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Oscillatory Solutions to Neutral Delay Differential Equations

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Abstract: This article aims to mark out new conditions for oscillation of the even-order Emden–Fowler neutral delay differential equations with neutral term $\left(\beta_1(\iota)\Phi_\alpha[\zeta^{(r-1)}(\iota)]\right)' + \beta_3(\iota)\Phi_\alpha[\varsigma(\xi(\iota))] = 0$. The obtained results extend, and simplify known conditions in the literature. The results are illustrated with examples.

Keywords: oscillation; even-order

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1. Introduction

Over the past few years, oscillation of Emden–Fowler-Type neutral delay differential equations with are attracting a lot of attention. As a matter of fact, natural of differential equation appear in the study of several real world problems such as biological systems, population dynamics, pharmacoki-netics, theoretical physics, biotechnology processes, chemistry, engineering, control, see [1–7].

In this manuscript, we investigate the oscillation of the following even-order Emden-Fowler neutral differential equations:

$$\left(\beta_1(\iota)\Phi_\alpha[\zeta^{(r-1)}(\iota)]\right)' + \beta_3(\iota)\Phi_\alpha[\varsigma(\xi(\iota))] = 0,\ \iota \geq \iota_0, \tag{1}$$

where $\zeta(\iota) := \varsigma(\iota) + \hat{\beta}(\iota)\varsigma(\varrho(\iota))$. Throughout this paper, we make the hypotheses as follows:

$$\begin{cases} \Phi_\alpha[s] = |s|^{\alpha-1}s, \beta_1 \in C[\iota_0,\infty), \beta_1(\iota) > 0, \beta_1'(\iota) \geq 0, \\ \varrho \in C^1[\iota_0,\infty), \xi \in C[\iota_0,\infty), \varrho'(\iota) > 0, \varrho(\iota) \leq \iota\ , \lim_{\iota\to\infty}\varrho(\iota) = \lim_{\iota\to\infty}\xi(\iota) = \infty, \\ \hat{\beta}, \beta_3 \in C[\iota_0,\infty), \beta_3(\iota) > 0, 0 \leq \hat{\beta}(\iota) < \hat{\beta}_0 < \infty, \\ r \geq 4 \text{ is an even natural number, } r \text{ is a quotient of odd positive integers.} \end{cases}$$

The following relations are satisfied

$$\int_{\iota_0}^{\infty} \beta_1^{-1/\alpha}(s)\mathrm{d}s = \infty. \tag{2}$$

Definition 1. *Let*

$$E = \{(\imath,s) \in \mathbb{R}^2 : \imath \geq s \geq \imath_0\} \text{ and } E_0 = \{(\imath,s) \in \mathbb{R}^2 : \imath > s \geq \imath_0\}.$$

Let $\varphi_i \in C(E,\mathbb{R})$ *for* $i = 1,2,$

(i) $\varphi_i(\imath,s) = 0$ *for* $\imath \geq \imath_0$, $\varphi_i(\imath,s) > 0$, $(\imath,s) \in E_0$;

(ii) *Let* $\partial\varphi_i/\partial s$ *on* E_0 *and there exist functions* $a_1, a_2 \in C^1([\imath_0,\infty),(0,\infty))$ *and* $\hat{\varphi}_i \in C(E_0,\mathbb{R})$ *such that*

$$\frac{\partial}{\partial s}\varphi_1(\imath,s) + \frac{a_1'(s)}{a_1(s)}\varphi(\imath,s) = \hat{\varphi}_1(\imath,s)\varphi_1^{\alpha/(\alpha+1)}(\imath,s) \tag{3}$$

and

$$\frac{\partial}{\partial s}\varphi_2(\imath,s) + \frac{a_2'(s)}{a_2(s)}\varphi_2(\imath,s) = \hat{\varphi}_2(\imath,s)\sqrt{\varphi_2(\imath,s)}. \tag{4}$$

In recent years, and in context of oscillation theory, many studies have been devoted to the oscillation conditions for non-linear delay differential equations; the reader can refer to [8–16].

Li et al. [17] discussed oscillation criteria for the equation

$$\begin{cases} \left(\alpha_1(\imath)|\zeta'''(\imath)|^{p-2}\zeta'''(\imath)\right)' + \beta(\imath)|\varsigma(\xi(\imath))|^{p-2}\varsigma(\xi(\imath)) = 0, \\ 1 < p < \infty,\ \imath \geq \imath_0 > 0, \end{cases}$$

where $\zeta(\imath) := \varsigma(\imath) + \hat{\beta}(\imath)\varsigma(\varrho(\imath))$.

Liu et al. [18] have obtained some oscillation conditions for equation

$$\begin{cases} \left(\alpha_1(\imath)\Phi\left(\zeta^{(r-1)}(\imath)\right)\right)' + \alpha_2(x)\Phi\left(\zeta^{(r-1)}(\imath)\right) + \beta(\imath)\Phi(\zeta(\xi(\imath))) = 0, \\ \Phi = |s|^{p-2}s,\ \imath \geq \imath_0 > 0,\ r \text{ is even}. \end{cases}$$

They used integral averaging technique.

Moaaz et al. [19] proved that equation

$$\left(\beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^{\alpha}\right)' + \beta_3(\imath)\varsigma^{\alpha}(\xi(\imath)) = 0, \tag{5}$$

is oscillatory if

$$\liminf_{\imath\to\infty}\int_{\varrho^{-1}(\delta(\imath))}^{\imath}\left(\frac{\left(\varrho^{-1}(\delta(s))\right)^{r-1}}{\beta_1^{1/\alpha}(\varrho^{-1}(\delta(s)))}\right)^{\alpha}\beta_3(s)F_r^{\alpha}(\xi(s))\mathrm{d}s > \frac{((r-1)!)^{\alpha}}{\mathrm{e}} \tag{6}$$

and

$$\liminf_{\imath\to\infty}\int_{\varrho^{-1}(\zeta(\imath))}^{\imath}\varrho^{-1}(\zeta(s))G_{r-3}(s)\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{7}$$

and used the Riccati method. The authors in [20] confirmed that (5) is oscillatory if

$$\left(\xi^{-1}(\imath)\right)' \geq \xi_0 > 0,\ \varrho'(\imath) \geq \varrho_0 > 0,\ \varrho^{-1}(\xi(\imath)) < \imath$$

and

$$\lim\ \inf_{\imath\to\infty}\int_{\varrho^{-1}(\xi(\imath))}^{\imath}\frac{\widehat{\beta_3}(s)}{\beta_1(s)}\left(s^{r-1}\right)^{\alpha}\mathrm{d}s > \left(\frac{1}{\xi_0} + \frac{\beta_0^{\alpha}}{\xi_0\varrho_0}\right) > \frac{((r-1)!)^{\alpha}}{\mathrm{e}}, \tag{8}$$

where $\widehat{\beta_3}(\imath) := \min\{\beta_3(\xi^{-1}(\imath)), \beta_3(\xi^{-1}(\varrho(\imath)))\}$. They used the comparison technique.

If we apply the results obtained by the authors in [19–22] to the equation

$$\left(\varsigma(\imath)+\frac{7}{8}\varsigma\left(\frac{1}{e}\imath\right)\right)^{(4)}+\frac{\jmath}{\imath^{4}}\varsigma\left(\frac{1}{e^{2}}\imath\right)=0,\ \imath\geq 1, \tag{9}$$

then we get that (9) is oscillatory if $\jmath > 113981.3$, $\jmath > 3561.9$, $\jmath > 3008.5$, $\jmath > 587.93$, respectively. Thus, [19] improved the results in [20–22].

This article purpose to establish new oscillation criteria for (1). The criteria obtained in this article complement the results in [19–22]. We provided an example to examine our main results.

These are some of the important Lemmas:

Lemma 1 ([3])**.** *If* $\varsigma^{(i)}(\imath) > 0$, $i = 0, 1, ..., r$, *and* $\varsigma^{(r+1)}(\imath) < 0$, *then*

$$\frac{\varsigma(\imath)}{\imath^{r}/r!}\geq\frac{\varsigma'(\imath)}{\imath^{r-1}/(r-1)!}.$$

Lemma 2 ([5])**.** *Let* $\varsigma \in C^{r}([\imath_0,\infty),(0,\infty))$, $\varsigma^{(r-1)}(\imath)\varsigma^{(r)}(\imath) \leq 0$ *and* $\lim_{\imath\to\infty}\varsigma(\imath) \neq 0$, *then for every* $\varepsilon \in (0,1)$ *there exists* $\imath_\varepsilon \geq \imath_1$ *such that*

$$\varsigma(\imath)\geq\frac{\varepsilon}{(r-1)!}\imath^{r-1}\left|\varsigma^{(r-1)}(\imath)\right| \text{ for } \imath\geq\imath_{\varepsilon}\geq\imath_{1},\ \varepsilon\in(0,1).$$

Lemma 3 ([4])**.** *Let* $\alpha \geq 1$, $L_2 > 0$. *Then*

$$L_{1}\varsigma-L_{2}\varsigma^{(\alpha+1)/\alpha}\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{L_{1}^{\alpha+1}}{L_{2}^{\alpha}}.$$

Lemma 4 ([8])**.** *Assume that*

$$\varsigma \text{ be an eventually positive solution of (1).} \tag{10}$$

Then, we have these cases:

$$\begin{array}{ll}(\mathbf{S}_1) & \zeta(\imath)>0,\ \zeta'(\imath)>0,\zeta''(\imath)>0,\ \ \zeta^{(r-1)}(\imath)>0,\ \ \zeta^{(r)}(\imath)<0,\\ (\mathbf{S}_2) & \zeta(\imath)>0,\zeta^{(j)}(\imath)>0,\zeta^{(j+1)}(\imath)<0 \text{ for all odd integer}\\ & j\in\{1,3,...,r-3\},\zeta^{(r-1)}(\imath)>0,\zeta^{(r)}(\imath)<0,\end{array}$$

for $\imath \geq \imath_1$, *where* $\imath_1 \geq \imath_0$ *is sufficiently large.*

Lemma 5. *Let (10) hold and*

$$\left(\varrho^{-1}\left(\varrho^{-1}(\imath)\right)\right)^{r-1}<\left(\varrho^{-1}(\imath)\right)^{r-1}\hat{\beta}\left(\varrho^{-1}\left(\varrho^{-1}(\imath)\right)\right). \tag{11}$$

Then

$$\varsigma(\imath)\geq\frac{\zeta(\varrho^{-1}(\imath))}{\hat{\beta}(\varrho^{-1}(\imath))}-\frac{1}{\hat{\beta}(\varrho^{-1}(\imath))}\frac{\zeta(\varrho^{-1}(\varrho^{-1}(\imath)))}{\hat{\beta}(\varrho^{-1}(\varrho^{-1}(\imath)))}. \tag{12}$$

Proof. Let (10) hold. From the definition of $\zeta(\imath)$, we have that

$$\hat{\beta}(\imath)\varsigma(\varrho(\imath))=\zeta(\imath)-\varsigma(\imath)$$

and so

$$\hat{\beta}\left(\varrho^{-1}(\imath)\right)\varsigma(\imath)=\zeta\left(\varrho^{-1}(\imath)\right)-\zeta\left(\varrho^{-1}(\imath)\right).$$

Repeating the same process, we obtain

$$\varsigma(\imath) = \frac{1}{\hat{\beta}(\varrho^{-1}(\imath))}\left(\zeta\left(\varrho^{-1}(\imath)\right) - \left(\frac{\zeta(\varrho^{-1}(\varrho^{-1}(\imath)))}{\hat{\beta}(\varrho^{-1}(\varrho^{-1}(\imath)))} - \frac{\varsigma(\varrho^{-1}(\varrho^{-1}(\imath)))}{\hat{\beta}(\varrho^{-1}(\varrho^{-1}(\imath)))}\right)\right),$$

which yields

$$\varsigma(\imath) \geq \frac{\zeta(\varrho^{-1}(\imath))}{\hat{\beta}(\varrho^{-1}(\imath))} - \frac{1}{\hat{\beta}(\varrho^{-1}(\imath))}\frac{\zeta(\varrho^{-1}(\varrho^{-1}(\imath)))}{\hat{\beta}(\varrho^{-1}(\varrho^{-1}(\imath)))}.$$

Thus, (12) holds. This completes the proof. □

Here, we define the next notations:

$$\begin{aligned}
F_\imath(\imath) &= \frac{1}{\hat{\beta}(\varrho^{-1}(\imath))}\left(1 - \frac{(\varrho^{-1}(\varrho^{-1}(\imath)))^{\imath-1}}{(\varrho^{-1}(\imath))^{\imath-1}\hat{\beta}(\varrho^{-1}(\varrho^{-1}(\imath)))}\right), \text{ for } \imath = 2, r,\\
G_0(\imath) &= \left(\frac{1}{\beta_1(\imath)}\int_\imath^\infty \beta_3(s)F_2^\alpha(\xi(s))\mathrm{d}s\right)^{1/\alpha},\\
\Theta(\imath) &= \alpha\frac{\varepsilon_1}{(r-2)!}\left(\frac{\beta_1(\imath)}{\beta_1(\varrho^{-1}(\xi(\imath)))}\right)^{1/\alpha}\frac{(\varrho^{-1}(\xi(\imath)))'(\varrho^{-1}(\xi(\imath)))^{r-2}}{(\beta_1 a_1)^{1/\alpha}(\imath)},\\
\tilde{\Theta}(\imath) &= \frac{\hat{\varphi}_1^{\alpha+1}(\imath,s)\varphi_1^\alpha(\imath,s)}{(\alpha+1)^{\alpha+1}}\frac{((r-2)!)^\alpha\beta_1(\varrho^{-1}(\xi(\imath)))a_1(\imath)}{\left(\varepsilon_1(\varrho^{-1}(\xi(\imath)))'(\varrho^{-1}(\xi(\imath)))^{r-2}\right)^\alpha}
\end{aligned}$$

and

$$G_m(\imath) = \int_\imath^\infty G_{m-1}(s)\mathrm{d}s, \quad m = 1, 2, ..., r-3.$$

Lemma 6. *Let (10) hold and*

$$\left(\beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^\alpha\right)' \leq -\zeta^\alpha\left(\varrho^{-1}(\xi(\imath))\right)\beta_3(\imath)F_r^\alpha(\xi(\imath)), \text{ if } \zeta \text{ satisfies } (\mathbf{S}_1) \tag{13}$$

and

$$\zeta''(\imath) + G_{r-3}(\imath)\zeta\left(\varrho^{-1}(\xi(\imath))\right) \leq 0, \text{ if } \zeta \text{ satisfies } (\mathbf{S}_2). \tag{14}$$

Proof. Let (10) hold. From Lemma 4, we have$(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

Let case $(\mathbf{S}_1)$ holds. Using Lemma 6, we get $\zeta(\imath) \geq \frac{1}{(r-1)}\imath\zeta'(\imath)$ and hence the function $\imath^{1-r}\zeta(\imath)$ is nonincreasing, which with the fact that $\varrho(\imath) \leq \imath$ gives

$$\left(\varrho^{-1}(\imath)\right)^{r-1}\zeta\left(\varrho^{-1}\left(\varrho^{-1}(\imath)\right)\right) \leq \left(\varrho^{-1}\left(\varrho^{-1}(\imath)\right)\right)^{r-1}\zeta\left(\varrho^{-1}(\imath)\right). \tag{15}$$

Combining (12) and (15), we conclude that

$$\begin{aligned}
\varsigma(\imath) &\geq \frac{1}{\hat{\beta}(\varrho^{-1}(\imath))}\left(1 - \frac{(\varrho^{-1}(\varrho^{-1}(\imath)))^{r-1}}{(\varrho^{-1}(\imath))^{r-1}\hat{\beta}(\varrho^{-1}(\varrho^{-1}(\imath)))}\right)\zeta\left(\varrho^{-1}(\imath)\right)\\
&= F_r(\imath)\zeta\left(\varrho^{-1}(\imath)\right).
\end{aligned} \tag{16}$$

From (1) and (16), we obtain

$$\begin{aligned}
\left(\beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^\alpha\right)' &\leq -\beta_3(\imath)F_r^\alpha(\xi(\imath))\zeta^\alpha\left(\varrho^{-1}(\xi(\imath))\right)\\
&\leq -\zeta^\alpha\left(\varrho^{-1}(\xi(\imath))\right)\beta_3(\imath)F_r^\alpha(\xi(\imath)).
\end{aligned}$$

Thus, (13) holds.

Let case $(\mathbf{S}_2)$ holds. Using Lemma 6, we get that

$$\zeta(\imath) \geq \imath\zeta'(\imath) \tag{17}$$

and thus the function $\imath^{-1}\zeta(\imath)$ is nonincreasing, eventually. Since $\varrho^{-1}(\imath) \leq \varrho^{-1}(\varrho^{-1}(\imath))$, we obtain

$$\varrho^{-1}(\imath)\zeta\left(\varrho^{-1}\left(\varrho^{-1}(\imath)\right)\right) \leq \varrho^{-1}\left(\varrho^{-1}(\imath)\right)\zeta\left(\varrho^{-1}(\imath)\right). \tag{18}$$

Combining (12) and (18), we find

$$\begin{aligned} \varsigma(\imath) &\geq \frac{1}{\hat{\beta}(\varrho^{-1}(\imath))}\left(1 - \frac{(\varrho^{-1}(\varrho^{-1}(\imath)))}{(\varrho^{-1}(\imath))\hat{\beta}(\varrho^{-1}(\varrho^{-1}(\imath)))}\right)\zeta\left(\varrho^{-1}(\imath)\right) \\ &= F_2(\imath)\zeta\left(\varrho^{-1}(\imath)\right), \end{aligned}$$

which with (1) yields

$$\left(\beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^{\alpha}\right)' + \beta_3(\imath)F_2^{\alpha}(\xi(\imath))\zeta^{\alpha}\left(\varrho^{-1}(\xi(\imath))\right) \leq 0. \tag{19}$$

Integrating the (19) from $\imath$ to ∞, we obtain

$$\zeta^{(r-1)}(\imath) \geq G_0(\imath)\zeta\left(\varrho^{-1}(\xi(\imath))\right).$$

NOW, integrating from $\imath$ to ∞ a total of $r-3$ times, we obtain

$$\zeta''(\imath) + G_{r-3}(\imath)\zeta\left(\varrho^{-1}(\xi(\imath))\right) \leq 0.$$

Thus, (14) holds. This completes the proof. □

2. Philos-Type Oscillation Criteria

Theorem 1. *Let $\xi(\imath) \leq \xi(\imath)$ and (11) holds. If the functions $a_1, a_2 \in^1 ([\imath_0,\infty),\mathbb{R})$ such that*

$$\limsup_{\imath\to\infty} \frac{1}{\varphi(\imath,\imath_1)}\int_{\imath_1}^{\imath}(\varphi(\imath,s)D(s) - \tilde{\Theta}(s))\mathrm{d}s = \infty \tag{20}$$

and

$$\limsup_{\imath\to\infty} \frac{1}{\varphi_2(\imath,\imath_1)}\int_{\imath_1}^{\imath}\left(\varphi_2(\imath,s)D^*(s) - \frac{a_2(s)\hat{\varphi}_2^2(\imath,s)}{4}\right)\mathrm{d}s = \infty, \tag{21}$$

where

$$D(s) = a_1(\imath)\beta_3(\imath)F_r^{\alpha}(\xi(\imath)),\ D^*(s) = a_2(\imath)G_{r-3}(\imath)\left(\frac{\varrho^{-1}(\xi(\imath))}{\imath}\right)$$

and

$$\tilde{\Theta}(s) = \frac{\hat{\varphi}_1^{\alpha+1}(\imath,s)\varphi_1^{\alpha}(\imath,s)}{(\alpha+1)^{\alpha+1}}\frac{((r-2)!)^{\alpha}\beta_1(\varrho^{-1}(\xi(\imath)))a_1(\imath)}{\left(\varepsilon_1(\varrho^{-1}(\xi(\imath)))'(\varrho^{-1}(\xi(\imath)))^{r-2}\right)^{\alpha}},$$

then (1) is oscillatory.

Proof. Let ς be a non-oscillatory solution of (1), then $\varsigma > 0$. Let $(\mathbf{S}_1)$ holds. Define

$$X(\imath) := a_1(\imath)\frac{\beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^{\alpha}}{\zeta^{\alpha}(\varrho^{-1}(\xi(\imath)))} > 0.$$

Differentiating and using (13), we obtain

$$\begin{aligned} X'(\imath) \leq & \frac{a_1'(\imath)}{a_1(\imath)}X(\imath) - a_1(\imath)\beta_3(\imath)F_r^\alpha(\xi(\imath)) \\ & -\alpha a_1(\imath)\frac{\beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^\alpha (\varrho^{-1}(\xi(\imath)))'\zeta_u'(\varrho^{-1}(\xi(\imath)))}{\zeta_u^{\alpha+1}(\varrho^{-1}(\xi(\imath)))}. \end{aligned} \tag{22}$$

Recalling that $\beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^\alpha$ is decreasing, we get

$$\beta_1\left(\varrho^{-1}(\xi(\imath))\right)\left(\zeta^{(r-1)}\left(\varrho^{-1}(\xi(\imath))\right)\right)^\alpha \geq \beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^\alpha.$$

This yields

$$\left(\zeta^{(r-1)}\left(\varrho^{-1}(\xi(\imath))\right)\right)^\alpha \geq \frac{\beta_1(\imath)}{\beta_1(\varrho^{-1}(\xi(\imath)))}\left(\zeta^{(r-1)}(\imath)\right)^\alpha. \tag{23}$$

It follows from Lemma 2 that

$$\zeta'\left(\varrho^{-1}(\xi(\imath))\right) \geq \frac{\varepsilon_1}{(r-2)!}\left(\varrho^{-1}(\xi(\imath))\right)^{r-2}\zeta^{(r-1)}\left(\varrho^{-1}(\xi(\imath))\right), \tag{24}$$

for all $\varepsilon_1 \in (0,1)$ and every sufficiently large $\imath$. Thus, by (22)–(24), we get

$$\begin{aligned} X'(\imath) \leq & \tfrac{a_1'(\imath)}{a_1(\imath)}X(\imath) - a_1(\imath)\beta_3(\imath)F_r^\alpha(\xi(\imath)) \\ & -\alpha a_1(\imath)\tfrac{\varepsilon_1}{(r-2)!}\left(\tfrac{\beta 1(\imath)}{\beta 1\left(\varrho^{-1}(\xi(\imath))\right)}\right)^{1/\alpha}\tfrac{\beta_1(\imath)\left(\zeta^{(r-1)}(\imath)\right)^{\alpha+1}\left(\varrho^{-1}(\xi(\imath))\right)'\left(\varrho^{-1}(\xi(\imath))\right)^{r-2}}{\zeta^{\alpha+1}\left(\varrho^{-1}(\xi(\imath))\right)}. \end{aligned}$$

Hence,

$$\begin{aligned} X'(\imath) \leq & \frac{a_1'(\imath)}{a_1(\imath)}X(\imath) - a_1(\imath)\beta_3(\imath)F_r^\alpha(\xi(\imath)) \\ & -\Theta(\imath)X^{\frac{\alpha+1}{\alpha}}(\imath). \end{aligned} \tag{25}$$

Multiplying (25) by $\varphi(\imath,s)$ and integrating from $\imath_1$ to $\imath$; we obtain

$$\begin{aligned} \int_{\imath_1}^{\imath}\varphi(\imath,s)D(s)\mathrm{d}s \leq & X(\imath_1)\varphi(\imath,\imath_1) + \int_{\imath_1}^{\imath}\left(\frac{\partial}{\partial s}\varphi(\imath,s) + \frac{a_1'(s)}{a_1(s)}\varphi(\imath,s)\right)X(s)\mathrm{d}s \\ & -\int_{\imath_1}^{\imath}\Theta(s)\varphi(\imath,s)X^{\frac{\alpha+1}{\alpha}}(s)\mathrm{d}s. \end{aligned}$$

From (3), we get

$$\begin{aligned} \int_{\imath_1}^{\imath}\varphi(\imath,s)D(s)\mathrm{d}s \leq & X(\imath_1)\varphi(\imath,\imath_1) + \int_{\imath_1}^{\imath}\hat{\varphi}_1(\imath,s)\varphi_1^{\alpha/(\alpha+1)}(\imath,s)X(s)\mathrm{d}s \\ & -\int_{\imath_1}^{\imath}\Theta(s)\varphi(\imath,s)X^{\frac{\alpha+1}{\alpha}}(s)\mathrm{d}s. \end{aligned} \tag{26}$$

Using Lemma 3 with $L_2 = \Theta(s)\varphi(\imath,s)$, $L_1 = \hat{\varphi}_1(\imath,s)\varphi_1^{\alpha/(\alpha+1)}(\imath,s)$ and $\varsigma = X(s)$, we get

$$\begin{aligned} & \hat{\varphi}_1(\imath,s)\varphi_1^{\alpha/(\alpha+1)}(\imath,s)X(s) - \Theta(s)\varphi(\imath,s)X^{\frac{\alpha+1}{\alpha}}(s) \\ \leq & \frac{\hat{\varphi}_1^{\alpha+1}(\imath,s)\varphi_1^{\alpha}(\imath,s)}{(\alpha+1)^{\alpha+1}}\frac{((r-2)!)^\alpha\beta_1(\varrho^{-1}(\xi(\imath)))a_1(\imath)}{\left(\varepsilon_1(\varrho^{-1}(\xi(\imath)))'(\varrho^{-1}(\xi(\imath)))^{r-2}\right)^\alpha}, \end{aligned}$$

which, with (26) gives

$$\frac{1}{\varphi(\imath,\imath_1)}\int_{\imath_1}^{\imath}\left(\varphi(\imath,s)D(s)-\tilde{\Theta}(s)\right)\mathrm{d}s\leq X(\imath_1),$$

which contradicts (20).

Let $(\mathbf{S}_2)$ holds. Define

$$Z(\imath)=a_2(\imath)\frac{\zeta'(\imath)}{\zeta(\imath)}. \tag{27}$$

Then $Z(\imath)>0$ for $\imath\geq\imath_1$. By differentiating Z and using (14), we find

$$\begin{aligned} Z'(\imath) &= \frac{a_2'(\imath)}{a_2(\imath)}Z(\imath)+a_2(\imath)\frac{\zeta''(\imath)}{\zeta(\imath)}-a_2(\imath)\left(\frac{\zeta'(\imath)}{\zeta(\imath)}\right)^2 \\ &\leq \frac{a_2'(\imath)}{a_2(\imath)}Z(\imath)-a_2(\imath)G_{r-3}(\imath)\frac{\zeta\left(\varrho^{-1}(\xi(\imath))\right)}{\zeta(\imath)}-\frac{1}{a_2(\imath)}Z^2(\imath). \end{aligned} \tag{28}$$

By using Lemma 1, we find that

$$\zeta(\imath)\geq\imath\zeta'(\imath). \tag{29}$$

From (29), we get that

$$\zeta\left(\varrho^{-1}(\xi(\imath))\right)\geq\frac{\varrho^{-1}(\xi(\imath))}{\imath}\zeta(\imath). \tag{30}$$

Thus, from (28) and (30), we obtain

$$Z'(\imath)\leq\frac{a_2'(\imath)}{a_2(\imath)}Z(\imath)-a_2(\imath)G_{r-3}(\imath)\left(\frac{\varrho^{-1}(\xi(\imath))}{\imath}\right)-\frac{1}{a_2(\imath)}Z^2(\imath). \tag{31}$$

Multiplying (31) by $\varphi_2(\imath,s)$ and integrating the resulting from $\imath_1$ to $\imath$, we see

$$\begin{aligned} \int_{\imath_1}^{\imath}\varphi_2(\imath,s)D^*(s)\mathrm{d}s &\leq Z(\imath_1)\varphi_2(\imath,\imath_1) \\ &\quad+\int_{\imath_1}^{\imath}\left(\frac{\partial}{\partial s}\varphi_2(\imath,s)+\frac{a_2'(s)}{a_2(s)}\varphi_2(\imath,s)\right)Z(s)\mathrm{d}s \\ &\quad-\int_{\imath_1}^{\imath}\frac{1}{a_2(s)}\varphi_2(\imath,s)Z^2(s)\mathrm{d}s. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\imath_1}^{\imath}\varphi_2(\imath,s)D^*(s)\mathrm{d}s &\leq Z(\imath_1)\varphi_2(\imath,\imath_1)+\int_{\imath_1}^{\imath}\hat{\varphi}_2(\imath,s)\sqrt{\varphi_2(\imath,s)}Z(s)\mathrm{d}s \\ &\quad-\int_{\imath_1}^{\imath}\frac{1}{a_2(s)}\varphi_2(\imath,s)Z^2(s)\mathrm{d}s \\ &\leq Z(\imath_1)\varphi_2(\imath,\imath_1)+\int_{\imath_1}^{\imath}\frac{a_2(s)\hat{\varphi}_2^2(\imath,s)}{4}\mathrm{d}s \end{aligned}$$

and so

$$\frac{1}{\varphi_2(\imath,\imath_1)}\int_{\imath_1}^{\imath}\left(\varphi_2(\imath,s)D^*(s)-\frac{a_2(s)\hat{\varphi}_2^2(\imath,s)}{4}\right)\mathrm{d}s\leq Z(\imath_1),$$

which contradicts (21). This completes the proof. □

Corollary 1. *Let (11) holds and* $a_1, a_2 \in^1 ([\imath_0, \infty), \mathbb{R})$ *such that*

$$\int_{\imath_0}^{\infty} \left(\varpi(s) - \frac{(r-2)!^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\beta_1(\varrho^{-1}(\xi(\imath)))\,(a_1'(\imath))^{\alpha+1}}{\left(\varepsilon_1 a_1(\imath)(\varrho^{-1}(\xi(\imath)))'(\varrho^{-1}(\xi(\imath)))^{r-2}\right)^{\alpha}} \right) \mathrm{d}s = \infty \tag{32}$$

and

$$\int_{\imath_0}^{\infty} \left(\theta(s) - \frac{(a_2'(s))^2}{4a_2(s)} \right) \mathrm{d}s = \infty, \tag{33}$$

for some $\varepsilon_1 \in (0,1)$*, where*

$$\varpi(\imath) := a_1(\imath)\beta_3(\imath)F_r^{\alpha}(\xi(\imath))$$

and

$$\theta(\imath) := F_1 a_2(\imath) \int_{\imath}^{\infty} \left(\frac{1}{\beta_1(\varrho)} \int_{\varrho}^{\infty} \beta_3(s) \left(\frac{\varrho^{-1}(\xi(s))}{s} \right)^{\alpha} \mathrm{d}s \right)^{1/\alpha} \mathrm{d}\varrho,$$

then (1) is oscillatory.

Proof. The proof of this theorem is the same as that of Theorem 1. □

Example 1. *Consider the equation*

$$\left(\varsigma(\imath) + \hat{\beta}_0 \varsigma(\delta\imath)\right)^{(r)} + \frac{\jmath}{\imath^r} \varsigma(\lambda\imath) = 0, \tag{34}$$

where $\imath \geq 1$*,* $\jmath > 0$*,* $\delta \in \left(\hat{\beta}_0^{-1/(r-1)}, 1\right)$*,* $\lambda \in (0,\delta)$*,* $\beta_1(\imath) = 1$*,* $\hat{\beta}(\imath) = \hat{\beta}_0$*,* $\varrho(\imath) = \delta\imath$*,* $\xi(\imath) = \lambda\imath$ *and* $\beta_3(\imath) = \jmath/\imath^r$*. Thus, we find*

$$F_1(\imath) = \frac{1}{\hat{\beta}_0}\left(1 - \frac{1}{\delta^3\hat{\beta}_0}\right),\ F_2(\imath) = \frac{1}{\hat{\beta}_0}\left(1 - \frac{1}{\delta\hat{\beta}_0}\right),\ \Psi(\imath) = \frac{F_1\jmath}{\imath}$$

and

$$B(\imath) = \frac{F_2\lambda\jmath}{6\delta\imath}.$$

Thus, (32) and (33) becomes

$$\int_{\imath_0}^{\infty} \left(\frac{F_1(\imath)\jmath}{s} - \frac{9\delta^4}{2\lambda^4}\frac{1}{s} \right) \mathrm{d}s = \left(F_1(\imath)\jmath - \frac{9\delta^4}{2\lambda^4} \right)(+\infty)$$

and

$$\int_{\imath_0}^{\infty} \left(B(s) - \frac{(a_2'(s))^2}{4a_2(s)} \right) \mathrm{d}s = \left(\frac{F_2\lambda}{6\delta}\jmath - \frac{1}{4} \right)(+\infty),$$

From Corollary 1, the equation (34) is oscillatory if

$$\jmath \frac{1}{\hat{\beta}_0}\left(1 - \frac{1}{\delta^3\hat{\beta}_0}\right) > \frac{9\delta^4}{2\lambda^4} \tag{35}$$

and

$$\jmath \frac{1}{\hat{\beta}_0}\left(1 - \frac{1}{\delta\hat{\beta}_0}\right) > \frac{3\delta}{2\lambda}. \tag{36}$$

Let $\hat{\beta}_0 = 16$*,* $\delta = 1/2$ *and* $\lambda = 1/3$*, Condition (35) yields* $\jmath > 41.14$*. Whereas, the criterion obtained from the results of [20] is* $\jmath > 4850.4$ *and [19] is* $\jmath > 587.93$*.*

Remark 1. *Hence, our results extend and simplify the results in [19–22].*

Example 2. *Consider the equation*

$$\left(\varsigma(\imath) + \frac{1}{3}\varsigma\left(\frac{\imath}{2}\right)\right)^{(4)} + \frac{\jmath}{\imath^4}\xi\left(\frac{\imath}{2}\right) = 0, \tag{37}$$

where $\imath \geq 1$ *and* $q_0 > 0$. *Let*

$$r = 4,\ \beta_1(\imath) = 1,\ \hat{\beta}(\imath) = 1/3,\ \varrho(\imath) = \xi(\imath) = \imath/2 \text{ and } \beta_3(\imath) = \jmath/\imath^4.$$

Then

$$\int_{\imath_0}^{\infty} \beta_1^{-1/\alpha}(s)\mathrm{d}s = \infty.$$

So, we see that the conditions (20) and (21) holds. By Theorem 1, all solution of (37) is oscillatory.

3. Conclusions

In this article, we give several oscillatory properties of differential equation of even-order with neutral term. The criteria obtained in this article complements the results in [19–22]. In our future work, and to supplement our results, we will present and discuss some oscillation theorems for differential equations of this type by using comparing technique with first/second-order delay differential equation.

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References

1. Bazighifan, O.; Abdeljawad, T.; Al-Mdallal, Q.M. Differential equations of even-order with p-Laplacian like operators: qualitative properties of the solutions. *Adv. Differ. Eq.* **2021**, *2021*, 96. [CrossRef]
2. Moaaz, O.; Elabbasy, E.M.; Muhib, A. Oscillation criteria for even-order neutral differential equations with distributed deviating arguments. *Adv. Differ. Eq.* **2019**, 297. [CrossRef]
3. Agarwal, R.P.; Bazighifan, O.; Ragusa, M.A. Nonlinear Neutral Delay Differential Equations of Fourth-Order: Oscillation of Solutions. *Entropy* **2021**, *23*, 129. [CrossRef] [PubMed]
4. Bazighifan, O.; Almutairi, A. Emden–Fowler-type neutral differential equations: oscillatory properties of solutions. *Adv. Differ. Eq.* **2021**, 131. [CrossRef]
5. Zhang, C.; Li, T.; Sun, B.; Thandapani, E. On the oscillation of higher-order half-linear delay differential equations. *Appl. Math. Lett.* **2011**, *24*, 1618–1621. [CrossRef]
6. Althobati, S.; Bazighifan, O.; Yavuz, M. Some Important Criteria for Oscillation of Non-Linear Differential Equations with Middle Term. *Mathematics* **2021**, *9*, 346. [CrossRef]
7. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
8. Nofal, T.A.; Bazighifan, O.; Khedher, K.M.; Postolache, M. More Effective Conditions for Oscillatory Properties of Differential Equations. *Symmetry* **2021**, *13*, 278. [CrossRef]
9. Santra, S.S.; Majumder, D.; Bhattacharjee, R.; Bazighifan, O.; Khedher, K.M.; Marin, M. New Theorems for Oscillations to Differential Equations with Mixed Delays. *Symmetry* **2021**, *13*, 367. [CrossRef]

10. Bazighifan, O. Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations. *Adv. Differ. Eq.* **2020**, *201*, 1–12. [CrossRef]
11. Bazighifan, O.; Ramos, H. On the asymptotic and oscillatory behavior of the solutions of a class of higher-order differential equations with middle term. *Appl. Math. Lett.* **2020**, *107*, 106431. [CrossRef]
12. Althobati, S.; Alzabut, J.; Bazighifan, O. Non-Linear Neutral Differential Equations with Damping: Oscillation of Solutions. *Symmetry* **2021**, *13*, 285. [CrossRef]
13. Dzurina, J.; Jadlovska, I. Oscillation theorems for fourth order delay differential equations with a negative middle term, Math. *Meth. Appl. Sci.* **2017**, *4563*, 1–13.
14. Grace, S.; Agarwal, R.; Graef, J. Oscillation theorems for fourth order functional differential equations. *J. Appl. Math. Comput.* **2009**, *30*, 75–88. [CrossRef]
15. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications*; Clarendon Press: Oxford, UK, 1991.
16. Bazighifan, O.; Alotaibi, H.; Mousa, A.A.A. Neutral Delay Differential Equations: Oscillation Conditions for the Solutions. *Symmetry* **2021**, *13*, 101. [CrossRef]
17. Li, T.; Baculikova, B.; Dzurina, J.; Zhang, C. Oscillation of fourth order neutral differential equations with p-Laplacian like operators. *Bound. Value Probl.* **2014**, *56*, 41–58. [CrossRef]
18. Liu, S.; Zhang, Q.; Yu, Y. Oscillation of even-order half-linear functional differential equations with damping. *Comput. Math. Appl.* **2011**, *61*, 2191–2196. [CrossRef]
19. Moaaz, O.; Awrejcewicz, J.; Bazighifan, O. A New Approach in the Study of Oscillation Criteria of Even-Order Neutral Differential Equations. *Mathematics* **2020**, *12*, 1–10. [CrossRef]
20. Xing, G.; Li, T.; Zhang, C. Oscillation of higher-order quasi linear neutral differential equations. *Adv. Differ. Eq.* **2011**, *2011*, 1–10. [CrossRef]
21. Zafer, A. Oscillation criteria for even order neutral differential equations. *Appl. Math. Lett.* **1998**, *11*, 21–25. [CrossRef]
22. Zhang, Q.; Yan, J. Oscillation behavior of even order neutral differential equations with variable coefficients. *Appl. Math. Lett.* **2006**, *19*, 1202–1206. [CrossRef]