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A Note on Some Definite Integrals of Arthur Erdélyi and George Watson

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Abstract: This manuscript concerns two definite integrals that could be connected to the Bose-Einstein and the Fermi-Dirac functions in the integrands, separately, with numerators slightly modified with a difference in two expressions that contain the Fourier kernel multiplied by a polynomial and its complex conjugate. In this work, we use our contour integral method to derive these definite integrals, which are given by $\int_0^\infty \frac{i\left(e^{-imx}(\log(a)-ix)^k - e^{imx}(\log(a)+ix)^k\right)}{2(e^{ax}-1)}dx$ and $\int_0^\infty \frac{i\left(e^{-imx}(\log(a)-ix)^k - e^{imx}(\log(a)+ix)^k\right)}{2(e^{ax}+1)}dx$ in terms of the Lerch function. We use these two definite integrals to derive formulae by Erdéyli and Watson. We derive special cases of these integrals in terms of special functions not found in current literature. Special functions have the property of analytic continuation, which widens the range of computation of the variables involved.

Keywords: entries in Erdélyi; Lerch function; hypergeometric function; incomplete beta function; definite integral; Mellin transform

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1. Introduction

In 1953, Erdélyi et al. [1] and Watson et al. [2] published famous books containing a significant number of definite integrals not containing formal derivations. In this manuscript, we focus on deriving a few of these integrals, which have interesting closed forms, including deriving new Mellin transforms in terms of special functions. Mellin transforms [3] are used in solving the potential problem in a wedge-shaped region and solving linear differential equations in electrical engineering. We apply our method to derive these closed-form solutions, which are continued analytically along with some special cases. The derivations follow the method used by us in [4–9]. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{1}$$

This method involves using a form of Equation (1), then multipliying both sides by a function, then taking a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then, we multiply both sides of Equation (1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. Definite Integral of the Contour Integrals

We use the method in [8]. The variable of integration in the contour integral is $z = m + w$. The cut and contour are in the second quadrant of the complex z-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and are on opposite sides of the cut.

2.1. The Hyperbolic Cotangent Contour Integral

Using a generalization of Cauchy's integral formula, we form two equations by replacing y by $ix + \log(a)$ and multiplying by e^{mxi} to form the first equation, followed by replacing x by $-x$ and multiplying by e^{-imx} to obtain the second, subtracting the two followed by multiplying by $\frac{1}{e^{\alpha x}-1}$, then taking the definite integral with respect $x \in [0, \infty)$ to obtain

$$\begin{aligned}
\int_0^\infty \frac{i\left(e^{-imx}(\log(a) - ix)^k - e^{imx}(\log(a) + ix)^k\right)}{2k!(e^{\alpha x} - 1)} dx &= \frac{1}{2\pi i}\int_0^\infty \int_C \frac{a^w w^{-k-1}\sin(x(m+w))}{e^{\alpha x} - 1} dw dx \\
&= \frac{1}{2\pi i}\int_C \int_0^\infty \frac{a^w w^{-k-1}\sin(x(m+w))}{e^{\alpha x} - 1} dx dw \\
&= \frac{1}{2\pi i}\int_C \left(\frac{\pi a^w w^{-k-1}\coth\left(\frac{\pi(m+w)}{\alpha}\right)}{2\alpha} - \frac{a^w w^{-k-1}}{2(m+w)}\right) dw \qquad (2)
\end{aligned}$$

from Equation (2.4.11) in [10]. The condition on the left-hand side of Equation (2) is $Re(\alpha) > |Im(m)|$ and the right-hand side of Equation (2) is $Re(\alpha) > Im(m+w)$. a and k are general complex numbers.

Here, we derive the additional contour integral in Equation (2). Use Equation (1), and replace y with $y + \log(a)$ and multiply both sides by $-\frac{e^{my}}{2}$, then take the definite integral over $y \in [0, \infty)$ to obtan

$$\frac{a^{-m}(-m)^{-k-1}\Gamma(k+1, -m\log(a))}{2k!} = -\frac{1}{2\pi i}\int_C \frac{a^w w^{-k-1}}{2(m+w)} dw \qquad (3)$$

from Equation (3.382.4) in [11], where $Re(m+w) < 0$ and $Re(m) < 0$.

2.2. The Hyperbolic Cosecant Contour Integral

Using a generalization of Cauchy's integral formula, we form two equations by replacing y by $ix + \log(a)$ and multiplying by e^{mxi} to form the first equation, followed by replacing x by $-x$ and multiplying by e^{-imx} to obtain the second. Subtracting the two followed by multiplying by $\frac{1}{e^{\alpha x}+1}$, then taking the definite integral with respect $x \in [0, \infty)$, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{i\left(e^{-imx}(\log(a) - ix)^k - e^{imx}(\log(a) + ix)^k\right)}{2k!(e^{\alpha x} + 1)} dx \\
&= \frac{1}{2\pi i}\int_0^\infty \int_C \frac{a^w w^{-k-1}\sin(x(m+w))}{e^{\alpha x} + 1} dw dx \\
&= \frac{1}{2\pi i}\int_C \int_0^\infty \frac{a^w w^{-k-1}\sin(x(m+w))}{e^{\alpha x} + 1} dx dw \\
&= \frac{1}{2\pi i}\int_C \left(\frac{a^w w^{-k-1}}{2(m+w)} - \frac{\pi a^w w^{-k-1}\operatorname{csch}\left(\frac{\pi(m+w)}{\alpha}\right)}{2\alpha}\right) dw \qquad (4)
\end{aligned}$$

from Equation (2.4.10) in [10].

3. The Lerch Function

The Lerch function given in (9.551) and (9.556) in [11] has a series representation given by

$$\Phi(z, s, v) = \sum_{n=0}^\infty (v+n)^{-s} z^n \qquad (5)$$

where $|z| < 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{6}$$

where $Re(v) > 0$, or $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4. Infinite Sum of the Contour Integral

4.1. The Hyperbolic Cotangent Contour Integral

Using Equation (1) and replacing y with $2\pi(y+1)/\alpha + \log(a)$, then multiplying both sides by $-\frac{\pi}{\alpha}\exp\left(\frac{2\pi m(y+1)}{\alpha}\right)$, we obtain

$$-\frac{2^k \pi^{k+1}\left(\frac{1}{\alpha}\right)^{k+1} e^{\frac{2\pi m}{\alpha}+\frac{2\pi m y}{\alpha}}\left(\frac{\alpha\log(a)}{2\pi}+y+1\right)^k}{k!} = -\frac{1}{2\pi i}\int_C \frac{\pi w^{-k-1}\exp\left(w\left(\log(a)+\frac{2\pi(y+1)}{\alpha}\right)+\frac{2\pi m(y+1)}{\alpha}\right)}{\alpha}dw \tag{7}$$

We then take the infinite sum over $y \in [0,\infty)$ and simplify in terms of the Lerch function to obtain

$$-\frac{2^k \pi^{k+1}\left(\frac{1}{\alpha}\right)^{k+1} e^{\frac{2\pi m}{\alpha}}\Phi\left(e^{\frac{2m\pi}{\alpha}},-k,\frac{\alpha\log(a)}{2\pi}+1\right)}{k!} = \frac{1}{2\pi i}\int_C \left(\frac{\pi a^w w^{-k-1}\coth\left(\frac{\pi(m+w)}{\alpha}\right)}{2\alpha}+\frac{\pi a^w w^{-k-1}}{2\alpha}\right)dw \tag{8}$$

from Equation (1.232.1) in [11] and $Re(m+w) < 0$.

Here, we derive the additional contour integral by using Equation (1) and replacing y with $\log(a)$ and multiplying both sides by $\frac{\pi}{2\alpha}$ to obtain

$$\frac{\pi\log^k(a)}{2\alpha k!} = \frac{1}{2\pi i}\int_C \frac{\pi a^w w^{-k-1}}{2\alpha}dw \tag{9}$$

4.2. The Hyperbolic Cosecant Contour Integral

Using Equation (1) and replacing y by

$$\pi(2y+1)/\alpha + \log(a)$$

then multiplying both sides by $\frac{\pi}{\alpha}e^{\frac{\pi m}{\alpha}+\frac{2\pi m y}{\alpha}}$; simplifying, we obtain

$$\frac{2^k \pi^{k+1}\left(\frac{1}{\alpha}\right)^{k+1} e^{\frac{\pi m}{\alpha}+\frac{2\pi m y}{\alpha}}\left(\frac{\alpha\log(a)}{2\pi}+\frac{1}{2}(2y+1)\right)^k}{k!} = \frac{1}{2\pi i}\int_C \frac{\pi w^{-k-1}\exp\left(w\left(\log(a)+\frac{\pi(2y+1)}{\alpha}\right)+\frac{\pi m(2y+1)}{\alpha}\right)}{\alpha}dw \tag{10}$$

We then take the infinite sum over $y \in [0,\infty)$ to obtain

$$\frac{2^k \pi^{k+1}\left(\frac{1}{\alpha}\right)^{k+1} e^{\frac{\pi m}{\alpha}}\Phi\left(e^{\frac{2m\pi}{\alpha}},-k,\frac{\alpha\log(a)+\pi}{2\pi}\right)}{k!} = -\frac{1}{2\pi i}\int_C \frac{\pi a^w w^{-k-1}\operatorname{csch}\left(\frac{\pi(m+w)}{\alpha}\right)}{2\alpha}dw \tag{11}$$

from Equation (1.232.3) in [11] and $Re(m+w) < 0$.

5. Definite Integrals in Terms of the Lerch Function

Theorem 1. *For all $a, k, \alpha \in \mathbb{C}$ and $Re(m) < 0$,*

$$\int_0^\infty \frac{i\left(e^{-imx}(\log(a) - ix)^k - e^{imx}(\log(a) + ix)^k\right)}{2(e^{\alpha x} - 1)} dx$$
$$= \frac{1}{2} a^{-m}(-m)^{-k-1}\Gamma(k+1, -m\log(a)) - 2^k \pi^{k+1} \left(\frac{1}{\alpha}\right)^{k+1} e^{\frac{2\pi m}{\alpha}} \Phi\left(e^{\frac{2m\pi}{\alpha}}, -k, \frac{\alpha \log(a)}{2\pi} + 1\right) - \frac{\pi \log^k(a)}{2\alpha} \quad (12)$$

Proof. Since the right hand-side of Equation (2) is equal to the sum of the right-hand sides of Equations (3), (8), and (9), we can equate the left-hand sides. □

Theorem 2. *For all $a, k, \alpha \in \mathbb{C}$ and $Re(m) < 0$,*

$$\int_0^\infty \frac{i\left(e^{-imx}(\log(a) - ix)^k - e^{imx}(\log(a) + ix)^k\right)}{2(e^{\alpha x} + 1)} dx$$
$$= 2^k \pi^{k+1} \left(\frac{1}{\alpha}\right)^{k+1} e^{\frac{\pi m}{\alpha}} \Phi\left(e^{\frac{2m\pi}{\alpha}}, -k, \frac{\alpha \log(a) + \pi}{2\pi}\right) - \frac{1}{2} a^{-m}(-m)^{-k-1}\Gamma(k+1, -m\log(a)) \quad (13)$$

Proof. Since the right-hand side of Equation (3) is equal to the sum of the right-hand sides of Equations (4) and (11), we can equate the left-hand sides. □

6. Table of Definite Integrals in and Special Cases

In this section, we derive formula in [11] and use Equations (12) and (13) with specific values of the parameters to derive definite integrals. Some of the definite integrals in this section can be viewed as either Mellin transforms or Laplace transforms.

6.1. Derivation of Entry (3.554.6)

Theorem 3. *For $Re(\beta) > 0$ and $Re(\mu) > 1$,*

$$\int_0^\infty e^{-\beta x} x^{\mu-1}(\coth(x) - 1) dx = 2^{1-\mu}\Gamma(\mu)\zeta\left(\mu, \frac{\beta}{2} + 1\right) \quad (14)$$

Proof. The derivation of this particular integral is a bit involved and we detail the steps below. First,using Equation (12), we replace m by $-m$ to form a second equation. Then, we take the difference between these two equations, setting $a = 1$ and replacing α with 2α, k with $s - 1$, and using $\Gamma(k, 0) = \Gamma(k)$ to obtain the Mellin transform

$$\begin{aligned}\int_0^\infty x^{s-1}\sin(mx)(\coth(\alpha x) - 1) dx = &-\frac{1}{2}(-m)^s\left(-m^2\right)^{-s}\csc\left(\frac{\pi s}{2}\right)\Gamma(s) \\ &+ \frac{1}{2} m^s \left(-m^2\right)^{-s}\csc\left(\frac{\pi s}{2}\right)\Gamma(s) \\ &+ \frac{1}{2}\pi^s\left(\frac{1}{\alpha}\right)^s\csc\left(\frac{\pi s}{2}\right)\mathrm{Li}_{1-s}\left(e^{-\frac{m\pi}{\alpha}}\right) \\ &- \frac{1}{2}\pi^s\left(\frac{1}{\alpha}\right)^s\csc\left(\frac{\pi s}{2}\right)\mathrm{Li}_{1-s}\left(e^{\frac{m\pi}{\alpha}}\right)\end{aligned} \quad (15)$$

Next, we take the partial derivative of Equation (15) with respect to m and replace s with $s-1$ and simplify to obtain

$$\begin{aligned}\int_0^\infty x^{s-1}\cos(mx)(\coth(\alpha x)-1)dx = &-\frac{1}{2}(-m)^s\left(-m^2\right)^{-s}\sec\left(\frac{\pi s}{2}\right)\Gamma(s)\\ &-\frac{1}{2}m^s\left(-m^2\right)^{-s}\sec\left(\frac{\pi s}{2}\right)\Gamma(s)\\ &+\frac{1}{2}\pi^s\left(\frac{1}{\alpha}\right)^s\sec\left(\frac{\pi s}{2}\right)\mathrm{Li}_{1-s}\left(e^{-\frac{m\pi}{\alpha}}\right)+\\ &\frac{1}{2}\pi^s\left(\frac{1}{\alpha}\right)^s\sec\left(\frac{\pi s}{2}\right)\mathrm{Li}_{1-s}\left(e^{\frac{m\pi}{\alpha}}\right)\end{aligned} \tag{16}$$

Next, we multiply Equation (15) by i and add to Equation (16), setting $\alpha=1$ and replacing m with $2\beta i$ and s with $1-\mu$ and simplifying to obtain

$$\begin{aligned}&\int_0^\infty e^{-2\beta x}x^{-\mu}(\coth(x)-1)dx\\ &=\frac{e^{-\frac{3}{2}i\pi\mu}\pi^{-\mu}}{\beta}(\cot(\pi\mu)+i)\left(\pi\beta\left(e^{i\pi\mu}\mathrm{Li}_\mu\left(e^{-2i\pi\beta}\right)+\mathrm{Li}_\mu\left(e^{2i\pi\beta}\right)\right)-e^{\frac{i\pi\mu}{2}}(2\pi)^\mu\beta^\mu\sin(\pi\mu)\Gamma(1-\mu)\right)\end{aligned} \tag{17}$$

Now, we simplify Equation (17) by applying Equation (1.11.16) in [1] given by

$$e^{i\pi s}\mathrm{Li}_s\left(e^{-2ia\pi}\right)+\mathrm{Li}_s\left(e^{2ia\pi}\right)=\frac{e^{\frac{i\pi s}{2}}(2\pi)^s\zeta(1-s,a)}{\Gamma(s)} \tag{18}$$

to yield

$$\int_0^\infty e^{-2\beta x}x^{-\mu}(\coth(x)-1)dx=\frac{\pi 2^\mu\csc(\pi\mu)(\beta\zeta(1-\mu,\beta)-\beta^\mu)}{\beta\Gamma(\mu)} \tag{19}$$

Next, we replace β with $\beta/2$ and μ with $1-\mu$, where $Re(\mu)>1$ and $Re(\beta)>0$, to obtain

$$\int_0^\infty e^{-\beta x}x^{\mu-1}(\coth(x)-1)dx=2^{1-\mu}\Gamma(\mu)\zeta\left(\mu,\frac{\beta}{2}+1\right) \tag{20}$$

and replacing β with $\beta-2$ in (20), we obtain

$$\int_0^\infty e^{-\beta x}x^{\mu-1}(\coth(x)+1)dx=2^{1-\mu}\Gamma(\mu)\zeta\left(\mu,\frac{\beta}{2}\right) \tag{21}$$

Replacing β with $\beta-1$ in (20), we obtain

$$\int_0^\infty e^{-\beta x}x^{\mu-1}\mathrm{csch}(x)dx=2^{1-\mu}\Gamma(\mu)\zeta\left(\mu,\frac{\beta+1}{2}\right) \tag{22}$$

Subtracting Equation (20) from (21), we obtain

$$\int_0^\infty e^{-\beta x}x^{\mu-1}dx=\beta^{-\mu}\Gamma(\mu) \tag{23}$$

from Equation (25.11.3) [12]. Using Equation (13) and repeating the process from Equations (15) to (22) and simplifying, we obtain

$$\int_0^\infty x^{s-1}e^{-\beta x}(1-\mathrm{sech}(x))dx=\frac{\pi 4^{-s}\beta^{-s}\csc(\pi s)\left(4^s-2\beta^s\left(\zeta\left(s,\frac{\beta+1}{4}\right)-\zeta\left(s,\frac{\beta+3}{4}\right)\right)\right)}{\Gamma(1-s)} \tag{24}$$

□

6.2. Derivation of Entry (3.554.1)

Theorem 4. *For $Re(\beta) > 0$,*

$$\int_0^\infty \frac{e^{-\beta x}(1 - sech(x))}{x} dx = \log\left(\frac{4}{\beta}\right) + 2\log\left(\frac{\Gamma\left(\frac{\beta+3}{4}\right)}{\Gamma\left(\frac{\beta+1}{4}\right)}\right) \tag{25}$$

Proof. Using Equation (24) given by

$$\int_0^\infty x^{s-1}e^{-tx}(1 - \operatorname{sech}(x))dx = \frac{\pi 4^{-s}t^{-s}\csc(\pi s)\left(4^s - 2t^s\left(\zeta\left(s, \frac{t+1}{4}\right) - \zeta\left(s, \frac{t+3}{4}\right)\right)\right)}{\Gamma(1-s)} \tag{26}$$

and applying L'Hopital's rule to the right-hand side as $s \to 1$, then taking the definite integral over $t \in [\beta, \infty)$ and simplifying, we obtain

$$\int_0^\infty \frac{e^{-\beta x}(1 - \operatorname{sech}(x))}{x} dx = \log\left(\frac{4}{\beta}\right) + 2\log\left(\frac{\Gamma\left(\frac{\beta+3}{4}\right)}{\Gamma\left(\frac{\beta+1}{4}\right)}\right) \tag{27}$$

where $Im(\beta) \neq 0$ and $Re(\beta) > 0$. □

6.3. Derivation of Entry (3.554.4)

Theorem 5. *For $Re(\beta) > 0$,*

$$\int_0^\infty \frac{e^{-\beta x}(1 - x\coth(x))}{x} dx = \frac{1}{\beta} + \log\left(\frac{2}{\beta}\right) + \psi^{(0)}\left(\frac{\beta}{2}\right) \tag{28}$$

Proof. We add Equations (20) and (21), then replace μ bwithy $\mu + 1$. We then take the difference of this new equation and (23) to obtain

$$\int_0^\infty e^{-\beta x}x^{\mu-1}(1 - x\coth(x))dx = \Gamma(\mu)\left(\beta^{-\mu-1}(\beta+\mu) - 2^{-\mu}\mu\zeta\left(\mu+1, \frac{\beta}{2}\right)\right) \tag{29}$$

Next, we apply L'Hopital's rule as $\mu \to 0$ to obtain

$$\int_0^\infty \frac{e^{-\beta x}(1 - x\coth(x))}{x} dx = \frac{1}{\beta} + \log\left(\frac{2}{\beta}\right) + \psi^{(0)}\left(\frac{\beta}{2}\right) \tag{30}$$

from Equation (13.21) in [2]. □

6.4. Derivation of Entry (3.554.2)

Theorem 6. *For $Re(\beta) > 0$,*

$$\int_0^\infty \frac{e^{-\beta x}(1 - xcsch(x))}{x} dx = \log\left(\frac{2}{\beta}\right) + \psi^{(0)}\left(\frac{\beta+1}{2}\right) \tag{31}$$

Proof. Using (22) and replacing μ with $\mu + 1$, then taking the difference from (23), we obtain

$$\int_0^\infty e^{-\beta x}x^{\mu-1}(1 - xcsch(x))dx = \Gamma(\mu)\left(\beta^{-\mu} - 2^{-\mu}\mu\zeta\left(\mu+1, \frac{\beta+1}{2}\right)\right) \tag{32}$$

Next ,we apply L'Hopital's rule to the right-hand side as $\mu \to 0$ to obtain

$$\int_0^\infty \frac{e^{-\beta x}(1 - xcsch(x))}{x} dx = \log\left(\frac{2}{\beta}\right) + \psi^{(0)}\left(\frac{\beta+1}{2}\right) \tag{33}$$

from Equation (13.21) in [2]. □

6.5. Derivation of Entry (3.554.5)

Theorem 7. *For $-1/2 < Re(m) < 1/2$,*

$$\int_0^\infty \frac{2me^{-x} - csch\left(\frac{x}{2}\right)\sinh(mx)}{x}dx = \log\left(\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-m\right)}\right) \tag{34}$$

Proof. Using Equation (20) and replacing β with $\beta + m - 1$ to form a second equation, we get

$$\int_0^\infty x^{\mu-1}\text{csch}(x)e^{-x(\beta+m)}dx = 2^{1-\mu}\Gamma(\mu)\zeta\left(\mu, \frac{1}{2}(m+\beta+1)\right) \tag{35}$$

Next, we replace m by $-m$ to obtain

$$\int_0^\infty x^{\mu-1}\text{csch}(x)e^{-x(\beta-m)}dx = 2^{1-\mu}\Gamma(\mu)\zeta\left(\mu, \frac{1}{2}(-m+\beta+1)\right) \tag{36}$$

Next, we take the difference of Equations (35) and (36) and simplifying to get

$$\begin{aligned}\int_0^\infty & e^{-\beta x}x^{\mu-1}\text{csch}(x)\sinh(mx)dx \\ &= 2^{-\mu}\Gamma(\mu)\left(\zeta\left(\mu, \frac{1}{2}(-m+\beta+1)\right) - \zeta\left(\mu, \frac{1}{2}(m+\beta+1)\right)\right)\end{aligned} \tag{37}$$

Next, we replace x by $x/2$, replace m with $2m$, and set $\beta = 0$; simplifying, we obtain

$$\int_0^\infty x^{\mu-1}\text{csch}\left(\frac{x}{2}\right)\sinh(mx)dx = \Gamma(\mu)\left(\zeta\left(\mu, \frac{1}{2}(1-2m)\right) - \zeta\left(\mu, \frac{1}{2}(2m+1)\right)\right) \tag{38}$$

Next, we take the difference of Equations (23) and (38) and simplify to get

$$\begin{aligned}\int_0^\infty & x^{\mu-1}\left(2me^{-\beta x} - \text{csch}\left(\frac{x}{2}\right)\sinh(mx)\right)dx \\ &= \Gamma(\mu)\left(-\zeta\left(\mu, \frac{1}{2}-m\right) + \zeta\left(\mu, m+\frac{1}{2}\right) + 2m\beta^{-\mu}\right)\end{aligned} \tag{39}$$

Next, we set $\beta = 1$ and apply L'Hopital's rule to the right-hand side as $\mu \to 0$ and simplify to get

$$\int_0^\infty \frac{2me^{-x} - \text{csch}\left(\frac{x}{2}\right)\sinh(mx)}{x}dx = \log\left(\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-m\right)}\right) \tag{40}$$

from Equation (13.21) in [2], where $-1/2 < Re(m) < 1/2$. This is equivalent to (3.554.5) in [11] by using the reflection formula and leads to a simpler formula. Note that the condition on q in [11] is in error even if q is real. □

6.6. Derivation of Entry (3.554.3)

Theorem 8. *For $0 < Re(\beta) < 1$,*

$$\int_0^\infty \frac{e^{-x}\left(2\beta + e^x csch\left(\frac{x}{2}\right)\sinh\left(\left(\frac{1}{2}-\beta\right)x\right) - 1\right)}{x}dx = \log\left(\frac{\Gamma(\beta)}{\Gamma(1-\beta)}\right) \tag{41}$$

Proof. We multiply Equation (23) by $1 - 2m$; using Equation (39), replace m by $1/2 - m$. Taking the difference, we get

$$\int_0^\infty e^{-\beta x} x^{\mu-1} \left(e^{\beta x} \operatorname{csch}\left(\frac{x}{2}\right) \sinh\left(\left(\frac{1}{2} - m\right)x\right) + 2m - 1 \right) dx$$
$$= \beta^{-\mu}\Gamma(\mu)(\beta^\mu(\zeta(\mu,m) - \zeta(\mu,1-m)) + 2m - 1) \quad (42)$$

Next, we set $\beta = 1$, replace the variable m with β to keep the equation consistent with [11], and apply L'Hopital's rule as $\mu \to 0$ to obtain

$$\int_0^\infty \frac{e^{-x}\left(2\beta + e^x \operatorname{csch}\left(\frac{x}{2}\right) \sinh\left(\left(\frac{1}{2} - \beta\right)x\right) - 1\right)}{x} dx = \log\left(\frac{\Gamma(\beta)}{\Gamma(1-\beta)}\right) \quad (43)$$

where $0 < Re(\beta) < 1$. This is equivalent to (3.554.3) in [11] by using the reflection formula and leads to a simpler formula. □

6.7. Special Cases Using Theorems

6.7.1. Example 1

Using (20) and setting $\mu = 1/2$ and $\beta = 2$, we get

$$\int_0^\infty \frac{e^{-2x}(\coth(x) - 1)}{\sqrt{x}} dx = \sqrt{2\pi}\left(\zeta\left(\frac{1}{2}\right) - 1\right) \quad (44)$$

6.7.2. Example 2

Using (32) and setting $\mu = 1/2$ and $\beta = 1$, we get

$$\int_0^\infty \frac{e^{-x}(1 - x\operatorname{csch}(x))}{\sqrt{x}} dx = -\frac{1}{4}\sqrt{\pi}\left(\sqrt{2}\zeta\left(\frac{3}{2}\right) - 4\right) \quad (45)$$

7. Definite Integrals in Terms of the Hypergeometric Function

In this section, we look at deriving definite integrals in terms of the hypergeometric function. These formula are not known in the current literature. Similar formula types are listed in [11]. The conditions for the hypergeometric function are given by Equation (15.2.1) in [12]. Since $\frac{2m\pi}{\alpha}$ and its negative both appear in the hypergeometric function, to satisfy the conditions on ${}_2F_1$, we must have $\frac{m}{\alpha}$ be purely imaginary.

7.1. Example 1

Theorem 9. *For all $a, \alpha \in \mathbb{C}$ and $Re(m) > 0$,*

$$\int_0^\infty \frac{\sin(mx)}{(e^{\alpha x} + 1)\left(\log^2(a) + x^2\right)} dx$$
$$= \frac{(-(a^m - a^{-m})Chi(m\log(a)) + i\pi)}{4\log(a)} + \frac{(a^m + a^{-m})Shi(m\log(a))}{4\log(a)}$$
$$- \frac{\pi e^{-\frac{\pi m}{\alpha}} {}_2F_1\left(1, \frac{\alpha\log(a)+\pi}{2\pi}; \frac{1}{2}\left(\frac{\alpha\log(a)}{\pi} + 3\right); e^{-\frac{2m\pi}{\alpha}}\right)}{2\log(a)(\alpha\log(a) + \pi)}$$
$$+ \frac{\pi e^{\frac{\pi m}{\alpha}} {}_2F_1\left(1, \frac{\alpha\log(a)+\pi}{2\pi}; \frac{1}{2}\left(\frac{\alpha\log(a)}{\pi} + 3\right); e^{\frac{2m\pi}{\alpha}}\right)}{2\log(a)(\alpha\log(a) + \pi)} \quad (46)$$

Proof. Use Equation (13) and replace m with $-m$ to form a second equation, then take the difference of the two equations and set $k = -1$ and simplify using Equations (8.359.1) and (9.559) in [11]. □

7.2. Example 2

Theorem 10. *For all $a, \alpha \in \mathbb{C}$ and $Re(m) > 0$,*

$$\int_0^\infty \frac{x\cos(mx)}{(e^{\alpha x}+1)\left(\log^2(a)+x^2\right)}dx$$
$$= \frac{1}{4}(-(a^m+a^{-m})Chi(m\log(a)) - i\pi) + \frac{1}{4}(a^m-a^{-m})Shi(m\log(a))$$
$$- \frac{\pi e^{-\frac{\pi m}{\alpha}}\, {}_2F_1\left(1, \frac{\alpha\log(a)+\pi}{2\pi}; \frac{1}{2}\left(\frac{\alpha\log(a)}{\pi}+3\right); e^{-\frac{2m\pi}{\alpha}}\right)}{2(\alpha\log(a)+\pi)}$$
$$- \frac{\pi e^{\frac{\pi m}{\alpha}}\, {}_2F_1\left(1, \frac{\alpha\log(a)+\pi}{2\pi}; \frac{1}{2}\left(\frac{\alpha\log(a)}{\pi}+3\right); e^{\frac{2m\pi}{\alpha}}\right)}{2(\alpha\log(a)+\pi)} \quad (47)$$

Proof. Use Equation (46) and take the first partial derivative with respect to m, then simplify using Equations (8.359.1) and (9.559) in [11]. □

7.3. Example 3

Theorem 11. *For all $a, \alpha \in \mathbb{C}$ and $Re(m) > 0$,*

$$\int_0^\infty \frac{\sin(mx)}{(e^{\alpha x}-1)\left(\log^2(a)+x^2\right)}dx$$
$$= \frac{(a^m-a^{-m})Chi(m\log(a))}{4\log(a)} - \frac{(a^m+a^{-m})Shi(m\log(a))}{4\log(a)} - \frac{i\pi a^{-m}}{4\log(a)}$$
$$+ \frac{\pi e^{-\frac{2\pi m}{\alpha}}\, {}_2F_1\left(1, \frac{\alpha\log(a)}{2\pi}+1; \frac{\alpha\log(a)}{2\pi}+2; e^{-\frac{2m\pi}{\alpha}}\right)}{2\log(a)(\alpha\log(a)+2\pi)}$$
$$- \frac{\pi e^{\frac{2\pi m}{\alpha}}\, {}_2F_1\left(1, \frac{\alpha\log(a)}{2\pi}+1; \frac{\alpha\log(a)}{2\pi}+2; e^{\frac{2m\pi}{\alpha}}\right)}{2\log(a)(\alpha\log(a)+2\pi)} \quad (48)$$

Proof. Use Equation (12) and form a second equation by replacing m with $-m$. Next, take the difference between the new equation and (12), then set $k = -1$ and simplify using Equations (8.359.1) and (9.559) in [11]. □

7.4. Example 4

Theorem 12. *For all $a, \alpha \in \mathbb{C}$ and $Re(m) > 0$,*

$$\int_0^\infty \frac{x\cos(mx)}{(e^{\alpha x}-1)\left(\log^2(a)+x^2\right)}dx$$
$$= \frac{1}{4}(a^m+a^{-m})Chi(m\log(a)) - \frac{1}{4}(a^m-a^{-m})Shi(m\log(a)) + \frac{1}{4}i\pi a^{-m} + \frac{\pi}{2\alpha\log(a)}$$
$$+ \frac{1}{4}\left(e^{-\frac{2\pi m}{\alpha}}\right)^{-\frac{\alpha\log(a)}{2\pi}} B_{e^{-\frac{2m\pi}{\alpha}}}\left(\frac{\alpha\log(a)}{2\pi}+1, 0\right) + \frac{1}{4}\left(e^{\frac{2\pi m}{\alpha}}\right)^{-\frac{\alpha\log(a)}{2\pi}} B_{e^{\frac{2m\pi}{\alpha}}}\left(\frac{\alpha\log(a)}{2\pi}+1, 0\right) \quad (49)$$

Proof. Use Equation (49) and take the first partial derivative with respect to m and simplify using Equations (8.359.1) in [11] and (6.6.8) in [13]. In this proof, we use the incomplete beta function given by Equation (8.39.1) in [11]. □

8. Definite Integral Involving Logarithmic and Sine Function and Quotient

In this section, we derive definite integrals in terms of the first partial derivative of the Lerch function with respect to the second parameter and the exponential function. The conditions for these functions are given in Sections (6.2) in [12] and (4.1) in [13].

8.1. Example 1

Theorem 13. *For all $a, \alpha \in \mathbb{C}$ and $Re(m) > 0$,*

$$\int_0^\infty \frac{\log\left(\log^2(a)+x^2\right)\sin(mx)}{e^{\alpha x}-1}dx$$
$$= -\frac{\pi e^{-\frac{2\pi m}{\alpha}}\Phi'\left(e^{-\frac{2\pi m}{\alpha}},0,\frac{\alpha\log(a)}{2\pi}+1\right)}{\alpha}+\frac{\pi e^{\frac{2\pi m}{\alpha}}\Phi'\left(e^{\frac{2\pi m}{\alpha}},0,\frac{\alpha\log(a)}{2\pi}+1\right)}{\alpha}$$
$$-\frac{a^{-m}E_1(-m\log(a))}{2m}-\frac{a^m E_1(m\log(a))}{2m}-\frac{\log(\log(a))}{m}+\frac{\pi\log\left(\frac{2\pi}{\alpha}\right)\coth\left(\frac{\pi m}{\alpha}\right)}{\alpha} \quad (50)$$

Proof. Use Equation (12) and formulate a second equation by replacing m with $-m$. Next, take the difference from the first equation to create a third. Use the third equation and take the first partial derivative with respect to k and set $k = 0$ and simplify using Equation (8.19.2) in [12], where $|Im(m)| < Re(\alpha)$. □

8.2. Example 2

Theorem 14. *For all $a, \alpha \in \mathbb{C}$ and $Re(m) > 0$,*

$$\int_0^\infty \frac{\log\left(\log^2(a)+x^2\right)\sin(mx)}{e^{\alpha x}+1}dx$$
$$= \frac{\pi e^{-\frac{\pi m}{\alpha}}\Phi'\left(e^{-\frac{2\pi m}{\alpha}},0,\frac{\alpha\log(a)+\pi}{2\pi}\right)}{\alpha}-\frac{\pi e^{\frac{\pi m}{\alpha}}\Phi'\left(e^{\frac{2\pi m}{\alpha}},0,\frac{\alpha\log(a)+\pi}{2\pi}\right)}{\alpha}$$
$$+\frac{a^{-m}E_1(-m\log(a))}{2m}+\frac{a^m E_1(m\log(a))}{2m}+\frac{\log(\log(a))}{m}-\frac{\pi\log\left(\frac{2\pi}{\alpha}\right)csch\left(\frac{\pi m}{\alpha}\right)}{\alpha} \quad (51)$$

Proof. Use Equation (13) and form a second equation by replacing m with $-m$. Next, take the difference from the first equation, followed by taking the first partial derivative with respect to k and setting $k = 0$ then simplify using Equation (8.19.2) in [12], where $|Im(m)| < Re(\alpha)$. □

9. Derivation of the Fourier Cosine Transform of Binet's Integral

In this section, we derive the Fourier cosine transform of Binet's integral. Binet's integral is given by Equation (2.24) in [14].

Theorem 15. *For all $a, k, \alpha \in \mathbb{C}$ and $Re(m) < 0$,*

$$\int_0^\infty \frac{i\cos(mx)\left((\log(a)-ix)^k-(\log(a)+ix)^k\right)}{e^{\alpha x}-1}dx$$
$$= \frac{1}{2}a^m m^{-k-1}\Gamma(k+1,m\log(a))-\frac{a^{-m}(-m)^{-k}\Gamma(k+1,-m\log(a))}{2m}$$
$$-2^k\pi^{k+1}\left(\frac{1}{\alpha}\right)^{k+1}e^{-\frac{2\pi m}{\alpha}}\Phi\left(e^{-\frac{2m\pi}{\alpha}},-k,\frac{\alpha\log(a)}{2\pi}+1\right)$$
$$-2^k\pi^{k+1}\left(\frac{1}{\alpha}\right)^{k+1}e^{\frac{2\pi m}{\alpha}}\Phi\left(e^{\frac{2m\pi}{\alpha}},-k,\frac{\alpha\log(a)}{2\pi}+1\right)-\frac{\pi\log^k(a)}{\alpha} \quad (52)$$

Proof. Use Equation (12) and replace m with $-m$ to form a second equation. Next, we add this new equation to Equation (12) and simplify. □

Theorem 16. *For all $a, \alpha \in \mathbb{C}$ and $Re(m) > 0$,*

$$\begin{aligned}
&\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{a}\right)\cos(mx)}{e^{\alpha x}-1}dx \\
&= \frac{\pi e^{-\frac{2\pi m}{\alpha}}\Phi'\left(e^{-\frac{2\pi m}{\alpha}},0,\frac{a\alpha}{2\pi}+1\right)}{2\alpha} + \frac{\pi e^{\frac{2\pi m}{\alpha}}\Phi'\left(e^{\frac{2\pi m}{\alpha}},0,\frac{a\alpha}{2\pi}+1\right)}{2\alpha} \\
&\quad - \frac{\pi\log(a)}{2\alpha} + \frac{e^{-am}Chi(am)}{4m} - \frac{e^{am}Chi(am)}{4m} \\
&\quad + \frac{e^{-am}Shi(am)}{4m} + \frac{e^{am}Shi(am)}{4m} + \frac{i\pi e^{-am}}{4m} + \frac{\pi\log\left(\frac{2\pi}{\alpha}\right)}{2\alpha}
\end{aligned} \tag{53}$$

Proof. Use Equation (53), take the first partial derivative with respect to k, then set $k = 0$, replace a with e^a, and simplify in terms of the arctangent function. □

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Using Equation (53), setting $a = \pi, \alpha = 2, m = -i$, and simplifying, we obtain

$$\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{\pi}\right)\cosh(x)}{e^{2x}-1}dx = \frac{1}{4}\left(-2\mathrm{Si}(\pi)+\pi+\pi\log\left(\frac{\pi}{2}\right)\right) \tag{54}$$

10. Discussion

In this work, we used our method to derive new and known definite integrals. We derived updated forms of a few of these previously published results. We also derived new Mellin transforms for the product of the exponential function and trigonometric functions. The implications of our work are that these new formulae may be added to existing published tables such as [1,15]. We will continue to use our method to add to other published tables.

11. Conclusions

In this manuscript, we evaluated a few definite integrals in [11]. We used analytic continuation where possible, resulting in a wider range of computation. The findings in this work were numerically evaluated for both real and imaginary values using Mathematica by Wolfram.

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