mathematics

Article

Boundary Stabilization of Heat Equation with Multi-Point Heat Source

Qing-Qing Hu, Feng-Fei Jin * and Bao-Qiang Yan

School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China; 2019020467@stu.sdnu.edu.cn (Q.-Q.H.); 106117@sdnu.edu.cn (B.-Q.Y.)
* Correspondence: jinfengfei@amss.ac.cn

Abstract: In this paper, we consider boundary stabilization problem of heat equation with multi-point heat source. Firstly, a state feedback controller is designed mainly by backstepping approach. Under the designed state controller, the exponential stability of closed-loop system is guaranteed. Then, an observer-based output feedback controller is proposed. We prove the exponential stability of resulting closed-loop system using operator semigroup theory. Finally, the designed state and output feedback controllers are effective via some numerical simulations.

Keywords: boundary stabilization; heat equation; multi-point heat source; backstepping transformation; parabolic equations

check for updates

Citation: Hu, Q.-Q.; Jin, F.-F.; Yan, B.-Q. Boundary Stabilization of Heat Equation with Multi-Point Heat Source. *Mathematics* **2021**, *9*, 834. https://doi.org/10.3390/math9080834

Academic Editor: Marin Marin

Received: 25 March 2021
Accepted: 9 April 2021
Published: 12 April 2021

1. Introduction

The parabolic partial differential equations (PDEs) have played a crucial role in the real-world practice over the past several decades. At early stages, Masi et al. [1] investigated the microscopic fluctuations of interacting particle systems on a lattice by the nonlinear parabolic PDE. In [2], Price et al. showed the analysis and enhancement on special types of images based on parabolic PDEs. Afterwards, under measuring the temperature changes of ovens' processing area on different points, the heating capability of forced convection reflow ovens and the effect of the ovens' construction on it, were considered in [3]. By the experimental data, the parabolic PDE was used to describe the specific characteristics. Besides, Pardo et al. [4] applied the three-dimensional parabolic PDE to the non-destructive evaluation of minefields. Recently, Gao et al. [5] proposed a method of parabolic equation modeling to address the problem of predicting electromagnetic wave propagation caused by atmospheric dust and rough sea surfaces in the maritime environment. Regarding the study of the parabolic PDEs, there are some other results and references [6–10] therein.

In the control engineering filed, many scholars have been keen on boundary stabilization problems of parabolic PDEs during the past decades. For boundary control problem of a class of unstable scalar parabolic PDEs, ref. [11] employed a gradient-based optimization method to parameterize the feedback kernel as a second-order polynomial, and made the optimized kernel generate closed-loop stability with restricting on the kernel coefficients. Liu [12] proposed the successive approximation to address the boundary stabilization problem of unstable parabolic PDE. Wu et al. [13] made use of fuzzy control approach to solve the stabilization problem of nonlinear parabolic PDEs. Vazquez and Krstic [14] adopted finite-dimension feedback linearization method to transform parabolic PDEs with Volterra nonlinearities into a stable system by the Volterra series nonlinear operators. Actually, the proposed finite-dimension feedback linearization method was infinite dimensional extension of the backstepping approach. It is well known that backstepping transformation is mainstream method to deal with boundary stabilization problems of parabolic PDEs, such as, linear parabolic PDEs [15–19], nonlinear parabolic PDEs [20–22], quasi-linear parabolic PDEs [23], coupled parabolic PDEs and ODE [24–27].

To the best of our knowledge, heat equation is an extremely classic class of parabolic PDEs. The boundary control results of heat equation with unstable term or source term have been proposed and references [19,28–34] therein. Krstic first proposed a separation principle result based on the passive and swapping identifiers and combined backstepping method to solve adaptive boundary control problem for heat equation with term $\lambda u(x,t)$ in [33], where λ is unknown parameter. Afterward, based on expressly parametrized control formula, adaptive controllers for parabolic PDEs with the term $\lambda u(x,t)$ were designed in [31], where λ is an unknown constant parameter. Additionally, Baccoli et al. [28] employed backstepping method to represent boundary stabilization result of the coupled reaction-diffusion processes with term $\Lambda Q(x,t)$, where $\Lambda \in \mathbb{R}^{n\times n}$ is a real-valued square matrix. In [19], heat equation system is considered by

$$\begin{cases} \Gamma_t(\xi,t) = \Gamma_{\xi\xi}(\xi,t) + \tau_0\Gamma(\xi_0,t),\ 0<\xi<1, t\in(0,+\infty), \\ \Gamma_\xi(0,t) = 0,\ t\geq 0, \\ \Gamma(1,t) = \hat{U}(t),\ t\geq 0, \\ \Gamma(\xi,0) = \Gamma_0(\xi), 0\leq\xi\leq 1, \end{cases} \tag{1}$$

where $\hat{U}(t)$ is boundary control, Γ_0 is initial value, τ_0 is a constant, $\xi_0\in(0,1)$. Zhou and Guo designed a state feedback controller based on backstepping approach for problem (1). Motivated by [19], we consider extension of heat system (1)

$$\begin{cases} \Gamma_t(\xi,t) = \Gamma_{\xi\xi}(\xi,t) + \sum\limits_{i=1}^{Z}\tau_i\Gamma(\xi_i,t),\ 0<\xi<1, t\in(0,+\infty), \\ \Gamma_\xi(0,t) = 0,\ t\geq 0, \\ \Gamma(1,t) = \hat{U}(t),\ t\geq 0, \\ \mathscr{Y} = \{\Gamma(\xi_1,t), \Gamma(\xi_2,t), ..., \Gamma(\xi_Z,t)\}, \\ \Gamma(\xi,0) = \Gamma_0(\xi), 0\leq\xi\leq 1, \end{cases} \tag{2}$$

where $0<\xi_i<1, i=1,2,...,Z$, $Z\in[2,+\infty)$ is a positive integer, τ_i $(i=1,2,...,Z)$ is a constant parameter, $\hat{U}(t)$ is control, Γ_0 is initial value, $\sum_{i=1}^{Z}\tau_i\Gamma(\xi_i,t)$ is non-local term. As a matter of convenience, let $\tau=\sum_{i=1}^{Z}\tau_i$. In our paper, owing to the existence of our multi-term τ, normal backstepping transformation is no longer applicable. A new backstepping transformation is constructed in our paper. Unlike the unstable terms mentioned above, term $\sum_{i=1}^{Z}\tau_i\Gamma(\xi_i,t)$ in system (1) is intermediate points related. In addition, $\sum_{i=1}^{Z}\tau_i\Gamma(\xi_i,t)$ can be seen as $\int_0^1\tau_i\delta(s-\xi_i)\Gamma(s,t)ds$ (a nice explanation in [19]). When $i=1$, system (2) becomes system (1). Actually, [35] adopted backstepping method to solve the output feedback problem of transport equation with non-local term $\mu u(x_0)$. Besides that there are other results about hyperbolic PDEs with integral term by backstepping approach in [36,37].

The rest of paper is organized as follows. In Section 2, we design a state feedback controller by backstepping approach to stabilize system (2). In Section 3, an output feedback controller is constructed. The resulting closed-loop system is proved to be exponentially stable. Section 4 illustrates the effectiveness of the proposed controller on the basis of some simulations. Finally, the concluding remarks are introduced in Section 5.

2. The State Feedback Controller Design by Backstepping

In this section, we introduce the backstepping transformation as below

$$\mathscr{P}(\xi,t) = \Gamma(\xi,t) - \sum_{i=1}^{Z} a_i(\xi)\int_0^{\xi_i} b_i(s)\Gamma(s,t)ds - \int_0^{\xi} c(\xi,s)\Gamma(s,t)ds, \tag{3}$$

where $a_i, b_i(i=1,2,...,Z)$ and c are kernels and to be determined later.

Under transformation (3), we convert system (2) into the following target system

$$\begin{cases} \mathscr{P}_t(\xi,t) = \mathscr{P}_{\xi\xi}(\xi,t) + \sigma\mathscr{P}(\xi,t), 0 < \xi < 1, t \in (0,+\infty), \\ \mathscr{P}_\xi(0,t) = 0,\ t \geq 0, \\ \mathscr{P}(1,t) = 0,\ t \geq 0, \end{cases} \tag{4}$$

where $\sigma \in (-\infty, 0]$ is any given number.

Now, we prove the kernels are unique. Firstly, define the triangular domain Λ

$$\Lambda := \left\{ (\xi,s) \in \mathbb{R}^2 | (\xi,s) \in [0,1] \times [0,\xi] \right\}.$$

For transformation (3), we take derivative of it with regard to ξ, there is

$$\mathscr{P}_\xi(\xi,t) = \Gamma_\xi(\xi,t) - \sum_{i=1}^{Z} a_i'(\xi) \int_0^{\xi_i} b_i(s)\Gamma(s,t)ds - c(\xi,\xi)\Gamma(s,t) - \int_0^{\xi} c_\xi(\xi,t)\Gamma(s,t)ds. \tag{5}$$

Take derivative of Equation (5) with regard to ξ

$$\begin{aligned} \mathscr{P}_{\xi\xi}(\xi,t) = \ & \Gamma_{\xi\xi}(\xi,t) - \sum_{i=1}^{Z} a_i''(\xi) \int_0^{\xi_i} b_i(s)\Gamma(s,t)ds - \frac{dc(\xi,\xi)}{d\xi}\Gamma(\xi,t) \\ & - c(\xi,s)\Gamma_{\xi\xi}(\xi,t) - c_\xi(\xi,\xi)\Gamma(\xi,t) - \int_0^{\xi} c''(\xi,s)\Gamma(s,t)ds. \end{aligned} \tag{6}$$

Next, we calculate the derivation of (3) with regard to t

$$\begin{aligned} \mathscr{P}_t(\xi,t) &= \Gamma_t(\xi,t) - \sum_{i=1}^{Z} a_i(\xi) \int_0^{\xi_i} b_i(s)\Gamma_t(s,t)ds - \int_0^{\xi} c(\xi,s)\Gamma_t(s,t)ds \\ &= \Gamma_{\xi\xi}(\xi,t) + \sum_{i=1}^{Z} \tau_i\Gamma(\xi_i,t) - \sum_{i=1}^{Z} a_i(\xi) \Big[b_i(\xi_i)\Gamma_\xi(\xi_i,t) - b_i(0)\Gamma_\xi(0,t) \\ &\quad - b_i'(\xi_i)\Gamma(\xi_i,t) + b_i'(0)\Gamma(0,t) + \int_0^{\xi_i} b''(s)\Gamma(s,t)ds + \int_0^{\xi_i} b_i(s)ds \sum_{j=1}^{Z} \tau_j\Gamma(\xi_j,t) \Big] \\ &\quad - c(\xi,\xi)\Gamma_\xi(\xi,t) + c_s(\xi,\xi)\Gamma(\xi,t) - c_s(\xi,0)\Gamma(0,t) - \int_0^{\xi} c_{ss}(\xi,s)\Gamma(s,t)ds \\ &\quad - \int_0^{\xi} c(\xi,s)ds \sum_{i=1}^{Z} \tau_i\Gamma(\xi_i,t). \end{aligned} \tag{7}$$

According to $\mathscr{P}_t(\xi,t) = \mathscr{P}_{\xi\xi}(\xi,t) + \sigma\mathscr{P}(\xi,t)$ in (4), one has

$$\begin{aligned} & \sum_{i=1}^{Z} \left[\int_0^{\xi_i} (a_i''(\xi)b_i(s) - a_i(\xi)b_i''(s) + \sigma a_i(\xi)b_i(s))\Gamma(s,t)ds \right] \\ & \quad + \int_0^{\xi} (c_{\xi\xi}(\xi,s) - c_{ss}(\xi,s) + \sigma c(\xi,s))\Gamma(s,t)ds + (2\frac{d}{d\xi}c(\xi,\xi) - \sigma)\Gamma(\xi,t) \\ & \quad + \sum_{i=1}^{Z} \Gamma(\xi_i) \left[\tau_i + a_i(\xi)b_i'(\xi_i) - \tau_i \sum_{j=1}^{Z} \int_0^{\xi_j} a_j(\xi)b_j(s)ds - \tau_i \int_0^{\xi} c(\xi,s)ds \right] \\ & \quad - \sum_{i=1}^{Z} a_i(\xi)b_i(\xi_i)\Gamma_\xi(\xi_i) - \left[\sum_{i=1}^{Z} a_i(\xi)b_i'(0) + c_{ss}(\xi,0) \right] \Gamma(0,t) = 0. \end{aligned} \tag{8}$$

In addition, by $\mathscr{P}_\xi(0,t) = 0$ and $\Gamma_\xi(0,t) = 0$,we get

$$\mathscr{P}_\xi(0,t) = \Gamma_\xi(0,t) - \sum_{i=1}^{Z} a_i'(0) \int_0^{\xi_i} b_i(s)\Gamma(s,t)ds - c(0,0)\Gamma(0,t). \tag{9}$$

By the (8) and (9), there is

$$\begin{cases} a_i''(\xi)b_i(s) - a_i(\xi)b_i''(s) + \sigma a_i(\xi)b_i(s) = 0, \\ 2\dfrac{d}{d\xi}c(\xi,\xi) - \sigma = 0, \\ c_{\xi\xi}(\xi,s) - c_{ss}(\xi,s) + \sigma c(\xi,s) = 0, \\ \sum\limits_{i=1}^{Z} a_i(\xi)b_i'(0) + c_s(\xi,0) = 0, \\ \tau_i + a_i(\xi)b_i'(\xi_i) - \tau_i \sum\limits_{j=1}^{Z}\displaystyle\int_0^{\xi_j} a_j(\xi)b_j(s)ds - \tau_i\int_0^{\xi} c(\xi,s)ds = 0, \\ b_i(\xi_i) = a_i'(0) = 0. \end{cases} \tag{10}$$

For our main result, the following assumption is given by

$$\begin{cases} \mathscr{S}_1 = 2 + \sum\limits_{i=1}^{Z} \tau_i\xi_i^2 \neq 0, & \text{if } \tau = 0; \\ \mathscr{S}_2 = \sum\limits_{i=1}^{Z} \tau_i \cosh(\sqrt{\tau}\xi_i) \neq 0, & \text{if } \tau > 0; \\ \mathscr{S}_3 = \sum\limits_{i=1}^{Z} \tau_i \cos(\sqrt{-\tau}\xi_i) \neq 0, & \text{if } \tau < 0. \end{cases} \tag{11}$$

Lemma 1. *Under assumption (11), for (10), there exist classical solutions $a_i(\cdot) \in C^2([0,1])$, $b_i(\cdot) \in C^2([0,\xi_i])$, and $c(\cdot,\cdot) \in C^2(\Lambda)$.*

Proof of Lemma 1. For convenience, we split (10) into

$$\begin{cases} a_i''(\xi)b_i(s) - a_i(\xi)b_i''(s) + \sigma a_i(\xi)b_i(s) = 0, \\ \tau_i + a_i(\xi)b_i'(\xi_i) - \tau_i \sum\limits_{j=1}^{Z}\displaystyle\int_0^{\xi_i} a_j(\xi)b_j(s)ds \\ \qquad\qquad -\tau_i\displaystyle\int_0^{\xi} c(\xi,s)ds = 0, \\ b_i(x_i) = a_i'(0) = 0, \end{cases} \tag{12}$$

and

$$\begin{cases} 2\dfrac{d}{d\xi}c(\xi,\xi) - \sigma = 0, \\ c_{\xi\xi}(\xi,s) - c_{ss}(\xi,s) + \sigma c(\xi,s) = 0, \\ \sum\limits_{i=1}^{Z} a_i(\xi)b_i'(0) + c_s(\xi,0) = 0. \end{cases} \tag{13}$$

Firstly, we prove that for (12), there exist classical solutions $a_i(\cdot) \in C^2([0,1])$, $b_i(\cdot) \in C^2([0,\xi_i])$. According to the first equation of (12), one has

$$\frac{a_i''(\xi) + \sigma a_i(\xi)}{a_i(\xi)} = \frac{a_i''(\xi) - \beta^2 a_i(\xi)}{a_i(\xi)} = \frac{b_i''(\xi)}{b_i(\xi)} = \tau, \tag{14}$$

where $\sigma = -\beta^2$. For computing (12) conveniently, we make $|\tau| = \lambda^2$. The proof is split the following three steps.

Step 1: $\tau = 0$.

There are the following solutions for Equation (14)

$$\begin{cases} a_i(\xi) = \eta_i ch\beta\xi, \\ b_i(s) = \zeta_i(s - \xi_i). \end{cases} \tag{15}$$

From the second of (12), one gives

$$\frac{a_i(\xi)b_i'(\xi_i)}{\tau_i} = \frac{a_1(\xi)b_1'(\xi_1)}{\tau_1}, \tag{16}$$

where $i = 2, 3, ..., Z$. Combining (15) and (16), we have

$$\eta_i\zeta_i = \frac{\eta_1\zeta_1\tau_i}{\tau_1}. \tag{17}$$

From (13), one has

$$\begin{aligned} \frac{d^2}{d\xi^2}\int_0^\xi c(\xi,s)ds &= \frac{d}{d\xi}\left[\int_0^\xi c_\xi(\xi,s)ds + c(\xi,\xi)\right] \\ &= \int_0^\xi c_\xi(\xi,s)ds + c_\xi(\xi,\xi) + \frac{d}{d\xi}c(\xi,\xi) \\ &= \sum_{i=1}^{Z} a_i(\xi)b_i'(0) - \sigma\int_0^\xi c(\xi,s)ds + \sigma. \end{aligned} \tag{18}$$

By (15) and (17), there is

$$\sum_{i=1}^{Z} a_i(\xi)b_i'(0) = \frac{\eta_1\zeta_1}{\tau_1}\sum_{i=1}^{Z}\tau_i ch\beta\xi = 0. \tag{19}$$

Then,

$$\frac{d^2}{d\xi^2}\int_0^\xi c(\xi,s)ds = -\alpha\int_0^\xi c(\xi,s)ds + \sigma. \tag{20}$$

For Equation (20), we obtain

$$\int_0^\xi c(\xi,s)ds - 1 = -\cosh\beta\xi. \tag{21}$$

Now, let

$$\mathscr{F}_i(\xi) = \tau_i + a_i(\xi)b_i'(\xi_i) - \tau_i\sum_{j=1}^{Z}\int_0^{\xi_j} a_j(\xi)b_j(s)ds - \tau_i\int_0^\xi c(\xi,s)ds. \tag{22}$$

By simple calculation, we can get

$$\mathscr{F}_i'(\xi) = a_i'(\xi)b_i'(\xi_i) - \tau_i\sum_{j=1}^{Z}\int_0^{\xi_j} a_j'(\xi)b_j(s)ds - \tau_i\int_0^\xi c_\xi(\xi,s)ds - \frac{\sigma}{2}\tau_i\xi, \tag{23}$$

and

$$\mathscr{F}_i''(\xi) = a_i''(\xi)b_i'(\xi_i) - \tau_i\sum_{j=1}^{Z}\int_0^{\xi_j} a_j''(\xi)b_j(s)ds - \tau_i\int_0^\xi c_{\xi\xi}(\xi,s)ds - \tau_i c_\xi(\xi,s) - \frac{\alpha}{2}\tau_i. \tag{24}$$

According to (14), (15), (17) and (24), one has $\mathscr{F}_i''(\xi) = 0$. That means $\mathscr{F}_i'(\xi)$ is constant. Since $a_i'(0) = 0$, there is $\mathscr{F}_i'(0) = 0$, which means $\mathscr{F}_i(\xi)$ is constant. Suppose that $\mathscr{F}_i(\xi) = 0$, when $\xi = 0$, we get

$$\mathscr{F}_i(0) = \tau_i + a_i(0)b_i'(\xi_i) - \tau_i\sum_{j=1}^{Z}\int_0^{\xi_i} a_j(0)b_j(s)ds = 0. \tag{25}$$

From (15), (17) ,(25), and the first equation of assumption (11), one has

$$\eta_i\zeta_i = \frac{-2\tau_1}{2+\sum\limits_{j=1}^{Z}\tau_j\xi_j^2} = \frac{-2\tau_1}{\mathscr{S}_1}. \tag{26}$$

Hence,

$$a_i(\xi)b_i(s) = -\frac{2\tau_1(s-\xi_i)\cosh\beta\xi}{2+\sum\limits_{j=1}^{Z}\tau_j\xi_j^2} = \frac{2\tau_1(\xi_i-s)\cosh\beta\xi}{\mathscr{S}_1}. \tag{27}$$

Step 2: $\tau > 0$.
By (14), one has

$$\begin{cases} a_i(\xi) = \eta_i\cosh(\sqrt{\lambda^2+\beta^2}\xi), \\ b_i(s) = \zeta_i(\sinh(\lambda s) - \tanh(\lambda\xi_i)\cosh(\lambda s)). \end{cases} \tag{28}$$

Similar calculation to step 1, we get

$$\eta_i\zeta_i = \frac{\eta_1\zeta_1\tau_i\cosh(\lambda\xi_i)}{\tau_1\cosh(\lambda\xi_1)}, \tag{29}$$

and

$$\int_0^\xi c(\xi,s)ds - 1 = \frac{\sum\limits_{i=1}^{Z} a_i(\xi)b_i'(0)}{\lambda^2}. \tag{30}$$

Substituting (14), (28), (29), (30) into (24), similarly, there is $\mathscr{F}_i''(\xi) = 0$. By the same some steps as step 1, we can also deduce $\mathscr{F}_i(\xi)$ is constant. Supposed that $\mathscr{F}_i(\xi) = 0$, when $\xi = 0$, it gives

$$\begin{aligned} \mathscr{F}_i(0) &= \tau_i + a_i(0)b_i'(\xi_i) - \tau_i\sum_{j=1}^{Z}\int_0^{\xi_j} a_j(0)b_j(y)dy \\ &= \tau_i + \frac{4\eta_1\zeta_1\tau_i\lambda}{\tau_1\cosh(\lambda\xi_1)} - \tau_i\sum_{j=1}^{Z}\frac{\eta_1\zeta_1\tau_j\cosh(\lambda\xi_j)}{\tau_1\cosh(\lambda\xi_1)}\left(\frac{4}{\cosh(\lambda\xi_j)} - 1\right) \\ &= \tau_i + \tau_i\sum_{j=1}^{Z}\frac{\eta_1\zeta_1\tau_j\cosh(\lambda\xi_j)}{\tau_1\cosh(\lambda\xi_1)} \\ &= 0. \end{aligned} \tag{31}$$

From (31) and the second equation of assumption (11), one has

$$\eta_1\zeta_1 = -\frac{\lambda\tau_1\cosh(\lambda\xi_1)}{\sum\limits_{j=1}^{N}\tau_j\cosh(\lambda\xi_j)} = \frac{-\lambda\tau_1\cosh(\lambda\xi_1)}{\mathscr{S}_2}. \tag{32}$$

By (29) and (32), it can be obtained that

$$\eta_i\zeta_i = -\frac{\lambda\tau_i\cosh(\lambda\xi_i)}{\mathscr{S}_2}. \tag{33}$$

Hence,

$$\begin{aligned} a_i(\xi)b_i(s) = -\ &\frac{1}{\mathscr{S}_2}\lambda\tau_i\cosh(\lambda\xi_i)\cosh(\sqrt{\lambda^2+\beta^2}\xi) \\ &(\sinh(\lambda s) - \tanh(\lambda\xi_i)\cosh(\lambda s)). \end{aligned} \tag{34}$$

Step 3: $\tau < 0$.

There is

$$\frac{a_i''(\xi)+\sigma a_i(\xi)}{a_i(\xi)}=\frac{a_i''(\xi)-\beta^2 a_i(\xi)}{a_i(\xi)}=\frac{b_i''(\xi)}{b_i(\xi)}=-\lambda^2. \tag{35}$$

By (35), one has

$$\begin{cases} a_i(\xi)=\eta_i\cos(\sqrt{\lambda^2-\beta^2}\xi), & \text{if} \quad \lambda^2-\beta^2>0, \\ a_i(\xi)=\eta_i, & \text{if} \quad \lambda^2-\beta^2=0, \\ a_i(\xi)=\eta_i\cosh\beta(\sqrt{\beta^2-\lambda^2}\xi), & \text{if} \quad \lambda^2-\beta^2<0, \\ b_i(s)=\zeta_i\sin(\lambda(\xi_i-s)). \end{cases} \tag{36}$$

Similar to step 1, we can get

$$\eta_i\zeta_i=\frac{\eta_1\zeta_1\tau_i}{\tau_1}, \tag{37}$$

and

$$\int_0^\xi c(\xi,s)ds-1=-\frac{\sum\limits_{i=1}^{Z}a_i(\xi)b_i'(0)}{\lambda^2}. \tag{38}$$

Substituting (14), (36), (37), (38) into (24), we can still ensure $\mathscr{F}_i''(\xi)=0$ and $\mathscr{F}_i(\xi)$ is constant. Supposed that $\mathscr{F}_i(\xi)=0$, when $\xi=0$, it gives

$$\begin{aligned} \mathscr{F}_i(0) &= \tau_i+a_i(0)b_i'(\xi_i)-\tau_i\sum_{j=1}^{Z}\int_0^{\xi_i}a_j(0)b_j(s)ds \\ &= \tau_i+\frac{\eta_1\zeta_1\tau_i\lambda}{\tau_1}+\tau_i\sum_{j=1}^{Z}\frac{\eta_1\zeta_1\tau_j}{\lambda\tau_1}(1-\cos(\lambda\xi_j)) \\ &= 0. \end{aligned} \tag{39}$$

From (39) and the last equation of assumption (11), there is

$$\eta_1\zeta_1=-\frac{\lambda\tau_1}{\sum\limits_{j=1}^{Z}\tau_j\cos(\lambda\xi_j)}=-\frac{\lambda\tau_1}{\mathscr{S}_3}, \tag{40}$$

which means

$$\eta_i\zeta_i=-\frac{\lambda\tau_i}{\mathscr{S}_3}. \tag{41}$$

Hence , we have

$$\begin{cases} a_i(\xi)b_i(s)=-\dfrac{\lambda\tau_i\cos(\sqrt{\lambda^2-\beta^2}\xi)\sin(\lambda(\xi_i-s))}{\mathscr{S}_3}, & \text{if} \quad \lambda^2-\beta^2>0, \\ a_i(\xi)b_i(s)=-\dfrac{\lambda\tau_i\sin(\lambda(\xi_i-s))}{\mathscr{S}_3}, & \text{if} \quad \lambda^2-\beta^2=0, \\ a_i(\xi)b_i(s)=-\dfrac{\lambda\tau_i\cosh(\sqrt{\beta^2-\lambda^2}\xi)\sin(\lambda(\xi_i-s))}{\mathscr{S}_3}, & \text{if} \quad \lambda^2-\beta^2<0. \end{cases} \tag{42}$$

Now, we show that for (13), there exists classical solution $c(\cdot,\cdot)\in C^2(\Lambda)$. Let $H(\xi)=\sum_{i=1}^{Z}a_i(\xi)b_i'(0)$, and

$$G(\hat{x},\hat{y})=c(\xi,s), \text{ govern by } \hat{x}=\xi+s,\ \hat{y}=\xi-s. \tag{43}$$

After a simple calculation, $G(\hat{x},\hat{y})$ can be written as

$$G(\hat{x},\hat{y}) = \frac{\sigma(\hat{x}+\hat{y})}{4} - \int_0^{\hat{y}} H(s)ds - \frac{\sigma}{4}\int_0^{\hat{y}}\int_0^{\alpha} G(\alpha,s)dsd\alpha - \frac{\sigma}{4}\int_0^{\hat{y}}\int_s^{\hat{x}} G(\alpha,s)d\alpha ds. \tag{44}$$

Next, we only need to show that there exists a unique solution $G(\cdot,\cdot) \in C([0,2]\times[0,1])$ for (44) under assumption (11). Define norm

$$\|G\| = \max_{(\hat{x},\hat{y})\in[0,2]\times[0,1]} e^{-\hat{a}\hat{x}}e^{-\hat{b}\hat{y}}|G(\hat{x},\hat{y})|, \tag{45}$$

where $\frac{\sigma}{2\hat{a}\hat{b}} \leq 1$. It is easy to see that norm $\|G\|$ is completed normed space. Let

$$\mathbb{T}G = \frac{\sigma(\hat{x}+\hat{y})}{4} - \int_0^{\hat{y}} H(s)ds - \frac{\sigma}{4}\int_0^{\hat{y}}\int_0^{\alpha} G(\tau,s)dsd\alpha - \frac{\sigma}{4}\int_0^{\hat{y}}\int_s^{\hat{x}} G(\tau,s)d\alpha ds. \tag{46}$$

Thus,

$$\begin{aligned}
|\mathbb{T}G_1(\hat{x},\hat{y}) - \mathbb{T}G_2(\hat{x},\hat{y})| &= \Bigg| -\frac{\sigma}{4}\int_0^{\hat{y}}\int_0^{\alpha}|G_1(\alpha,s)-G_2(\alpha,s)|dsd\alpha \\
&\quad -\frac{\sigma}{4}\int_0^{\hat{y}}\int_s^{\hat{x}}|G_1(\alpha,s)-G_2(\alpha,s)|d\alpha ds\Bigg| \\
&\leq \frac{\sigma}{2}\int_0^{\hat{y}}\int_0^{\hat{x}}|G_1(\tau,s)-G_2(\alpha,s)|dsd\alpha \\
&= \frac{\sigma}{2}\int_0^{\hat{y}}\int_0^{\hat{x}}|G_1(\tau,s)-G_2(\alpha,s)|dsd\alpha \\
&= \frac{\sigma}{2}\int_0^{\hat{y}}\int_0^{\hat{x}} e^{\hat{a}s}e^{\hat{b}\tau}e^{-\hat{a}s}e^{-\hat{b}\tau}|G_1(\tau,s)-G_2(\alpha,s)|dsd\alpha \\
&\leq \frac{\sigma}{2}\int_0^{\hat{y}}\int_0^{\hat{x}} e^{\hat{a}s}e^{\hat{b}\alpha}e^{-\hat{a}s}e^{-\hat{b}\alpha}dsd\alpha\|G_1(\hat{x},\hat{y})-G_1(\hat{x},\hat{y})\| \\
&\leq \frac{\sigma}{2\hat{a}\hat{b}}e^{\hat{a}s}e^{\hat{b}\alpha}\|G_1(\hat{x},\hat{y})-G_1(\hat{x},\hat{y})\|.
\end{aligned} \tag{47}$$

Hence, we have $e^{-\hat{a}\hat{x}}e^{-\hat{b}\hat{y}}|\mathbb{T}G_1(\hat{x},\hat{y}) - \mathbb{T}G_2(\hat{x},\hat{y})| \leq \frac{\sigma}{2\hat{a}\hat{b}}\|G_1(\hat{x},\hat{y})-G_1(\hat{x},\hat{y})\|$, then

$$\|\mathbb{T}G_1(\hat{x},\hat{y}) - \mathbb{T}G_2(\hat{x},\hat{y})\| \leq \frac{\sigma}{2\hat{a}\hat{b}}\|G_1(\hat{x},\hat{y})-G_1(\hat{x},\hat{y})\|.$$

By contraction mapping principle, unique solution $G(\cdot,\cdot) \in C([0,2]\times[0,1])$ for (44) holds. □

Remark 1. *From the above proof, we cannot explain that there are unique solutions $a_i(\cdot) \in C^2([0,1])$, $b_i(\cdot) \in C^2([0,\xi_i])$ for Equation (12). However, $a_i(\xi)b_i(s)$ is unique, which indicates transformation (3) is unique.*

Next, we show that transformation (3) is reversible. Assume that inverse backstepping transformation is governed by

$$\Gamma(\xi,t) = \mathscr{P}(\xi,t) + \sum_{i=1}^{Z} d_i(\xi)\int_0^{\xi_i} h_i(s)\mathscr{P}(s,t)ds + \int_0^{\xi} k(\xi,s)\mathscr{P}(s,t)ds. \tag{48}$$

Analogously, from (6) to (7), we get

$$\begin{aligned}
\Gamma_{\xi\xi}(\xi,t) = \quad & \mathscr{P}_{\xi\xi}(\xi,t) + \sum_{i=1}^{Z} d_i''(\xi)\int_0^{\xi_i} h_i(s)\Gamma(s,t)ds - \frac{d}{d\xi}(k(\xi,\xi))\Gamma(\xi,t) \\
& + k(\xi,\xi)\Gamma_{\xi}(\xi,t) - k_{\xi}(\xi,\xi)\Gamma_{\xi}(\xi,t) - \int_0^{\xi} k_{\xi\xi}(\xi,s)\Gamma(s,t)ds,
\end{aligned} \tag{49}$$

and

$$
\begin{aligned}
\Gamma_t(\xi,t) &= \mathscr{P}_t(\xi,t)+\sigma\mathscr{P}(\xi,t)+\sum_{i=1}^{Z} d_i(\xi)\int_0^{\xi_i} h_i(s)(\mathscr{P}_{ss}(s,t)+\sigma\mathscr{P}(s,t))ds \\
&\quad -\int_0^{\xi} k(\xi,s)(\mathscr{P}_{ss}(s,t)+\sigma\mathscr{P}(s,t))ds \\
&= \mathscr{P}_t(\xi,t)+\sigma\mathscr{P}(\xi,t)+\sum_{i=1}^{Z} d_i(\xi)\Big[h_i(\xi_i)\mathscr{P}_{\xi}(\xi_i,t)-h_i'(\xi_i)\mathscr{P}_{\xi}(\xi_i,t) \\
&\quad -h_i'(0)\mathscr{P}_{\xi}(0,t)+\int_0^{\xi_i} h_i''(s)\mathscr{P}(s,t)ds+\sigma\int_0^{\xi_i} h_i(s)\mathscr{P}(s,t)ds\Big] \\
&\quad +k(\xi,\xi)\mathscr{P}_{\xi}(\xi,t)-k_s(\xi,\xi)\mathscr{P}(\xi,t)+k_s(\xi,0)\mathscr{P}(0,t) \\
&\quad +\int_0^{\xi} k_{ss}(\xi,s)\mathscr{P}(s,t)ds+\sigma\int_0^{\xi} k(\xi,s)\mathscr{P}(s,t)ds.
\end{aligned}
\tag{50}
$$

Owing to $\Gamma_t(\xi,t)=\Gamma_{\xi\xi}(\xi,t)+\sum_{i=1}^{Z}\tau_i\Gamma(\xi_i,t)$, one has

$$
\begin{aligned}
&\sum_{i=1}^{Z}\Big[\int_0^{\xi_i}(-d_i''(\xi)h_i(s)+d_i(\xi)h_i''(s)+\sigma d_i(\xi)h_i(s)-\sum_{j=1}^{Z}\tau_i d_j(\xi_i)h_j(s) \\
&-\tau_i k(\xi_i,s))\mathscr{P}(s,t)ds\Big]+\int_0^{\xi}(k_{ss}(\xi,s)-k_{\xi\xi}(\xi,s)+\sigma k(\xi,s))\mathscr{P}(s,t)ds \\
&-(2\frac{d}{d\xi}k(\xi,\xi)-\sigma)\mathscr{P}(\xi,t)+\sum_{i=1}^{Z}\mathscr{P}(\xi_i)(\tau_i+d_i(\xi)h_i'(\xi_i))+(\sum_{i=1}^{Z}d_i(\xi)h_i'(0) \\
&+k_s(\xi,0))\mathscr{P}(0,t)+\sum_{i=1}^{Z}d_i(\xi)h_i(\xi_i)\mathscr{P}_{\xi}(\xi_i,t)=0.
\end{aligned}
\tag{51}
$$

Additionally,

$$
\Gamma_{\xi}(0,t)=\mathscr{P}_{\xi}(0,t)-\sum_{i=1}^{Z}d'(0)\int_0^{\xi_i}h_i(s)\mathscr{P}(s,t)ds-k(0,0)\mathscr{P}(0,t). \tag{52}
$$

By (51) and (52) , we get

$$
\begin{cases}
k_{ss}(\xi,s)-k_{\xi\xi}(\xi,s)+\sigma k(\xi,s)=0, \\
2\dfrac{d}{d\xi}k(\xi,\xi)-\sigma=0, \\
\sum\limits_{i=1}^{Z}d_i(\xi)h_i'(0)+k_s(\xi,0)=0, \\
\tau_i+d_i(\xi)h_i'(\xi_i)=0, \\
-d_i''(\xi)h_i(s)+d_i(\xi)h_i''(s)+\sigma d_i(\xi)h_i(s) \\
\qquad\qquad -\tau_i k(\xi_i,s)-\sum\limits_{j=1}^{Z}\tau_i d_j(\xi_i)h_j(s)=0, \\
h_i(\xi_i)=d'(0)=0.
\end{cases}
\tag{53}
$$

Lemma 2. *Under assumption (11), for (53), there exist classical solutions* $d_i(\cdot)\in C^2([0,1])$, $h_i(\cdot)\in C^2([0,\xi_i])$, *and* $k(\cdot,\cdot)\in C^2(\Lambda)$.

Proof of Lemma 2. Similarly, let (53) split into the following

$$
\begin{cases}
\tau_i+d_i(\xi)h_i'(\xi_i)=0, \\
-d_i''(\xi)h_i(s)+d_i(\xi)h_i''(s)+\sigma d_i(\xi)h_i(s) \\
\qquad\qquad -\tau_i k(\xi_i,s)-\sum\limits_{j=1}^{Z}\tau_i d_j(\xi_i)h_j(s)=0, \\
h_i(\xi_i)=d'(0)=0,
\end{cases}
\tag{54}
$$

and

$$\begin{cases} k_{ss}(\xi,s) - k_{\xi\xi}(\xi,s) + \sigma k(\xi,s) = 0, \\ 2\dfrac{d}{d\xi}k(\xi,\xi) - \sigma = 0, \\ \sum\limits_{i=1}^{Z} d_i(\xi)h_i'(0) + k_s(\xi,0) = 0. \end{cases} \tag{55}$$

Firstly, we prove that for (54), there exist classical solutions $d_i(\cdot) \in C^2([0,1])$, $h_i(\cdot) \in C^2([0,\xi_i])$. By $\tau_i + d_i(\xi)h_i'(\xi_i) = 0$ in (54), one has

$$d_i(\xi) = -\frac{\tau_i}{h_i'(\xi_i)}. \tag{56}$$

Combining the first equation of (54) and (56), we have

$$\begin{cases} h_i''(s) + (\sigma - \tau)h_i(s) + h_i'(\xi_i)k(\xi_i,s) = 0, \\ h_i(\xi_i) = 0. \end{cases} \tag{57}$$

We discuss (57) by the following three cases.
Case 1: $\tau > \sigma$
By (57), we have

$$\begin{aligned} h_i(s) =& h_i'(\xi_i)\Bigg[\frac{1}{2\sqrt{\tau-\sigma}}(e^{\sqrt{\tau-\sigma}s-\sqrt{\tau-\sigma}\xi_i} - e^{-\sqrt{\tau-\sigma}s+\sqrt{\tau-\sigma}\xi_i}) \\ & - \frac{e^{\sqrt{\tau-\sigma}s}}{2\sqrt{\tau-\sigma}}\int_{\xi_i}^{s} e^{-\sqrt{\tau-\sigma}s}k(\xi_i,s)ds \\ & + \frac{e^{-\sqrt{\tau-\sigma}s}}{2\sqrt{\tau-\sigma}}\int_{\xi_i}^{s} e^{\sqrt{\tau-\sigma}s}k(\xi_i,s)ds\Bigg]. \end{aligned} \tag{58}$$

Then, it gives

$$\begin{aligned} h_i(s)d_i(\xi) =& -\tau_i\Bigg[\frac{1}{2\sqrt{\tau-\sigma}}(e^{\sqrt{\tau-\sigma}s-\sqrt{\tau-\sigma}\xi_i} - e^{-\sqrt{\tau-\sigma}s+\sqrt{\tau-\sigma}\xi_i}) \\ & - \frac{e^{\sqrt{\tau-\sigma}s}}{2\sqrt{\tau-\sigma}}\int_{\xi_i}^{s} e^{-\sqrt{\tau-\sigma}s}k(\xi_i,s)ds \\ & + \frac{e^{-\sqrt{\tau-\sigma}s}}{2\sqrt{\tau-\sigma}}\int_{\xi_i}^{s} e^{\sqrt{\tau-\sigma}s}k(\xi_i,s)ds\Bigg]. \end{aligned} \tag{59}$$

Case 2: $\tau = \sigma$
According to (57), we obtain

$$h_i(s) = h_i'(\xi_i)(s-\xi_i) - h_i'(\xi_i)\int_{\xi_i}^{s}\int_{\xi_i}^{y} k(\xi_i,x)dxdy. \tag{60}$$

So we have

$$h_i(s)d_i(\xi) = -\tau_i(s-\xi_i) + \tau_i\int_{\xi_i}^{s}\int_{\xi_i}^{y} k(\xi_i,x)dxdy. \tag{61}$$

Case 3: $\tau < \sigma$

Similarly, by (57), one has

$$\begin{aligned} h_i(s) =& h_i'(\xi_i)\frac{1}{\sqrt{\sigma-\tau}}\Big[(-\sin(\sqrt{\sigma-\tau}\xi_i)\cos(\sqrt{\sigma-\tau}s) \\ &+\sin(\sqrt{\sigma-\tau}s)\cos(\sqrt{\sigma-\tau}\xi_i)) \\ &+\cos(\sqrt{\sigma-\tau}s)\int_{x_i}^{s}\sin(\sqrt{\sigma-\tau}x)k(\xi_i,x)dx \\ &-\sin(\sqrt{\sigma-\tau}s)\int_{x_i}^{s}\cos(\sqrt{\sigma-\tau}x)k(\xi_i,x)dx\Big]. \end{aligned} \tag{62}$$

Then,

$$\begin{aligned} h_i(s)d_i(\xi) =& -\tau_i\frac{1}{\sqrt{\sigma-\tau}}\Big[(-\sin(\sqrt{\sigma-\tau}\xi_i)\cos(\sqrt{\sigma-\tau}s) \\ &+\sin(\sqrt{\sigma-\tau}s)\cos(\sqrt{\sigma-\tau}\xi_i)) \\ &+\cos(\sqrt{\sigma-\tau}s)\int_{x_i}^{s}\sin(\sqrt{\sigma-\tau}x)k(\xi_i,x)dx \\ &-\sin(\sqrt{\sigma-\tau}s)\int_{x_i}^{s}\cos(\sqrt{\sigma-\tau}x)k(\xi_i,x)dx\Big]. \end{aligned} \tag{63}$$

Now, we prove that there exists classical solution $k(\cdot,\cdot)\in C^2(\Lambda)$ for (55). We firstly show that kernel $k(\xi,s)$ is independent of $h_i(s)$. In fact, we obtain $h_i'(x_i)=h_i'(0)$ from Case 1–Case 3. For convenience, let $A=\frac{h_i'(0)}{h_i'(\xi_i)}$. By (58), we get

$$\begin{aligned} h_i'(0) =& h_i'(\xi_i)\Big[\frac{1}{2}(e^{-\sqrt{\tau-\sigma}\xi_i}+e^{\sqrt{\tau-\sigma}\xi_i}) \\ &+\frac{1}{2}\int_0^{\xi_i}e^{-\sqrt{\tau-\sigma}x}k(\xi_i,x)dx \\ &+\frac{1}{2}\int_0^{x_i}e^{\sqrt{\tau-\sigma}x}k(\xi_i,x)dx\Big]. \end{aligned} \tag{64}$$

Define

$$\begin{cases} f(\xi)=\int_0^{\xi_i}e^{-\sqrt{\tau-\sigma}x}k(\xi_i,x)dx, \\ g(\xi)=\int_0^{\xi_i}e^{\sqrt{\tau-\sigma}x}k(\xi_i,x)dx. \end{cases} \tag{65}$$

Then, combining (55), we obtain

$$\begin{cases} f''(x)-\tau f(\xi)=\alpha e^{-\sqrt{\tau-\sigma}x}-k_s(\xi,0)-\sqrt{\tau-\sigma}k(\xi,0), \\ f(0)=0, f'(0)=0, \end{cases} \tag{66}$$

and

$$\begin{cases} g''(x)-\tau g(\xi)=\alpha e^{-\sqrt{\tau-\sigma}\xi}-k_s(\xi,0)+\sqrt{\tau-\sigma}k(\xi,0), \\ g(0)=0, g'(0)=0. \end{cases} \tag{67}$$

Let $F(\xi)=f(\xi)+g(\xi)$, one has

$$F(\xi)=\frac{2k_s(\xi,0)}{\tau}k(\xi,s)-e^{\sqrt{\tau-\sigma}\xi_i}-e^{-\sqrt{\tau-\sigma}\xi_i}. \tag{68}$$

Then,

$$A=\frac{k_s(\xi,0)}{\tau}. \tag{69}$$

Hence, A is any constant in this case, moreover, $k_s(\xi,0) = A\tau$. By (60), we have

$$h_i'(0) = h_i'(\xi_i) + h_i'(\xi_i)\int_0^{\xi_i} k(\xi_i,x)dx. \tag{70}$$

By (66) with $\tau = \sigma$, we can get $\int_0^{\xi_i} k(\xi_i,x)dx = 0$. So $A = 1$ holds by (66). By Equation (63), there is

$$h_i'(0) = h_i'(\xi_i)\left[\cos(\sqrt{\sigma-\tau}\xi_i) + \frac{1}{\sigma-\tau}\int_0^{\xi_i}\cos(\sqrt{\sigma-\tau}x)k(\xi_i,x)dx\right]. \tag{71}$$

Let

$$\gamma(\xi) = \int_0^{\xi}\cos(\sqrt{\sigma-\tau}x)k(\xi,x)dx. \tag{72}$$

From (72), one has

$$\begin{cases} \gamma''(\xi) - \tau\gamma(\xi) = \sigma\cos(\sqrt{\sigma-\tau}\xi) - k_s(\xi,0), \\ \gamma(0) = 0, \gamma'(0) = 0. \end{cases} \tag{73}$$

There is a unique solution in (73), which is written by

$$\gamma(\xi) = -\cos(\sqrt{\sigma-\tau}\xi) + \frac{k_s(\xi,0)}{\tau}. \tag{74}$$

Combining (72) and (74) at $\xi = \xi_i$, $i = 1,2,...,Z$, we obtain

$$\gamma(\xi_i) = \int_0^{\xi_i}\cos(\sqrt{\sigma-\tau}x)k(\xi_i,x)dx = -\cos(\sqrt{\sigma-\tau}\xi_i) + \frac{k_s(\xi_i,0)}{\tau}. \tag{75}$$

Then, substituting (75) into (71), we have

$$A = \frac{k_s(\xi,0)}{\tau}. \tag{76}$$

We can see A is also any constant in this case.

Next, we prove that $A = 1$. Substituting (48) into (3), we make the kernels satisfying

$$\begin{aligned} &a_i(\xi)\left[b_i(s) + \int_s^{\xi_i} b_i(x)k(x,s)dx\right] \\ &= h_i(s)\left[d_i(\xi) - \sum_{j=1}^{Z} a_j(\xi)\int_0^{\xi_i} b_j(s)d_j(s)ds - \int_0^{\xi} c(\xi,x)d_j(s)dx\right], \end{aligned} \tag{77}$$

and

$$k(\xi,s) = c(\xi,s) + \int_s^{\xi} c(\xi,x)k(x,s)dx. \tag{78}$$

From (77), we have

$$\frac{h_i'(0)}{h_i'(\xi_i)} = \frac{1}{b_i'(\xi_i)}(b_i'(0) + \int_0^{\xi_i} b_i(x)k_s(x,0)dx). \tag{79}$$

For simplifying calculation, we only give the result under case $\tau > 0$. In fact, case $\tau > 0$ is the same as $\tau < 0$. Substituting $b_i(s) = \zeta_i(\sinh(\lambda s) - \tanh(\lambda\xi_i)\cosh(\lambda s))$ of (28) into (79), one has $A = 1$.

Finally, we can see that (55) is the same as (13) except for the last equations. However, it can be transformed into form "$H(\xi)$" for the last of Equation (55). So the remaining steps are similar from (43) to (47). □

Remark 2. *Similar to Lemma 1, from the above proof, we cannot explain that there are unique solutions $d_i(\cdot) \in C^2([0,1])$, $h_i(\cdot) \in C^2([0,\xi_i])$ for Equation (54). However, $d_i(\xi)h_i(s)$ is unique, which indicates transformation (48) is unique.*

We design the state feedback controller

$$\hat{U}(t) = \sum_{i=1}^{Z} a_i(1) \int_0^{\xi_i} b_i(s)\Gamma(s,t)ds + \int_0^1 c(1,s)\Gamma(s,t)ds. \tag{80}$$

The closed-loop system (2) corresponding with controller (80) is described by

$$\begin{cases} \Gamma_t(\xi,t) = \Gamma_{\xi\xi}(\xi,t) + \sum_{i=1}^{Z} \tau_i \Gamma(\xi_i,t),\ 0 < \xi < 1, t \in (0,+\infty), \\ \Gamma_\xi(0,t) = 0,\ t \geq 0, \\ \Gamma(1,t) = \sum_{i=1}^{Z} a_i(1) \int_0^{\xi_i} b_i(s)\Gamma(s,t)ds + \int_0^1 c(1,s)\Gamma(s,t)ds,\ t \geq 0, \\ \Gamma(\xi,0) = \Gamma_0(\xi), 0 \leq \xi \leq 1. \end{cases} \tag{81}$$

The system (81) is considered in the state space $\mathscr{H} = L^2(0,1)$. We define system operator $\mathscr{A} : D(\mathscr{A}) \to \mathscr{H}$ for closed-loop system (81)

$$\begin{cases} \mathscr{A}\phi = \phi'' + \sum_{i=1}^{Z} \tau_i \phi(\xi_i,t),\ \forall \phi \in \mathscr{A}, \\ \mathscr{A} = \{\phi \in \mathscr{H}^2(0,1) \mid \mathscr{A}\phi \in \mathscr{H}, \phi(1) = \sum_{i=1}^{Z} a_i(1) \int_0^{\xi_i} b_i(s)\Gamma(s,t)ds \\ \qquad + \int_0^1 c(1,s)\Gamma(s,t)ds\}. \end{cases} \tag{82}$$

Theorem 1. *Under assumption (11), for each initial value $\Gamma_0(\cdot) \in \mathscr{H}$, the closed-loop system (81) admits a unique solution $\Gamma(\cdot,t) \in C([0,+\infty);\mathscr{H})$. Moreover, the closed-loop system generates an exponentially stable C_0-semigroup such that*

$$\|e^{\mathscr{A}t}\| \leq L_{\mathscr{A}} e^{-\rho t}, t \geq 0, \tag{83}$$

where $L_{\mathscr{A}}$ and ρ two positive constants.

Proof. Define linear operator $\mathscr{A}_0$: $D(\mathscr{A}_0) \to \mathscr{H}$ for the target system (4)

$$\begin{cases} \mathscr{A}_0\varphi = \varphi'' + \sigma\varphi;\ \forall \varphi \in D(\mathscr{A}_0), \\ D(\mathscr{A}_0) = \{\varphi \in \mathscr{H}^2(0,1) \mid \varphi'(0) = 0, \varphi(1) = 0\}. \end{cases} \tag{84}$$

For any $\omega \in D(\mathscr{A}_0)$,

$$\mathrm{Re}\langle \mathscr{A}_0\varphi, \varphi\rangle = \mathrm{Re}\langle \varphi'' + \sigma\varphi, \varphi\rangle \leq -\|\varphi'\|^2 + \sigma\|\varphi\|^2 \leq 0,$$

which means $\mathscr{A}_0$ is dissipative on $\mathscr{H}$. On the other hand, for any $\hat{\varphi} \in \mathscr{H}$, $\mathscr{A}_0(\varphi) = \hat{\varphi}$, by $\hat{\varphi} = \varphi'' + \sigma\varphi$, we have

$$\begin{aligned} \varphi(\xi) = {} & \frac{-1}{\sigma}\left[\coth(\sqrt{-\sigma})\int_0^1 \cosh(\sqrt{-\sigma}s)\hat{\varphi}(s)ds - \int_0^1 \sinh(\sqrt{-\sigma}s)\hat{\varphi}(s)ds\right] \\ & \cosh(\sqrt{-\sigma}\xi) + \frac{1}{\sqrt{-\sigma}}\sinh(\sqrt{-\sigma}\xi)\int_0^\xi \cosh(\sqrt{-\sigma}s)\hat{\varphi}(s)ds \\ & -\frac{1}{\sqrt{-\sigma}}\cosh(\sqrt{-\sigma}\xi)\int_0^\xi \sinh(\sqrt{-\sigma}s)\hat{\varphi}(s)ds, \end{aligned}$$

which shows that $\mathscr{A}_0^{-1} \in \mathscr{L}(\mathscr{H})$ is compact on $\mathscr{H}$. By the Lumer Phillips theorem [38], $\mathscr{A}_0$ generates a compressed C_0-semigroup $e^{\mathscr{A}_0}$ on $\mathscr{H}$. That is to say,

$$||e^{\mathscr{A}_0 t}|| \leq L_{\mathscr{A}_0} e^{-\delta t},$$

where $L_{\mathscr{A}_0}$ are δ two positive constants. Define Lyapunov function

$$E(t) = \frac{1}{2}\int_0^1 \mathscr{P}^2(\xi,t)d\xi. \tag{85}$$

Find the derivative of (85), and the calculus of the derivatives gives as the following quantity

$$\dot{E}(t) = -\int_0^1 \mathscr{P}_\xi^2 dt + 2\sigma E(t) \leq 2\sigma E(t),$$

which means

$$E(t) \leq e^{2\sigma t} E(0).$$

Thus, the target system (4) is exponentially stable. Based on the transformation (3) and reversible transformation (48), we define the following bounded invertible operator $\mathscr{K} : \mathscr{H} \to \mathscr{H}$

$$\begin{cases} \mathscr{K}\phi = \varphi, \\ \varphi = \phi - \sum\limits_{i=1}^{Z} a_i(1)\int_0^{\xi_i} b_i(s)\Gamma(s,t)ds - \int_0^1 c(1,s)\Gamma(s,t)ds, \\ \mathscr{K}^{-1}\varphi = \phi, \\ \phi = \varphi + \sum\limits_{i=1}^{Z} d_i(1)\int_0^{\xi_i} h_i(s)\mathscr{P}(s,t)ds + \int_0^1 k(1,s)\mathscr{P}(s,t)ds. \end{cases} \tag{86}$$

So there exists a bounded reversible operator $\mathscr{K}$ satisfying $\mathscr{A} = \mathscr{K}^{-1}\mathscr{A}_0\mathscr{K}$. Hence, the operator $\mathscr{A}$ yields an exponentially stable C_0-semigroup on $\mathscr{H}$, that is,

$$e^{\mathscr{A}t} = \mathscr{K}^{-1}e^{\mathscr{A}_0 t}\mathscr{K}.$$

The Theorem 1 holds by the exponential stability of $e^{\mathscr{A}_0 t}$. □

3. Output Feedback Controller Design

In this section, an observer-based output feedback control for system (2) is designed as follows

$$\begin{cases} \hat{\Gamma}_t(\xi,t) = \hat{\Gamma}_{\xi\xi}(\xi,t) + \sum\limits_{i=1}^{Z} \tau_i\Gamma(\xi_i,t),\ 0<\xi<1, t\in(0,+\infty), \\ \hat{\Gamma}(1,t) = \hat{U}(t),\ t\geq 0, \\ \hat{\Gamma}_\xi(0,t) = 0,\ t\geq 0, \\ \hat{\Gamma}(\xi,0) = \hat{\Gamma}_0(\xi), 0\leq\xi\leq 1. \end{cases} \tag{87}$$

Let $\bar{\Gamma} = \hat{\Gamma} - \Gamma$. Error system $\bar{\Gamma}$ is given by

$$\begin{cases} \bar{\Gamma}_t(\xi,t) = \bar{\Gamma}_{\xi\xi}(\xi,t), 0<\xi<1, t\in(0,+\infty), \\ \bar{\Gamma}(1,t) = 0,\ t\geq 0, \\ \bar{\Gamma}_\xi(0,t) = 0,\ t\geq 0, \\ \bar{\Gamma}(\xi,0) = \bar{\Gamma}_0. \end{cases} \tag{88}$$

It is well known that system (88) yields a unique exponentially stable solution in state space $\mathscr{H}$. We propose the following output feedback controller

$$\hat{U}(t) = \sum_{i=1}^{Z} a_i(1)\int_0^{x_i} b_i(s)\hat{\Gamma}(s,t)ds + \int_0^1 c(1,s)\hat{\Gamma}(s,t)ds, \tag{89}$$

where a_i, b_i and c are determined by (10).

The closed-loop system composed of system (2) and system (87) under observer (89), is governed by

$$\begin{cases} \Gamma_t(\xi,t) = \Gamma_{\xi\xi}(\xi,t) + \sum\limits_{i=1}^{Z} \tau_i \Gamma(\xi_i,t), \ 0<\xi<1, t\in(0,+\infty), \\ \Gamma(1,t) = \sum\limits_{i=1}^{Z} a_i(1)\int_0^{x_i} b_i(s)\hat{\Gamma}(s,t)ds + \int_0^1 c(1,s)\hat{\Gamma}(s,t)ds, \ t\geq 0, \\ \Gamma_\xi(0,t) = 0, \ t\geq 0, \\ \hat{\Gamma}_t(\xi,t) = \hat{\Gamma}_{\xi\xi}(\xi,t) + \sum\limits_{i=1}^{Z} \tau_i \Gamma(\xi_i,t), \ 0<\xi<1, t\in(0,+\infty), \\ \hat{\Gamma}(1,t) = \sum\limits_{i=1}^{Z} a_i(1)\int_0^{\xi_i} b_i(s)\hat{\Gamma}(s,t)ds + \int_0^1 c(1,s)\hat{\Gamma}(s,t)ds, \ t\geq 0, \\ \hat{\Gamma}_\xi(0,t) = 0, \ t\geq 0, \\ \Gamma(\xi,0) = \Gamma_0(\xi), \hat{\Gamma}(\xi,0) = \hat{\Gamma}_0(\xi), 0\leq\xi\leq 1. \end{cases} \tag{90}$$

Theorem 2. *Under assumption (11), for any initial datum $(\Gamma(\cdot,0),\hat{\Gamma}(\cdot,0))^\top \in \mathscr{H}\times\mathscr{H}$, the closed-loop system (90) admits a unique solution $(\Gamma(\cdot,t),\hat{\Gamma}(\cdot,t))^\top \in C([0,+\infty);\mathscr{H}\times\mathscr{H})$. Moreover, the closed-loop system (90) is exponentially stable, that is, for any $t>0$, there exist $L_{\hat{\mathscr{A}}}>0, \hat{\delta}>0$ such that*

$$\|(\Gamma(\cdot,t),\hat{\Gamma}(\cdot,t))\|_{\mathscr{H}} \leq L_{\hat{\mathscr{A}}} e^{-\hat{\delta}t} \|(\Gamma(\cdot,0),\hat{\Gamma}(\cdot,0))\|_{\mathscr{H}}. \tag{91}$$

Proof. To prove our result, we aim to transform system (90) into an equivalent one. Then, we divide the equivalent system into two sub-systems which are proved to be well-posed and exponentially stable.

Owing to

$$\begin{bmatrix} \hat{\Gamma} \\ \bar{\Gamma} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Gamma \\ \hat{\Gamma} \end{bmatrix}, \tag{92}$$

the closed-loop system (90) is equivalent to PDEs as follows

$$\begin{cases} \hat{\Gamma}_t(\xi,t) = \hat{\Gamma}_{\xi\xi}(\xi,t) + \sum\limits_{i=1}^{Z} \tau_i \Gamma(\xi_i,t), \ x\in(0,1), t\in(0,+\infty), \\ \hat{\Gamma}(1,t) = \sum\limits_{i=1}^{Z} a_i(1)\int_0^{\xi_i} b_i(s)\hat{\Gamma}(s,t)ds + \int_0^1 c(1,s)\hat{\Gamma}(s,t)ds, \ t\geq 0, \\ \hat{\Gamma}_\xi(0,t) = 0, \ t\geq 0, \\ \bar{\Gamma}_t(\xi,t) = \bar{\Gamma}_{\xi\xi}(\xi,t), \xi\in(0,1), t\in(0,+\infty), \\ \bar{\Gamma}(1,t) = 0, \ t\geq 0, \\ \bar{\Gamma}_\xi(0,t) = 0, \ t\geq 0, \\ \hat{\Gamma}(\xi,0) = \hat{\Gamma}_0(\xi), \bar{\Gamma}(\xi,0) = \bar{\Gamma}_0, 0\leq\xi\leq 1. \end{cases} \tag{93}$$

Next, we prove that system(93) is exponentially stable. Notice that "$\bar{\Gamma}$-part" of system (93) is special case at $\sigma=0$ for system (4), so "$\bar{\Gamma}$-part" has a unique solution which is exponentially stable. Now, we only need to prove that "$\hat{\Gamma}$-part" of system (93) has a unique solution and is exponentially stable. Similar to transformation (3), we give the following transformation

$$p(\xi,t) = \hat{\Gamma}(\xi,t) - \sum_{i=1}^{Z} a_i(\xi)\int_0^{\xi_i} b_i(s)\hat{\Gamma}(s,t)ds - \int_0^{\xi} c(\xi,s)\hat{\Gamma}(s,t)ds, \tag{94}$$

under which "$\hat{\Gamma}$-part" of system (93) is mapped to

$$\begin{cases} p_t(\xi,t) = p_{\xi\xi}(\xi,t) + \sigma p(\xi,t) + \sum\limits_{i=1}^{Z} \bar{\Gamma}(\xi_i,t)\Big[-1 \\ \qquad\qquad + \sum\limits_{j=1}^{Z} a_j(\xi)\int_0^{\xi_j} b_j(s)ds + \int_0^1 c(\xi,s)ds\Big], \xi \in (0,1), t \in (0,+\infty), \\ p(1,t) = 0,\ t \geq 0, \\ p_\xi(0,t) = 0,\ t \geq 0. \end{cases} \tag{95}$$

Now, we are in a position to prove that system (95) admits a unique solution and is exponentially stable. Based on (84), the solution of system (95) can be obtained as

$$p(\cdot,t) = e^{\mathscr{A}_0 t}p(\cdot,0) + \int_0^t e^{\mathscr{A}_0(t-s)}F_p(\xi)\sum_{i=1}^{Z}\bar{\Gamma}(\xi_i,s)ds, \tag{96}$$

where $F_p(\xi) = -1 + \sum_{j=1}^{Z} a_j(\xi)\int_0^{\xi_j} b_j(s)ds + \int_0^1 c(\xi,s)ds$. Define the following energy function $E_1(t)$ of system (88)

$$E_1(t) = \frac{1}{2}\int_0^1 (\bar{\Gamma}(\xi,t))^2 d\xi. \tag{97}$$

A simple computation of derivative of (97) with respect to t shows that

$$\dot{E}_1(t) = -\int_0^1 (\bar{\Gamma}_\xi(\xi,t))^2 d\xi \leq -\int_0^1 (\bar{\Gamma}(\xi,t))^2 d\xi = -2E_1(t), \tag{98}$$

which means

$$E_1(t) \leq e^{-2t}E_1(0).$$

For any $t_2 > t_1 > 0$, one has

$$\begin{aligned} \int_{t_1}^{t_2} |-\bar{\Gamma}_\xi(\xi,s)|^2 ds &= \int_{t_1}^{t_2} |\int_\xi^1 \bar{\Gamma}_\xi(\xi,s)d\xi|^2 ds \\ &\leq \int_{t_1}^{t_2} |\int_0^1 \bar{\Gamma}_\xi(\xi,s)d\xi|^2 ds \\ &\leq \int_{t_1}^{t_2}\int_0^1 |\bar{\Gamma}_\xi(\xi,s)|^2 d\xi ds \\ &= -\int_{t_1}^{t_2} \dot{E}_1(s)ds \\ &= E_1(t_1) - E_1(t_2) \\ &\leq E_1(0)(e^{-2\delta_1 t_1} + e^{-2\delta_1 t_2}). \end{aligned} \tag{99}$$

Owing to the proof process of Theorem 1, there is

$$\|e^{\mathscr{A}_0 t}p(\cdot,0)\| \leq \|e^{\mathscr{A}_0 t}\|\|p(\cdot,0)\| \leq L_{\mathscr{A}_0}e^{-\delta t}\|p(\cdot,0)\|. \tag{100}$$

Simultaneously,

$$\|\int_0^t e^{\mathscr{A}_0(t-s)}F_p(\xi)\sum_{i=1}^{Z}\bar{\Gamma}(\xi_i,s)ds\| \leq \int_0^t \|e^{\mathscr{A}_0(t-s)}\|\|F_p(\xi)\|\|\sum_{i=1}^{Z}\bar{\Gamma}(\xi_i,s)ds\|. \tag{101}$$

Additionally, from (99), for any $i = 1, 2, ..., Z$, we obtain(see [39])

$$\begin{aligned}\int_0^t e^{\delta_1(t-s)}|\bar{\Gamma}(\xi_i,s)|ds &= \int_0^{\frac{t}{2}} e^{-\delta_1(t-s)}|\bar{\Gamma}(\xi_i,s)|ds + \int_{\frac{t}{2}}^t e^{-\delta_1(t-s)}|\bar{\Gamma}(\xi_i,s)|ds \\ &= \int_0^{\frac{t}{2}} e^{-\delta_1\varepsilon}|\bar{\Gamma}(\xi_i,t-\varepsilon)|d\varepsilon + \int_{\frac{t}{2}}^t e^{-\delta_1(t-s)}|\bar{\Gamma}(\xi_i,s)|ds \\ &\le [\int_0^{\frac{t}{2}} e^{-2\delta_1\varepsilon}d\varepsilon]^{\frac{1}{2}}[\int_0^{\frac{t}{2}} |\bar{\Gamma}(\xi_i,t-\varepsilon)|^2 d\varepsilon]^{\frac{1}{2}} \\ &\quad +[\int_{\frac{t}{2}}^t e^{-2\delta_1(t-s)}ds]^{\frac{1}{2}}[\int_{\frac{t}{2}}^t |\bar{\Gamma}(\xi_i,s)|^2 ds]^{\frac{1}{2}} \\ &\le \tfrac{1}{2\delta_1}e^{-2\delta_1 t}[\int_0^\infty |\bar{\Gamma}(\xi_i,\varepsilon)|^2 d\varepsilon] + \frac{1}{2\delta_1}\sqrt{E_1(0)e^{\frac{-\delta_1 t}{4}}}.\end{aligned} \tag{102}$$

Then, it can be obtained that

$$\int_0^t e^{\delta_1(t-s)}|\sum_{i=1}^{Z}\bar{\Gamma}(\xi_i,s)|ds \le \frac{Ze^{-2\delta_1 t}}{2\delta_1}[\int_0^\infty |\bar{\Gamma}(\xi_i,\varepsilon)|^2 d\varepsilon] + \frac{Z}{2\delta_1}e^{\frac{-\delta_1 t}{4}}\sqrt{E_1(0)}. \tag{103}$$

Therefore, system (95) is exponentially stable, which means Theorem 2 holds. □

4. Simulation Results

In this section, some simulation results are presented to explain the effectiveness of proposed controller by the finite element method. For the open-loop system of (2) and the closed-loop system (90), we choose parameters

$$\begin{cases} \tau_1 = 1, \tau_2 = -1, \tau_3 = 3; \\ \xi_1 = 0.3, \xi_2 = 0.4, \xi_3 = 0.5; \\ \sigma = 0, \end{cases}$$

and initial values $\Gamma_0 = 6\sin(\pi\xi)$, $\hat{\Gamma}_0 = 4\cos(\pi\xi)$.

Figure 1 displays the solution of open-loop system (2). We can see that the solution of open-loop system (2) is growing fast and is unstable. Figure 2a,b display the solutions of "Γ-part" and "$\hat{\Gamma}$-part" of closed-loop system (90), respectively. Trajectory of the controller (89) is displayed in Figure 3. So it is clearly seen that the solution of closed-loop system (90) decays to zero and is stable under controller (89).

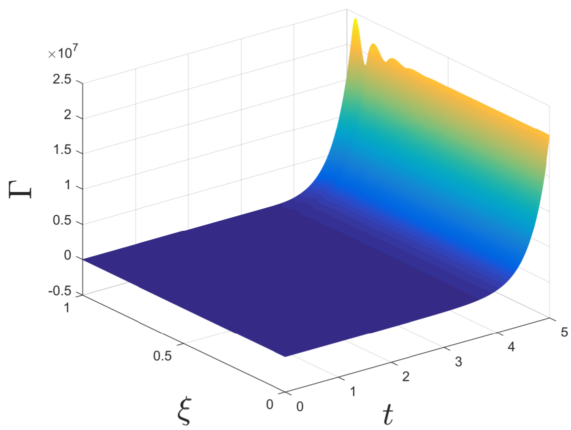

Figure 1. Solution of open-loop system (2).

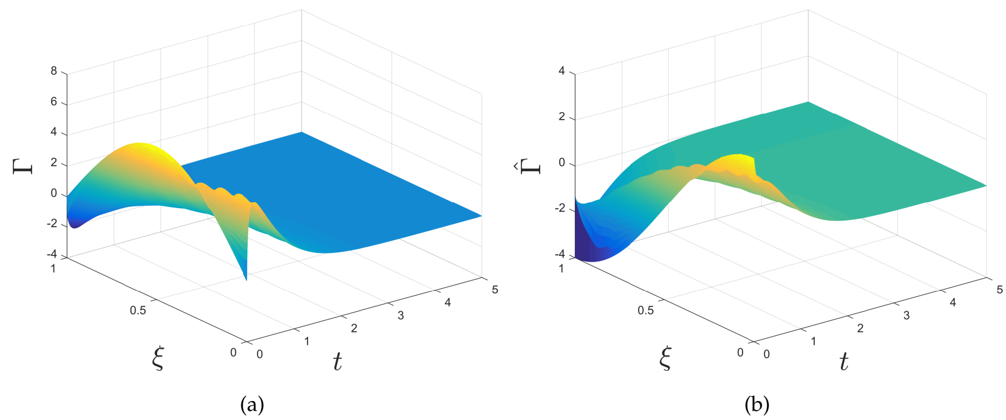

Figure 2. Solution of closed-loop system (90): (**a**) "Γ-part" of system (90); (**b**) "$\hat{\Gamma}$-part" of system (90).

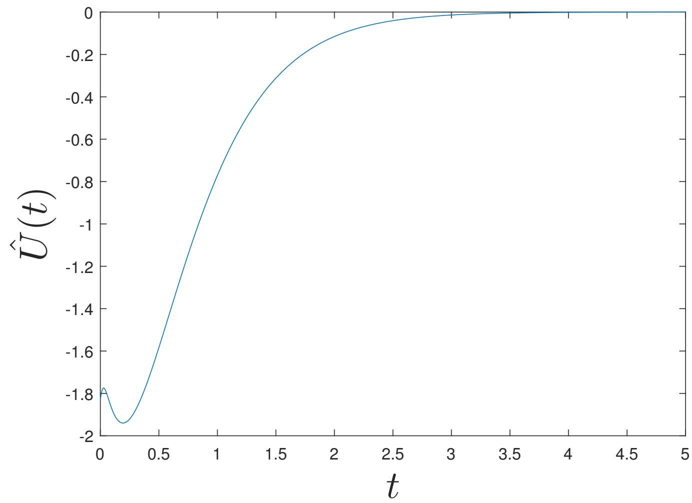

Figure 3. Trajectory of the controller (89).

5. Concluding Remarks

The paper is presented the boundary output feedback stabilization of a heat equation with multi-point heat source mainly by backstepping approach. We first design the state feedback controller based on backstepping transformation. The exponential stability of closed-loop system is guaranteed. Secondly, the observer-based output feedback controller is constructed for infinite-dimensional systems. Furthermore, we prove that closed-loop system is exponentially stable. In the future, we will consider the result without assumption (11) and the case that τ_i is a variable function or a more general continuous function hold.

Author Contributions: Methodology, F.-F.J.; investigation, B.-Q.Y.; formal analysis, Q.-Q.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the NSFC of China (62073203).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Masi, A.D.; Ferrari, P.A.; Lebowitz, J.L. Reaction-diffusion equations for interacting particle systems. *J. Stat. Phys.* **1986**, *44*, 589–644. [CrossRef]
2. Price, C.B.; Wambacq, P.; Oosterlinck, A. Applications of reaction-diffusion equations to image processing. In Proceedings of the Third International Conference on Image Processing and its Applications, Warwick, UK, 18–20 July 1989.
3. Illés, B.; Harsányi, G. Investigating direction characteristics of the heat transfer coefficient in forced convection reflow oven. *Exp. Therm. Fluid Sci.* **2009**, *33*, 642–650. [CrossRef]
4. Pardo, F.; López, P.; Cabello, D.; Balsi, M. Efficient software-hardware 3D heat equation solver with applications on the non-destructive evaluation of minefields. *Comput. Geosci.* **2009**, *35*, 2239–2249. [CrossRef]
5. Gao, Y.; Shao, Q.; Yan, B.Z.; Li, Q.F.; Guo, S.X. Parabolic equation modeling of electromagnetic wave propagation over rough sea surfaces. *Sensors* **2019**, *19*, 1252. [CrossRef] [PubMed]
6. Cheng, H.M.; Yuan, R. The spreading property for a prey-predator reaction-diffusion system with fractional diffusion. *Frac. Calc. Appl. Anal.* **2015**, *3*, 565–579. [CrossRef]
7. Guo, W.; Jin, F.F. Adaptive error feedback regulator design for one-dimensional heat equation with unknown harmonic disturbance anticollocated with control. *IEEE Trans. Autom. Control* **2020**, *2*, 824–830. [CrossRef]
8. Jin, F.F.; Guo, B.Z. Performance boundary output tracking for one-dimensional heat equation with boundary unmatched disturbance. *Automatica* **2018**, *96*, 1–10. [CrossRef]
9. Kumar, S.; Cao, J.D.; Li, X.D. A numerical method for time-fractional reaction-diffusion and integro reaction-diffusion equation based on quasi-wavelet. *Complexity* **2020**, *2020*, 3291723. [CrossRef]
10. Ma, T.F.; Yan, B.Q. Positive global solutions of nonlocal boundary value problems for the nonlinear convection reaction-diffusion equations. *Bound. Value Probl.* **2017**, *2017*, 9. [CrossRef]
11. Ren, Z.G.; Xu, C.; Zhou, Z.C.; Wu, Z.Z.; Chen, T.H. Boundary stabilization of a class of reaction-advection-diffusion systems via a gradient-based optimization approach. *J. Frankl. Inst.* **2019**, *356*, 173–195. [CrossRef]
12. Liu, W. Boundary feedback stabilization of an unstable heat equation. *SIAM J. Control Optim.* **2003**, *42*, 1033–1043. [CrossRef]
13. Wu, H.N.; Wang, J.W.; Li, H.X. Exponential stabilization for a class of nonlinear parabolic PDE systems via fuzzy control approach. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 318–329. [CrossRef]
14. Vazquez, R.; Krstic, M. Control of 1-D parabolic PDEs with Volterra nonlinearities, Part I: Design. *Automatica* **2008**, *44*, 2778–2790. [CrossRef]
15. Bošković, D.; Balogh, A.; Krstic, M. Backstepping in infinite dimension for a class of parabolic distributed parameter systems. *Math. Control Signal Syst.* **2003**, *16*, 44–75. [CrossRef]
16. Deutsche, J.; Gabrie, J. Fredholm backstepping control of coupled linear parabolic PDEs with input and output delays. *IEEE Trans. Autom. Control* **2014**, *65*, 3128–3135. [CrossRef]
17. Kerschbaum, S.; Deutscher, J. Backstepping control of coupled linear parabolic PDEs with space and time dependent coefficients. *IEEE Trans. Autom. Control* **2020**, *65*, 3060–3067. [CrossRef]
18. Sano, H. Neumann boundary stabilization of one-dimensional linear parabolic systems with input delay. *IEEE Trans. Autom. Control* **2018**, *63*, 3105–3111. [CrossRef]
19. Zhou, Z.C.; Guo, C.L. Stabilization of linear heat equation with a heat source at intermediate point by boundary control. *Automatica* **2013**, *49*, 448–456. [CrossRef]
20. Cerpa, E.; Coron, J.M. Rapid Stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition. *IEEE Trans. Autom. Control* **2013**, *58*, 1688–1695. [CrossRef]
21. Si, Y.C.; Xie, C.K.; Zhen, Z.Y.; Zhao, A.L. Local stabilization of coupled nonlinear parabolic equations by boundary control. *J. Frankl. Inst.* **2018**, *355*, 5592–5612. [CrossRef]
22. Yu, X.; Xu, C.; Chu, J. Local exponential stabilization of Fisher's equation using the backstepping technique. *Syst. Control Lett.* **2014**, *74*, 1–7. [CrossRef]
23. Jadachowski, L.; Meurer, T.; Kugi, A. Backstepping observers for periodic quasi-linear parabolic PDEs. In Proceedings of the Proceedings of the 19th World Congress the International Federation of Automatic Control, Cape Town, South Africa, 24–29 August 2014.
24. Benabdallah, A. Stabilization of a class of nonlinear uncertain ordinary differential equation by parabolic partial differential equation controller. *Int. J. Robust Nonlinear Control* **2020**, *30*, 3023–3038. [CrossRef]
25. Tang, S.X.; Xie, C.K. Stabilization for a coupled PDE-ODE control system. *J. Frankl. Inst.* **2011**, *348*, 2142–2155. [CrossRef]
26. Tang, S.X.; Xie, C.K.; Zhou, Z.C. Stabilization for a class of delayed coupled PDE-ODE systems with boundary control. In Proceedings of the 2011 Chinese Control and Decision Conference, Mianyang, China, 23–25 May 2011.
27. Wang, J.; Krstic, M. Output feedback boundary control of a heat PDE sandwiched between two ODEs. *IEEE Trans. Autom. Control* **2009**, *64*, 4653–4660. [CrossRef]
28. Baccoli, A.; Pisanoa, A.; Orlov, Y. Boundary control of coupled reaction-diffusion processes with constant parameters. *Automatica* **2015**, *54*, 80–90. [CrossRef]
29. Dai, J.G.; Ren, B.B. UDE-based robust boundary control for an unstable parabolic PDE with unknown input disturbance. *Automatica* **2018**, *93*, 363–368. [CrossRef]

30. Kang, W.; Guo, B.Z. Stabilization of unstable cascaded heat PDE system subject to boundary disturbance. *IET Control Theo. Appl.* **2016**, *10*, 1027–1039. [CrossRef]
31. Krstic, M.; Smyshlyaev, A. Adaptive boundary control for unstable parabolic PDEs-Part I: Lyapunov design. *IEEE Trans. Autom. Control.* **2008**, *53*, 1575–1591. [CrossRef]
32. Ragusa, M.A. Dirichlet problem associated to divergence form parabolic equations. *Commun. Contemp. Math.* **2005**, *7*, 551. [CrossRef]
33. Smyshlyaev, A.; Krstic, M. Adaptive boundary control for unstable parabolic PDEs-Part II: Estimation-based designs. *Automatica* **2007**, *43*, 1543–1556. [CrossRef]
34. Zhan, H.; Feng, Z. Stability of polytropic filtration equation with variable exponents. *Adv. Differ. Equ.* **2020**, *25*, 255–278. [CrossRef]
35. Wang, L.P.; Jin, F.F. Boundary output feedback stabilization of transport equation with non-local term. *IMA J. Math. Control Inf.* **2020**, *37*, 752–764. [CrossRef]
36. Argomedo, F.B.; Krstic, M. Backstepping-forwarding control and observation for hyperbolic PDEs with fredholm integrals. *IEEE Trans. Autom. Control* **2015**, *60*, 2145–2160.
37. Bernard, P.; Krstic, M. Adaptive output-feedback stabilization of non-local hyperbolic PDEs. *Automatica* **2014**, *50*, 2692–2699. [CrossRef]
38. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983. [CrossRef]
39. Su, L.L.; Guo, W.; Wang, J.M.; Krstic, M. Boundary stabilization of wave equation with velocity recirculation. *IEEE Trans. Autom. Control.* **2017**, *62*, 4760–4767. [CrossRef]