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Nonlocal Sequential Boundary Value Problems for Hilfer Type Fractional Integro-Differential Equations and Inclusions

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Abstract: In the present research, we study boundary value problems for fractional integro-differential equations and inclusions involving the Hilfer fractional derivative. Existence and uniqueness results are obtained by using the classical fixed point theorems of Banach, Krasnosel'skiĭ, and Leray–Schauder in the single-valued case, while Martelli's fixed point theorem, a nonlinear alternative for multivalued maps, and the Covitz–Nadler fixed point theorem are used in the inclusion case. Examples are presented to illustrate our results.

Keywords: fractional differential equations and inclusions; Hilfer fractional derivative; Riemann–Liouville fractional derivative; Caputo fractional derivative; boundary value problems; existence and uniqueness; fixed point theory

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1. Introduction

In last few decades, fractional differential equations with initial/boundary conditions have been studied by many researchers. This is because fractional differential equations describe many real world processes more accurately compared to classical order differential equations. Therefore, the fractional-order models become more practical and realistic compared to the integer-order models. Fractional differential equations arise in lots of engineering and clinical disciplines, including biology, physics, chemistry, economics, signal and image processing, control theory, and so on; see the monographs [1–8].

There exist several different definitions of fractional integrals and derivatives in the literature, such as the Riemann–Liouville and Caputo fractional derivatives, the Hadamard fractional derivative, the Erdeyl–Kober fractional derivative, and so on. Both Riemann–Liouville and Caputo fractional derivatives were generalized by Hilfer in [9]. This generalization, known as the Hilfer fractional derivative of order α and a type $\beta \in [0,1]$, interpolates between the Riemann–Liouville and Caputo derivative, since it can be reduced to the Riemann–Liouville and Caputo fractional derivatives when $\beta = 0$ and $\beta = 1$, respectively. See [9–11] and the references cited therein for some properties and applications of the Hilfer derivative.

Several authors have studied initial value problems involving Hilfer fractional derivatives, see, for example, [12–14] and the references included therein. Boundary value problems for the Hilfer fractional derivative and nonlocal boundary conditions were initiated in [15].

Motivated by the research going in this direction, in the present paper, we study existence and uniqueness of solutions for the following new class of boundary value problems consisting of fractional-order sequential Hilfer-type differential equations supplemented with nonlocal integro-multipoint boundary conditions of the form

$$\left({}^{H}D^{\alpha,\beta} + k^{H}D^{\alpha-1,\beta}\right)x(t) = f(t, x(t), I^{\delta}x(t)), \quad t \in [a,b], \tag{1}$$

$$x(a) = 0, \quad x(b) = \sum_{i=1}^{m-2} \zeta_i I^{\phi_i} x(\theta_i), \tag{2}$$

where ${}^{H}D^{\alpha,\beta}$ denote the Hilfer fractional derivative operator of order $\alpha, 1 < \alpha < 2$ and parameter $\beta, 0 \leq \beta \leq 1, f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, given constant $k \in \mathbb{R}, I^{\psi}$ as the Riemann–Liouville fractional integral of order $\psi > 0, \psi \in \{\phi_i, \delta\}, a \geq 0, a < \theta_1 < \theta_2 < \cdots < \theta_{m-2} < b$ and $\zeta_i \in \mathbb{R}, i = 1, 2, \ldots, m-2$.

By using standard tools from functional analysis, we study existence and uniqueness results for the sequential boundary value in Problems (1) and (2). We establish the existence of a unique solution via Banach's fixed point theorem [16], while nonlinear alternatives of Leray–Schauder-type [17] and Krasnosel'skiĭ's fixed point theorem [18] are applied to obtain the existence results.

After that, we look at the corresponding multivalued problem by studying existence of solutions for a new class of sequential boundary value problems of Hilfer-type fractional differential inclusions with nonlocal integro-multipoint boundary conditions of the form

$$\left({}^{H}D^{\alpha,\beta} + k^{H}D^{\alpha-1,\beta}\right)x(t) \in F(t, x(t), I^{\delta}x(t)), \quad t \in [a,b], \tag{3}$$

$$x(a) = 0, \quad x(b) = \sum_{i=1}^{m-2} \zeta_i I^{\phi_i} x(\theta_i), \tag{4}$$

where $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

Existence results for the sequential boundary value Problems (3) and (4) with convex-valued maps are derived by applying a fixed point theorem according to Martelli's [19] nonlinear alternative for multivalued maps. For possible nonconvex-valued maps, we obtain an existence result by using Covitz–Nadler fixed point theorem [20] for contractive maps.

The organization of this paper is as follows: Section 2 contains some preliminary concepts related to our problem. We present our main work for Problems (1) and (2) in Section 3, while the main results for the multivalued Problems (3) and (4) are presented in Section 4. Examples are constructed to illustrate the main results.

2. Preliminaries

We present the preliminary results from fractional calculus needed in our proofs as follows [2,5].

Definition 1. *The fractional integral of Riemann–Liouville, of a continuous function of order $\alpha > 0$ is defined by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \qquad n-1 < \alpha < n,$$

where $n = [\alpha] + 1$ ($[\alpha]$ is the integer part of $\alpha \in \mathbb{R}$), provided the right-hand side is pointwise defined on (a, ∞).

Definition 2. *For a continuous function, the Riemann–Liouville fractional derivative of order $\alpha > 0$ is defined by*

$${}^{RL}D^{\alpha}u(t) := D^n I^{n-\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s) ds, \quad n-1 < \alpha < n,$$

where $D^n = d^n/dt^n$, provided the right-hand side is pointwise defined on (a, ∞).

Definition 3. *For a continuous function, the Caputo fractional derivative of order $\alpha > 0$ is defined by*

$$^C D^\alpha u(t) := I^{n-\alpha} D^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s) ds, \quad n-1 < \alpha < n,$$

provided the right-hand side is pointwise defined on (a, ∞).

Another new definition of fractional derivative was introduced in [9] (see also [11]).

Definition 4. *The Hilfer fractional derivative of order α and parameter β is defined by*

$$^H D^{\alpha,\beta} u(t) = I^{\beta(n-\alpha)} D^n I^{(1-\beta)(n-\alpha)} u(t),$$

where $n-1 < \alpha < n,\ 0 \leq \beta \leq 1,\ t > a,\ D = \dfrac{d}{dt}$.

Remark 1. *When $\beta = 0$, the Hilfer fractional derivative gives the Riemann–Liouville fractional derivative*

$$^H D^{\alpha,0} u(t) = D^n I^{n-\alpha} u(t),$$

while when $\beta = 1$ the Hilfer fractional derivative gives the Caputo fractional derivative

$$^H D^{\alpha,1} u(t) = I^{n-\alpha} D^n u(t).$$

The following basic lemma is used in the sequel.

Lemma 1 ([11])**.** *Let $f \in L(a,b), n-1 < \alpha \leq n, n \in \mathbb{N}, 0 \leq \beta \leq 1, I^{(n-\alpha)(1-\beta)} f \in AC^k[a,b]$. Then,*

$$\left(I^\alpha\, {}^H D^{\alpha,\beta} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \lim_{t \to a^+} \frac{d^k}{dt^k} \left(I^{(1-\beta)(n-\alpha)} f\right)(t).$$

3. Main Results

The following lemma concerns a linear variant of the sequential boundary value Problems (1) and (2).

Lemma 2. *Let $h \in C([a,b], \mathbb{R})$ and $a \geq 0, 1 < \alpha < 2, \gamma = \alpha + 2\beta - \alpha\beta$. Assume that*

$$\Lambda := (b-a)^{\gamma-1} - \Gamma(\gamma) \sum_{i=1}^{m-2} \zeta_i \frac{(\theta_i - a)^{\gamma+\phi_i-1}}{\Gamma(\gamma+\phi_i)} \neq 0. \tag{5}$$

Then, the function x is a solution of the sequential boundary value problem

$$\left({}^H D^{\alpha,\beta} + k\, {}^H D^{\alpha-1,\beta}\right) x(t) = h(t),\ t \in [a,b], \tag{6}$$

$$x(a) = 0, \quad x(b) = \sum_{i=1}^{m-2} \zeta_i I^{\phi_i} x(\theta_i), \tag{7}$$

if and only if

$$x(t) = I^{\alpha}h(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}h(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + k\int_a^b x(s)ds - I^{\alpha}h(b)\right]. \tag{8}$$

Proof. Assume that x is a solution of the sequential nonlocal boundary value Problems (6) and (7). Applying the operator I^{α} to both sides of Equation (6) and using Lemma 1, we obtain

$$\begin{aligned} x(t) &= c_0\frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))} + c_1\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - kI^1x(t) + I^{\alpha}h(t) \\ &= c_0\frac{(t-a)^{\gamma-2}}{\Gamma(\gamma-1)} + c_1\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} - kI^1x(t) + I^{\alpha}h(t), \end{aligned}$$

where $c_0, c_1 \in \mathbb{R}$ and $(1-\beta)(2-\alpha) = 2-\gamma$.

From the boundary condition $x(a) = 0$, we obtain $c_0 = 0$. Then, we get

$$x(t) = c_1\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} - kI^1x(t) + I^{\alpha}h(t). \tag{9}$$

From the boundary condition $x(b) = \sum_{i=1}^{m-2}\zeta_i I^{\phi_i}x(\theta_i)$, we find

$$c_1 = \frac{\Gamma(\gamma)}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}h(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + k\int_a^b x(s)ds - I^{\alpha}h(b)\right].$$

Substituting the value of c_1 into (9), we obtain the solution (8). Conversely, it is easily shown that the solution x given by (8) satisfies the sequential nonlocal boundary value Problems (6) and (7). This completes the proof. □

Let $C([a,b],\mathbb{R})$ denote the Banach space endowed with the sup-norm $\|x\| = \sup\{|x(t)| : t \in [a,b]\}$. In view of Lemma 2, we define an operator $\mathcal{A} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$(\mathcal{A}x)(t) = I^{\alpha}f_x^{\delta}(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}f_x^{\delta}(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + k\int_a^b x(s)ds - I^{\alpha}f_x^{\delta}(b)\right], \tag{10}$$

where

$$f_x^{\delta}(t) = f(t, x(t), I^{\delta}x(t)).$$

It is obvious that the sequential nonlocal boundary value problem has a solution if and only if the operator $\mathcal{A}$ has fixed points.

To simplify the computations, we use the following notations:

$$\Omega = \frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \tag{11}$$

and

$$\Omega_1 = \frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[|k|\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\phi_i+1}}{\Gamma(\phi_i+2)} + |k|(b-a)\right] + |k|(b-a). \tag{12}$$

By using classical fixed point theorems, we prove in the next subsections, for the sequential boundary value Problems (1) and (2), our main existence and uniqueness results.

3.1. Existence and Uniqueness Result for Problems (1) and (2)

Our first result is an existence and uniqueness result, based on Banach's fixed point theorem [16].

Theorem 1. *Assume that*

(H_1) *there exists a constant* $L > 0$ *such that*

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq L(|x_1 - y_1| + |x_2 - y_2|)$$

for each $t \in [a,b]$ *and* $x_i, y_i \in \mathbb{R}, i = 1,2.$

If

$$LL_1\Omega + \Omega_1 < 1, \tag{13}$$

where Ω *and* Ω_2 *are defined by* (11) *and* (12)*, respectively, and* $L_1 = 1 + \dfrac{(b-a)^\delta}{\Gamma(\delta+1)}$*, then the sequential boundary value Problems* (1) *and* (2) *have a unique solution on* $[a,b]$*.*

Proof. Consider the operator $\mathcal{A}$ defined in (10). The sequential boundary value Problems (1) and (2) are then transformed into a fixed point problem $x = \mathcal{A}x$. We shall show that $\mathcal{A}$ has a unique fixed point by applying the Banach contraction mapping principle.

We set $\sup_{t\in[a,b]} |f(t,0,0)| = M < \infty$, and choose $r > 0$ such that

$$r \geq \frac{M\Omega}{1 - LL_1\Omega - \Omega_1}. \tag{14}$$

Now, we show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{x \in C([a,b],\mathbb{R}) : \|x\| \leq r\}$. By using the assumption (H_1), we have

$$\begin{aligned}
f_x^\delta(t) := |f(t,x(t),I^\delta x(t))| &\leq |f(t,x(t),I^\delta x(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\
&\leq L(|x(t)| + |I^\delta x(t)|) + M \\
&\leq L\left(\|x\| + \frac{(b-a)^\delta}{\Gamma(\delta+1)}\|x\|\right) + M \\
&= L\|x\|\left(1 + \frac{(b-a)^\delta}{\Gamma(\delta+1)}\right) + M \\
&= LL_1\|x\| + M.
\end{aligned}$$

For any $x \in B_r$, we have

$$\begin{aligned}
|(\mathcal{A}x)(t)| \leq \sup_{t\in[a,b]} &\left\{ I^\alpha |f_x^\delta(t)| + |k| I^1 |x(t)| + \frac{(t-a)^{\gamma-1}}{|\Lambda|}\left(\sum_{i=1}^{m-2} |\zeta_i| I^{\alpha+\phi_i} |f_x^\delta(\theta_i)| \right.\right. \\
&\left.\left. + |k| \sum_{i=1}^{m-2} |\zeta_i| I^{\phi_i+1} |x(\theta_i)| + |k| I^1 |x(b)| + I^\alpha |f_x^\delta(b)| \right)\right\} \\
\leq\ & (LL_1\|x\| + M) I^\alpha(1)(t) + |k| I^1 |x(t)| \\
&+ \frac{(t-a)^{\gamma-1}}{|\Lambda|}\left\{ (LL_1\|x\| + M) \sum_{i=1}^{m-2} |\zeta_i| I^{\alpha+\phi_i}(1)(\theta_i) \right. \\
&\left. + |k| \sum_{i=1}^{m-2} |\zeta_i| I^{\phi_i+1} |x(\theta_i)| + |k| I^1 |x(b)| + (LL_1\|x\| + M) I^\alpha(1)(b) \right\} \\
\leq\ & (LL_1\|x\| + M)\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + |k|(b-a)\|x\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg\{(LL_1\|x\|+M)\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} \\
&+|k|\|x\|\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\phi_i+1}}{\Gamma(\phi_i+2)}+|k|(b-a)\|x\|+(LL_1\|x\|+M)\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Bigg\} \\
=&\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right]+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right\}(LL_1\|x\|+M) \\
&+\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[|k|\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\phi_i+1}}{\Gamma(\phi_i+2)}+|k|(b-a)\right]+|k|(b-a)\right\}\|x\| \\
\leq&(LL_1r+M)\Omega+\Omega_1 r\leq r,
\end{aligned}
$$

which implies that $\mathcal{A}B_r\subset B_r$.

Next, we let $x,y\in C([a,b],\mathbb{R})$. Then, for $t\in[a,b]$, we have

$$
\begin{aligned}
&|(\mathcal{A}x)(t)-(\mathcal{A}y)(t)| \\
&\leq I^{\alpha}|f_x^{\delta}(t)-f_y^{\delta}(t)|+|k|I^{1}|x(t)-y(t)|+\frac{(t-a)^{\gamma-1}}{|\Lambda|}\Bigg\{\sum_{i=1}^{m-2}|\zeta_i|I^{\alpha+\phi_i}|f_x^{\delta}(\theta_i)-f_y^{\delta}(\theta)| \\
&\quad+|k|\sum_{i=1}^{m-2}|\zeta_i|I^{\phi_i+1}|x(\theta_i)-y(\theta)|+|k|I^{1}|x(b)-y(b)|+I^{\alpha}|f_x^{\delta}(b)-f_y^{\delta}(b)|\Bigg\} \\
&\leq\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right]+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right\}LL_1\|x-y\| \\
&\quad+\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[|k|\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\phi_i+1}}{\Gamma(\phi_i+2)}+|k|(b-a)\right]+|k|(b-a)\right\}\|x-y\| \\
&=(LL_1\Omega+\Omega_1)\|x-y\|,
\end{aligned}
$$

which implies that $\|\mathcal{A}x-\mathcal{A}y\|\leq(LL_1\Omega+\Omega_1)\|x-y\|$. As $LL_1\Omega+\Omega_1<1$, $\mathcal{A}$ is a contraction. Thus, by the Banach's contraction mapping principle, we deduce that $\mathcal{A}$ has a unique fixed point, which is the unique solution of the sequential boundary value Problems (1) and (2). The proof is finished. $\square$

Example 1. *Consider the sequential nonlocal boundary value problem with Hilfer fractional differential equation of the form:*

$$
\begin{cases}
\left({}^{H}D^{\frac{5}{3},\frac{3}{4}}+\dfrac{1}{10}{}^{H}D^{\frac{2}{3},\frac{3}{4}}\right)x(t)=f(t,x(t),I^{\frac{2}{3}}x(t)),\quad t\in\left[\dfrac{1}{4},\dfrac{7}{4}\right], \\
x\left(\dfrac{1}{4}\right)=0,\quad x\left(\dfrac{7}{4}\right)=\dfrac{1}{5}I^{\frac{1}{4}}x\left(\dfrac{1}{2}\right)+\dfrac{2}{7}I^{\frac{1}{3}}x\left(\dfrac{3}{4}\right)+\dfrac{3}{11}I^{\frac{1}{2}}x(1) \\
\qquad\qquad\qquad+\dfrac{4}{13}I^{\frac{3}{2}}x\left(\dfrac{5}{4}\right)+\dfrac{5}{16}I^{\frac{5}{4}}x\left(\dfrac{3}{2}\right),
\end{cases}
\tag{15}
$$

where

$$
f(t,x(t),I^{\frac{2}{3}}x(t))=\frac{4}{4t+159}\left(\frac{x^2(t)+2|x(t)|}{1+|x(t)|}\right)+\frac{4}{4t+83}\sin|I^{\frac{2}{3}}x(t)|+\frac{1}{3}e^{-t}. \tag{16}
$$

Here, $\alpha=5/3,\beta=3/4,k=1/10,\delta=2/3,a=1/4,b=7/4,m=7,\zeta_1=1/5,\zeta_2=2/7,\zeta_3=3/11,\zeta_4=4/13,\zeta_5=5/16,\phi_1=1/4,\phi_2=1/3,\phi_3=1/2,\phi_4=3/2,\phi_5=5/4,\theta_1=1/2,\theta_2=3/4,\theta_3=1,\theta_4=5/4,\theta_5=3/2$, and $\gamma=(5/3)+(2-(5/3))(3/4)=23/12$. From these constants, we can verify that $\Lambda\approx 0.8651080717,\Omega\approx 3.928448219,\Omega_1\approx 0.4944426735$, and $L_1\approx 2.451539773$.

Since

$$|f(t,w_1,w_2)-f(t,z_1,z_2)| \le \frac{1}{20}(|w_1-z_1|+|w_2-z_2|), \quad \forall t \in \left[\frac{1}{4},\frac{7}{4}\right],\ w_i,z_i \in \mathbb{R},\ i=1,2,$$

we set $L = 1/20$, which leads to $LL_1\Omega + \Omega_1 \approx 0.9759800261 < 1$. Then, inequality (13) holds. Therefore, by applying Theorem 1, the Hilfer fractional differential equation with nonlocal Conditions (15) and (16) has a unique solution on $[1/4, 7/4]$.

3.2. Existence Results

Two existence results are presented in this subsection. The first is based on the well-known Krasnosel'skiĭ's fixed point theorem [18].

Theorem 2. *Let $f : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following assumption:*
$(H_2)\ |f(t,x,y)| \le \varphi(t), \quad \forall (t,x,y) \in [a,b]\times\mathbb{R}\times\mathbb{R}$, *and* $\varphi \in C([a,b],\mathbb{R}^+)$.
Then, if

$$\Omega_1 < 1, \tag{17}$$

the sequential boundary value Problems (1) and (2) have at least one solution on $[a,b]$.

Proof. We set $\sup_{t\in[a,b]} \varphi(t) = \|\varphi\|$ and choose $\rho > 0$ such that

$$\rho \ge \frac{\|\varphi\|\Omega}{1-\Omega_1}, \tag{18}$$

(where Ω, Ω_1 are defined by (11) and (12), respectively), and consider $B_\rho = \{x \in C([a,b],\mathbb{R}) : \|x\| \le \rho\}$. On B_ρ, we define the operators $\mathcal{A}_1, \mathcal{A}_2$ by

$$(\mathcal{A}_1x)(t) = I^\alpha f_x^\delta(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i} f_x^\delta(\theta_i) - I^\alpha f_x^\delta(b)\right], \quad t \in [a,b]$$

and

$$(\mathcal{A}_2x)(t) = -kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[-k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b)\right], \quad t \in [a,b].$$

For any $x, y \in B_\rho$, we have

$$\begin{aligned}
|(\mathcal{A}_1x)(t)+(\mathcal{A}_2y)(t)| &\le \Bigg\{ I^\alpha|f_x^\delta(t)| + |k|I^1|x(t)| + \frac{(t-a)^{\gamma-1}}{|\Lambda|}\Big(\sum_{i=1}^{m-2}|\zeta_i|I^{\alpha+\phi_i}|f_x^\delta(\theta_i)| \\
&\quad + |k|\sum_{i=1}^{m-2}|\zeta_i|I^{\phi_i+1}|x(\theta_i)| + |k|I^1|x(b)| + I^\alpha|f_x^\delta(b)|\Bigg\} \\
&\le \left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right] + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right\}\|\varphi\| \\
&\quad + \left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[|k|\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\phi_i+2}}{\Gamma(\phi_i+1)} + |k|(b-a)\right] + |k|(b-a)\right\}\|x\| \\
&\le \|\varphi\|\Omega + \Omega_1\rho \le \rho.
\end{aligned}$$

This shows that $\mathcal{A}_1x + \mathcal{A}_2y \in B_\rho$. As we have proved with little difficulty, using (17), operator $\mathcal{A}_2$ is a contraction mapping.

$\mathcal{A}_1$ is continuous, since f is continuous. In addition, $\mathcal{A}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{A}_1x\| \le \left\{ \frac{(b-a)^{\gamma-1}}{|\Lambda|} \left[\sum_{i=1}^{m-2} |\zeta_i| \frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right\} \|\varphi\|.$$

Now, we prove the compactness of the operator $\mathcal{A}_1$. Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
|(\mathcal{A}_1x)(t_2) - (\mathcal{A}_1x)(t_1)| &= \frac{1}{\Gamma(\alpha)} \bigg| \int_a^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] f(s,x(s),I^\delta x(s))ds \\
&\quad + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s,x(s),I^\delta x(s))ds \bigg| \\
&\quad + \frac{(t_2-a)^{\gamma-1} - (t_1-a)^{\gamma-1}}{|\Lambda|} \left[\sum_{i=1}^{m-2} \zeta_i I^{\alpha+\phi_i} |f_x^\delta(\theta_i)| + I^\alpha |f_x^\delta(b)| \right] \\
&\le \frac{\|\phi\|}{\Gamma(\alpha+1)} [2(t_2-t_1)^\alpha + |(t_2-a)^\alpha - (t_1-a)^\alpha|] \\
&\quad + \frac{(t_2-a)^{\gamma-1} - (t_1-a)^{\gamma-1}}{|\Lambda|} \left[\sum_{i=1}^{m-2} \zeta_i I^{\alpha+\phi_i} |f_x^\delta(\theta_i)| + I^\alpha |f_x^\delta(b)| \right],
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Hence, $\mathcal{A}_1$ is equicontinuous. The operator $\mathcal{A}_1$ is compacted on B_ρ by the Arzelá–Ascoli theorem. Thus, all the assumptions of Krasnosel'skiĭ's fixed point theorem are satisfied. So, by its conclusion, the sequential boundary value Problems (1) and (2) have at least one solution on $[a,b]$. The proof is finished. □

Our second existence result is proved by using Leray–Schauder's Nonlinear Alternative [17].

Theorem 3. *Assume that (17) holds. Moreover, we suppose that*

(H_3) there exists a continuous, nondecreasing, subhomogeneous (that is, $\psi(\mu x) \le \mu\psi(x)$ for all $\mu \ge 1$ and $x \in \mathbb{R}^+$) function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([a,b],\mathbb{R}^+)$ such that

$$|f(t,u,v)| \le p(t)\psi(|u|+|v|) \quad \text{for each} \quad (t,u,v) \in [a,b] \times \mathbb{R} \times \mathbb{R};$$

(H_4) there exists a constant $K > 0$ such that

$$\frac{(1-\Omega_1)K}{L_1\psi(K)\|p\|\Omega} > 1,$$

where Ω and Ω_1 are defined by (11) and (12), respectively, and L_1 is defined in Theorem 1. Then, the sequential boundary value Problems (1) and (2) have at least one solution on $[a,b]$.

Proof. Consider the operator $\mathcal{A}$ given by (10). In the first step, we shall show that bounded sets (balls) are mapped by $\mathcal{A}$ into bounded sets in $C([a,b],\mathbb{R})$. For $r > 0$, let a bounded ball in $C([a,b],\mathbb{R})$ given by $B_r = \{x \in C([a,b],\mathbb{R}) : \|x\| \le r\}$. Then, for $t \in [a,b]$, we have

$$\begin{aligned}
|(\mathcal{A}x)(t)| &\le \sup_{t\in[a,b]} \Bigg\{ I^\alpha |f_x^\delta(t)| + |k| I^1 |x(t)| + \frac{(t-a)^{\gamma-1}}{|\Lambda|} \Bigg(\sum_{i=1}^{m-2} |\zeta_i| I^{\alpha+\phi_i} |f_x^\delta(\theta_i)| \\
&\quad + |k| \sum_{i=1}^{m-2} |\zeta_i| I^{\phi_i+1} |x(\theta_i)| + |k| I^1 |x(b)| + I^\alpha |f_x^\delta(b)| \Bigg) \Bigg\} \\
&\le \left\{ \frac{(b-a)^{\gamma-1}}{|\Lambda|} \left[\sum_{i=1}^{m-2} |\zeta_i| \frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \right] + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \right\} \psi(L_1\|x\|)\|p\|
\end{aligned}$$

$$+\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[|k|\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\phi_i+1}}{\Gamma(\phi_i+2)}+|k|(b-a)\right]+|k|(b-a)\right\}\|x\|$$

$$\leq L_1\psi(\|x\|)\|p\|\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right]+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right\}$$

$$+\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[|k|\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\phi_i+1}}{\Gamma(\phi_i+2)}+|k|(b-a)\right]+|k|(b-a)\right\}\|x\|,$$

and consequently,

$$\|\mathcal{A}x\|\leq L_1\psi(r)\|p\|\Omega+\Omega_1 r.$$

Next, we will show that bounded sets are mapped by $\mathcal{A}$ into equicontinuous sets of $C([a,b],\mathbb{R})$. Let $\tau_1,\tau_2\in[a,b]$ with $\tau_1<\tau_2$ and $x\in B_r$. Then, we have

$$\begin{aligned}|(\mathcal{A}x)(\tau_2)-(\mathcal{A}x)(\tau_1)| &\leq \frac{1}{\Gamma(\alpha)}\left|\int_a^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1 s)^{\alpha-1}]f_x^{\delta}(s)ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}f_x^{\delta}(s)ds\right| \\ &\quad + r(t_2-t_1)+\frac{(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}|f_x^{\delta}(\theta_i)|+I^{\alpha}|f_x^{\delta}(b)|\right. \\ &\quad \left.+|k|\sum_{i=1}^{m-2}|\zeta_i|I^{\phi_i+1}|x(\theta_i)|+|k|I^1|x(b)|\right] \\ &\leq \frac{\|p\|L_1\psi(r)}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^{\alpha}+|(\tau_2-a)^{\alpha}-(\tau_1-a)^{\alpha}|]+r(\tau_2-\tau_1) \\ &\quad +\frac{(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}|f_x^{\delta}(\theta_i)|+I^{\alpha}|f_x^{\delta}(b)|\right. \\ &\quad \left.+|k|\sum_{i=1}^{m-2}|\zeta_i|I^{\phi_i+1}|x(\theta_i)|+|k|I^1|x(b)|\right].\end{aligned}$$

In the above inequality, the right-hand side is independent of $x\in B_r$ tends to zero as $\tau_2-\tau_1\to 0$. By the Arzelá–Ascoli theorem, the operator $\mathcal{A}:C([a,b],\mathbb{R})\to C([a,b],\mathbb{R})$ is completely continuous.

Now, to finish the proof, we must prove that the set of all solutions to equations $x=\lambda\mathcal{A}x$ for $\lambda\in(0,1)$ is bounded.

Let x be a solution. By using the computations in the first step, we have for $t\in[a,b]$,

$$|x(t)|\leq L_1\psi(\|x\|)\|p\|\Omega+\Omega_1\|x\|,$$

and consequently,

$$\frac{(1-\Omega_1)\|x\|}{L_1\psi(\|x\|)\|p\|\Omega}\leq 1.$$

By (H_4), there exists K such that $\|x\|\neq K$. Consider the set

$$U=\{x\in C([a,b],\mathbb{R}):\|x\|<K\}.$$

We have proved that the operator $\mathcal{A}:\bar{U}\to C([a,b],\mathbb{R})$ is completely continuous. There is no $x\in\partial U$ such that $x=\lambda\mathcal{A}x$ for some $\lambda\in(0,1)$, by the choice of U. By the nonlinear alternative of Leray–Schauder type ([17]), we have as result that $\mathcal{A}$ has a fixed point $x\in\bar{U}$, which is a solution of the sequential boundary value Problems (1) and (2). The proof is completed. □

Example 2. *Consider the sequential nonlocal Hilfer fractional boundary value problem of the form:*

$$\begin{cases} \left({}^{H}D^{\frac{7}{5},\frac{3}{8}} + \frac{1}{9}{}^{H}D^{\frac{2}{5},\frac{3}{8}}\right)x(t) = f(t, x(t), I^{\frac{3}{2}}x(t)), \quad t \in \left[\frac{1}{5}, \frac{8}{5}\right], \\ x\left(\frac{1}{5}\right) = 0, \quad x\left(\frac{8}{5}\right) = \frac{2}{11}I^{\frac{1}{3}}x\left(\frac{2}{5}\right) + \frac{4}{13}I^{\frac{2}{3}}x\left(\frac{3}{5}\right) + \frac{6}{17}I^{\frac{4}{3}}x\left(\frac{4}{5}\right) \\ \qquad\qquad + \frac{8}{19}I^{\frac{5}{3}}x(1) + \frac{10}{23}I^{\frac{7}{3}}x\left(\frac{6}{5}\right) + \frac{12}{29}I^{\frac{8}{3}}x\left(\frac{7}{5}\right). \end{cases} \tag{19}$$

Here, $\alpha = 7/5, \beta = 3/8, k = 1/9, \delta = 3/2, a = 1/5, b = 8/5, m = 8, \zeta_1 = 2/11, \zeta_2 = 4/13, \zeta_3 = 6/17, \zeta_4 = 8/19, \zeta_5 = 10/23, zeta_6 = 12/29, \phi_1 = 1/3, \phi_2 = 2/3, \phi_3 = 4/3, \phi_4 = 5/3, \phi_5 = 7/3, \phi_6 = 8/3, \theta_1 = 2/5, \theta_2 = 3/5, \theta_3 = 4/5, \theta_4 = 1, \theta_5 = 6/5, \theta_6 = 7/5$ and $\gamma = (7/5) + (2 - (7/5))(3/8) = 65/40$. From the given data, we find that $\Lambda \approx 0.8346353019, \Omega \approx 3.404450305, \Omega_1 \approx 0.4283560839$, and $L_1 \approx 2.246108486$.

(i) Let the nonlinear function $f(t, x, I^{\frac{3}{2}}x)$ in Problem (19) be defined by

$$f(t, x, I^{\frac{3}{2}}x) = t^3 e^{-x^4} + \frac{1}{2}t^2 \frac{x^{12}}{1 + x^{12}} \sin^6\left(I^{\frac{3}{2}}x\right) + t\tan^{-1}\left(I^{\frac{3}{2}}x\right) + \frac{1}{4}. \tag{20}$$

Then, we see that the function f satisfies condition (H_2) in Theorem 2 as

$$|f(t, w, z)| \le t^3 + \frac{1}{2}t^2 + \frac{\pi}{2}t + \frac{1}{4} := \varphi(t), \quad \forall(t, w, z) \in \left[\frac{1}{5}, \frac{8}{5}\right] \times \mathbb{R}^2.$$

By the benefit of the conclusion in Theorem 2, it implies that the nonlocal Hilfer fractional boundary value Problems (19) and (20), with f given by (20), have at least one solution on $[1/5, 8/5]$.

(ii) Let the nonlinear function $f(t, x, I^{\frac{3}{2}}x)$ in Problem (19) be given by

$$\begin{aligned} f(t, x, I^{\frac{3}{2}}x) = \frac{5}{5t + 74}\left(\frac{|x|^{23}}{1 + x^{22}} + 1\right) + \frac{5}{5t + 79}\left(|I^{\frac{3}{2}}x|\cos^8\left(|I^{\frac{3}{2}}x|\right)\right) \\ + \frac{5}{5t + 84}\left(\frac{1}{|x| + \left|I^{\frac{3}{2}}x\right| + 1}\right). \end{aligned} \tag{21}$$

Thus, we can compute that the above function f satisfies inequality

$$|f(t, w, z)| \le \frac{5}{5t + 74}\left(|w| + |z| + 1 + \frac{1}{|w| + |z| + 1}\right).$$

Setting $p(t) = 5/(5t + 74)$ and $\psi(u) = u + 1 + (1/(u + 1)), u = |w| + |z|$, we obtain $\|p\| = 1/15$ and $\psi(\mu u) \le \mu\psi(u), \forall\mu \ge 1$, which is a subhomogeneous function. Then, there exists a constant $K > 9.060160134$ satisfying condition (H_4) in Theorem 3. Therefore, all assumptions of Theorem 3 are fulfilled. Then, the boundary value Problems (19)–(21), with f given by (21), have at least one solution on $[1/5, 8/5]$.

4. Existence Results for Problems (3) and (4)

In this section, we will establish existence results for a new class of sequential boundary value problems of Hilfer-type fractional differential inclusions with nonlocal integro-multipoint boundary Conditions (3) and (4).

In the following, by $\mathcal{P}_p$, we denote the set of all nonempty subsets of X that have the property "p", where "p" will be bounded (b), closed (cl), convex (c), compact (cp), etc. Thus, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$, and $\mathcal{P}_{b,cl,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded, closed, and convex}\}$.

For details on multivalued maps, we refer the interested reader to the books by Castaing and Valadier [21], Deimling [16], Gorniewicz [22], and Hu and Papageorgiou [23]. For a brief summary of the needed results of multivalued analysis for fractional differential equations, we refer to [7,24,25].

For the sequential boundary value Problems (3) and (4), we will give the definition of its solution.

Definition 5. *A function $x \in C^2([a,b],\mathbb{R})$ is a solution of the sequential boundary value Problems* (3) *and* (4) *if there exists a function $v \in L^1(J,\mathbb{R})$ with $v \in F(t,x,y)$ a.e. on $[a,b]$ such that x satisfies the sequential fractional differential equation $D^\alpha x(t) = v(t)$ on $[a,b]$ and the nonlocal integro-multipoint boundary Condition* (4)*.*

4.1. The Upper Semicontinuous Case

Let us discuss first the case when the multivalued F has convex values and we give an existence result based on Martelli's fixed point theorem [19]. For the reader's convenience, we state the following form of Martelli's fixed point, which is the multivalued version of Schaefer's fixed point theorem.

Lemma 3. (Martelli's fixed point theorem) *[19] Let X be a Banach space and $T : X \to \mathcal{P}_{b,cl,c}(X)$ be a completely continuous multivalued map. If the set $\mathcal{E} = \{x \in X : \lambda x \in T(x),\ \lambda > 1\}$ is bounded, then T has a fixed point.*

Theorem 4. *Assume that* (17) *holds. In addition, we assume that*

(A_1) *$F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory i.e.,*

(i) $t \longmapsto F(t,x,y)$ *is measurable for each $y \in \mathbb{R}$;*

(ii) $(x,y) \longmapsto F(t,x,y)$ *is upper semicontinuous for almost all $t \in [a,b]$;*

(iii) for each $r > 0$, there exists $\phi_r \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y\} \le \phi_r(t)$$

for all $x, y \in \mathbb{R}$ with $\|x\|, \|y\| \le r$ and for a.e. $t \in [a,b]$;

(A_2) *there exists a function $q \in C([a,b],\mathbb{R})$ such that*

$$\|F(t,x,y)\| \le q(t), \quad \text{for a.e. } t \in [a,b] \text{ and each } x,y \in \mathbb{R}.$$

Then, the sequential boundary value Problems (3) *and* (4) *have at least one solution on $[a,b]$.*

Proof. We consider the multivalued map $N : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ in order to transform Problems (3) and (4) into a fixed point problem:

$$N(x) = \left\{ \begin{array}{l} h \in C([a,b],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} I^\alpha v(t) - kI^1 x(t) \\ + \dfrac{(t-a)^{\gamma-1}}{\Lambda} \left[\displaystyle\sum_{i=1}^{m-2} \zeta_i I^{\alpha+\phi_i} v(\theta_i) \right. \\ \left. -k \displaystyle\sum_{i=1}^{m-2} \zeta_i I^{\phi_i+1} x(\theta_i) + kI^1 x(b) - I^\alpha v(b) \right], \quad v \in S_{F,x,y} \end{array} \right. \end{array} \right\}.$$

Obviously, the solutions of the sequential boundary value Problems (3) and (4) are fixed points of N. We will show that the operator N satisfies all conditions of Martelli's fixed point theorem (Lemma 3). The proof is constructed in several steps.

Step 1. *$N(x)$ is convex for each $x \in C([a,b],\mathbb{R})$.*

Indeed, if h_1, h_2 belongs to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in [a,b]$, we have

$$h_i(t) = I^\alpha v_i(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v_i(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^\alpha v_i(b)\Bigg], \quad i = 1,2.$$

Let $0 \le \lambda \le 1$. Then, for each $t \in [a,b]$, we have

$$\begin{aligned}[\lambda h_1 + (1-\lambda)h_2](t) &= I^\alpha[\lambda v_1(s) + (1-\lambda)v_2(s)](t) - kI^1x(t) \\ &+ \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}[\lambda v_1(s) + (1-\lambda)v_2(s)](\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) \\ &+ kI^1x(b) - I^\alpha[\lambda v_1(s) + (1-\lambda)v_2(s)](b)\Bigg], \quad i = 1,2.\end{aligned}$$

Then, we have

$$\lambda h_1 + (1-\lambda)h_2 \in N(x),$$

since F has convex values.

Step 2. *Bounded sets are mapped by N into bounded sets in* $C([a,b],\mathbb{R})$.

For $r > 0$, let a bounded ball in $C([a,b],\mathbb{R})$, defined by $B_r = \{x \in C([a,b],\mathbb{R}) : \|x\| \le r\}$. Then, for each $h \in N(x)$, $x \in B_r$, there exists $v \in S_{F,x}$ such that

$$h(t) = I^\alpha v(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^\alpha v(b)\Bigg].$$

Then, for $t \in [a,b]$, we have

$$\begin{aligned}|h(t)| &\le I^\alpha|v(t)| + |k|I^1|x(t)| + \frac{(t-a)^{\gamma-1}}{|\Lambda|}\Bigg\{\sum_{i=1}^{m-2}|\zeta_i|I^{\alpha+\phi_i}|v(\theta_i)| \\ &+ |k|\sum_{i=1}^{m-2}|\zeta_i|I^{\phi_i+1}|x(\theta_i| + |k|I^1|x(b)| + I^\alpha|v(b)|\Bigg\} \\ &\le \Bigg\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\Bigg] + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\Bigg\}\|q\| \\ &+ \Bigg\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\phi_i+1}}{\Gamma(\phi_i+2)} + |k|(b-a)\Bigg] + |k|(b-a)\Bigg\}\|x\|,\end{aligned}$$

and consequently,

$$\|N(x)\| \le \|q\|\Omega + \Omega_1 r.$$

Step 3. *Bounded sets are mapped by N into equicontinuous sets of* $C([a,b],\mathbb{R})$.

Let $x \in B_r$ and $h \in N(x)$. For each $t \in [a,b]$, there exists a function $v \in S_{F,x}$ such that

$$h(t) = I^{\alpha}v(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v(b)\right].$$

Let $t_1,\ t_2 \in [a,b],\ t_1 < t_2$. Then, we have

$$\begin{aligned}
|h(t_2) - h(t_1)| \leq{}& \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]v(s)ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}v(s)ds\right| \\
&+ r(t_2-t_1) + \frac{(t_2-a)^{\gamma-1} - (t_1-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}|v(\theta_i)| + I^{\alpha}|v(b)|\right. \\
&\left. + |k|\sum_{i=1}^{m-2}|\zeta_i| I^{\phi_i+1}|x(\theta_i)| + |k|I^1|x(b)|\right] \\
\leq{}& \frac{\|q\|}{\Gamma(\alpha+1)}[2(t_2-t_1)^{\alpha} + |(t_2-a)^{\alpha} - (t_1-a)^{\alpha}|] \\
&+ r(t_2-t_1) + \frac{(\tau_2-a)^{\gamma-1} - (\tau_1-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}|v(\theta_i)| + I^{\alpha}|v(b)|\right. \\
&\left. + |k|\sum_{i=1}^{m-2}|\zeta_i| I^{\phi_i+1}|x(\theta_i)| + |k|I^1|x(b)|\right].
\end{aligned}$$

The right side of the above inequality clearly tends to zero independently of $x \in B_r$ as $t_1 \to t_2$. As a consequence of Steps 1–3 together with Arzelá–Ascoli's theorem, we conclude that $N : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ is completely continuous.

Now, we show that the operator N is upper semicontinuous. To prove this, [16] [Proposition 1.2] is enough to show that N has a closed graph.

Step 4. *N has a closed graph.*

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. We will show that $h_* \in N(x_*)$. Now, $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that for each $t \in [a,b]$,

$$h_n(t) = I^{\alpha}v_n(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v_n(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v_n(b)\right].$$

We show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [a,b]$,

$$h_*(t) = I^{\alpha}v_*(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v_*(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v_*(b)\right].$$

Consider the continuous linear operator $\Theta : L^1([a,b],\mathbb{R}) \to C([a,b])$ by

$$v \to \Theta(v)(t) = I^{\alpha}v(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v(b)\Bigg].$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$, and thus, it follows from a closed graph theorem [26] that $\Theta \circ S_{F,x}$ is a closed graph operator. Moreover, we have

$$h_n \in \Theta(S_{F,x_n}).$$

Since $x_n \to x_*$, the closed graph theorem [26] implies that

$$h_*(t) = I^{\alpha}v_*(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v_*(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v_*(b)\Bigg],$$

for some $v_* \in S_{F,x_*}$.

Step 5. *We show the boundedness of the set*

$$\mathcal{E} = \{x \in C([a,b],\mathbb{R}) : \lambda x \in N(x),\ \lambda > 1.\}$$

.

Let $x \in \mathcal{E}$, then, $\lambda x \in N(x)$ for some $\lambda > 1$ and there exists a function $v \in S_{F,x}$ such that

$$x(t) = \frac{1}{\lambda}\, I^{\alpha}v(t) - k\frac{1}{\lambda}\, I^1x(t) + \frac{1}{\lambda}\,\frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v(b)\Bigg].$$

For each $t \in [a,b]$, we have from Step 2 that

$$\|x\| \le \|q\|\Omega + \Omega_1\|x\|,$$

and consequently,

$$\|x\| \le \frac{\|q\|\Omega}{1-\Omega_1},$$

which means that the set $\mathcal{E}$ is bounded. By using Lemma 3, we conclude that N has at least one fixed point and consequently, the sequential boundary value Problems (3)–(4) have a solution on $[a,b]$. $\square$

Our second existence result is proved via Leray–Schauder nonlinear alternative for multivalued maps [17].

Theorem 5. *Assume that* (17) *and* (A_1) *hold. In addition we assume that:*

(A_3) *there exists a continuous, nondecreasing, subhomogeneous function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([a,b],\mathbb{R}^+)$ *such that*

$$\|F(t,x,y)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(|x|+|y|) \text{ for each } (t,x,y) \in [a,b]\times\mathbb{R}\times\mathbb{R};$$

(A_4) *there exists a constant* $M > 0$ *such that*

$$\frac{(1-\Omega_1)M}{L_1\psi(M)\|p\|\Omega} > 1.$$

Then the sequential boundary value Problems (3) *and* (4) *has at least one solution on* $[a,b]$.

Proof. Consider the operator N defined in the proof of Theorem 4. Let $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. We show there exists $U \subseteq C([a,b],\mathbb{R})$, U is an open set, with $x \notin N(x)$ for all $x \in \partial U$ and for any $\lambda \in (0,1)$. Let $\lambda \in (0,1)$ and $x \in \lambda N(x)$. Then, there exists $v \in L^1([a,b],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in J$, we have

$$x(t) = I^{\alpha}v(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v(b)\Bigg].$$

In view of (A_3), we have for each $t \in [a,b]$, as in Theorem 3, that

$$|x(t)| \le L_1\psi(\|x\|)\|p\|\Omega + \Omega_1\|x\|$$

or

$$\frac{(1-\Omega_1)\|x\|}{L_1\psi(\|x\|)\|p\|\Omega} \le 1.$$

In view of (A_4), there exists M such that $\|x\| \neq M$. Consider the set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

Proceeding as in the proof of Theorem 4, we have that the operator $N : \overline{U} \to \mathcal{P}(C([a,b],\mathbb{R}))$ is a compact, upper semicontinuous, multivalued map with convex closed values. There is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$, by the choice of U. Thus, N has a fixed point $x \in \overline{U}$, which is a solution of the sequential boundary value Problems (3) and (4), by the nonlinear alternative of Leray–Schauder type ([17]). This completes the proof. □

4.2. The Lipschitz Case

In this subsection, we establish the existence of solutions for the sequential boundary value Problems (3) and (4) with a possible nonconvex-valued right-hand side by using a fixed point theorem for multivalued maps suggested by Covitz and Nadler [20].

Theorem 6. *Assume that*

(A_4) $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot,x,y) : [a,b] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x, y \in \mathbb{R}$.

(A_5) $H_d(F(t,x,y),F(t,\bar{x},\bar{y})) \le m(t)(|x-\bar{x}| + |y-\bar{y}|)$ *for almost all* $t \in [a,b]$ *and* $x,y,\bar{x},\bar{y} \in \mathbb{R}$ *with* $m \in C([a,b],\mathbb{R}^+)$ *and* $d(0,F(t,0,0)) \le m(t)$ *for almost all* $t \in [a,b]$.

Then, if

$$L_1\Omega\|m\| + \Omega_1 < 1,$$

the sequential boundary value Problems (3) *and* (4) *have at least one solution on* $[a,b]$.

Proof. We transform the boundary value Problems (3) and (4) into a fixed point problem by using the operator $N : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ defined in Theorem 4. We will show that the operator N satisfies the conditions of the Covitz–Nadler theorem.

Step I. N is nonempty and closed for every $v \in S_{F,x}$.

The set-valued map $F(\cdot, x(\cdot))$ admits a measurable selection $v : [a,b] \to \mathbb{R}$ by the measurable selection theorem ([21] Theorem III.6).

We have, by the assumption (A_5), that

$$|v(t)| \leq m(t) + m(t)(|x(t)| + |I^{\delta}x(t)|) \leq m(t) + L_1 m(t)|x(t)|,$$

i.e., $v \in L^1([a,b],\mathbb{R})$ and hence, F is integrably bounded, which means $S_{F,x} \neq \emptyset$. Further, $N(x) \in \mathcal{P}_{cl}(C([a,b],\mathbb{R}))$ for each $x \in C([a,b],\mathbb{R})$. Let $\{u_n\}_{n\geq 0} \in N(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([a,b],\mathbb{R})$. Then, $u \in C([a,b],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [a,b]$,

$$h_n(t) = I^{\alpha}v_n(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v_n(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v_n(b)\Bigg].$$

v_n converges to v in $L^1([a,b],\mathbb{R})$, as F has compact values. Thus, $v \in S_{F,x}$ and for each $t \in [a,b]$, we have

$$u_n(t) \to v(t) = I^{\alpha}v(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v(b)\Bigg].$$

Hence, $u \in N(x)$.

Step II. Next, we show that there exists $0 < \theta < 1$ $(\theta = L_1\Omega\|m\| + \Omega_1)$ such that

$$H_d(N(x), N(\bar{x})) \leq \theta\|x - \bar{x}\| \text{ for each } x, \bar{x} \in C(J,\mathbb{R}).$$

Let $x, \bar{x} \in C([a,b],\mathbb{R})$ and $h_1 \in N(x)$. Then, there exists $v_1(t) \in F(t, x(t), y(t))$ such that, for each $t \in [a,b]$,

$$h_1(t) = I^{\alpha}v_1(t) - kI^1x(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{m-2}\zeta_i I^{\alpha+\phi_i}v_1(\theta_i) - k\sum_{i=1}^{m-2}\zeta_i I^{\phi_i+1}x(\theta_i) + kI^1x(b) - I^{\alpha}v_1(b)\Bigg].$$

By (A_5), we have

$$H_d(F(t,x,y), F(t,\bar{x},\bar{y})) \leq m(t)(|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|).$$

So, there exists $w(t) \in F(t, \bar{x}(t), \bar{y}(t))$ such that

$$|v_1(t) - w| \leq m(t)(|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|), \quad t \in [a,b].$$

Define $U : J \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)(|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|)\}.$$

There exists a function $v_2(t)$, which is a measurable selection for U, since the multivalued operator $U(t) \cap F(t, \bar{x}(t), \bar{y}(t))$ is measurable ([21] Proposition III.4). So, $v_2(t) \in F(t, \bar{x}(t), \bar{y}(t))$ and for each $t \in [a,b]$, we have $|v_1(t) - v_2(t)| \leq m(t)(|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|)$.

For each $t \in [a,b]$, let us define

$$h_2(t) = I^{\alpha} v_2(t) - kI^1 \bar{x}(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2} \zeta_i I^{\alpha+\phi_i} v_2(\theta_i) - k\sum_{i=1}^{m-2} \zeta_i I^{\phi_i+1}\bar{x}(\theta_i) + kI^1\bar{x}(b) - I^{\alpha} v_2(b)\right].$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &= I^{\alpha}|v_2(t) - v_1(t)| + |k|(b-a)\|x-\bar{x}\| + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2} \zeta_i I^{\alpha+\phi_i}|v_2(\theta_i) - v_1(\theta_i)|\right. \\ &\quad \left. + |k|\sum_{i=1}^{m-2} |\zeta_i|\frac{(\theta_i - a)^{\phi_i+1}}{\Gamma(\phi_i+2)}\|x-\bar{x}\| + |k|(b-a)\|x-\bar{x}\| + I^{\alpha}|v_2(b) - v_1(b)|\right] \\ &\le (L_1\Omega\|m\| + \Omega_1)\|x-\overline{x}\| \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le (L_1\Omega\|m\| + \Omega_1)\|x - \overline{x}\|.$$

Interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(N(x), N(\bar{x})) \le (L_1\Omega\|m\| + \Omega_1)\|x - \overline{x}\|.$$

Hence, N is a contraction. By the Covitz–Nadler fixed point theorem [20], N has a fixed point x, which is a solution of sequential boundary value Problems (3) and (4). This finishes the proof. □

Example 3. *Consider the nonlocal sequential Hilfer inclusion boundary value problem of the form:*

$$\begin{cases} \left({}^{H}D^{\frac{5}{4},\frac{3}{4}} + \frac{1}{8}{}^{H}D^{\frac{1}{4},\frac{3}{4}}\right)x(t) \in F(t, x(t), I^{\frac{4}{3}}x(t)), \quad t \in \left[\frac{1}{6}, \frac{3}{2}\right], \\ x\left(\frac{1}{6}\right) = 0, \quad x\left(\frac{3}{2}\right) = \frac{1}{6}I^{\frac{1}{4}}x\left(\frac{1}{3}\right) + \frac{3}{26}I^{\frac{1}{2}}x\left(\frac{1}{2}\right) + \frac{5}{36}I^{\frac{3}{4}}x\left(\frac{2}{3}\right) + \frac{7}{46}I^{\frac{5}{4}}x\left(\frac{5}{6}\right) \\ \qquad + \frac{9}{56}I^{\frac{3}{2}}x(1) + \frac{13}{66}I^{\frac{7}{4}}x\left(\frac{7}{6}\right) + \frac{15}{76}I^{\frac{9}{4}}x\left(\frac{4}{3}\right), \end{cases} \tag{22}$$

where

$$F(t, x, I^{\frac{4}{3}}x) = \left[0, \frac{6}{6t+71}\left(\frac{x^6 e^{-\left(I^{\frac{4}{3}}x\right)^2}}{1+|x|^5}\right) + \frac{6}{6t+77}\left(\left|\sin\left(I^{\frac{4}{3}}x\right)\right| + \frac{3x^2}{1+x^2}\right) + \frac{6}{6t+83}\left(\frac{1}{|x| + \left|I^{\frac{4}{3}}x\right| + 3}\right)\right]. \tag{23}$$

Now, we set constants $\alpha = 5/4, \beta = 3/4, k = 1/8, \delta = 4/3, a = 1/6, b = 3/2, m = 9, \zeta_1 = 1/6, \zeta_2 = 3/26, \zeta_3 = 5/36, \zeta_4 = 7/46, \zeta_5 = 9/56, \zeta_6 = 13/66, \zeta_7 = 15/76, \phi_1 = 1/4, \phi_2 = 1/2, \phi_3 = 3/4, \phi_4 = 5/4, \phi_5 = 3/2, \phi_6 = 7/4, \phi_7 = 9/4, \theta_1 = 1/3, \theta_2 = 1/2, \theta_3 = 2/3, \theta_4 = 5/6, \theta_5 = 1, \theta_6 = 7/6, \theta_7 = 4/3, \gamma = (5/4) + (2-(5/4))(3/4) = 29/16$. From the information, we find that $\Lambda \approx 1.024874752, \Omega \approx 2.992145342, \Omega_1 \approx 0.4023896011$, and $L_1 \approx 2.232550581$.

From (23), we obtain

$$\|F(t,w,z)\|_{\mathcal{P}} \le \frac{6}{6t+71}\left(|w|+|z|+3+\frac{1}{|w|+|z|+3}\right),$$

which can be set as $p(t) = 6/(6t+71)$ and $\psi(u) = u+3+(1/(u+3)), u = |w|+|z|$. Consequently, we get $\|p\| = 1/12$ and $\psi(\mu u) \le \mu\psi(u), \forall \mu \ge 1$, which is the subhomogeneous function. Therefore, the condition (A_3) of Theorem 5 is fulfilled. Indeed, there exists a constant $M > 41.10633915$, which makes condition (A_4) true. Thus, the boundary value problem of Hilfer fractional differential Inclusions (22) and (23) have at least one solution on $[1/6, 3/2]$.

5. Conclusions

In the present paper, we discussed a new class of boundary value problems for sequential fractional differential equations and inclusions involving Hilfer fractional derivatives, supplemented with nonlocal integro-multipoint boundary conditions. In the single-valued case, existence and uniqueness results are established by using the classical fixed point theorems of Banach and Krasnosel'skiĭ and the nonlinear alternative of Leray–Schauder. In the multivalued case and convex-valued multivalued maps, we proved two existence results by applying Martelli's fixed point theorem and the nonlinear alternative for Kakutani maps. In the case of possible nonconvex-valued maps, we obtained an existence result via the Covitz–Nadler fixed point theorem for contractive maps. The obtained results are well illustrated by numerical examples. The results obtained in the present paper are new and significantly contribute to the existing literature on the topic.

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