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Constructing C_0-Semigroups via Picard Iterations and Generating Functions: An Application to a Black–Scholes Integro-Differential Operator

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Abstract: An alternative approach is proposed for constructing a strongly continuous semigroup based on the classical method of successive approximations, or Picard iterations, together with generating functions. An application to a Black–Scholes integro-differential operator which arises in the pricing of European options under jump-diffusion dynamics is provided. The semigroup is expressed as the Mellin convolution of time-inhomogeneous jump and Black–Scholes kernel functions. Other applications to the heat and transport equations are also given. The connection of the proposed approach to the Adomian decomposition method is explored.

Keywords: strongly continuous semigroup; Picard iterations; generating functions; Black–Scholes theory; jump-diffusion process; partial integro-differential equation; Adomian method

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1. Introduction

Let $(X, \| \cdot \|)$ be a Banach space and $A : \mathscr{D}(A) \to X$ a linear operator with domain $\mathscr{D}(A) \subset X$. Fix $f \in \mathscr{D}(A)$ and consider the abstract Cauchy problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \quad u(0) = f. \tag{1}$$

It is well known [1–3] that if A is the generator of a C_0-semigroup $\{T(t) : t \geq 0\}$, then the unique solution of (1) is given by

$$u(t) = T(t)f.$$

Moreover, the abstract Cauchy problem (1) is uniformly well posed.

Given an operator A satisfying the conditions of the Hille–Yosida Theorem, one of the fundamental problems in the theory of linear semigroups is how to construct the C_0-semigroup $\{T(t) : t \geq 0\}$ generated by A. There are several such constructions (see [1–5] for more details). A special case is if A is a bounded operator, so that

$$T(t) = \exp(tA) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n. \tag{2}$$

For an arbitrary generator A, there are at least three constructions. One is

$$T(t)f = \lim_{h \to 0^+} \exp\left(\frac{t}{h}[T(h) - I]\right)f, \quad t \geq 0, \quad f \in X, \tag{3}$$

where I is the identity operator. The major drawback of (3) is that $T(h)$ has to be known at least for small values of $h > 0$. Hille's construction is

$$T(t)f = \lim_{n\to\infty} \left[\left(I - \frac{t}{n}A\right)^{-1}\right]^n f, \quad t \geqslant 0, \quad f \in X, \tag{4}$$

while Yosida's construction is

$$T(t)f = \lim_{h\to 0^+} \exp\left(\frac{t}{h}\left[(I - hA)^{-1} - I\right]\right)f, \quad t \geqslant 0, \quad f \in X. \tag{5}$$

Equations (3)–(5) provide three ways of obtaining the semigroup $\{T(t) : t \geqslant 0\}$ generated by an arbitrary A. However, as pointed out in [1], it is only rarely that a closed-form expression can be obtained for $T(t)$.

In this article we propose an alternative approach to constructing a strongly continuous semigroup based on the classical method of successive approximations together with generating functions. When A is a differential operator in (1), successive approximations are also referred to as Picard iterations. Some authors [6–9] refer to the method of successive approximation as the Adomian decomposition method (ADM) [10] although for linear differential equations the ADM is basically a Picard iteration technique.

The main application in this article is on finding the strongly continuous semigroup of a Black–Scholes integro-differential operator. To motivate the problem, let us first recall some relevant background material on option pricing. Let $S = \{S_t : t \geqslant 0\}$ and $W = \{W_t : t \geqslant 0\}$, respectively, denote the underlying asset price process and a Wiener process with respect to the risk-neutral measure. A popular model [11] for the asset price dynamics is given by

$$\frac{dS_t}{S_t} = (r - D)\,dt + \sigma\,dW_t. \tag{6}$$

Here, the risk-free rate r, the dividend yield D and the volatility σ are assumed to be constants with $r, \sigma > 0$ and $D \geqslant 0$. Denote the generic European option price at time t by V_t and the corresponding payoff function by f. At the expiry date T there holds $V_T = f(S_T)$. It can be shown [12] that $V_t = v(S_t, t)$, where the option pricing function $v = v(x, t)$ satisfies the Black–Scholes partial differential equation (PDE)

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + (r - D)x\frac{\partial v}{\partial x} - rv = 0, \quad x > 0, \quad 0 \leqslant t < T,$$
$$v(x, T) = f(x), \quad x > 0. \tag{7}$$

It is well known that geometric Brownian motion as assumed in the Black–Scholes asset price model (6) cannot capture many of the features of asset price returns such as the skew/smile features of the implied volatility surface. Merton [13] considered a jump-diffusion process that incorporates the possibility of the asset price to change at large magnitudes irrespective of the time interval between successive observations. As pointed out in [14], the jumps in the asset price can be accommodated by including an additional source of uncertainty into the asset price dynamics. Later empirical studies have demonstrated that the asset price is best described by a process with a discontinuous sample path [15–18]. The modified Merton asset price model is therefore

$$\frac{dS_t}{S_{t-}} = [r - D - \lambda E(Y - 1)]\,dt + \sigma\,dW_t + (Y - 1)\,dN_t, \tag{8}$$

where $S_{t-} = \lim_{u\to t-} S_u$, Y is a nonnegative continuous random variable with $Y - 1$ denoting the impulse change in the asset price from S_{t-} to $S_t = YS_{t-}$ as a consequence of the jump, E is the expectation operator and $N = \{N_t : t \geqslant 0\}$ is a Poisson process with constant intensity $\lambda \geqslant 0$ and such that $dN_t = 1$ (respectively, $dN_t = 0$) with probability $\lambda\,dt$ (respectively, $1 - \lambda\,dt$). Analogous to the Black–Scholes PDE (7), it can be shown [13]

that the European option pricing function v satisfies the Black–Scholes partial integro-differential equation (PIDE)

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + (r - D)x\frac{\partial v}{\partial x} - rv + \lambda \mathscr{I}(v; f_Y) = 0, \quad x > 0, \quad 0 \leqslant t < T,$$

$$v(x, T) = f(x), \quad x > 0,$$

(9)

where $\mathscr{I}(v; f_Y)$ is the integral operator defined by

$$\mathscr{I}(v; f_Y)(x, t) = \int_0^\infty [v(xy, t) - v(x, t)]f_Y(y) \, \mathrm{d}y$$

(10)

and f_Y is the probability density function of Y. When $\lambda = 0$, the Black–Scholes PIDE (9) reduces to the Black–Scholes PDE (7).

The outline of this paper is as follows. In Section 2 an outline of the construction method for C_0-semigroups is provided and illustrated for an initial value problem for the heat equation on the real line, thus recovering the Gauss–Weierstrass semigroup. A Black–Scholes integro-differential operator is considered in Section 3. To be able to find the associated generating function, an auxiliary problem is first analysed, whose solution is expressed in terms of time-homogeneous jump and Black–Scholes kernel functions. With the aid of the generating function, the C_0-semigroup is found and is expressed this time in terms of time-inhomogeneous jump and Black–Scholes kernel functions. Two further examples are studied in Section 4, in particular, how to handle initial-boundary value problems. Brief concluding remarks are given in Section 5.

2. A Method of Constructing C_0-Semigroups via Picard Iterations and Generating Functions

We can rewrite (1) into the equivalent expression

$$u = Bu, \quad Bu(t) = f + \int_0^t Au(\tau) \, \mathrm{d}\tau.$$

(11)

The method of successive approximations (or Picard iterations) entails the construction of a Cauchy sequence $(u_n)_{n=0}^\infty$ in an appropriate Banach space and defined recursively by $u_{n+1} = Bu_n$ for $n \geqslant 0$. This would imply that the sequence $(u_n)_{n=0}^\infty$ would converge to the solution u of (11) and therefore of the abstract Cauchy problem (1).

Let us see how we can define, at least formally, such a sequence $(u_n)_{n=0}^\infty$. Suppose that $(v_k)_{k=0}^\infty \subset \mathscr{D}(A)$ is the sequence given recursively by

$$v_0 = f, \quad v_{k+1} = Av_k, \quad k \geqslant 0.$$

(12)

Then define the sequence $(u_n)_{n=0}^\infty$ by

$$u_n(t) = \sum_{k=0}^n \frac{t^k}{k!}v_k, \quad n \geqslant 0.$$

(13)

Note that each summand (13) is in "separable" form. We claim that (13) satisfies $u_{n+1} = Bu_n$ for $n \geqslant 0$. Indeed, (11), the linearity of A and (12) imply that

$$Bu_n(t) = f + \int_0^t \sum_{k=0}^n \frac{\tau^k}{k!}Av_k \, \mathrm{d}\tau = f + \sum_{k=0}^n \frac{t^{k+1}}{(k+1)!}Av_k$$

$$= f + \sum_{k=0}^n \frac{t^{k+1}}{(k+1)!}v_{k+1} = \sum_{k=0}^{n+1} \frac{t^k}{k!}v_k = u_{n+1}(t)$$

for all $t \geqslant 0$ and thus proves the claim. The next step is to determine the sequence $(v_k)_{k=0}^{\infty}$ satisfying (12).

Remark 1. *In the Introduction the ADM was briefly mentioned. The idea of this method is to define a sequence $(w_k)_{k=0}^{\infty}$ through the recursion relation*

$$w_0 = f, \quad w_{k+1}(t) = f + \int_0^t A w_k(\tau) \, d\tau, \quad k \geqslant 0.$$

The solution of (1) *is sought in the form*

$$u(t) = \sum_{k=0}^{\infty} w_k(t). \tag{14}$$

We observe that this is essentially the limit of (13) *as $n \to \infty$ if we identify*

$$w_k(t) = \frac{t^k}{k!} v_k.$$

Under certain conditions, the Adomian series solution (14) can be shown to converge [19]. One disadvantage of the Adomian approach (see [20] for a critical review) is that it is often difficult to find a non-recursive formula for $w_k(t)$. Indeed, although from (12) we deduce the non-recursive formula

$$w_k(t) = \frac{t^k}{k!} A^k f,$$

for a complicated operator A such as in the Black–Scholes PIDE (9), it is not easy to calculate $A^k f$. To address this problem, here we introduce a generating function with respect to a parameter, say s, and whose coefficients are the elements of the sequence $(v_k)_{k=0}^{\infty}$. By determining the generating function, a non-recursive formula for v_k for all $k \geqslant 0$ (and thus of $w_k(t)$) can also be obtained.

Define the generating function

$$g(s) = \sum_{k=0}^{\infty} s^k v_k, \quad s \geqslant 0.$$

Suppose that we can find a solution $g(s)$ of the nonhomogeneous linear operator equation

$$Ag(s) - \frac{1}{s} g(s) = -\frac{1}{s} f \quad \text{or} \quad (I - sA)g(s) = f, \quad s > 0. \tag{15}$$

Then $g(s) = (I - sA)^{-1} f$ and therefore

$$v_k = \lim_{s \to 0^+} \frac{1}{k!} g^{(k)}(s) = \lim_{s \to 0^+} \frac{1}{k!} \frac{d^k}{ds^k}[(I - sA)^{-1} f], \quad k \geqslant 0. \tag{16}$$

Assuming that $(u_n)_{n=0}^{\infty}$ is a Cauchy sequence in an appropriate Banach space, then the semigroup $\{T(t) : t \geqslant 0\}$ generated by A is formally given by

$$\begin{aligned}
T(t)f = u(t) &= \lim_{n \to \infty} u_n(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} v_k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \lim_{s \to 0^+} \frac{1}{k!} g^{(k)}(s) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{(k!)^2} \lim_{s \to 0^+} \frac{d^k}{ds^k}[(I - sA)^{-1} f].
\end{aligned} \tag{17}$$

This formula gives an alternative representation of the semigroup (cf. (2)–(5)). Although the final form in (17) does not involve the generating function g, the latter is useful for perform-

ing explicit calculations when considering particular examples of the operator A, as will be shown later.

Rather than prove the above calculations rigorously, let us illustrate the main ideas with several examples beginning with the classical initial value problem for the heat equation on the real line:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = f(x), \quad x \in \mathbb{R}.$$

Here, $A = \mathrm{d}^2/\mathrm{d}x^2$ and

$$g(x, s) = \sum_{k=0}^{\infty} v_k(x) s^k,$$

where

$$\frac{\partial^2 g}{\partial x^2}(x, s) - \frac{1}{s} g(x, s) = -\frac{1}{s} f(x) \tag{18}$$

from (15). Using the method of variation of parameters (see [21] for instance), we obtain

$$\begin{aligned} g(x, s) &= \frac{1}{2\sqrt{s}} \left[\int_x^{\infty} \mathrm{e}^{(x-y)/\sqrt{s}} f(y) \, \mathrm{d}y + \int_{-\infty}^x \mathrm{e}^{-(x-y)/\sqrt{s}} f(y) \, \mathrm{d}y \right] \\ &= \frac{1}{2\sqrt{s}} \int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|/\sqrt{s}} f(y) \, \mathrm{d}y. \end{aligned} \tag{19}$$

Suppose first that f is analytic, i.e.

$$f(y) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} (y - x)^j, \quad x, y \in \mathbb{R}.$$

Then we can write

$$\begin{aligned} g(x, s) &= \frac{1}{2\sqrt{s}} \int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|/\sqrt{s}} \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} (y - x)^j \, \mathrm{d}y \\ &= \frac{1}{2\sqrt{s}} \sum_{j=0}^{\infty} (-1)^j \frac{f^{(j)}(x)}{j!} \int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|/\sqrt{s}} (x - y)^j \, \mathrm{d}y. \end{aligned}$$

Introducing the substitutions

$$z = \frac{x - y}{\sqrt{s}}, \quad \mathrm{d}z = -\frac{1}{\sqrt{s}} \, \mathrm{d}y,$$

we see that

$$\begin{aligned} g(x, s) &= \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \frac{f^{(j)}(x)}{j!} s^{j/2} \int_{-\infty}^{\infty} z^j \mathrm{e}^{-|z|} \, \mathrm{d}z \\ &= \frac{1}{2} \sum_{k=0}^{\infty} (-1)^{2k} \frac{f^{(2k)}(x)}{(2k)!} s^k \int_{-\infty}^{\infty} z^{2k} \mathrm{e}^{-|z|} \, \mathrm{d}z. \end{aligned}$$

But

$$\int_{-\infty}^{\infty} z^{2k} \mathrm{e}^{-|z|} \, \mathrm{d}z = 2 \int_0^{\infty} z^{2k} \mathrm{e}^{-|z|} \, \mathrm{d}z = 2(2k)!,$$

so that

$$g(x,s) = \sum_{k=0}^{\infty} f^{(2k)}(x) s^k.$$

This implies that $v_k(x) = f^{(2k)}(x)$ and thus (17) yields

$$(T(t)f)(x) = u(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(2k)}(x).$$

Now suppose that f is not analytic. Consider the Fourier transform

$$\hat{g}(\xi,s) = \mathscr{F}\{g(x,s); \xi\} = \int_{-\infty}^{\infty} e^{-i\xi x} g(x,s)\,dx.$$

Taking the Fourier transform of (18), we have

$$\hat{g}(\xi,s) = \frac{\hat{f}(\xi)}{1 + \xi^2 s} = \left(1 + \xi^2 s\right)^{-1} \hat{f}(\xi).$$

Differentiating with respect to s, we see that

$$\frac{\partial^k \hat{g}}{\partial s^k}(\xi,s) = (-1)^k k! \left(1 + \xi^2 s\right)^{-(k+1)} \xi^{2k} \hat{f}(\xi), \quad k \geqslant 0.$$

Hence (16) in transform space gives

$$\widehat{v_k}(\xi) = \lim_{s \to 0} \frac{1}{k!} \frac{\partial^k \hat{g}}{\partial s^k}(\xi,s) = (-1)^k \xi^{2k} \hat{f}(\xi) = (i\xi)^{2k} \hat{f}(\xi). \tag{20}$$

However, (17) in transform space is

$$\hat{u}(\xi,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \widehat{v_k}(\xi) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (i\xi)^{2k} \hat{f}(\xi) = e^{-\xi^2 t} \hat{f}(\xi). \tag{21}$$

Recalling that

$$\mathscr{F}^{-1}\left\{e^{-\xi^2/(4a)}; x\right\} = \sqrt{\frac{a}{\pi}} e^{-ax^2}, \quad a > 0$$

and taking $a = 1/(4t)$, we deduce from (17), (21) and the Fourier convolution property that

$$(T(t)f)(x) = u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)} f(y)\,dy, \tag{22}$$

which is the well-known Gauss–Weierstrass semigroup.

Remark 2. *We considered the case when f is analytic separately because it is possible to express (19) as a power series in s, from which it is easy to recover $v_k(x)$ for all $k \geqslant 0$. If f is not analytic, then it is not straightforward to find the derivatives of (19) and take the limit as $s \to 0^+$. Hence it is convenient to work in transform space, eventually arriving at (20). However, we should not invert (20) yet, because we would have to assume that f is infinitely differentiable, but wait until we have summed over all k in (21) and then invert $\hat{u}(\xi,t)$ from there. This "trick" will be applied again in the next section when we look at the Black–Scholes PIDE.*

3. A C_0-Semigroup for a Black–Scholes Integro-Differential Operator

To introduce more flexibility with the results, let us consider a slightly more general PIDE

$$\frac{\partial u}{\partial t} + ax^2 \frac{\partial^2 u}{\partial x^2} + bx \frac{\partial u}{\partial x} + cu + \lambda \mathscr{I}(u; f_Y) = 0, \quad x > 0, \quad 0 \leqslant t < T,$$
$$u(x, T) = f(x), \quad x > 0,$$
(23)

where a, b and c are constants ($a > 0$, $c < 0$) and $\mathscr{I}(u; f_Y)$ is as defined in (10). For example, if the underlying asset dynamics is given by (8), then $a = \sigma^2/2$, $b = r - D$ and $c = -r$. The solution of (23) is the price of a European option on an underlying stock. On the other hand, if $a = \sigma^2/2$, $b = 0$ and $c = -r$, then the solution of (23) is the price of a European option on an underlying futures contract. See [22] for a more detailed exposition.

To obtain an abstract Cauchy problem, we must convert the final value problem (23) to an initial value problem. Replacing t by $T - t$, we obtain the initial value problem

$$\frac{\partial u}{\partial t} = ax^2 \frac{\partial^2 u}{\partial x^2} + bx \frac{\partial u}{\partial x} + cu + \lambda \mathscr{I}(u; f_Y), \quad x > 0, \quad 0 < t \leqslant T,$$
$$u(x, 0) = f(x), \quad x > 0,$$
(24)

where t here now represents time to maturity rather than physical time. Defining the operator A by

$$Af(x) = ax^2 f''(x) + bx f'(x) + cf(x) + \lambda \int_0^\infty [f(xy) - f(x)] f_Y(y) \, dy, \quad x > 0,$$
(25)

then the initial value problem (24) can be reformulated as an abstract Cauchy problem

$$\frac{du}{dt} = Au, \quad u(0) = f.$$

3.1. An Auxiliary Problem

Let us consider the auxiliary problem

$$Av(x) = h(x), \quad x > 0,$$
(26)

where $h = h(x)$ is given and $v = v(x)$ is to be determined. We will solve (26) using the Mellin transform. Let

$$\hat{v}(\xi) = \mathscr{M}\{v(x); \xi\} = \int_0^\infty x^{\xi - 1} v(x) \, dx$$

denote the Mellin transform of v. Taking the transform of (26) and using standard properties [23,24], we obtain

$$\hat{v}(\xi) = \frac{\hat{h}(\xi)}{a\xi^2 + (a - b)\xi + c + \lambda E(Y^{-\xi}) - \lambda},$$
(27)

where $\hat{h}(\xi) = \mathscr{M}\{h(x); \xi\}$. Let $\mathscr{K} = \mathscr{K}(x)$ and $\mathscr{J} = \mathscr{J}(x)$ be functions that are to be determined such that

$$\hat{\mathscr{K}}(\xi) = -\frac{1}{a\xi^2 + (a - b)\xi + c}, \quad \hat{\mathscr{J}}(\xi) = \frac{1}{1 - \lambda [E(Y^{-\xi}) - 1] \hat{\mathscr{K}}(\xi)},$$
(28)

respectively. Then we can express (27) as

$$\hat{v}(\xi) = -\hat{\mathscr{J}}(\xi) \hat{\mathscr{K}}(\xi) \hat{h}(\xi).$$
(29)

Our goal here is to invert (29). Recall the Mellin convolution of f and g (with a slight abuse of notation):

$$(f * g)(x) = \int_0^\infty \frac{1}{y} f\left(\frac{x}{y}\right) g(y) \, dy.$$

It is not difficult to show that the convolution operator $*$ is commutative and associative. Moreover, the Mellin convolution property is

$$\widehat{(f * g)}(\xi) = \mathcal{M}\{(f * g)(x); \xi\} = \mathcal{M}\{f(x); \xi\} \mathcal{M}\{g(x); \xi\} = \hat{f}(\xi)\hat{g}(\xi).$$

We first determine the "time-homogeneous Black–Scholes kernel" $\mathcal{K}$ and the "time-homogeneous jump function" $\mathcal{J}$. The choice of terminology will be explained later.

3.1.1. Time-Homogeneous Black–Scholes Kernel

Partial fraction decomposition yields

$$\frac{1}{a\xi^2 + (a-b)\xi + c} = \frac{1}{a(\xi - \alpha_1)(\xi - \alpha_2)} = \frac{1}{a(\alpha_2 - \alpha_1)}\left(\frac{1}{\xi - \alpha_2} - \frac{1}{\xi - \alpha_1}\right),$$

where

$$\alpha_1 = \frac{b - a - \sqrt{(b-a)^2 - 4ac}}{2a}, \quad \alpha_2 = \frac{b - a + \sqrt{(b-a)^2 - 4ac}}{2a}.$$

It is easily shown that $\alpha_1 < 0$ and $\alpha_2 > 0$ since $a > 0$ and $c < 0$ by hypothesis. By a straightforward integration, we have that

$$\mathcal{M}\{x^{-\alpha_1}\mathbf{1}_{(0,1]}(x); \xi\} = \frac{1}{\xi - \alpha_1}, \quad \mathcal{M}\{-x^{-\alpha_2}\mathbf{1}_{(1,\infty)}(x); \xi\} = \frac{1}{\xi - \alpha_2}, \quad \alpha_1 < \text{Re}(\xi) < \alpha_2.$$

Hence

$$\mathcal{M}^{-1}\left\{\frac{1}{a\xi^2 + (a-b)\xi + c}; x\right\} = -\frac{1}{a(\alpha_2 - \alpha_1)}\left[x^{-\alpha_1}\mathbf{1}_{(0,1]}(x) + x^{-\alpha_2}\mathbf{1}_{(1,\infty)}(x)\right]$$

and from (28) we obtain the time-homogeneous Black–Scholes kernel

$$\mathcal{K}(x) = \mathcal{M}^{-1}\left\{-\frac{1}{a\xi^2 + (a-b)\xi + c}; x\right\} = \frac{1}{a(\alpha_2 - \alpha_1)}\left[x^{-\alpha_1}\mathbf{1}_{(0,1]}(x) + x^{-\alpha_2}\mathbf{1}_{(1,\infty)}(x)\right]. \tag{30}$$

3.1.2. Time-Homogeneous Jump Function

Define the sequence $(h_n)_{n=0}^\infty$ by

$$h_0(x) = \delta(x - 1), \quad h_1(x) = \frac{1}{x}f_Y\left(\frac{1}{x}\right), \quad h_{n+1}(x) = (h_1 * h_n)(x), \quad n \geq 1, \tag{31}$$

where f_Y is the probability density function of the random variable Y (see (8) and (10)) and δ is the Dirac delta function. It was shown in [14,25] that

$$\hat{h}_n(\xi) = \mathcal{M}\{h_n(x); \xi\} = \left[E(Y^{-\xi})\right]^n, \quad n \geq 0. \tag{32}$$

Using (28), we see that

$$\hat{\mathcal{J}}(\xi) = \frac{1}{1 - \lambda\left[E(Y^{-\xi}) - 1\right]\hat{\mathcal{K}}(\xi)} = \sum_{n=0}^\infty \lambda^n \left[\hat{\mathcal{K}}(\xi)\right]^n \left[E(Y^{-\xi}) - 1\right]^n$$

provided that λ is sufficiently small. The binomial theorem and (32) give

$$[E(Y^{-\xi}) - 1]^n = \sum_{j=0}^{n} \binom{n}{j} [E(Y^{-\xi})]^j (-1)^{n-j} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \widehat{h}_j(\xi),$$

implying that

$$\widehat{\mathcal{J}}(\xi) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \lambda^n \binom{n}{j} \widehat{h}_j(\xi) [\widehat{\mathcal{K}}(\xi)]^n.$$

Hence from the Mellin convolution property we get

$$\mathcal{J}(x) = \mathcal{M}^{-1}\{\widehat{\mathcal{J}}(\xi); x\} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \lambda^n \binom{n}{j} (h_j * \overbrace{\mathcal{K} * \cdots * \mathcal{K}}^{n \text{ factors}})(x).$$

Defining the operator

$$\mathcal{K}^n = \overbrace{\mathcal{K} * \cdots * \mathcal{K}}^{n \text{ factors}}, \quad n \geqslant 1,$$

where $\mathcal{K}^0$ is the identity operator, we obtain the time-homogeneous jump function

$$\mathcal{J}(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \lambda^n \binom{n}{j} (h_j * \mathcal{K}^n)(x). \tag{33}$$

3.1.3. Solution of the Auxiliary Problem

We are now ready to invert (29). Define

$$v_0(x) = -(\mathcal{K} * h)(x) = -\int_0^{\infty} \frac{1}{y} \mathcal{K}\left(\frac{x}{y}\right) h(y)\, \mathrm{d}y. \tag{34}$$

The Mellin convolution property implies that $\widehat{v}_0(\xi) = -\widehat{\mathcal{K}}(\xi)\widehat{h}(\xi)$ and (29) can be rewritten as $\widehat{v}(\xi) = \widehat{\mathcal{J}}(\xi)\widehat{v}_0(\xi)$. Using the Mellin convolution property and (33), we have

$$v(x) = (\mathcal{J} * v_0)(x) = \int_0^{\infty} \frac{1}{y} \mathcal{J}\left(\frac{x}{y}\right) v_0(y)\, \mathrm{d}y = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \lambda^n \binom{n}{j} (h_j * \mathcal{K}^n * v_0)(x) \tag{35}$$

as the explicit solution of the auxiliary problem (26).

Having obtained the solution to the auxiliary problem, let us return to the problem of finding the semigroup generated by A in (25). The nonhomogeneous linear operator equation (15) in this case is

$$ax^2 \frac{\partial^2 g}{\partial x^2}(x,s) + bx \frac{\partial g}{\partial x}(x,s) + \left(c - \frac{1}{s}\right) g(x,s) + \lambda \int_0^{\infty} [g(xy,s) - g(x,s)] f_Y(y)\, \mathrm{d}y = -\frac{1}{s} f(x).$$

Note that this has the form of the auxiliary problem (26) but with c replaced by $c - 1/s$ and $h(x) = -f(x)/s$.

If we let

$$p(\xi) = -a\xi^2 + (b - a)\xi - c, \tag{36}$$

then (30) and (33) become

$$\mathcal{K}(x,s) = \mathcal{M}^{-1}\left\{-\frac{1}{a\xi^2 + (a - b)\xi + c - 1/s}; x\right\} = \mathcal{M}^{-1}\left\{\frac{s}{1 + p(\xi)s}; x\right\}$$

and

$$\mathscr{J}(x,s) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \lambda^n \binom{n}{j} \left(h_j * \mathscr{K}(\cdot,s)^n\right)(x), \tag{37}$$

respectively. Moreover, (34) and (35) simplify to

$$g_0(x,s) = \frac{1}{s}(\mathscr{K}(\cdot,s) * f)(x)$$

and

$$g(x,s) = (\mathscr{J}(\cdot,s) * g_0(\cdot,s))(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \lambda^n \binom{n}{j} \left(h_j * \mathscr{K}(\cdot,s)^n * g_0(\cdot,s)\right)(x),$$

respectively. Equations (28) and (29) imply that

$$\hat{g}(\xi,s) = \frac{1}{s} \hat{\mathscr{J}}(\xi,s) \hat{\mathscr{K}}(\xi,s) \hat{f}(\xi) = \hat{f}(\xi) \left[1 + \Phi_\lambda(\xi)s\right]^{-1}, \quad \Phi_\lambda(\xi) = p(\xi) + \lambda \left[1 - E(Y^{-\xi})\right].$$

Differentiating with respect to s yields

$$\frac{\partial^k \hat{g}}{\partial s^k}(\xi,s) = (-1)^k k! \hat{f}(\xi) \left[1 + \Phi_\lambda(\xi)s\right]^{-(k+1)} \left[\Phi_\lambda(\xi)\right]^k, \quad k \geqslant 0.$$

Hence (16) in transform space implies that $\hat{v}_k(\xi) = (-1)^k [\Phi_\lambda(\xi)]^k \hat{f}(\xi)$, which when substituted into (17) in transform space gives

$$\begin{aligned} \hat{u}(\xi,t) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \hat{v}_k(\xi) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (-1)^k [\Phi_\lambda(\xi)]^k \hat{f}(\xi) \\ &= e^{-\Phi_\lambda(\xi)t} \hat{f}(\xi) = e^{\lambda[E(Y^{-\xi})-1]t} e^{-p(\xi)t} \hat{f}(\xi). \end{aligned} \tag{38}$$

We recall some preliminary results. Let

$$\mathscr{K}_t(x) = \frac{x e^{(b+c)t}}{\sqrt{2at}} N'(z_1(x,t)) = \frac{e^{ct}}{\sqrt{2at}} N'(z_2(x,t)), \quad x > 0, \quad t > 0, \tag{39}$$

where

$$z_1(x,t) = \frac{\log(x) + (b+a)t}{\sqrt{2at}}, \quad z_2(x,t) = \frac{\log(x) + (b-a)t}{\sqrt{2at}}$$

and N is the cumulative distribution function of a standard normal random variable. Equation (39) is precisely the time-inhomogeneous Black–Scholes kernel defined and studied in [14,22,24–27]. Note that t here is time to maturity. It was shown in [14,22] that the Mellin transform of the time-inhomogeneous Black–Scholes kernel is

$$\widehat{\mathscr{K}_t}(\xi) = \mathscr{M}\{\mathscr{K}_t(x); \xi\} = e^{-p(\xi)t}, \tag{40}$$

where p is as defined in (36). A time-inhomogeneous jump function was also defined in [14,22,25], namely

$$\mathscr{J}_t(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} h_n(x), \tag{41}$$

where $(h_n)_{n=0}^{\infty}$ is the same as the sequence given in (31). We observe that $\mathscr{J}_t(x) = \delta(x-1)$ when $\lambda = 0$. It was shown in [25] that

$$\widehat{\mathscr{J}_t}(\xi) = \mathscr{M}\{\mathscr{J}_t(x); \xi\} = e^{\lambda[E(Y^{-\xi})-1]t}. \tag{42}$$

Substituting (40) and (42) into (38), invoking the Mellin convolution property and recalling (17), we finally obtain the following C_0-semigroup generated by the Black–Scholes integro-differential operator A in (25):

$$(T(t)f)(x) = u(x,t) = (\mathscr{J}_t * \mathscr{K}_t * f)(x), \tag{43}$$

where $\mathscr{J}_t$ and $\mathscr{K}_t$ are defined in (41) and (39), respectively. This is analogous to the Gauss–Weierstrass semigroup (22).

Remark 3. *In the absence of jumps in the underlying asset (i.e., $\lambda = 0$), the Black–Scholes PIDE (9) reduces to the Black–Scholes PDE (7). Then (43) simplifies to*

$$(T(t)f)(x) = u_0(x,t) = (\mathscr{K}_t * f)(x) = (f * \mathscr{K}_t)(x) = \int_0^\infty \frac{1}{y} f\left(\frac{x}{y}\right) \mathscr{K}_t(y) \, dy. \tag{44}$$

Furthermore, suppose that f is analytic, so that

$$f(y) = \sum_{n=0}^\infty \frac{f^{(n)}(x)}{n!}(y-x)^n, \quad x > 0, \quad y > 0.$$

This implies that

$$f\left(\frac{x}{y}\right) = \sum_{n=0}^\infty \frac{f^{(n)}(x)}{n!}\left(\frac{x}{y}\right)^n (1-y)^n = \sum_{n=0}^\infty \frac{f^{(n)}(x)}{n!} \frac{x^n}{y^n} \sum_{j=0}^n \binom{n}{j}(-1)^{n-j} y^{n-j}.$$

Substituting into (44), we see that

$$u_0(x,t) = \int_0^\infty \frac{1}{y} \sum_{n=0}^\infty \frac{f^{(n)}(x)}{n!} \frac{x^n}{y^n} \sum_{j=0}^n \binom{n}{j}(-1)^{n-j} y^{n-j} \mathscr{K}_t(y) \, dy$$

$$= \sum_{n=0}^\infty \frac{x^n f^{(n)}(x)}{n!} \sum_{j=0}^n \binom{n}{j}(-1)^{n-j} \int_0^\infty y^{-j-1} \mathscr{K}_t(y) \, dy.$$

However, we deduce from (40) that

$$\int_0^\infty y^{-j-1} \mathscr{K}_t(y) \, dy = \mathcal{M}\{\mathscr{K}_t(y); -j\} = e^{-p(-j)t} = \sum_{k=0}^\infty [-p(-j)]^k \frac{t^k}{k!}.$$

Thus

$$u_0(x,t) = \sum_{n=0}^\infty \frac{x^n f^{(n)}(x)}{n!} \sum_{j=0}^n \binom{n}{j}(-1)^{n-j} \sum_{k=0}^\infty [-p(-j)]^k \frac{t^k}{k!}$$

$$= \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{n=0}^\infty \frac{x^n f^{(n)}(x)}{n!} \sum_{j=0}^n \binom{n}{j}(-1)^{n-j} [-p(-j)]^k.$$

If $F(x) = [-p(-x)]^k$ and Δ^n is the nth-order forward difference operator, then

$$\Delta^n F(x) = \sum_{j=0}^n \binom{n}{j}(-1)^{n-j} F(x+j) = \sum_{j=0}^n \binom{n}{j}(-1)^{n-j}[-p(-x-j)]^k.$$

In particular,

$$\Delta^n F(0) = \sum_{j=0}^n \binom{n}{j}(-1)^{n-j}[-p(-j)]^k.$$

However, $[-p(-j)]^k$ is a polynomial of degree $2k$ in j, so $\Delta^n F(0) = 0$ for all $n \geqslant 2k + 1$. Therefore the solution of (24) when $\lambda = 0$ is

$$u_0(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{n=0}^{2k} \frac{x^n f^{(n)}(x)}{n!} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [-p(-j)]^k. \tag{45}$$

As a special case, if we choose $a = \sigma^2/2$, $b = r - D$ and $c = -r$, then (45) is the solution of the Black–Scholes PDE (7) obtained by Bohner and Zheng [6] using the ADM; cf. (2.3) in [6]. Note that f is assumed to be analytic although in financial practice f is a piecewise linear function, which is non-analytic in general. Several authors [7,28,29] replaced the non-analytic call/put payoff functions by analytic approximations. Ke et al. [8] adapted the Adomian decomposition method to handle the non-differentiable call payoff functions by "transferring" the non-singular point to infinity. However, as the calculations leading to (43) show, all of the above are not necessary because the non-differentiability of the payoff function f at a finite number of points is not an issue if we work with the Adomian decomposition method (i.e., Picard iterations with generating functions) in Mellin transform space and only invert until the penultimate step (38) after setting up the series. Examples of how to evaluate (43) for specific payoff functions using the so-called Black–Scholes kernel identities can be found in [14,22,24–27].

Remark 4. *In the presence of jumps in the underlying asset (i.e., $\lambda \neq 0$) and if f is analytic, then from (43) and (41) we get that the solution of the Black–Scholes PIDE (24) is*

$$u(x,t) = (\mathscr{J}_t * u_0(\cdot, t))(x) = e^{-\lambda t} \sum_{m=0}^{\infty} \frac{\lambda^m t^m}{m!} (h_m * u_0(\cdot, t))(x),$$

where u_0 is given by (45). This extends the result of [6] for an analytic f to the jump-diffusion case.

4. Discussion

The construction of strongly continuous semigroups through the method of successive approximations and generating functions proposed in this article can also be used for other problems. Here we take a look at two more classical problems. Note that the results given here are well known and are only chosen to illustrate how generating functions can be useful for explicit calculations.

Example 1. *Consider the heat equation defined on a bounded interval with homogeneous Dirichlet boundary conditions:*

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\
u(x,0) &= f(x), \quad 0 \leqslant x \leqslant L, \\
u(0,t) &= u(L,t) = 0, \quad t > 0.
\end{aligned}$$

As before, $A = \mathrm{d}^2/\mathrm{d}x^2$. Equation (15) in this case is the boundary value problem

$$\frac{\partial^2 g}{\partial x^2}(x,s) - \frac{1}{s} g(x,s) = -\frac{1}{s} f(x), \quad g(0,s) = g(L,s) = 0. \tag{46}$$

Let

$$\hat{g}(n,s) = \mathscr{S}\left\{ \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) g(x,s) \, \mathrm{d}x; n \right\}$$

denote the Fourier sine transform of *g* with respect to the first argument. Taking the transform of (46) gives

$$\hat{g}(n,s) = \frac{\hat{f}(n)}{1 + (n\pi/L)^2 s}, \quad \hat{f}(n) = \mathscr{S}\{f(x); n\} = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx.$$

Using standard formulas, the inverse Fourier sine transform is

$$g(x,s) = \sum_{n=1}^{\infty} \hat{g}(n,s)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \frac{\hat{f}(n)}{1 + (n\pi/L)^2 s}\sin\left(\frac{n\pi x}{L}\right).$$

Differentiating with respect to *s*, we get

$$\frac{\partial^k g}{\partial s^k}(x,s) = \sum_{n=1}^{\infty} \hat{f}(n)(-1)^k k!\left[1 + \left(\frac{n\pi}{L}\right)^2 s\right]^{-(k+1)}\left(\frac{n\pi}{L}\right)^{2k}\sin\left(\frac{n\pi x}{L}\right), \quad k \geqslant 0.$$

Hence (16) gives

$$v_k(x) = \lim_{s\to 0^+}\frac{1}{k!}\frac{\partial^k g}{\partial s^k}(x,s) = (-1)^k \sum_{n=1}^{\infty} \hat{f}(n)\left(\frac{n\pi}{L}\right)^{2k}\sin\left(\frac{n\pi x}{L}\right)$$

and we deduce from (17) that

$$(T(t)f)(x) = u(x,t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}(-1)^k \sum_{n=1}^{\infty}\hat{f}(n)\left(\frac{n\pi}{L}\right)^{2k}\sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty}\hat{f}(n)\sin\left(\frac{n\pi x}{L}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{k!}\left(\frac{n\pi}{L}\right)^{2k} = \sum_{n=1}^{\infty}\hat{f}(n)e^{-n^2\pi^2 t/L^2}\sin\left(\frac{n\pi x}{L}\right).$$

Remark 5. *The same procedure can be performed for other boundary conditions, e.g., homogeneous Neumann boundary conditions. In general, when reformulating an initial-boundary value problem for a PDE as an abstract Cauchy problem* (1), *the boundary conditions are included with the defining Equation* (15) *of the generating function. This avoids the complications arising from the use of 'partial solutions' in the ADM* [9,30]. *In the context of the Black–Scholes PIDE* (9), *pricing of barrier options with jumps in the underlying asset* [22] *is an example of an initial-boundary value problem that is amenable to the proposed method. The details are nontrivial and will be presented in a forthcoming article by the author.*

Example 2. *Consider the transport equation problem*

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad x > 0, \quad t > 0,$$
$$u(x,0) = f(x), \quad x > 0.$$

Here, $A = d/dx$. The defining Equation (15) *for the generating function is now*

$$\frac{\partial g}{\partial x}(x,s) - \frac{1}{s}g(x,s) = -\frac{1}{s}f(x). \tag{47}$$

Denote the Laplace transform of g with respect to the first argument by

$$\hat{g}(p,s) = \mathscr{L}\{g(x,s); p\} = \int_0^{\infty} e^{-px} g(x,s)\,dx.$$

Taking the Laplace transform of (47), *we have*

$$\hat{g}(p,s) = -\frac{1}{s}\frac{\hat{f}(p)}{p-1/s} = (1-ps)^{-1}\hat{f}(p), \quad \hat{f}(p) = \mathscr{L}\{f(x);p\} = \int_0^\infty \mathrm{e}^{-px}f(x)\,\mathrm{d}x.$$

It is straightforward to show that

$$\frac{\partial^k \hat{g}}{\partial s^k}(p,s) = (-1)^k k!(1-ps)^{-(k+1)}(-p)^k\hat{f}(p)$$

and (16) *in transform space is* $\widehat{v_k}(p) = p^k\hat{f}(p)$. *Hence* (17) *in transform space becomes*

$$\hat{u}(p,t) = \sum_{k=0}^\infty \frac{t^k}{k!}p^k\hat{f}(p) = \mathrm{e}^{pt}\hat{f}(p).$$

Letting H denote the usual Heaviside function, we deduce from inverting $\hat{u}(p,t)$ *and* (17) *that*

$$(T(t)f)(x) = u(x,t) = H(x+t)f(x+t) = f(x+t).$$

This recovers the translation semigroup.

5. Concluding Remarks

In this article we proposed an alternative construction method for a strongly continuous group using the classical method of successive approximations (or Picard iterations) together with the employment of generating functions. The calculations were kept at a formal level in the sense that the correct function spaces were not carefully considered, but this can of course be done (e.g., with the aid of the theory of entire vectors; see [31,32] and the references therein) but is outside the scope of this article. Aside from illustrating the construction method to classical problems involving the heat equation and the transport equation, the main contribution of this article is the determination of the C_0-semigroup for the Black–Scholes integro-differential operator (25), which includes the Black–Scholes differential operator as a special case. This C_0-semigroup is expressed as a Mellin convolution of the time-inhomogeneous jump function and the time-inhomogeneous Black–Scholes kernel, and is analogous to the Gauss–Weierstrass and Poisson semigroups.

The results of this article serve as a foundation for constructing other semigroups of operators arising in option pricing when the underlying asset essentially follows geometric Brownian motion. Barrier options were already mentioned, but other types of options such as Asian options, American options and European options with stochastic volatility in the underlying asset are currently under investigation. Another interesting direction to explore is the connection of the approach given in this article with singularly perturbed problems. Indeed, in the semigroup formula given in (17), the crucial step is to calculate the generating function g in (15) and its derivatives with respect to s, and then take their limits as $s \to 0^+$. If $g(s)$ turns out to be very difficult to calculate exactly, then one could try to approximate it using matched asymptotic expansions, adapted for operator equations, by taking advantage of the smallness of s.

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References

1. Belleni-Morante, A.; McBride, A.C. *Applied Nonlinear Semigroups*; John Wiley & Sons: Chichester, UK, 1998.
2. Engel, K.-J.; Nagel, R. *A Short Course on Operator Semigroups*; Springer: New York, NY, USA, 2006.
3. Hille, E.; Phillips, R.S. *Functional Analysis and Semi-Groups*; American Mathematical Society: Charles St, RI, USA, 1957.

4. McBride, A.C. *Semigroups of Linear Operators: An Introduction*; Longman: Harlow, UK, 1987.
5. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983.
6. Bohner, M.; Zheng, Y. On analytical solutions of the Black–Scholes equation. *Appl. Math. Lett.* **2009**, *22*, 309–313. [CrossRef]
7. El-Wakil, S.A.; Abdou, M.A.; Elhanbaly, A. Adomian decomposition method for solving the diffusion-convection-reaction equations. *Comput. Math. Appl.* **2006**, *177*, 729–736. [CrossRef]
8. Ke, Z.; Goard, J.; Zhu, S.-P. An appropriate approach to pricing European-style options with the Adomian decomposition method. *ANZIAM J.* **2017**, *59*, 349–369.
9. Luo, X.-G.; Wu, Q.-B.; Zhang, B.-Q. Revisit on partial solutions in the Adomian decomposition method: Solving heat and wave equations. *J. Math. Anal. Appl.* **2006**, *321*, 353–363. [CrossRef]
10. Adomian, G. *Nonlinear Stochastic Operator Equations*; Academic Press: Cambridge, MA, USA, 1986.
11. Wilmott, P.; Dewynne, J.; Howison, S. *Option Pricing: Mathematical Models and Computation*; Oxford University Press: Oxford, UK, 1994.
12. Björk, T. *Arbitrage Theory in Continuous Time*; Oxford University Press: Oxford, UK, 2009.
13. Merton, R.C. Option pricing when underlying stock returns are discontinuous. *J. Financ. Econ.* **1976**, *3*, 125–144. [CrossRef]
14. Rodrigo, M.R.; Goard, J.M. Pricing of general European options on discrete dividend-paying assets with jump-diffusion dynamics. *Appl. Math. Model.* **2018**, *64*, 47–54. [CrossRef]
15. Ball, C.A.; Torous, W.N. On jumps in common stock prices and their impact on call option pricing. *J. Financ.* **1985**, *40*, 155–173. [CrossRef]
16. Brown, S.; Dybvig, P. The empirical implications of the Cox-Ingersoll-Ross theory of the term structure of interest rates. *J. Financ.* **1986**, *41*, 617–630. [CrossRef]
17. Jarrow, R.; Rosenfeld, E. Jump risks and the intertemporal capital asset pricing model. *J. Bus.* **1984**, *57*, 337–351. [CrossRef]
18. Rosenfeld E. Stochastic Processes of Common Stock Returns: An Empirical Examination. Ph.D. Thesis, Massachussetts Institute of Technology, Cambridge, MA, USA, 1980.
19. Hosseini, M.M.; Nasabzadeh, H. On the convergence of Adomian decomposition method. *Appl. Math. Comput.* **2006**, *182*, 536–543. [CrossRef]
20. Nelson, P. Adomian's method of decomposition: Critical review and examples of failure. *J. Comput. Appl. Math.* **1988**, *23*, 389–393. [CrossRef]
21. Fasano, A.; Herrero, M.A.; Rodrigo, M.R. Slow and fast invasion waves in a model of acid-mediated tumour growth. *Math. Biosci.* **2009**, *220*, 45–56. [CrossRef]
22. Rodrigo, M.R. Pricing of barrier options on underlying assets with jump-diffusion dynamics: A Mellin transform approach. *Mathematics* **2020**, *8*, 1271. [CrossRef]
23. Oberhettinger, O. *Tables of Mellin Transforms*; Springer: Berlin, Germany, 1974.
24. Rodrigo, M.R. Approximate ordinary differential equations for the optimal exercise boundaries of American put and call options. *Eur. J. Appl. Math.* **2014**, *25*, 27–43. [CrossRef]
25. Li, T.R.; Rodrigo, M.R. Alternative results for option pricing and implied volatility in jump-diffusion models using Mellin transforms. *Eur. J. Appl. Math.* **2016**, *28*, 789. [CrossRef]
26. Guardasoni, C.; Rodrigo, M.R.; Sanfelici, S. A Mellin transform approach to barrier option pricing. *IMA J. Manag. Math.* **2020**, *31*, 49–67. [CrossRef]
27. Rodrigo, M.R.; Mamon, R.S. An application of Mellin transform techniques to a Black–Scholes equation problem. *Anal. Appl.* **2007**, *5*, 51–66. [CrossRef]
28. Bohner, M.; Marín, F.; Rodríguez, S. European call option pricing using the Adomian decomposition method. *Adv. Dynam. Syst. Appl.* **2014**, *9*, 75–85.
29. González-Gaxiola, O.; Ruíz de Chávez, J.; Santiago, J.A. A nonlinear option pricing model through the Adomian decomposition method. *Int. J. Appl. Comput. Math.* **2016**, *2*, 453–467. [CrossRef]
30. Adomian, G.; Rach, R. A further consideration of partial solutions in the decomposition method. *Comput. Math. Appl.* **1992**, *23*, 51–64. [CrossRef]
31. Gorbachuk, V.M.; Gorbachuk, M.L. The representation of a C_0-semigroup of linear operators in a Banach space on the set of entire vectors of its generator. *Integral Eq. Oper. Theory* **2016**, *85*, 497–512. [CrossRef]
32. Horbachuk, V.M.; Horbachuk, M.L. Representations of a group of linear operators in a Banach space on the set of entire vectors of its generator. *Ukr. Math. J.* **2015**, *67*, 668–679. [CrossRef]