary set,* $*$ *is a continuous t-norm and* $\mathcal{M}_r$ *is a fuzzy set defined on* $U^2 \times int(\mathrm{P})$, *satisfying the following conditions;*

(fcm)-1 $\mathcal{M}_r(\mu, \nu, \varsigma) > 0$ *and* $\mathcal{M}_r(\mu, \nu, \varsigma) = 1 \Leftrightarrow \mu = \nu$;
(fcm)-2 $\mathcal{M}_r(\mu, \nu, \varsigma) = \mathcal{M}_r(\nu, \mu, \varsigma)$;
(fcm)-3 $\mathcal{M}_r(\mu, \nu, \varsigma) * \mathcal{M}_r(\nu, \omega, s) \leq \mathcal{M}_r(\mu, \omega, \varsigma + s)$;
(fcm)-4 $\mathcal{M}_r(\mu, \nu, .) : int(P) \to [0, 1]$ *is continuous.*

$\forall\, \mu, \nu, \omega \in U$ *and* $\varsigma, s \in int(\mathrm{P})$. *It is worth to note that* $0 < \mathcal{M}_r(\mu, \nu, \varsigma) < 1$ *(for all* $\varsigma > 0$,*) provided* $\mu \neq \nu$. *If we take* $E = \Re, \mathrm{P} = [0, \infty)$, *and* $a * b = ab$, *then every fuzzy metric space becomes fuzzy cone metric space.*

Definition 4 ([27]). *(i)* *Let* $(U, \mathcal{M}_r, *)$ *be a FCM-space,* $\mu \in U$ *and a sequence* (μ_j) *in U converges to* μ *if* $c \in (0, 1)$ *and* $\varsigma \gg \theta$ $\exists\, j_1 \in \mathbb{N}$ *such that* $\mathcal{M}_r(\mu_j, \mu, \varsigma) > 1 - c$, $\forall\, j \geq j_1$. *We may write this* $\lim_{j\to\infty} \mu_j = \mu$ *or* $\mu_j \to \mu$ *as* $j \to \infty$.

(ii) *A sequence* $\{\mu_n\}$ *in U is Cauchy sequence if* $c \in (0, 1)$ *and* $\varsigma \gg \theta$ $\exists\, j_1 \in \mathbb{N}$ *such that* $\mathcal{M}_r(\mu_j, \mu_k, \varsigma) > 1 - c$, $\forall\, j, k \geq j_1$.
(iii) *Fuzzy cone metric space is complete if Cauchy sequences in U are convergent.*
(iv) *A sequence* $\{\mu_n\}$ *in U is a* $\mathcal{G}$*-Cauchy sequence iff* $\lim_{n\to+\infty} \mathcal{M}_r(\mu_{n+p}, \mu_n, \varsigma) = 1$, *for any* $p > 0$ *and* $\varsigma > 0$.
(v) *The fuzzy metric space is called* $\mathcal{G}$*-complete if every* $\mathcal{G}$*-Cauchy sequence is convergent.*

Lemma 2 ([27]). *Let* $(U, \mathcal{M}_r, *)$ *be a FCM-space and let a sequence* (μ_j) *in U converges to a point* $\mu \in U$ *if and only if* $\mathcal{M}_r(\mu_j, \mu, \varsigma) \to 1$ *as* $j \to \infty$, *for each* $\varsigma \gg \theta$.

Example 1. *A function* $\mathcal{M}_r$ *be defined as;*

$$\mathcal{M}_r(\mu, \nu, \varsigma) = \frac{g_1(\varsigma)}{g_1(\varsigma) + \acute{m}.d(\mu, \nu)}, \quad \acute{m} > 0,$$

Then $(\mathcal{M}_r, \cdot)$ is a fuzzy metric on U. As a particular case if we take $g_1(\varsigma) = \varsigma^n$, $n \in \mathbb{N}$, and $\acute{m} = 1$,

$$\mathcal{M}_r(\mu, \nu, \varsigma) = \frac{\varsigma^n}{\varsigma^n + d(\mu, \nu)}$$

A well-known standard fuzzy metric is obtained for $n = 1$. If we are using g_1 as a constant function, $g_1(\varsigma) = k > 0$, and $\acute{m} = 1$, we get

$$\mathcal{M}_r(\mu, \nu, \varsigma) = \frac{k}{k + d(\mu, \nu)}$$

and so $(\mathcal{M}_r, \cdot)$ is a standard fuzzy metric on U.

Definition 5 ([28]). *Let $(U, \mathcal{M}_r, *)$ be a FCM-space. The fuzzy cone metric $\mathcal{M}_r$ is triangular, if*

$$\frac{1}{\mathcal{M}_r(\mu, \omega, \varsigma)} - 1 \leq \left(\frac{1}{\mathcal{M}_r(\mu, \nu, \varsigma)} - 1\right) + \left(\frac{1}{\mathcal{M}_r(\nu, \omega, \varsigma)} - 1\right), \quad \forall\, \mu, \nu, \omega \in U,\ \varsigma \gg \theta.$$

Definition 6 ([27]). *Let $(U, \mathcal{M}_r, *)$ be fuzzy cone metric space. A mapping $T : U \to U$ is said to be fuzzy cone contractive if $\exists\, a \in (0, 1)$, such that;*

$$\frac{1}{\mathcal{M}_r(T\mu, T\nu, \varsigma)} - 1 \leq a\left(\frac{1}{\mathcal{M}_r(\mu, \nu, \varsigma)} - 1\right), \quad \mu, \nu \in U,\ \varsigma \gg \theta. \tag{3}$$

Definition 7 ([29]). *A function $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function if $\psi(t)$ is monotone non-decreasing and continuous and $\psi(t) = 0$ if and only if $t = 0$.*

The following "Fuzzy cone Banach contraction theorem" is obtained in [27].

Theorem 1. *Let $(U, \mathcal{M}_r, *)$ be a complete FCM-space with Cauchy fuzzy cone contractive sequences and a fuzzy cone contractive mapping with $T : U \to U$. Then T has a unique fixed point.*

Recently Choudhry [26] have introduced the following weak-contractive condition in metric spaces.

Definition 8 ([26]). *Let (U, d) be a complete metric space. A mapping $T : U \to U$ is said to be weakly-contractive, if*

$$d(T\mu, T\nu) \leq d(\mu, \nu) - \psi(d(\mu, \nu)) \tag{4}$$

where $\mu, \nu \in U$, mapping $\psi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing, $\psi(\mu) = 0$ if and only if $\mu = 0$ and $\lim_{\mu \to \infty} \psi(\mu) = \infty$. If we take $\psi(\mu) = k\mu$, where $0 < k < 1$, then (4) reduces to (1).

Under this new scenario, we modify the definition of weak-contraction by Choudhry [26] from metric space to fuzzy cone metric space as follows:

Definition 9. *A mapping $T : U \to U$ in a FCM-space is said to be weakly contractive, if;*

$$\frac{1}{\mathcal{M}_r(T\mu, T\nu, \varsigma)} - 1 \leq \left(\frac{1}{\mathcal{M}_r(\mu, \nu, \varsigma)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_r(\mu, \nu, \varsigma)} - 1\right), \tag{5}$$

$\forall \mu, \nu \in U, \varsigma \gg \theta$, ψ is an altering distance function, $\psi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing, if $\psi(\mu) = 0, \Leftrightarrow \mu = 0$, and $\lim_{\mu \to \infty} \psi(\mu) = k\mu, \quad 0 < k < 1, \mu \in U$.

Theorem 2 ([30]). *Let $(U, \mathcal{M}_r, *)$ be FCM-space. A sequence $\{\mu_n\}$ in U is called convergent if for any $\varsigma \gg \theta$ and any $r \in [0,1]$, $\exists$ a natural number n_0 such that $\mathcal{M}_r(\mu_n, \mu, t) > 1 - r, \quad \forall n \geq n_0$. We denote this by $\lim_{n\to\infty} \mu_n = \mu$ or $\mu_n \to \mu$, as $n \to \infty$.*

It is clear ς-uniformly continuous and if mapping T is a fuzzy contractive mapping similar to those in [11,17], following the principles of the graphs.

Let Δ denote the diagonal of the Cartesian product $U \times U$. Consider the graph $\hat{\mathcal{G}}$ so that the collection of its vertices $V(\hat{\mathcal{G}})$ coincides with U, and the set of its edges $E(\hat{\mathcal{G}})$ contains all its loops, i.e., $E(\hat{\mathcal{G}}) \supseteq \Delta$. We assume that $\hat{\mathcal{G}}$ has no parallel edges. Therefore, we have

$$E\left(\hat{\mathcal{G}}^{-1}\right) = \{(\mu, \nu) \in U \times U : (\nu, \mu) \in E(\hat{\mathcal{G}})\}$$

The $\widetilde{\hat{\mathcal{G}}}$ character refers to the undirected graph obtained from $\hat{\mathcal{G}}$ ignoring the edge path. In fact, it would be more convenient for us to consider $\widetilde{\hat{\mathcal{G}}}$ as a graph that is symmetrical to the set of its edges. According to this convention,

$$E(\widetilde{\hat{\mathcal{G}}}) = E(\hat{\mathcal{G}}) \cup E\left(\hat{\mathcal{G}}^{-1}\right)$$

If μ and ν are vertices in a graph $\hat{\mathcal{G}}$, then a path in $\hat{\mathcal{G}}$ from μ to ν of length l is a sequence $(\mu_i)_{i=0}^{l}$ of $l+1$ vertices such that $\mu_0 = \mu$, $\mu_l = \nu$ and $(\mu_{i-1}, \mu_i) \in E(\hat{\mathcal{G}})$ for $i = 1, \ldots, l$.

If there is a path between any two vertices of $\hat{\mathcal{G}}$, the graph $\hat{\mathcal{G}}$ is called connected. A graph $\hat{\mathcal{G}}$ is weakly connected if $\widetilde{\hat{\mathcal{G}}}$ is connected. The subgraph consists of all edges and vertices which are contained in some path of $\hat{\mathcal{G}}$. In this case $V(\hat{\mathcal{G}}_\mu) = [\mu]_{\hat{\mathcal{G}}}$, where $[\mu]_{\hat{\mathcal{G}}}$ is the equivalence class of a relation R defined on $V(\hat{\mathcal{G}})$ by the rule:

$$\nu R \omega \text{ if there is a path in } \hat{\mathcal{G}} \text{ from } \nu \text{ to } \omega.$$

Clearly, $\hat{\mathcal{G}}_\mu$ is connected.

3. Fixed Point Results of Weak-Fuzzy Graphic Contractions

We now determine that a weak-contraction has a unique fixed point endowed with a graph in a complete FCM-space. Before the main outcome, we define the related definitions. We assume that U is a non-empty set in this section, $\hat{\mathcal{G}}$ is a graph directed to $V(\hat{\mathcal{G}}) = U$, and $E(\hat{\mathcal{G}}) \supseteq \Delta$. First, in the setting of fuzzy metric spaces, we define the Cauchy equivalent sequence and Weak-fuzzy contraction.

Definition 10 ([31]). *A mapping $T : U \to U$ is called Banach $\hat{\mathcal{G}}$-contraction or simply $\hat{\mathcal{G}}$-contraction if T preserves edges of $\hat{\mathcal{G}}$, i.e.,*

$$(\mu, \nu) \in E(\hat{\mathcal{G}}) \Rightarrow (T(\mu), T(\nu)) \in E(\hat{\mathcal{G}}), \quad \forall \mu, \nu \in U, \tag{6}$$

and mapping T reduces the edge weights of $\hat{\mathcal{G}}$ as follows, $\exists \alpha \in (0,1)$, $\forall \mu, \nu \in U$

$$(\mu, \nu) \in E \Rightarrow d(T(\mu), T(\nu)) \leqslant \alpha d(\mu, \nu) \tag{7}$$

Definition 11 ([32]). *A mapping $T : U \to U$ is said to be $\hat{\mathcal{G}}$-continuous, if for given $\mu \in U$ and sequence $\{\mu_n\}$, $\mu_n \to \mu$ as $n \to \infty$ and $(\mu_n, \mu_{n+1}) \in E(\hat{\mathcal{G}}), \forall n \in N \Longrightarrow T\mu_n \to T\mu$.*

Remark 1 ([31]). *For any sequence $(\mu_n)_{n \in N}$ in U, if $\mu_n \longrightarrow \mu$ and $(\mu_n, \mu_{n+1}) \in E(\hat{\mathcal{G}})$, for $n \in N$, then $(\mu_n, \mu) \in E(\hat{\mathcal{G}})$.*

Definition 12. *Let $(U, \mathcal{M}_r, *)$ be a fuzzy metric space and $\hat{\mathcal{G}}$ be a graph. Two sequences $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ in U are said to be Cauchy equivalent if each sequence is Cauchy and $\lim_{n\to\infty} \mathcal{M}_r(\mu_n, \nu_n, \varsigma) = 1$, $\forall \varsigma > 0$.*

Definition 13. *Let $(U, \mathcal{M}_r, *)$ be a fuzzy metric space and $\hat{\mathcal{G}}$ be a graph. The mapping $T : U \to U$ is said to be a weak-fuzzy graphical contraction (wfgc), if the following conditions are hold:*

(wfgc)-1 $\forall \mu, \nu \in U$, $(T\mu, T\nu) \in E(\hat{\mathcal{G}})$,, *i.e., T is edge-preserving;*

(wfgc)-2 $\left(\frac{1}{\mathcal{M}_r(T\mu,T\nu,\varsigma)} - 1\right) \leq \left(\frac{1}{\mathcal{M}_r(\mu,\nu,\varsigma)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_r(\mu,\nu,\varsigma)} - 1\right)$, $\varsigma > 0$, $(\mu, \nu) \in E(\hat{\mathcal{G}})$.

ψ is an altering distance function. $\psi : [0, \infty) \to [0, \infty)$ is monotone non-decreasing, continuous and $\psi(t) = 0 \Leftrightarrow t = 0$, $\lim_{\mu \to \infty} \psi(\mu) = k\mu$, $0 < k < 1, \mu \in U$.

Remark 2. *If T is weak-fuzzy graphical contraction mapping, then it is a fuzzy contraction of both $\mathcal{G}^{\hat{}-1}$-fuzzy and $\tilde{\hat{\mathcal{G}}}$-fuzzy.*

Definition 14. *Let $(U, \mathcal{M}_r, *)$ be a fuzzy metric space and $T : U \to U$ be a mapping. We denote the n^{th} iterate of T on $\mu \in U$ by $T^n\mu$ and $T^n\mu = TT^{n-1}\mu$, $\forall\, n \in \mathbb{N}$ with $T^0\mu = \mu$. T is called a Picard Operator (PO), if T has a unique fixed point u and*

$$\lim_{n\to\infty} \mathcal{M}_r(T^n\mu, u, \varsigma) = 1, \quad \forall \mu \in U,\ \varsigma > 0.$$

T is called Weakly Picard Operator (WPO) if there exists a fixed point $u_\mu \in U$ such that $\lim_{n\to\infty} \mathcal{M}_r\big(T^n\mu, u_\mu, \varsigma\big) = 1$, for all $\varsigma > 0$. Note that every Picard Operator is Weakly Picard Operator. Furthermore, the fixed point of (WPO) need not be unique. We will denote the set of all fixed points of T by Fix-T. A subset $A \subset U$ is said to be T-invariant if $T(A) \subset A$. The following lemma will be useful in this sequel.

Lemma 3. *Let $T : U \to U$ be a weak-fuzzy contraction, then given $\mu \in U$ and $\nu \in [\mu]_{\tilde{\hat{\mathcal{G}}}}$, we have* $\lim_{n\to\infty} \mathcal{M}_r(T^n\mu, T^n\nu, \varsigma) = 1, \forall \varsigma > 0$.

Proof. Let $\mu \in U$ and $\nu \in [\mu]_{\tilde{\hat{\mathcal{G}}}}$. Then by definition there exists a path $(\mu_i)_{i=0}^m$ in $\tilde{\hat{\mathcal{G}}}$ from μ to ν,, i.e., $\mu_0 = \mu$, $\mu_m = \nu$. We define $\mathcal{M}_r(T^n\mu, T^n\nu, t) = \mathcal{M}_r(T^n\mu_0, T^n\mu_m, t)$ and $(\mu_i, \mu_{i-1}) \in E(\tilde{\hat{\mathcal{G}}})$, for $i = 1, 2, \ldots, m$. From Definition 13, we assume that $\mu_{i+1} \neq \mu_i, i \in N$. Since the mapping T is weakly-contractive, $(T^n\mu_i, T^n\mu_{i-1}) \in E(\tilde{\hat{\mathcal{G}}})$ for $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\left(\frac{1}{\mathcal{M}_r(\mu_i, \mu_{i+1}, t)} - 1\right) &\leq \left(\frac{1}{\mathcal{M}_r(T\mu_i, T\mu_{i+1}, t)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_r(T\mu_i, T\mu_{i+1}, t)} - 1\right) \\ &< \left(\frac{1}{\mathcal{M}_r(\mu_{i-1}, \mu_i, t)} - 1\right)\end{aligned}$$

considering that the ψ function is non-decreasing, implies that $\mathcal{M}_r(\mu_i, \mu_{i+1}, t) > \mathcal{M}_r(\mu_{i-1}, \mu_i, t)$, $\forall i \in \mathbb{N}$ and hence $\{\mathcal{M}_r(\mu_{i-1}, \mu_i, t)\}$ is an increasing sequence of positive real numbers in $(0, 1]$. We can now choose a series that strictly decreases $(S_n)_{n\in\mathbb{N}}$ of positive numbers, such that

$$\sum_{i=1}^{\infty} S_i(t) = 1$$

Let $S(t) = \lim_{i\to\infty} \mathcal{M}_r(\mu_{i-1}, \mu_i, t)$, we show that $S(t) = 1$, for all $t \gg \theta$. If not, there exists $t \gg \theta$ such that $S(t) < 1$, then from the above inequality on taking $i \to \infty$, we obtain

$$\left(\frac{1}{S(t)} - 1\right) \leq \left(\frac{1}{S(t)} - 1\right) - \psi\left(\frac{1}{S(t)} - 1\right)$$

which is a contradiction. Therefore $M_r(\mu_i, \mu_{i+1}, t) \to 1$ as $i \to \infty$. Now, for each positive integer p, we have

$$\mathcal{M}_r(\mu_i, \mu_{i+p}, t) \geq \mathcal{M}_r\left(\mu_i, \mu_{i+1}, \frac{t}{p}\right) * \mathcal{M}_r\left(\mu_{i+1}, \mu_{i+2}, \frac{t}{p}\right) * \cdots * \mathcal{M}_r\left(\mu_{i+p-1}, \mu_{i+p}, \frac{t}{p}\right)$$

It follows that

$$\lim_{n\to\infty} \mathcal{M}_r(\mu_i, \mu_{i+p}, t) \geq 1 * 1 * \cdots * 1 = 1$$

thus we conclude that for $\varsigma \gg \theta$.

$$\lim_{n\to\infty} \mathcal{M}_r(\mu_n, \mu_{n+1}, \varsigma) = 1,$$

□

Theorem 3. *Equivalent to the following statements:*

1. *The graph $\hat{\mathcal{G}}$ is weakly connected;*
2. *For any weakly-fuzzy graphical contraction mapping $T : U \to U$, given $\mu, \nu \in U$ the sequences $(T^n\mu)_{n\in\mathbb{N}}$ and $(T^n\nu)_{n\in\mathbb{N}}$ are Cauchy equivalent;*
3. *For any weak-fuzzy graphical contraction mapping $T : U \to U, \text{card}(\text{Fix} - T) \leq 1$.*

Proof. $\underline{(1) \Rightarrow (2):}$

Let T be a weak-fuzzy graphical contraction and $\mu, \nu \in U$ then by hypothesis graph $\hat{\mathcal{G}}$ is weakly connected, therefore $[\mu]_{\tilde{\mathcal{G}}} = U$ and so $T^p\mu \in [\mu]_{\tilde{\mathcal{G}}}$, for all $p \in \mathbb{N}$. Now by Lemma 3, we have $(T^n\mu)_{n\in\mathbb{N}}$ is a Cauchy sequence. Similarly, $(T^n\nu)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $[\mu]_{\tilde{\mathcal{G}}} = U$. Therefore, $\lim_{n\to\infty} \mathcal{M}_r(T^n\mu, T^n\nu, \varsigma) = 1$, for all $\varsigma > 0$, follows from Lemma 3. Hence the sequences $(T^n\mu)_{n\in\mathbb{N}}$ and $(T^n\nu)_{n\in\mathbb{N}}$ are Cauchy equivalent.

$\underline{(2) \Rightarrow (3):}$

Let $\mu, \nu \in$ Fix-T, where T is a weak-fuzzy contraction. Since $\mu, \nu \in$ Fix-T^n and we have $\mathcal{M}_r(\mu, \nu, \varsigma) = \mathcal{M}_r(T^n\mu, T^n\nu, \varsigma)$. So by assumption $\mu = \nu$, $\text{card}(\text{Fix} - T) \leq 1$.

$\underline{(3) \Rightarrow (1):}$

Suppose (3) holds, but graph $\hat{\mathcal{G}}$ does not have a weak connection, i.e., it disconnects $\widetilde{\mathcal{G}}$. Let $u \in U$, be non-empty of both $[u]_{\tilde{\mathcal{G}}}$ and $U \backslash [u]_{\tilde{\mathcal{G}}}$ sets. Let $v \in U \backslash [u]_{\tilde{\mathcal{G}}}$, and define a mapping $T : U \to U$

$$T\mu = \begin{cases} u, & \text{if} \quad \mu \in [u]_{\tilde{\mathcal{G}}} \\ v, & \text{if} \quad \mu \in U \backslash [u]_{\tilde{\mathcal{G}}} \end{cases}$$

Now clearly Fix-$T = \{u, v\}$. We show that T is a weak-fuzzy contraction. If $(\mu, \nu) \in E(\hat{\mathcal{G}})$ then by the definition, we have $[\mu]_{\tilde{\mathcal{G}}} = [\nu]_{\tilde{\mathcal{G}}}$. Thus, either $\mu, \nu \in [u]_{\tilde{\mathcal{G}}}$ or $u, vs. \in U \backslash [u]_{\tilde{\mathcal{G}}}$. In both the cases, we have $T\mu = T\nu$, and so $(T\mu, T\nu) \in E(\hat{\mathcal{G}})$ since $E(\hat{\mathcal{G}}) \supseteq \Delta)$. Also, $\mathcal{M}_r(T\mu, T\nu, \varsigma) = 1, \forall\, \varsigma > 0$, so Definition 13 is satisfied. Thus, T is a weakly-fuzzy graphical contraction and $\text{card}(\text{Fix} - T) = 2 > 1$. This contradiction proves the result. □

Corollary 1. *Let $(U, \mathcal{M}_r, *)$ be a complete fuzzy metric space. Then following assertions hold:*

1. *The graph $\hat{\mathcal{G}}$ is weakly connected;*
2. *For any weak-fuzzy graph contraction mapping $T : U \to U$, there is $\mu^* \in U$ such that $\lim_{n\to\infty} T^n\mu = \mu^*, \forall \mu \in U$.*

Definition 15 ([33])**.** *A fuzzy metric $(U, \mathcal{M}_r, *)$ on an abstract (i.e., not necessarily topological) group G is said to be left invariant (respectively, right invariant), if $\mathcal{M}_r(\mu, \nu, \varsigma) = \mathcal{M}_r(a\mu, a\nu, \varsigma)$ (respectively, $\mathcal{M}_r(\mu, \nu, \varsigma) = \mathcal{M}_r(\mu a, \nu a, \varsigma)$), whenever $a, \mu, \nu \in G$ and $\varsigma > 0$.*

Proposition 1. *Assume that $T : U \to U$ is a weak-fuzzy graphical contraction such that for some $\mu_0 \in U$ we have $T\mu_0 \in [\mu_0]_{\tilde{\mathcal{G}}}$. Let $\widetilde{\mathcal{G}}_{\mu_0}$ be the component of $\widetilde{\mathcal{G}}$ containing μ_0. Then $[\mu_0]_{\tilde{\mathcal{G}}}$ is T-invariant and $T|_{[\mu_0]_{\tilde{\mathcal{G}}}}$ is a $\widetilde{\mathcal{G}}_{\mu_0}$ fuzzy contraction. Moreover, if $\mu, \nu \in [\mu_0]_{\tilde{\mathcal{G}}}$, then the sequences $(T^n\mu)_{n\in\mathbb{N}}$ and $(T^n\nu)_{n\in\mathbb{N}}$ are Cauchy equivalent.*

Proof. Let $\mu \in [\mu_0]_{\widetilde{\hat{\mathcal{G}}}}$. Then there is a path $(\mu_i)_{i=0}^N$ in $\widetilde{\hat{\mathcal{G}}}$ from μ_0 to μ,, i.e., $\mu_N = \mu$ and $(\mu_{i-1}, \mu_i) \in E(\widetilde{\hat{\mathcal{G}}})$, for $i = 1, \ldots, N$. Since T is a $\hat{\mathcal{G}}$-contraction which yields $(T\mu_{i-1}, T\mu_i) \in E(\widetilde{\hat{\mathcal{G}}})$ for $i = 1, \ldots, N$,, i.e., $(T\mu_i)_{i=0}^N$ is a path in $\widetilde{\hat{\mathcal{G}}}$ from $T\mu_0$ to $T\mu$. Thus, $T\mu \in [T\mu_0]_{\widetilde{\hat{\mathcal{G}}}}$. Since, by hypothesis, $T\mu_0 \in [\mu_0]_{\widetilde{\hat{\mathcal{G}}}}$,, i.e., $[T\mu_0]_{\widetilde{\hat{\mathcal{G}}}} = [\mu_0]_{\widetilde{\hat{\mathcal{G}}}}$, we infer $T\mu \in [\mu_0]_{\widetilde{\hat{\mathcal{G}}}}$. Thus, $[\mu_0]_{\widetilde{\hat{\mathcal{G}}}}$ is T-invariant.

Now, let $(\mu, \nu) \in E\left(\widetilde{\hat{\mathcal{G}}}_{\mu_0}\right)$. This means there is a path $(\mu_i)_{i=0}^N$ in $\widetilde{\hat{\mathcal{G}}}$ from μ_0 to ν such that $\mu_{N-1} = \mu$. Let $(\nu_i)_{i=0}^N$ be a path in $\widetilde{\hat{\mathcal{G}}}$ from μ_0 to $T\mu_0$. Repeating the argument from the first part of the proof, we infer $(\nu_0, \nu_1, \ldots, \nu_N, T\mu_1, \ldots, T\mu_N)$ is a path in $\widetilde{\hat{\mathcal{G}}}$ from μ_0 to $T\nu$; in particular, $(T\mu_{N-1}, T\mu_N) \in E\left(\widetilde{\hat{\mathcal{G}}}_{\mu_0}\right)$,, i.e., $(T\mu, T\nu) \in E\left(\widetilde{\hat{\mathcal{G}}}_{\mu_0}\right)$.

Moreover, $E\left(\widetilde{\hat{\mathcal{G}}}_{\mu_0}\right) \subseteq E(\widetilde{\hat{\mathcal{G}}})$ and T is a $\hat{\mathcal{G}}$-contraction. Thus, $T|_{[\mu_0]_{\widetilde{\hat{\mathcal{G}}}}}$ is a $\widetilde{\hat{\mathcal{G}}}_{\mu_0}$-contraction.

Finally, in view of Theorem 3, the second statement follows immediately from the first one since $\widetilde{\hat{\mathcal{G}}}_{\mu_0}$ is connected. □

Definition 16. *Let $(U, \mathcal{M}_r, *)$ be a fuzzy metric space and $\hat{\mathcal{G}}$ be a directed graph, $T : U \to U$ be a mapping and $\mu, \mu^* \in U$. Then we say that the quadruple $(U, \mathcal{M}_r, *, \hat{\mathcal{G}})$ have the property $(\mathcal{P}_T)$ if for any sequence $(T^n\mu)_{n\in\mathbb{N}}$, which converges to μ^* with $\left(T^n\mu, T^{n+1}\mu\right) \in E(\hat{\mathcal{G}})$, $\forall n \in \mathbb{N}$ there exists is a sequence $\left(T^{k_n}\mu\right)_{n\in\mathbb{N}}$ with $\left(T^{k_n}\mu, \mu^*\right) \in E(\hat{\mathcal{G}})$, for $n \in \mathbb{N}$.*

Theorem 4. *Let $(U, \mathcal{M}_r, *)$ be complete fuzzy cone metric space and $\hat{\mathcal{G}}$ be a directed graph. Assume that quadruple $(U, \mathcal{M}_r, *, \hat{\mathcal{G}})$ have the property $(\mathcal{P}_T)$. Let $T : U \to U$ be a weak-fuzzy graph contraction and $U_T = \{\mu \in U : (\mu, T\mu) \in E(\hat{\mathcal{G}})\}$, then the following assertions hold:*

(A) If $\mu \in U_T$, then $T|_{[\mu]_{\widetilde{\hat{\mathcal{G}}}}}$ is a Picard Operator;

(B) If $U_T \neq \emptyset$ and $\hat{\mathcal{G}}$ is weakly connected, then T is a Picard Operator. Furthure, for any weakly-fuzzy graph contraction $T : U \to U$, there is $\mu^* \in U$ such that $\lim_{n\to\infty} T^n\mu = \mu^*$ $\forall \mu \in U$.

(C) Fix-$T \neq \emptyset$, if and only if $U_T \neq \emptyset$.

(D) If $T \subseteq E(\hat{\mathcal{G}})$, then T is a Weakly Picard Operator (WPO).

Proof. To prove (A) : Let $\mu \in U_T$. By definition of U_T, $(\mu, T\mu) \in E(\hat{\mathcal{G}})$ and so we have $T\mu \in [\mu]_{\widetilde{\hat{\mathcal{G}}}}$. Now by Proposition 1 we have $T : [\mu]_{\widetilde{\hat{\mathcal{G}}}} \to [\mu]_{\widetilde{\hat{\mathcal{G}}}}$ and T is a $\widetilde{\hat{\mathcal{G}}}_\mu$-fuzzy contraction and if $\nu \in \widetilde{\hat{\mathcal{G}}}_\mu$ then $(T^n\mu)_{n\in\mathbb{N}}$ and $(T^n\nu)_{n\in\mathbb{N}}$ are Cauchy equivalent and so $(T^n\mu)_{n\in\mathbb{N}}$ is a Cauchy sequence. By completeness of U, $\exists \mu^* \in U$ such that

$$\lim_{n\to\infty} \mathcal{M}_r(T^n\mu, \mu^*, \varsigma) = 1, \quad \forall \varsigma > 0,$$

since $(\mu^*, T\mu) \in E(\hat{\mathcal{G}})$ we have $(\mu^*, T\mu) \in E(\widetilde{\hat{\mathcal{G}}})$ and so by Definition 13

$$\left(T^n\mu, T^{n+1}\mu^*\right) \in E(\hat{\mathcal{G}}), \quad \forall n \in \mathbb{N}$$

Now by property $(\mathcal{P}_T)$ $\exists$ a subsequence $\left(T^{k_n}\mu\right)_{n\in\mathbb{N}}$ such that $\left(T^{k_n}\mu, \mu\right) \in E(\hat{\mathcal{G}}), \forall n \in \mathbb{N}$. Hence, $\left(\mu, T\mu, T^2\mu, \ldots, T^{k_n}\mu, \mu^*\right)$ is a path in $\hat{\mathcal{G}}$ and so in $\widetilde{\hat{\mathcal{G}}}$. Therefore, $\mu^* \in [\mu]_{\widetilde{\hat{\mathcal{G}}}}$. Using Definition 13 ((wfgc)-2), we have

$$\frac{1}{\mathcal{M}_r\left(T^{k_n+1}\mu, T\mu^*, \varsigma\right)} - 1 \leq \left[\frac{1}{\mathcal{M}_r\left(T^{k_n}\mu, \mu^*, \varsigma\right)} - 1\right]$$

for all $\varsigma > 0$. In order to show that $\{\mu_n\}$ is a Cauchy sequence, if otherwise, there exist $0 < c < 1$, and increasing sub-sequence of integers $\{m_i\}, \{n_i\}$, such that for all integers

$m, n \leq i$, $n(i) > m(i) > i$. For each $c > 0$, and each $\varsigma > 0$, we can choose $\mathrm{i}_0 \in \mathrm{N}$ such that $\mathcal{M}_{\mathrm{r}}\left(\mu_{m(i)}, \mu_{n(i)}, \varsigma\right) > 1 - c, \forall\ \mathrm{m} \geq \mathrm{n}$. Then we have,

$$\mathcal{M}_r(\mu_{m(i)}, \mu_{n(i)}, \varsigma) \leq (1 - c)$$

and

$$\mathcal{M}_r(\mu_{m(i)}, \mu_{n(i)-1}, \varsigma) > (1 - c).$$

Now from the triangular property of FCM-space for all $i \geq 1, 0 < \varsigma < \frac{\acute{\epsilon}}{2}$, we obtain

$$\begin{aligned} (1 - c) &\geq \mathcal{M}_r(\mu_{m(i)}, \mu_{n(i)}, \acute{\epsilon}) \\ &\geq \mathcal{M}_r(\mu_{m(i)}, \mu_{m(i)-1}, \varsigma) * \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon} - 2\varsigma) * \mathcal{M}_r(\mu_{n(i)-1}, \mu_{n(i)}, \varsigma) \end{aligned} \tag{8}$$

We simplify the above terms in terms of ς, and using the fact that,

$$\lim_{n \to \infty} \mathcal{M}_r(\mu_n, \mu_{(n)+1}, \varsigma) = 1,$$

applying $\lim\limits_{i \to \infty}$, from inequality Definition 14, we have the following

$$(1 - c) \geq 1 * \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon} - 2\varsigma) * 1.$$

Let

$$T_1(\varsigma) = \overline{\lim_{i \to \infty}} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \varsigma), \quad \varsigma > 0$$

$$(1 - c) \geq 1 * \overline{\lim_{i \to \infty}} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon} - 2\varsigma) * 1 = T_1(\acute{\epsilon} - 2\varsigma).$$

We know that $\mathcal{M}_r$ is bounded with range $[0, 1]$, continuous and monotonically increasing in the third variable ς. Applying the limit supremum, and letting $\varsigma \to 0$ in above, we get

$$\overline{\lim_{i \to \infty}} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon}) \leq (1 - c) \tag{9}$$

Again, let

$$T_2(\varsigma) = \underline{\lim_{i \to \infty}} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \varsigma), \quad \varsigma > 0$$

for all $i \geq 1, \varsigma > 0$

$$\mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon} + \varsigma) \geq \mathcal{M}_r(\mu_{m(i)-1}, \mu_{m(i)}, \varsigma) * \mathcal{M}_r(\mu_{m(i)}, \mu_{n(i)-1}, \acute{\epsilon}).$$

Taking limit infimum in the above inequality, and by virtue of inequality Definition 14, we have

$$\begin{aligned} T_2(\acute{\epsilon} + \varsigma) &\geq \underline{\lim_{i \to \infty}} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{m(i)}, \varsigma) * (1 - c) \\ &= 1 * (1 - c) = (1 - c). \end{aligned}$$

Since, again we know that $\mathcal{M}_r$ is bounded with range $[0, 1]$, continuous and monotonically increasing in third variable ς, taking $\varsigma \to 0$ in above inequality

$$\underline{\lim_{i \to \infty}} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon}) \geq (1 - c) \tag{10}$$

Combining (9) and (10), we get

$$\lim_{i \to \infty} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon}) = (1 - c).$$

Now again from (9)

$$\overline{\lim_{i\to\infty}} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon}) \leq (1-c) \tag{11}$$

and for all $i \geq 1, \varsigma > 0$, we obtain

$$\begin{aligned}&\mathcal{M}_r(\mu_{m(i)}, \mu_{n(i)}, \acute{\epsilon} + 2\varsigma)\\ &\geq \mathcal{M}_r(\mu_{m(i)}, \mu_{m(i)-1}, \varsigma) * \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon}) * \mathcal{M}_r(\mu_{n(i)-1}, \mu_{n(i)}, \varsigma).\end{aligned}$$

Taking limit in above inequality and using the fact $\lim_{n\to\infty} \mathcal{M}_r(\mu_n, \mu_{n+1}, \varsigma) = 1$,

$$\underline{\lim}_{i\to\infty} \mathcal{M}_r(\mu_{m(i)}, \mu_{n(i)}, \acute{\epsilon} + 2\varsigma) \geq 1 * \underline{\lim}_{i\to\infty} \mathcal{M}_r(\mu_{m(i)-1}, \mu_{n(i)-1}, \acute{\epsilon}) * 1 = (1-c).$$

Since $\mathcal{M}_r$ is bounded with range $[0,1]$, continuous and monotonically increasing in third variable ς, taking $\varsigma \to 0$ in above inequality

$$\underline{\lim}_{i\to\infty} \mathcal{M}_r(\mu_{m(i)}, \mu_{n(i)}, \acute{\epsilon}) \geq (1-c). \tag{12}$$

Combining (11) and (12) we get

$$\lim_{i\to\infty} \mathcal{M}_r(\mu_{m(i)}, \mu_{n(i)}, \acute{\epsilon}) = (1-c).$$

Set $\mu = \mu_{m(i)-1}, \nu = \mu_{n(i)-1}$ and $\varsigma = \acute{\epsilon}$ in weak-contraction mapping, we deduce

$$\lim_{i\to\infty} \mu_n = \mu^* \in U.$$

Since we have $\{\mu_n\}$ is Cauchy, $\exists\ \mu \in U$ such that $\lim_{n\to\infty} \mu_n \to \mu^*$,, i.e.,

$$\lim_{n\to\infty} \mathcal{M}_r(\mu_n, \mu^*, \varsigma) = 1, \quad \varsigma \gg \theta$$

So we conclude that $T\mu_n = \mu, \forall\ n \geq 1$, obviously μ^* is fixed point of T, i.e, $\mu^* \in [\mu]_{\tilde{\mathcal{G}}}$. Let μ_1 and μ_2 are two fixed points of mapping T, we find a unique fixed point of T.

$$\begin{aligned}\frac{1}{\mathcal{M}_r(\mu_1, \mu_2, \varsigma)} - 1 &= \frac{1}{\mathcal{M}_r(T\mu_1, T\mu_2, \varsigma)} - 1\\ &\leq \left(\frac{1}{\mathcal{M}_r(\mu_1, \mu_2, \varsigma)} - 1\right) - \psi\left(\frac{1}{\mathcal{M}_r(\mu_1, \mu_2, \varsigma)} - 1\right).\end{aligned}$$

That is by property of ψ it is contradiction unless

$$\left(\frac{1}{\mathcal{M}_r(\mu_1, \mu_2, \varsigma)} - 1\right) = 0, \text{ or } \mathcal{M}_r(\mu_1, \mu_2, \varsigma) = 1,$$

that is $\mu_1 = \mu_2$. Hence T has a unique fixed point. This completes the proof of uniqueness of the fixed point.

Letting $n \to \infty$ in the above inequality we obtain $\mathcal{M}_r(\mu^*, T\mu^*, \varsigma) = 1, \forall \varsigma > 0$. Thus, $T\mu^* = \mu^*$,, i.e., $\mu^* \in [\mu]_{\tilde{\mathcal{G}}}$ is a fixed point of T and so by Theorem 3 $T|_{[\mu]_{\tilde{\mathcal{G}}}}$ is a Picard Operator.

To prove (B) : Let $U_T \neq \emptyset$ and graph $\hat{\mathcal{G}}$ is weak connected then $[\mu]_{\tilde{\mathcal{G}}} = U, \forall \mu \in U_T$ and so by (A), mapping T is a Picard Operator (PO).

To prove (C) : Note that if Fix $- T \neq \emptyset$ then $\exists$ some $\mu \in$ Fix $- T$, i.e., $T\mu = \mu$ and $E(\hat{\mathcal{G}}) \supseteq \Delta$ we have $(\mu, T\mu) \in E(\hat{\mathcal{G}})$. So $\mu \in U_T$ and Fix $-T \subseteq U_T \neq \emptyset$. If $U_T \neq \emptyset$, then by (A) for any $\mu \in U_T$, $T|_{[\mu]_{\tilde{\mathcal{G}}}}$ is a Picard Operator and so Fix $T \neq \emptyset$.

To prove (D) : If $T \subseteq E(\hat{\mathcal{G}})$, then $(\mu, T\mu) \in E(\hat{\mathcal{G}})\ \forall\ \mu \in U$, so $U = U_T$. The result follows from (A). $\square$

Example 2. *Let $(U, \mathcal{M}_r, *)$ be a complete fuzzy cone metric space and let $U = \Re$, $*$ be a minimum norm. Let $\mathcal{M}_r$ be defined by*

$$\mathcal{M}_r(\mu, \nu, \varsigma) = \begin{cases} \frac{\varsigma}{\varsigma + |\mu - \nu|}, & \text{if } \varsigma > 0; \\ 0, & \text{if } \varsigma = 0, \end{cases}$$

$\forall\ \mu, \nu \in U$. Furthermore, define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(s) = \frac{s}{8}$, for all $s \gg \theta$, $T(\mu) = \frac{\mu}{4}$. Obviousely, $T(\mu)$ and ψ are continuous functions. Then we have

$$\begin{aligned} \left(\frac{1}{\mathcal{M}_r(T\mu, T\nu, \varsigma)} - 1 \right) &= \frac{|\mu - v|}{4\varsigma} \\ &\geq \frac{|\mu - \nu|}{8\varsigma} \\ &= \left(\frac{1}{\mathcal{M}_r(\mu, \nu, \varsigma)} - 1 \right) - \psi\left(\frac{1}{\mathcal{M}_r(\mu, \nu, \varsigma)} - 1 \right) \end{aligned}$$

From the above inequality we conclude that (5) is satisfied. Thus, mapping T is a weak-fuzzy contraction.

Let $\hat{\mathcal{G}}$ be a directed graph with $V(\hat{\mathcal{G}}) = U$ and $E(\hat{\mathcal{G}}) = (U_o \times U_o) \cup (U_e \times U_e)$, where U_o and U_e be two subsets of even numbers and odd numbers from the set of $\Re$. Then it is easy to see that mapping T is weak-fuzzy graphical contraction (wfgc).

By definition of T, Fix $- T = (\mu, \nu) \in E(\hat{\mathcal{G}})$. *Furthermore, $T \subseteq E(\hat{\mathcal{G}})$ and T is a Weakly Picard Operator. Let μ_1, μ_2 be two fixed points of T, then*

$$\lim_{n \to \infty} \left(\frac{1}{\mathcal{M}_r(T\mu_1, T\mu_2, \varsigma)} - 1 \right) \leq \lim_{n \to \infty} \left| \frac{|\mu_1 - \mu_2|}{\varsigma} \right|, \quad \forall \mu_1, \mu_2 \in \Re$$

$T(\mu_1, \mu_2) = 0$, *implies that $\mu_1 = \mu_2 = 0, \forall\ \Re$. All the conditions of Theorem 4 are satisfied, then T have a unique common fixed point.*

4. An Application to Existence of Solution of Integral Equations

We will now establish a new result of the existence and uniqueness of solution nonlinear integral equation via weak-contraction mapping:

Let

$$\mu(\varsigma) = \int_a^b \acute{K}(\varsigma, s)\mu(s)ds + f(\varsigma) \tag{13}$$

where $a, b \in \Re, a < b, \acute{K} \in C\big([a, b] \times [a, b] \times \Re^2\big), f \in C[a, b]$ are given functions, and $\mu \in C[a, b]$ is an unknown function. Let $U = C[a, b]$, and metric $\mathcal{M}_r : U \times U \to [0, 1]$ given by;

$$\mathcal{M}_r(\mu, \nu, \varsigma) := \frac{\varsigma}{\varsigma + \left(\sup_{\varsigma \in [a,b]} |\mu(\varsigma)| + \sup_{\varsigma \in [a,b]} |\nu(\varsigma)| \right)} \quad \text{for all} \quad \mu, \nu \in C[a, b]$$

is a complete FCM-space, and

$$\frac{1}{\mathcal{M}_r(\mu, \nu, \varsigma)} - 1 := \frac{\sup_{\varsigma \in [a,b]} |\mu(\varsigma)| + \sup_{\varsigma \in [a,b]} |\nu(\varsigma)|}{\varsigma}, \quad \forall\ \mu, \nu \in C[a, b].$$

Furthermore, assume this FCM-space endowed with a graph $\hat{\mathcal{G}}$ and It is easy to see that T on $C[a,b]$ is a *weak*-contraction. Next, we attach to this integral equation with the operator $T : C[a,b] \to C[a,b]$ defined by;

$$T(\mu(\varsigma)) := \int_a^b \acute{K}(\varsigma, s)\mu(s)\, ds + f(\varsigma) \quad \forall \mu \in C[a,b],\ \varsigma \in [a,b]$$

We show that operator T satisfies the contraction condition Definition 13.

Furthermore, we define a continuous and non-decreasing function; $\psi : [0,+\infty) \to [0,+\infty)$ such that $\psi(\varsigma) < \varsigma, \quad \forall \varsigma > 0$, and $\psi(\varsigma) = 0 \Leftrightarrow \varsigma = 0$, such that $\forall\ s, \varsigma \in [a,b]$, $\forall \mu, \nu \in E(\hat{\mathcal{G}}) \Longrightarrow (T(\mu), T(\nu)) \in E(\hat{\mathcal{G}})$, and taking $\psi(\varsigma) = \frac{\varsigma}{2}$.

Since $\acute{K}$ is continuous, such that $|\acute{K}(\varsigma, s)| \le P,\ \forall \varsigma, s \in [a,b]$. We have the following approximation to illustrate that $\forall\ \mu_1, \mu_2 \in U$ operator T satisfies the contraction condition Definition 13:

$$\begin{aligned}
|T(\mu_1(\varsigma)) - T(\mu_2(\varsigma))| &= \lambda \left| \int_a^b (\acute{K}_1(\varsigma, s) - \acute{K}_2(\varsigma, s)) \right| (\mu_1(s) - \mu_2(s)) ds \\
&= \lambda \big| (\acute{K}_1(\varsigma, s) - \acute{K}_2(\varsigma, s)) \big| (\mu_1(s) - \mu_2(s)) \int_a^b ds \\
&\le |\lambda| (P(\varsigma, s)(b-a)) d(\mu_1, \mu_2), \quad \forall \varsigma \in [a,b].
\end{aligned}$$

Since $T^2(\mu_1) - T^2(\mu_2) = T(T(\mu_1) - T(\mu_2))$, we have

$$\begin{aligned}
\left| T^2(\mu_1(\varsigma)) - T^2(\mu_2(\varsigma)) \right| &= \lambda \int_a^b \acute{K}(\varsigma, s)\ |T(\mu_1(\varsigma)) - T(\mu_2(\varsigma))|\ ds \\
&\le \frac{|\lambda|^2\ P^2(\varsigma, s)(b-a)}{2}\ d(\mu_1, \mu_2)
\end{aligned}$$

Continuing this iterative process, we obtain

$$|T^n(\mu_1(\varsigma)) - T^n(\mu_2(\varsigma))| \le \frac{|\lambda|^n\ P^n(\varsigma, s)(b-a)}{n!}\ d(\mu_1, \mu_2), \quad \forall \varsigma \in [a,b]$$

As, $\frac{r^n}{n!} \to 0$ as $n \to \infty$ for any r is real number. Hence we conclude that $\exists n$ such that T^n is a contraction mapping. By taking n sufficiently large we have

$$\frac{[|\lambda| \| P(\varsigma, s)(b-a) \|]^n}{n!} < 1$$

where $0 < \lambda < 1$ is contraction constant.

Therefore, for all $\mu, \nu \in C[a,b] \in E(\hat{\mathcal{G}})$, i.e., the operator T satisfies the contraction condition of weak-fuzzy graphical contraction Definition 13. In addition, for each $\mu_0 \in U$, the successive approximation sequence $\{\mu_n\}$, defined by $\mu_n = T^n(\mu_0), \forall n \in \mathbb{N}$ converges to a unique fixed point of nonlinear integral Equation (13) with the operator T.

(1) There exist $\mu_0 \in U$ such that $(\mu_0, T(\mu_0)) \in E(\hat{\mathcal{G}})$.

(2) If $\{\mu_n\}$ is a sequence in U such that $(\mu_n, \mu_{n+1}) \in E(\hat{\mathcal{G}}),\ \forall n \in \mathbb{N}$ and $\mu_n \to \mu$ as $n \to +\infty$, then $(\mu_n, \mu) \in E(\hat{\mathcal{G}}),\ \forall n \in \mathbb{N}$.

(3) For any weakly-fuzzy graphical contraction $T : U \to U, \mathrm{card}(\mathrm{Fix} - T) \le 1$.

Thus, all the conditions of Theorem 3 are fulfilled, and therefore the mapping T has a fixed point, that is the solution in $U = C([a,b], \Re)$ of the integral Equation (13).

Example 3. *Let $U = [0,1]$ and the following integral equation be of the form*

$$\mu(\varsigma) = \int_0^1 \frac{1}{3(\varsigma + 1 + \mu(s))} ds + \frac{\varsigma}{3}$$

where $f = \frac{\varsigma}{3}$ *and* $\varsigma \in [0,1]$, *and*

$$\acute{K}_i(\varsigma, s, \mu_i(s)) = \frac{1}{3(\varsigma + 1 + \mu_i(s))}, \quad \text{where } i = 1,2$$

Then we have

$$\begin{aligned}\left\| \int_0^1 \acute{K}_1(\varsigma, s, \mu_1(s)) - \int_0^1 \acute{K}_2(\varsigma, s, \mu_2(s)) \right\| &= \int_0^1 \left\| \frac{1}{3(\varsigma + 1 + \mu_1(s))} - \frac{1}{3(\varsigma + 1 + \mu_2(s))} \right\| ds \\ &= \frac{1}{3} \left\| \frac{\mu_1(s) - \mu_2(s)}{(\varsigma + 1 + \mu_1(s))(\varsigma + 1 + \mu_2(s))} \right\| \int_0^1 ds \\ &\leq \frac{1}{3} P(\mu_1, \mu_2),\end{aligned}$$

where, $\lambda = \frac{1}{3} < 1$.

5. Conclusions

We have introduced the concept of weak-fuzzy graphical contractions in the framework of fuzzy cone metric spaces, which is a new expansion to the current writing in the context of fuzzy cone metric spaces. The obtained results revamp and extend some well-known results in the existing state-of-art, by relaxing continuity of the mapping involved. With the help of some novel lemmas, we provide that our proofs are straight forward. Non-trivial examples are presented to show the novelty of the established results. We conclude that research in fixed point theory with the graphical structure of contraction mappings is a field of active research in seeking the presence and uniqueness of fixed point for mappings that fulfill various contractive conditions. Any interested researchers may use this opportunity to carry out their future research in this field.

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