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Large Time Decay of Solutions to a Linear Nonautonomous System in Exterior Domains

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Abstract: In this expository paper, we study L^q-L^r decay estimates of the evolution operator generated by a perturbed Stokes system in n-dimensional exterior domains when the coefficients are time-dependent and can be unbounded at spatial infinity. By following the approach developed by the present author for the physically relevant case where the rigid motion of the obstacle is time-dependent, we clarify that some decay properties of solutions to the same system in whole space $\mathbb{R}^n$ together with the energy relation imply the desired estimates in exterior domains provided $n \geq 3$.

Keywords: decay estimate; evolution operator; perturbed Stokes system; exterior domain

1. Introduction

This paper studies the large time decay of solutions to the initial value problem for a linear nonautonomous system arising from fluid dynamics, specifically a viscous incompressible flow past an obstacle. Let Ω be an exterior domain in $\mathbb{R}^n$, $n \geq 3$, with $C^{1,1}$-boundary $\partial\Omega$. Complement $\mathbb{R}^n \setminus \Omega$ is identified with the obstacle (rigid body) immersed in a fluid, and it is assumed to be a compact set in $B_1(0)$ with nonempty interior. Towards the understanding of the stability or attainability of time-dependent Navier–Stokes flow past the obstacle whose motion could also be time-dependent, an essential step is to deduce some decay properties of a linearized system of form

$$\begin{aligned} \partial_t u - \Delta u + \nabla p - b(t) \cdot \nabla u + M(t)u = 0, \\ \operatorname{div} u = 0, \end{aligned} \tag{1}$$

in $\Omega \times (s, \infty)$, where vector field $u = (u_1(x,t), \cdots, u_n(x,t))^\top \in \mathbb{R}^n$ and scalar function $p = p(x,t) \in \mathbb{R}$ are unknowns denoting fluid velocity and pressure, while $b(x,t) \in \mathbb{R}^n$ and $M(x,t) \in \mathbb{R}^{n\times n}$ are prescribed functions. When $b = 0$ and $M = O$, (1) is just the well-known Stokes system. We consider perturbed Stokes system (1) subject to homogeneous Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \qquad u \to 0 \quad \text{as } |x| \to \infty \tag{2}$$

and initial condition

$$u(x,s) = f(x) \tag{3}$$

at initial time $s \geq 0$. The adjoint of the solution operator (evolution operator) $T(t,s)$: $f \mapsto u(t)$ to (1)–(3) provides solution operator $T(t,s)^*$: $g \mapsto v(s)$ to backward problem

$$\begin{aligned} -\partial_s v - \Delta v - \nabla\sigma + b(s) \cdot \nabla v + \Big[M(s)^\top + \big(\operatorname{div} b(s)\big)\,\mathbb{I}\Big] v = 0, \\ \operatorname{div} v = 0, \end{aligned} \tag{4}$$

in $\Omega \times [0,t)$, with $\mathbb{I}$ being the $n \times n$ identity matrix, where $v(y,s)$ and $\sigma(y,s)$ are unknowns, subject to

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$$v|_{\partial\Omega} = 0, \qquad v \to 0 \quad \text{as } |y| \to \infty \tag{5}$$

and final condition

$$v(y, t) = g(y) \tag{6}$$

at final time $t > 0$. In what follows, let us assume that $\operatorname{div} b = 0$ for simplicity. This condition is actually satisfied for a typical example, (9) below. Initial and final velocities f and g are taken from class $L^q_\sigma(\Omega)$ of solenoidal L^q-vector fields, $1 < q < \infty$, with vanishing normal trace at boundary $\partial\Omega$.

In [1,2], the present author established the L^q-L^r estimate

$$\|\nabla^j u(t)\|_{L^r(\Omega)} \le C(t-s)^{-(n/q-n/r)/2-j/2}\|f\|_{L^q(\Omega)}, \qquad t > s \ge 0, \quad j = 0, 1, \tag{7}$$

of solution $u(t)$ to (1)–(3) and the same estimate for (4)–(6) when $b = \eta(t) + \omega(t) \times x$ (rigid motion consisting of translation and rotation) and $Mu = \omega(t) \times u$ (one half of the Coriolis force) in 3D, where $1 < q \le r \le \infty$ $(q \ne \infty)$ for $j = 0$ and $1 < q \le r \le n = 3$ for $j = 1$. We interpret (7) as the linearized stability with definite rate. Since the only assumption on (η, ω) is

$$\eta,\ \omega \in C^\theta([0, \infty); \mathbb{R}^3) \cap L^\infty(0, \infty; \mathbb{R}^3) \tag{8}$$

with some $\theta \in (0, 1)$, the result of [1,2] completely recovers Estimate (7) for the Stokes and Oseen semigroups, those semigroups with rotating effect due to [3–7]. In order to study the attainability (relating to the celebrated starting problem raised by Finn [8]) and stability of the steady (or even time-periodic) Navier–Stokes flow, especially within the L^q framework, see [6,9–16], it is always crucial to make full use of (7) when $n \ge 3$, whereas even more linear analysis is needed in [17] by Maekawa for 2D cases. Since the equation is nonautonomous without any specific structure of (η, ω), such as time periodicity, one can no longer carry out spectral analysis that is standard as a strategy of obtaining time-decay estimates of semigroups for the autonomous case; see [3,5–7,11,18]. Moreover, as observed in [19,20], drift operator $(\omega \times x) \cdot \nabla$ with unbounded coefficient arising from the presence of rotation of the obstacle prevents the usual analysis on the basis of the theory of parabolic evolution operators; see, for instance, Tanabe [21], which corresponds to analytic semigroups for the autonomous case.

The aim of this expository paper is to clarify how one can deduce decay estimate (7) of the solution to (1)–(3) if we adapt the approach developed in [1,2]. A typical example that we have in mind is 3D case

$$b \cdot \nabla u = (\eta + \omega \times x - V) \cdot \nabla u, \qquad Mu = \omega \times u + u \cdot \nabla V \tag{9}$$

with $V = V(x, t)$ being time-dependent Navier–Stokes flow induced from rigid motion (8) of the obstacle. Roughly speaking, we make Assumptions (i)–(iii):

(i) Initial value problem (1)–(3) and backward adjoint problem (4)–(6) are well-posed in $L^q_\sigma(\Omega)$. A unique solution $u(t)$ to (1)–(3) satisfies smoothing estimate (7) only for $t - s \le \tau_*$ and

$$\|\partial_t u(t)\|_{W^{-1,q}(\Omega_3)} \le C(t-s)^{-\alpha}\|f\|_{L^q(\Omega)}, \qquad 0 \le s < t \le s + \tau_*, \tag{10}$$

with some $\alpha \in [1/2, 1)$, where $\tau_* \in (0, \infty)$ is arbitrary, constants C may depend on τ_*, and we set $\Omega_3 = \Omega \cap B_3(0)$. See (A3)–(A5) in the next section.

(ii) Initial value problem (1) in $\mathbb{R}^n \times (s, \infty)$ subject to (3) and backward adjoint problem (4) in $\mathbb{R}^n \times [0, t)$ subject to (6) are well-posed in $L^q_\sigma(\mathbb{R}^n)$. A unique solution $u(t)$ to the former enjoys (7) (both smoothing action and decay, in which Ω is replaced by $\mathbb{R}^n$) and

$$\|\partial_t u(t)\|_{W^{-1,q}(B_3)} \le C\|f\|_{L^q(\mathbb{R}^n)} \begin{cases} (t-s)^{-1/2}, & 0 < t - s \le 1, \\ (t-s)^{-n/2q}, & t - s > 1, \end{cases} \tag{11}$$

and we have the same estimates for the latter (backward problem), where $B_3 = B_3(0)$. See (A6)–(A7) in the next section.

(iii) Initial value problem (1)–(3) and backward adjoint problem (4)–(6) fulfil the energy relations with dissipation

$$\frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 + \delta \int_\tau^t \|\nabla u(\sigma)\|_{L^2(\Omega)}^2 \, d\sigma \leq \frac{1}{2}\|u(\tau)\|_{L^2(\Omega)}^2, \qquad t \geq \tau \geq s \geq 0, \tag{12}$$

$$\frac{1}{2}\|v(s)\|_{L^2(\Omega)}^2 + \delta \int_s^\tau \|\nabla v(\sigma)\|_{L^2(\Omega)}^2 \, d\sigma \leq \frac{1}{2}\|v(\tau)\|_{L^2(\Omega)}^2, \qquad t \geq \tau \geq s \geq 0, \tag{13}$$

with some $\delta > 0$, where the second term of each exhibits the dissipative effect. See (A8) in the next section.

Several comments on Assumptions (i)–(iii) above are in order. The well-posedness in Assumption (i), in other words, the generation of the evolution operator, is never obvious; however, this is a different issue from what we address in this paper. When $b = \eta + \omega \times x$ and $Mu = \omega \times u$ with (8), the generation of the evolution operator with (7) for $0 \leq s < t \leq \mathcal{T}$ was successfully proved by Hansel and Rhandi [22] for every $\mathcal{T} \in (0, \infty)$; then, it was verified by [1] that constant $C = C(\tau_*)$ in (7) can be taken uniformly in (t,s) with $t - s \leq \tau_*$, and by [2] that smoothing estimate (10) of $\partial_t u(t)$ near the obstacle holds with $\alpha = (1 + 1/q)/2$. The latter estimate is closely related to the asymptotic behavior of pressure (in a bounded domain near the obstacle) and very crucial in [2] on account of the lack of smoothing action exhibited by analytic semigroups since the evolution operator is not parabolic. The remarkable smoothing rate $\alpha = (1 + 1/q)/2$ was already found by [6,7,23] for the Stokes and Oseen semigroups with rotating effect, and it was a slight improvement of the rate deduced by Noll ans Saal [24] in another context. Assumption (ii) on the L^q-L^r estimate (7) for the whole space problem is nontrivial, but it is the starting point of analysis in this paper. When $b = \eta + \omega \times x$ and $Mu = \omega \times u$ with (8), the solution in the whole space can be explicitly described in terms of the heat semigroup in which a change of variable is made, so that Estimates (7) and (11) are in fact available. Energy Relations (12) and (13) in (iii) are reasonable assumptions that play several roles, especially for deduction of $(7)_{j=0}$. When $b = \eta + \omega \times x$ and $Mu = \omega \times u$, (12) and (13) are obvious on account of the skew–symmetry of $b \cdot \nabla u - Mu$ without any smallness condition. Concerning case (9) as well, we can easily see (12) and (13), provided that V is small enough in $L^\infty(0,\infty; L^{n,\infty}(\Omega))$, see, for instance, [25], where $L^{n,\infty}$ is the weak-L^n space (a Lorentz space). Except for Assumption (iii), we do not have useful higher energy estimates, which play an important role in [4] by Maremonti and Solonnikov for the Stokes semigroup.

As the substitution of analysis of a parametrix of the resolvent in exterior domains for the autonomous case, the key of our approach is how we make use of energy Relations (12) and (13) to deduce $(7)_{j=0}$, see Proposition 1. Here and in what follows, by $(7)_{j=0}$ we denote estimate (7) with $j = 0$. Case $r = q$ (uniform boundedness) for large $t - s > 0$ is our main task, yielding the other cases by use of the energy inequality of the differential form. It is reasonable under Assumption (ii) to regard the solution to (1)–(3) as a perturbation from (a modification of) $\mathbb{R}^n$–flow by means of a cut-off procedure. The desired uniformly boundedness of the perturbation is discussed by a bootstrap argument and by duality argument with the aid of Assumption (iii) on the energy relations, and the use of duality is why we need to simultaneously study adjoint evolution operator $T(t,s)^*$ with $T(t,s)$. Unfortunately, this step does not work when $n = 2$. If Dirichlet Condition (2) is replaced by another boundary condition, a core part of this step does not follow, even for $n \geq 3$.

With $(7)_{j=0}$ at hand, we are able to proceed to the decay estimates of $\|\partial_t u(t)\|_{W^{-1,q}(\Omega_3)}$ and $\|u(t)\|_{W^{1,q}(\Omega_3)}$ near the obstacle that we call the local energy decay, as in several papers for the autonomous case. Among other papers, the method of local energy decay is traced back to [3] by Iwashita on the Stokes semigroup in the context of mathematical fluid dynamics, and the origin would be even back to studies of hyperbolic equations with dissipation by Shibata. The final step with another cut-off procedure is to derive the decay estimate of $\|\nabla u(t)\|_{L^q(\mathbb{R}^n\setminus B_3)}$ near spatial infinity by the use of Assumption (ii) combined with the local energy decay obtained in the preceding step. The cut-off remainder consists of several terms; among them, two terms are delicate: one is $\partial_t u(t)$, and the other is pressure. What we need is both decay for $t-s\to\infty$ and smoothing rate for $t-s\to 0$ of those terms; see (78). The latter of the temporal derivative is the assumption (10) in (i), and the deduction of (10) was actually one of the main tasks in [2] on case $b=\eta+\omega\times x$ and $Mu=\omega\times u$ with (8). For more general case, as in [2], we have to look into details about the construction of a parametrix of the evolution operator to verify (10). If it is constructed with the use of evolution operators for the whole space problem and for the interior one near the obstacle by a cut-off technique as in [22], smoothing rate (10) of $\partial_t u(t)$ is determined by the one of pressure for the interior problem.

To sum up, we claim that some decay properties of solutions to the same system in whole space $\mathbb{R}^n$ together with the energy relation imply the desired estimates in exterior domains provided $n\geq 3$, and that we need to find (10) through analysis of pressure to justify this statement. Let us close the introductory section with a remark on Case (9). With the results for case $V=0$ obtained in [1,2] at hand, it is actually possible to show the stability or attainability of scale-critical Navier–Stokes flow $V\in L^\infty(0,\infty;\,L^{n,\infty}(\Omega))$ by an interpolation technique due to Yamazaki [26] as long as it is small enough; see, for instance, Takahashi [15] on the attainability of steady flow for the purely rotating case. However, we have less information about the asymptotic behavior of disturbance. If we intend to show some decay properties of gradient of the disturbance, we have to know $(7)_{j=1}$ for Problem (1)–(3) with (9). If we adapt the approach developed in ([25], Section 4) to case (9) with the aforementioned V that is sufficiently small, we could verify Assumption (ii), but only partially.

In the next section, we precisely formulate the problem and provide the main theorem. Section 3 is devoted to the proof. We close the paper with a conclusion in the final section.

2. Result

Let us fix the notation. Given a domain $D\subset\mathbb{R}^n$, $q\in[1,\infty]$ and integer $k\geq 0$, the standard Lebesgue and Sobolev spaces are denoted by $L^q(D)$ and by $W^{k,q}(D)$. We abbreviate norm $\|\cdot\|_{q,D}=\|\cdot\|_{L^q(D)}$ and even $\|\cdot\|_q=\|\cdot\|_{q,\Omega}$, where Ω is the exterior domain with $C^{1,1}$-boundary $\partial\Omega$ under consideration. We assumed that $\mathbb{R}^n\setminus\Omega\subset B_1$, where $B_\rho=B_\rho(0)$ denotes the open ball centered at the origin with radius $\rho>0$. We set $\Omega_\rho=\Omega\cap B_\rho$ for $\rho\geq 1$. By $C_0^\infty(D)$ we denote the class of all C^∞ functions with compact support in D, and by $W_0^{k,q}(D)$ the completion of $C_0^\infty(D)$ in $W^{k,q}(D)$. We set $W^{-1,q}(D)=W_0^{1,q'}(D)^*$, where $1/q'+1/q=1$ and $q\in(1,\infty)$. By $\langle\cdot,\cdot\rangle_D$ we denote various duality pairings over domain D. In what follows, we use the same symbols for denoting the scalar and vector functions if there is no confusion. Let X be a Banach space. Then, $\mathcal{L}(X)$ stands for the Banach space consisting of all bounded linear operators from X into itself.

Let $D\in\{\Omega,\mathbb{R}^n\}$, where Ω is the exterior domain under consideration. Class $C_{0,\sigma}^\infty(D)$ consists of all solenoidal vector fields in $C_0^\infty(D)$. Let $1<q<\infty$. We define space $L_\sigma^q(D)$ by the completion of $C_{0,\sigma}^\infty(D)$ in $L^q(D)$. For $D=\Omega$, it is characterized as

$$L_\sigma^q(\Omega)=\{u\in L^q(\Omega);\ \mathrm{div}\ u=0,\ \nu\cdot u|_{\partial\Omega}=0\},$$

where ν stands for the outer unit normal to $\partial\Omega$. When $D=\mathbb{R}^n$, the boundary condition in the sense of a normal trace is absent. The space of the L^q-vector fields admits Helmholtz decomposition

$$L^q(D)=L_\sigma^q(D)\oplus\{\nabla p\in L^q(D);\ p\in L_{\mathrm{loc}}^q(\overline{D})\},$$

see [27–29]; Simader and Sohr [29] established decomposition under condition $\partial\Omega\in C^1$ when $D=\Omega$. By $P_D=P_{D,q}:L^q(D)\to L_\sigma^q(D)$ we denote the Fujita–Kato projection associated with the decomposition above. We then observe that $P_D\in\mathcal{L}(W^{1,q}(D))$ as well as $P_D\in\mathcal{L}(L^q(D))$. Note the duality relation $(P_{D,q})^*=P_{D,q'}$ and $L_\sigma^q(D)^*=L_\sigma^{q'}(D)$, where

$1/q' + 1/q = 1$. We simply write $P = P_\Omega$ for exterior domain Ω under consideration. We easily see that $P_{\mathbb{R}^n} = I + \mathcal{R} \otimes \mathcal{R}$, where $\mathcal{R} = \nabla(-\Delta)^{-1/2}$ is the Riesz transform.

Suppose that

- (A1) The coefficients of (1)

$$(x,t) \mapsto b(x,t) \in \mathbb{R}^n, \qquad (x,t) \mapsto M(x,t) \in \mathbb{R}^{n\times n}$$

are measurable in $x \in \mathbb{R}^n$ and continuous in $t \geq 0$ with the property

$$\|(b,M)\| := \sup\left\{\frac{|b(x,t)|}{1+|x|} + |M(x,t)|_{\mathbb{R}^{n\times n}};\ x \in \mathbb{R}^n,\ t \geq 0\right\} < \infty, \tag{14}$$

where $|\cdot|$ denotes the $\mathbb{R}^n$–norm. For simplicity, $b(t)$ is assumed to be a solenoidal vector field, that is, div $b(t) = 0$ in the sense of distributions for each $t \geq 0$.

Let $1 < q < \infty$. For $D \in \{\Omega, \mathbb{R}^n\}$ and (b, M) given above, we introduce linear operator $L_{D,b,M}(t)$ on $L^q_\sigma(\Omega)$ by

$$\begin{cases} D_q(L_{D,b,M}(t)) = \{u \in W^{2,q}(D) \cap W^{1,q}_0(D) \cap L^q_\sigma(D);\ b(t)\cdot\nabla u \in L^q(D)\}, \\ L_{D,b,M}(t)u = -P_D[\Delta u + b(t)\cdot\nabla u - M(t)u]. \end{cases} \tag{15}$$

Then initial value Problem (1)–(3) and backward adjoint problem (4)–(6) (with div b = 0) are formulated, respectively, as

$$\frac{du}{dt} + L_{\Omega,b,M}(t)u = 0, \quad t \in (s,\infty); \qquad u(s) = f, \tag{16}$$

$$-\frac{dv}{ds} + L_{\Omega,-b,M^\top}(s)v = 0, \quad s \in [0,t); \qquad v(t) = g. \tag{17}$$

It is easily seen that

$$\langle L_{D,b,M}(t)u, v\rangle_D = \langle u, L_{D,-b,M^\top}(t)v\rangle_D \tag{18}$$

for all $u \in D_q(L_{D,b,M}(t))$ and $v \in D_{q'}(L_{D,-b,M^\top}(t))$, where $1/q' + 1/q = 1$. In fact, we fix $\zeta \in C^\infty([0,\infty))$ such that $\zeta(\rho) = 1$ for $\rho \leq 1$ and $\zeta(\rho) = 0$ for $\rho \geq 2$, and set $\phi_R(x) = \zeta(|x|/R)$ for $R \geq 1$; then, we find

$$\int_D \{(b\cdot\nabla u)\cdot v + u\cdot(b\cdot\nabla v)\}\phi_R\,dx = -\int_{R<|x|<2R}(u\cdot v)b\cdot\nabla\phi_R\,dx.$$

By (14) and by $u\cdot v \in L^1(D)$, passing to the limit as $R \to \infty$ justifies

$$\int_D (b\cdot\nabla u)\cdot v\,dx = -\int_D u\cdot(b\cdot\nabla v)\,dx \tag{19}$$

for $u \in D_q(L_{D,b,M}(t))$ and $v \in D_{q'}(L_{D,-b,M^\top}(t))$, yielding (18) and, therefore, $L_{D,-b,M^\top}(t) \subset L_{D,b,M}(t)^*$, where the adjoint is well-defined since $L_{D,b,M}(t)$ is densely defined. At this point, we need the maximality in the sense that

- (A2) For each $t \geq 0$ there is $\lambda(t) > 0$ such that both $\lambda(t) + L_{D,b,M}(t)$ and $\lambda(t) + L_{D,-b,M^\top}(t)$ are surjective, where $D \in \{\Omega, \mathbb{R}^n\}$.

This means the solvability of the associated elliptic problem, which implies that $\lambda(t) + L_{D,b,M}(t)^*$ is injective. Under this condition, (18) leads to duality relation

$$L_{D,-b,M^\top}(t) = L_{D,b,M}(t)^*. \tag{20}$$

Hence, $L_{D,b,M}(t) = L_{D,-b,M^\top}(t)^*$ is a closed operator. In what follows, given (b, M), we abbreviate $L_D(t) = L_{D,b,M}(t)$ and even $L(t) = L_\Omega(t)$. For case $b = \eta(t) + \omega(t) \times x$

and $Mu = \omega(t) \times u$, Condition (A2) follows from [23] by Shibata along with the Mozzi–Chasles transform ([30], Chapter VIII) (to reduce the problem to the particular case when the translational and angular velocities are parallel each other).

As in [2,22], auxiliary spaces

$$\begin{aligned} Y_q(D) &= \{u \in W^{2,q}(D) \cap W_0^{1,q}(D) \cap L^q_\sigma(D);\ |x|\nabla u \in L^q(D)\}, \\ Z_q(D) &= \{u \in W^{1,q}(D) \cap L^q_\sigma(D);\ |x|\nabla u \in L^q(D)\}, \end{aligned} \tag{21}$$

for $D \in \{\Omega, \mathbb{R}^n\}$ play a role to describe the regularity of solutions, since domain $D_q(L_D(t))$ varies as t goes on. Note that $C^\infty_{0,\sigma}(D) \subset Y_q(D) \subset D_q(L_D(t))$ for every $t \geq 0$, and that the homogeneous Dirichlet condition at $\partial\Omega$ is not involved in the space $Z_q(\Omega)$.

We further make Assumptions (A3)–(A5):

- (A3) Let $1 < q < \infty$. Operator family $\{L(t)\}_{t\geq 0}$ generates an evolution operator $\{T(t,s)\}_{t\geq s\geq 0}$ on $L^q_\sigma(\Omega)$ such that $T(t,s)$ is a bounded linear operator from $L^q_\sigma(\Omega)$ into itself with the semigroup property

$$T(t,\tau)T(\tau,s) = T(t,s) \quad (t \geq \tau \geq s \geq 0); \qquad T(s,s) = I, \tag{22}$$

in $\mathcal{L}(L^q_\sigma(\Omega))$ and that the map

$$\{t \geq s \geq 0\} \ni (t,s) \mapsto T(t,s)f \in L^q_\sigma(\Omega)$$

is continuous for every $f \in L^q_\sigma(\Omega)$. Moreover, we have the following properties:

1. Let $n' < q \leq r < \infty$, where $1/n' + 1/n = 1$. For every $f \in Z_q(\Omega)$ and $t \in (s,\infty)$, we have

$$T(t,s)f \in Y_q(\Omega) \cap Z_r(\Omega) \tag{23}$$

and

$$T(\cdot,s)f \in C^1((s,\infty);\ L^q_\sigma(\Omega))$$

with

$$\partial_t T(t,s)f + L(t)T(t,s)f = 0, \qquad t \in (s,\infty), \tag{24}$$

in $L^q_\sigma(\Omega)$ (condition $q > n'$ is consistent with Lemma 3);

2. Let $1 < q < \infty$. For every $f \in Y_q(\Omega)$, we have

$$T(t,\cdot)f \in C^1([0,t];\ L^q_\sigma(\Omega))$$

with

$$\partial_s T(t,s)f = T(t,s)L(s)f, \qquad s \in [0,t], \tag{25}$$

in $L^q_\sigma(\Omega)$.

- (A4) Let $1 < q < \infty$ and $q \leq r < \infty$. Given $\tau_* \in (0,\infty)$ and $m \in (0,\infty)$, there is a constant $c_1 = c_1(\tau_*, m, q, r, n, \Omega) > 0$, such that evolution operator $T(t,s)$ satisfies

$$\|\nabla^j T(t,s)f\|_r \leq c_1(t-s)^{-(n/q-n/r)/2-j/2}\|f\|_q \tag{26}$$

for all $(t,s) \in \Lambda(\tau_*)$, $f \in L^q_\sigma(\Omega)$ and $j = 0,1$ whenever $\|(b,M)\| \leq m$, where

$$\Lambda(\tau_*) = \{(t,s);\ t > s \geq 0,\ t-s \leq \tau_*\}. \tag{27}$$

In addition, for every $f \in L^q_\sigma(\Omega)$, we have

$$T(\cdot,s)f \in C^1((s,\infty);\ W^{-1,q}(\Omega_3))$$

and there is a constant $\alpha \in [1/2, 1)$ with the following property: Given $\tau_* \in (0, \infty)$ and $m \in (0, \infty)$, there is a constant $c_2 = c_2(\tau_*, m, q, n, \Omega) > 0$ such that

$$\|\partial_t T(t,s)f\|_{W^{-1,q}(\Omega_3)} \le c_2(t-s)^{-\alpha}\|f\|_q \tag{28}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in L^q_\sigma(\Omega)$ whenever $\|(b, M)\| \le m$.

- (A5) Let $1 < q < \infty$. Given $t > 0$, operator family $\{L(t-\tau)^*\}_{\tau\in[0,t]}$ (see (20)) generates an evolution operator $\{\widetilde{T}(\tau, s; t)\}_{0\le s\le\tau\le t}$ on $L^q_\sigma(\Omega)$ in the same sense as in Assumption (A3) (with obvious change).

Assumption (A5) is related to the solution operator to backward adjoint problem (17). In fact, we set

$$S(t,s) := \widetilde{T}(t-s, 0; t), \qquad t \ge s \ge 0, \tag{29}$$

then, for every $g \in Z_q(\Omega)$ and $s \in [0, t)$, we have

$$S(t,s)g \in Y_q(\Omega) \cap Z_r(\Omega), \tag{30}$$

where $n' < q \le r < \infty$, as well as

$$S(t, \cdot)g \in C^1([0,t);\, L^q_\sigma(\Omega))$$

with

$$-\partial_s S(t,s)g + L(s)^* S(t,s)g = 0, \qquad s \in [0,t), \tag{31}$$

in $L^q_\sigma(\Omega)$. Given $f, g \in C^\infty_{0,\sigma}(\Omega)$, we compute $\partial_\tau \langle T(\tau,s)f, S(t,\tau)g\rangle_\Omega$ with use of (24) and (31) to find

$$\langle T(t,s)f, g\rangle_\Omega = \langle f, S(t,s)g\rangle_\Omega \tag{32}$$

where $\langle\cdot,\cdot\rangle_\Omega$ should be understood for the pair of $L^{q'}_\sigma(\Omega)$ and $L^q_\sigma(\Omega)$ with $q \in (n', n)$ in this computation; nevertheless, once we have (32), continuity argument justifies (32) for every $q \in (1, \infty)$, $f \in L^q_\sigma(\Omega)$ and $g \in L^{q'}_\sigma(\Omega)$ as well. We thus verified duality relation

$$T(t,s)^* = S(t,s), \qquad S(t,s)^* = T(t,s) \tag{33}$$

in $\mathcal{L}(L^q_\sigma(\Omega))$ for $t \ge s \ge 0$, which, together with (22), leads to backward semigroup property

$$T(\tau,s)^* T(t,\tau)^* = T(t,s)^* \quad (t \ge \tau \ge s \ge 0); \qquad T(t,t)^* = I, \tag{34}$$

in $\mathcal{L}(L^q_\sigma(\Omega))$. By duality, it follows from $(26)_{j=0}$ that

$$\|T(t,s)^* g\|_r \le c_1'(t-s)^{-(n/q-n/r)/2}\|g\|_q \tag{35}$$

for all $(t,s) \in \Lambda(\tau_*)$, see (27), and $g \in L^q_\sigma(\Omega)$ whenever $\|(b,M)\| \le m$, where $1 < q \le r < \infty$ and $c_1' = c_1(\tau_*, m, r', q', n, \Omega)$. If one wishes to claim (43) below for $\nabla T(t,s)^*$ as well, additional assumptions $(26)_{j=1}$ and (28) in which $T(t,s)$ is replaced by $T(t,s)^*$ are needed.

We next make the following assumption on the evolution operators for the whole space problem.

- (A6) Let $1 < q < \infty$. Operator family $\{L_{\mathbb{R}^n}(t)\}_{t\ge0}$ generates an evolution operator $\{T_{\mathbb{R}^n}(t,s)\}_{t\ge s\ge0}$ on $L^q_\sigma(\mathbb{R}^n)$ in the same sense as in (A3) (with obvious change). Given $t > 0$, operator family $\{L_{\mathbb{R}^n}(t-\tau)^*\}_{\tau\in[0,t]}$ (see (20)) generates an evolution operator $\{\widetilde{T}_{\mathbb{R}^n}(\tau,s;t)\}_{0\le s\le\tau\le t}$ on $L^q_\sigma(\mathbb{R}^n)$ in the same sense as in Assumptions (A3) and (A5) (with obvious change).

In the same manner as for the exterior problem mentioned above, we observe

$$T_{\mathbb{R}^n}(t,s)^* = \widetilde{T}_{\mathbb{R}^n}(t-s, 0; t), \qquad t \ge s \ge 0.$$

We need the following estimates of $T_{\mathbb{R}^n}(t,s)^*$ as well as $T_{\mathbb{R}^n}(t,s)$.

- (A7) Let $1<q<\infty$ and $q\le r\le\infty$ ($r=\infty$ is not needed for case $j=1$ below). Given $m\in(0,\infty)$, there is a constant $c_3=c_3(m,q,r,n)>0$, such that evolution operator $T_{\mathbb{R}^n}(t,s)$ satisfies

$$\|\nabla^j T_{\mathbb{R}^n}(t,s)f\|_{r,\mathbb{R}^n}\le c_3(t-s)^{-(n/q-n/r)/2-j/2}\|f\|_{q,\mathbb{R}^n} \tag{36}$$

for all $t>s\ge 0$, $f\in L^q_\sigma(\mathbb{R}^n)$ and $j=0,1$ whenever $\|(b,M)\|\le m$. In addition, for every $f\in L^q_\sigma(\mathbb{R}^n)$, we have

$$T_{\mathbb{R}^n}(\cdot,s)f\in C^1((s,\infty);\,W^{-1,q}(B_3))$$

and, given $m\in(0,\infty)$, there is a constant $c_4=c_4(m,q,n)>0$, such that

$$\|\partial_t T_{\mathbb{R}^n}(t,s)f\|_{W^{-1,q}(B_3)}\le c_4\|f\|_{q,\mathbb{R}^n}\begin{cases}(t-s)^{-1/2}, & 0<t-s\le 1,\\ (t-s)^{-n/2q}, & t-s>1,\end{cases} \tag{37}$$

for all $f\in L^q_\sigma(\mathbb{R}^n)$ whenever $\|(b,M)\|\le m$. We have the same estimates, (36) and (37), for adjoint $T_{\mathbb{R}^n}(t,s)^*$ as well.

Lastly, we suppose

- (A8) There is a constant $\delta>0$, such that

$$\langle L(t)u,u\rangle_\Omega=\langle u,L(t)^*u\rangle_\Omega\ge\delta\|\nabla u\|_2^2 \tag{38}$$

for all $u\in D_2(L(t))=D_2(L(t)^*)$ and $t\ge 0$.

This, combined with (24) and (31), implies energy relations

$$\frac{d}{dt}\frac{1}{2}\|T(t,s)f\|_2^2+\delta\|\nabla T(t,s)f\|_2^2\le 0, \tag{39}$$

$$-\frac{d}{ds}\frac{1}{2}\|T(t,s)^*g\|_2^2+\delta\|\nabla T(t,s)^*g\|_2^2\le 0, \tag{40}$$

and, thereby,

$$\frac{1}{2}\|T(t,s)f\|_2^2+\delta\int_\tau^t\|\nabla T(\sigma,s)f\|_2^2\,d\sigma\le\frac{1}{2}\|T(\tau,s)f\|_2^2,\qquad t\ge\tau\ge s\ge 0, \tag{41}$$

$$\frac{1}{2}\|T(t,s)^*g\|_2^2+\delta\int_s^\tau\|\nabla T(t,\sigma)^*g\|_2^2\,d\sigma\le\frac{1}{2}\|T(t,\tau)^*g\|_2^2,\qquad t\ge\tau\ge s\ge 0, \tag{42}$$

as long as $f,\,g\in Z_2(\Omega)$. On account of (19), condition

$$\left|\int_\Omega(Mu)\cdot u\,dx\right|\le c_5\|\nabla u\|_2^2$$

with some $c_5\in[0,1)$ independent of $u\in W^{1,2}_0(\Omega)$ is sufficient for (38) with $\delta=1-c_5$. This is accomplished for case (9) with small $V\in L^\infty(0,\infty;\,L^{n,\infty}(\Omega))$ by the Lorentz-Hölder inequality and the Lorentz–Sobolev embedding relation.

We are now in a position to give our main theorem.

Theorem 1. *Let $n\ge 3$. Suppose (A1)–(A8). Given $m\in(0,\infty)$, there exists a constant $C=C(m,q,r,n,\Omega)>0$, such that*

$$\|\nabla^j T(t,s)f\|_r\le C(t-s)^{-\kappa(j)}\|f\|_q \tag{43}$$

for all (t,s) *with* $t-s>1$ *and* $f\in L^q_\sigma(\Omega)$ *whenever* $\|(b,M)\|\le m$, *see* (14), *where*

$$\kappa(j) := \min\left\{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)+\frac{j}{2},\ \frac{n}{2q}\right\}, \quad 1<q\le r<\infty, \quad j=0,1 \tag{44}$$

and $r=\infty$ *is also allowed for* $j=0$.

Remark 1. *When* $b=0$ *and* $M=O$ *(Stokes semigroup), decay rate* (44) *is optimal even for case* $j=1$; *see [4,31].*

Remark 2. *For case* $b=\eta+\omega\times x$ *and* $Mu=\omega\times u$ *with* (8) *studied in [1,2], the global Hölder continuity condition is needed to prove Assumptions (A3)–(A5), so that constants* c_1 *and* c_2 *(see* (26) *and* (28)*) involve the Hölder seminorm of* (η,ω), *which is thereby hidden in* (43) *through* c_1, c_2. *If we consider* (η,ω) *of which the Hölder seminorm (and* L^∞*-norm) is bounded from above by* $m>0$, *constant* C *in* (43) *can be taken uniformly with respect to such* (η,ω).

Remark 3. *It is clearly hopeless to verify (A2)–(A8) for* (b,M), *which merely satisfies (A1). The verification of those conditions for* (b,M) *with more properties such as Hölder continuity (in* $t\ge 0$*) mentioned in Remark 2 is an important but different issue, and it very much depends on how a parametrix of evolution operator* $T(t,s)$ *is constructed. For case* $b=\eta+\omega\times x$ *and* $Mu=\omega\times u$ *with* (8), *Conditions (A1)–(A8) are met due to [1,2,22].*

3. Proof of Theorem 1

We start with a quite elementary but useful lemma on optimal growth rate of the integral (see (46) below) under a condition on the decay of the square integral.

Lemma 1. *Let* $\mu\in(0,1)$. *Suppose that* $z=z(\tau)$ *is a real-valued function in* $L^2_{\rm loc}((1,\infty))$, *and that*

$$\int_{1+\sigma}^{1+2\sigma} z(\tau)^2\,d\tau \le K\sigma^{-\mu} \tag{45}$$

for all $\sigma>0$ *with some constant* $K>0$. *Then, we have* $z\in L^1_{\rm loc}([1,\infty))$ *with*

$$\int_1^{1+\sigma} |z(\tau)|\,d\tau \le C\sqrt{K}\sigma^{(1-\mu)/2} \tag{46}$$

for all $\sigma>0$ *with some constant* $C=C(\mu)>0$.

Proof. From (45), it follows that

$$\int_{1+\sigma}^{1+2\sigma} |z(\tau)|\,d\tau \le \sqrt{K}\sigma^{(1-\mu)/2}$$

for all $\sigma>0$, which leads to (46) since

$$\int_1^{1+\sigma} |z(\tau)|\,d\tau = \sum_{j=0}^\infty \int_{1+\sigma/2^{j+1}}^{1+\sigma/2^j} |z(\tau)|\,d\tau \le \sqrt{K}\sigma^{(1-\mu)/2}\sum_{j=0}^\infty \left(2^{(\mu-1)/2}\right)^{j+1}.$$

The proof is complete. □

Our proof of (43) with $j=0$ is based on the following lemma with the aid of energy Relations (39)–(42).

Lemma 2. *Suppose (A1)–(A5) and (A8). Let* $r_0\in(2,\infty)$ *and* $1/r_0'+1/r_0=1$.

1. *Assume that, given* $m\in(0,\infty)$,

$$\|T(t,s)f\|_{r_0}\le C\|f\|_{r_0}, \qquad t-s>3, \tag{47}$$

for all $f \in C^\infty_{0,\sigma}(\Omega)$ with some constant $C = C(m, n, \Omega) > 0$ whenever $\|(b, M)\| \le m$. Then, we have (43)$_{j=0}$ for all $t > s \ge 0$ and $f \in L^q_\sigma(\Omega)$ provided $2 \le q \le r \le r_0$ as well as

$$\|T(t,s)^* g\|_r \le C(t-s)^{-(n/q-n/r)/2}\|g\|_q \tag{48}$$

for all $t > s \ge 0$ and $g \in L^q_\sigma(\Omega)$ with some constant $C = C(m, q, r, n, \Omega) > 0$ provided $r_0' \le q \le r \le 2$, whenever $\|(b, M)\| \le m$.

2. *Assume that, given $m \in (0, \infty)$,*

$$\|T(t,s)^* g\|_{r_0} \le C\|g\|_{r_0}, \qquad t-s > 3, \tag{49}$$

for all $g \in C^\infty_{0,\sigma}(\Omega)$ with some constant $C = C(m, n, \Omega) > 0$ whenever $\|(b, M)\| \le m$. Then, we have (48) for all $t > s \ge 0$ and $g \in L^q_\sigma(\Omega)$ provided $2 \le q \le r \le r_0$ as well as (43)$_{j=0}$ for all $t > s \ge 0$ and $f \in L^q_\sigma(\Omega)$ provided $r_0' \le q \le r \le 2$, whenever $\|(b, M)\| \le m$.

Proof. It suffices to show (48). The other part of the first assertion follows from duality, and the second assertion is similarly proved. By (26)$_{j=0}$ with $q = r = r_0$ for $t - s \le 3$ and by duality together with the energy relation (42) with $\tau = t$, we obtain (48) with $r = q$ for all $q \in [r_0', 2]$, $t > s \ge 0$ and $g \in L^q_\sigma(\Omega)$. We arbitrarily fix $t > 0$ and $q \in [r_0', 2)$, and set $v(s) = T(t,s)^* g$ for $g \in C^\infty_{0,\sigma}(\Omega) \setminus \{0\}$. We then find

$$\|v(s)\|_2 \le \|v(s)\|_q^\theta \|v(s)\|_{2_*}^{1-\theta} \le C\|g\|_q^\theta \|\nabla v(s)\|_2^{1-\theta} \tag{50}$$

where $1/2 = \theta/q + (1-\theta)/2_*$ and $1/2_* = 1/2 - 1/n$. As in ([4], Section 5), we only have to solve differential inequality

$$\frac{d}{ds}\|v(s)\|_2^2 \ge \frac{C\|v(s)\|_2^{2/(1-\theta)}}{\|g\|_q^{2\theta/(1-\theta)}} \qquad (0 \le s < t)$$

that follows from (50) combined with energy Relation (40) to furnish

$$\|v(s)\|_2 \le C(t-s)^{-(n/q-n/2)/2}\|g\|_q$$

for all $t > s \ge 0$. This, together with the uniform boundedness in $L^q_\sigma(\Omega)$ implies (48) with $r_0' \le q \le r \le 2$. □

The following proposition provides us with (43)$_{j=0}$ for all $t > s \ge 0$ except the case $r = \infty$. Note that $\kappa(0) = (n/q - n/r)/2$.

Proposition 1. *Suppose* (A1)–(A8). *The assertion of Theorem 1 is true when $j = 0$ and $r < \infty$.*

Proof. Given $f \in C^\infty_{0,\sigma}(\Omega)$, we set $u(t) = T(t,s)f$ and denote by $p(t)$ the pressure associated with $u(t)$. Let us take $T_{\mathbb{R}^n}(t,s)f$ with $T_{\mathbb{R}^n}(t,s)$ being the evolution operator in the whole space $\mathbb{R}^n$ given by (A6) and single out associated pressure $p_{\mathbb{R}^n}(t)$, such that

$$\int_{B_3} p_{\mathbb{R}^n}(x,t)\,dx = 0. \tag{51}$$

We regard (u, p) as a perturbation from (a modification of) the $\mathbb{R}^n$-flow, to be precise,

$$u(t) = \widetilde{u}(t) + v(t), \qquad p(t) = \widetilde{p}(t) + p_v(t)$$

where

$$\widetilde{u}(t) := (1-\phi)T_{\mathbb{R}^n}(t,s)f + \mathbb{B}[(T_{\mathbb{R}^n}(t,s)f)\cdot\nabla\phi], \qquad \widetilde{p}(t) = (1-\phi)p_{\mathbb{R}^n}(t), \tag{52}$$

and $\phi \in C_0^\infty(B_3)$ is a fixed cut-off function satisfying $\phi = 1$ on B_2. Here, $\mathbb{B}$ is called the Bogovskii operator [30,32–34] in the domain $G := B_3 \setminus \overline{B_1}$ that provides a particular solution (among many solutions) of the boundary value problem for the equation of continuity subject to homogeneous Dirichlet boundary condition, and enjoys optimal regularity estimates ($1 < q < \infty, k = 0, 1, 2, \cdots$)

$$\begin{aligned} \mathbb{B} : W_0^{k,q}(G) \to W_0^{k+1,q}(G)^n, \qquad & \|\nabla^{k+1}\mathbb{B}g\|_{q,G} \le C\|\nabla^k g\|_{q,G} \\ \mathbb{B} : W^{1,q/(q-1)}(G)^* \to L^q(G)^n, \qquad & \|\mathbb{B}g\|_{q,G} \le C\|g\|_{W^{1,q/(q-1)}(G)^*} \end{aligned} \tag{53}$$

along with

$$\text{div}\,(\mathbb{B}g) = g \qquad \text{if } \langle g, 1\rangle_G = \int_G g(x)\,dx = 0. \tag{54}$$

Let $2 < r < \infty$, and let us show (47) (with $r_0 = r$). Since we have the desired estimate for $\widetilde{u}(t)$ above by (36), together with (53), our task is to find uniform boundedness for $v(t)$. In what follows, all constants can be taken uniformly in (b, M) with $\|(b, M)\| \le m$, see (14). By (A3) and (A6), we deduce from $f \in C_{0,\sigma}^\infty(\Omega)$ that $v \in C^1((s,\infty); L_\sigma^r(\Omega))$ along with $v(t) \in Y_r(\Omega)$ since $r > 2 > n'$. One can write the initial boundary value problem that (v, p_v) obeys, and by (25), it is then converted into

$$v(t) = T(t,s)\widetilde{f} + \int_s^t T(t,\tau)PF(\tau)\,d\tau \tag{55}$$

in $L_\sigma^r(\Omega)$, where $\widetilde{f} = \phi f - \mathbb{B}[f \cdot \nabla\phi]$ and

$$\begin{aligned} F(x,t) = &-2\nabla\phi \cdot \nabla T_{\mathbb{R}^n}(t,s)f - (\Delta\phi + b\cdot\nabla\phi)T_{\mathbb{R}^n}(t,s)f \\ &- \mathbb{B}[(\partial_t T_{\mathbb{R}^n}(t,s)f)\cdot\nabla\phi] + \Delta\mathbb{B}[(T_{\mathbb{R}^n}(t,s)f)\cdot\nabla\phi] \\ &+ b\cdot\nabla\mathbb{B}[(T_{\mathbb{R}^n}(t,s)f)\cdot\nabla\phi] - M\mathbb{B}[(T_{\mathbb{R}^n}(t,s)f)\cdot\nabla\phi] \\ &+ (\nabla\phi)p_{\mathbb{R}^n}(t). \end{aligned} \tag{56}$$

We note that the pressure $p_{\mathbb{R}^n}(t)$ with (51) is trivial for the case (9) with $V = 0$ [1,2], however, this is not the case here. Therefore, we need well-known estimate

$$\|w - \overline{w}\|_{r,B_3} \le C\|\nabla w\|_{W^{-1,r}(B_3)}, \qquad \overline{w} := \frac{1}{|B_3|}\int_{B_3} w(x)\,dx, \tag{57}$$

for all $w \in L^r(B_3)$, which follows from (53) (for $\mathbb{B}'$), together with

$$\begin{aligned} |\langle w - \overline{w}, \varphi\rangle_{B_3}| = |\langle w, \varphi - \overline{\varphi}\rangle_{B_3}| &= |\langle w, \text{div}\,(\mathbb{B}'(\varphi - \overline{\varphi}))\rangle_{B_3}| \\ &\le C\|\nabla w\|_{W^{-1,r}(B_3)}\|\mathbb{B}'(\varphi - \overline{\varphi})\|_{W_0^{1,r/(r-1)}(B_3)} \end{aligned}$$

for every $\varphi \in L^{r/(r-1)}(B_3)$, where $1 < r < \infty$ and $\mathbb{B}'$ denotes the Bogovskii operator in domain B_3. By (51), (53), and (57) together with Equation (1), we find

$$\begin{aligned} &\|\mathbb{B}[(\partial_t T_{\mathbb{R}^n}(t,s)f)\cdot\nabla\phi]\|_{r,G} + \|(\nabla\phi)p_{\mathbb{R}^n}(t)\|_{r,G} \\ &\le C\|(\partial_t T_{\mathbb{R}^n}(t,s)f)\cdot\nabla\phi\|_{W^{1,r/(r-1)}(G)^*} + C\|p_{\mathbb{R}^n}(t)\|_{r,B_3} \\ &\le C\|\partial_t T_{\mathbb{R}^n}(t,s)f\|_{W^{-1,r}(B_3)} + C\|\nabla p_{\mathbb{R}^n}(t)\|_{W^{-1,r}(B_3)} \\ &\le C\|\partial_t T_{\mathbb{R}^n}(t,s)f\|_{W^{-1,r}(B_3)} + C\|\nabla T_{\mathbb{R}^n}(t,s)f\|_{r,B_3} + C\|T_{\mathbb{R}^n}(t,s)f\|_{r,B_3}, \end{aligned}$$

while the other terms of $F(t)$ are harmless. Estimates (36) and (37) thus imply that

$$\|F(t)\|_r \le C(t-s)^{-1/2}(1+t-s)^{-n/2r+1/2}\|f\|_r \tag{58}$$

for all $t > s \geq 0$ (and $1 < r < \infty$). Since the support of $\widetilde{f}$ is compact in Ω, it follows from (26), along with energy relation (41) that

$$\|T(t,s)\widetilde{f}\|_r \leq C\|T(t-1,s)\widetilde{f}\|_2 \leq C\|\widetilde{f}\|_2 \leq C\|f\|_r. \tag{59}$$

To estimate the Duhamel term, it is convenient to adopt the duality argument. Let $\psi \in C_{0,\sigma}^\infty(\Omega)$. It is easy to see from (35), (42) and (58) together with backward semigroup property (34) that

$$\begin{aligned}
&\left|\left(\int_s^{s+1} + \int_{t-1}^t\right)\langle F(\tau), T(t,\tau)^*\psi\rangle\, d\tau\right| \\
&\leq \int_s^{s+1} \|F(\tau)\|_{2,\Omega_3}\|T(t,\tau)^*\psi\|_{2,\Omega_3}\, d\tau + \int_{t-1}^t \|F(\tau)\|_r\|T(t,\tau)^*\psi\|_{r'}\, d\tau \\
&\leq C\int_s^{s+1} \|F(\tau)\|_r\, d\tau\, \|T(t,t-1)^*\psi\|_2 + C(t-s-1)^{-n/2r}\|f\|_r\|\psi\|_{r'} \\
&\leq C\|f\|_r\|\psi\|_{r'}
\end{aligned} \tag{60}$$

for $t - s > 2$ and every $r \in (2,\infty)$, where $r' = r/(r-1)$. Let us discuss term

$$J := \int_{s+1}^{t-1} \langle F(\tau), T(t,\tau)^*\psi\rangle\, d\tau.$$

By (30) and (33) we know $T(t,\tau)^*\psi \in Y_q(\Omega)$ for all $q \in (n',\infty)$ and $\tau \in [0,t)$, so that $(T(t,\tau)^*\psi)|_{\partial\Omega} = 0$. We thus have Poincaré inequality

$$\|T(t,\tau)^*\psi\|_{2,\Omega_3} \leq C\|\nabla T(t,\tau)^*\psi\|_{2,\Omega_3}.$$

From this along with (35), (42) and (58) it follows that

$$\begin{aligned}
|J| &\leq C\left(\int_{s+1}^{t-1} \|F(\tau)\|_r^2\, d\tau\right)^{1/2}\left(\int_{s+1}^{t-1} \|\nabla T(t,\tau)^*\psi\|_2^2\, d\tau\right)^{1/2} \\
&\leq C\left(\int_{s+1}^{t-1} (\tau-s)^{-n/r}\, d\tau\right)^{1/2} \|f\|_r\|T(t,t-1)^*\psi\|_2 \\
&\leq C\|f\|_r\|\psi\|_{r'}
\end{aligned} \tag{61}$$

provided $r \in (2,n)$, which, together with (59) and (60), implies (47) for such r. Hence, we obtain (48) for $n' < q \leq r \leq 2$.

With this at hand, we proceed to the next step in which $r > n$ is assumed. This time, we need to split above integral J into

$$|J| \leq C\|f\|_r\left(\int_{s+1}^{(s+t)/2} + \int_{(s+t)/2}^{t-1}\right)(\tau-s)^{-n/2r}\|\nabla T(t,\tau)^*\psi\|_2\, d\tau. \tag{62}$$

Given r such that

$$\begin{cases} 3 < r < 6, & \text{if } n = 3, \\ n < r < \infty, & \text{if } n \geq 4, \end{cases} \tag{63}$$

we set $\mu := 1 - \frac{n}{r} \in (0,1)$ and define exponent q by $\frac{1}{q} - \frac{1}{2} = \frac{\mu}{n}$. Then, we have $q \in (n',2)$. In view of Assumption (A5), (29), (31) and (33), let us define $w(\tau)$ for $\tau \in [0,t]$ by

$$w(t-\tau) = T(t,\tau)^*\psi, \qquad \tau \in [0,t],$$

that satisfies

$$\frac{dw}{d\tau} + L(t-\tau)^*w = 0, \quad \tau \in (0,t]; \qquad w(0) = \psi.$$

Then, by virtue of energy Relation (42) along with (35) and by the decay estimate obtained in the previous step, we find

$$\begin{aligned}\int_{(t-s)/2}^{t-s-1} \|\nabla w(\tau)\|_2^2\,d\tau &= \int_{s+1}^{(s+t)/2} \|\nabla T(t,\tau)^*\psi\|_2^2\,d\tau \\ &\le C\|T(t,(s+t)/2)^*\psi\|_2^2 \\ &\le C(t-s-2)^{-\mu}\|T(t,t-1)^*\psi\|_q^2 \\ &\le C\|\psi\|_{r'}^2\,(t-s-2)^{-\mu}\end{aligned} \tag{64}$$

for $t-s>2$. As for the former integral of (62), we use (64) to furnish

$$\int_{s+1}^{(s+t)/2} \le C(t-s)^{1/2-n/2r}\left(\int_{s+1}^{(s+t)/2} \|\nabla T(t,\tau)^*\psi\|_2^2\,d\tau\right)^{1/2} \le C\|\psi\|_{r'}$$

for $t-s>3$. The latter integral of (62) must be comparable with that. In fact, due to Lemma 1 with $\sigma=(t-s-2)/2$, it follows from (64) that

$$\begin{aligned}\int_{(s+t)/2}^{t-1} &\le C(t-s)^{-n/2r}\int_{(s+t)/2}^{t-1} \|\nabla T(t,\tau)^*\psi\|_2\,d\tau \\ &= C(t-s)^{-n/2r}\int_1^{(t-s)/2} \|\nabla w(\tau)\|_2\,d\tau \\ &\le C\|\psi\|_{r'}\end{aligned}$$

for $t-s>3$. In this way we accomplish (47) for all $r_0\in(n,\infty)$ when $n\ge 4$, while we still need to repeat the procedure once more when $n=3$. This procedure in 3D is indeed possible for every $r<\infty$ by the same manner as before with the aid of (48) for $6/5<q\le r\le 2$. We completed the proof of $(43)_{j=0}$ for all $t>s\ge 0$ and $2\le q\le r<\infty$.

Uniform boundedness (49) for the adjoint is similarly proven by use of (26), (A7) for $T_{\mathbb{R}^n}(t,s)^*$ and (41). Here, the Duhamel formula can be justified merely in weak form by (18) and (24), but it is enough for a duality argument. We thus employ Lemma 2 to obtain $(43)_{j=0}$ for all $t>s\ge 0$ and $1<q\le r\le 2$. Remaining case $q<2<r$ is easily filled on account of semigroup property (22) to conclude $(43)_{j=0}$ for all $t>s\ge 0$ and $1<q\le r<\infty$. □

The proof of this proposition would be a novelty among other arguments in [1,2]. However, the first step of the proof above does not work well when $n=2$; in fact, (61) cannot be bounded for $r>2$.

For the proof of $(43)_{j=1}$, it suffices to show

$$\|\nabla T(t,s)f\|_r \le C(t-s)^{-\min\{1/2,n/2r\}}\|f\|_r \tag{65}$$

for all (t,s) with $t-s>1$, $r\in(1,\infty)$ and $f\in L^r_\sigma(\Omega)$ since we combine (65) with $(43)_{j=0}$ to obtain the desired estimates. As in several papers for the autonomous case mentioned in Section 1, it is standard to split (65) into $\|\nabla T(t,s)f\|_{r,\Omega_3}$ and $\|\nabla T(t,s)f\|_{r,\mathbb{R}^n\setminus B_3}$. The former is given by Proposition 2 below, for which the following lemma is needed.

Lemma 3. *Let $n'<q<\infty$, where $1/n'+1/n=1$.*

1. *There is a constant $C=C(q,n,\Omega)>0$, such that*

$$\||x|\nabla Pg\|_q \le C\big(\||x|\nabla g\|_q + \|g\|_{W^{1,q}(\Omega)}\big) \tag{66}$$

for all $g\in W^{1,q}(\Omega)^n$ with $|x|\nabla g\in L^q(\Omega)^{n\times n}$.

2. *Let $g \in W^{1,q}(\Omega)^n$ satisfy $|x|\nabla g \in L^q(\Omega)^{n\times n}$. Then, we have*
$$T(t,s)Pg \in Y_q(\Omega) \cap Z_r(\Omega)$$
for all $r \in (q,\infty)$ and $t \in (s,\infty)$.

Proof. The second assertion follows from (23) because (66) implies $Pg \in Z_q(\Omega)$. Let us consider Neumann problem

$$-\Delta w = \mathrm{div}\, g \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega} = -\nu \cdot g|_{\partial\Omega}.$$

It then suffices to show

$$\| |x|\nabla^2 w\|_q \le C\big(\| |x|(\mathrm{div}\, g)\|_q + \|g\|_{W^{1,q}(\Omega)}\big), \tag{67}$$

which implies (66) since $Pg = g + \nabla w$. We take the same cut-off function $\phi \in C_0^\infty(B_3)$ as in the proof of Proposition 1, and choose a solution w satisfying $\int_{\Omega_3} w\, dx = 0$, so that

$$\|w\|_{q,\Omega_3} \le C\|\nabla w\|_{q,\Omega_3} \le C\|\nabla w\|_q \le C\|g\|_q \tag{68}$$

where the last inequality is due to [28,29]. Then, ϕw obeys

$$-\Delta(\phi w) = \phi(\mathrm{div}\, g) - 2\nabla\phi\cdot\nabla w - (\Delta\phi)w \;\text{ in } \Omega_3, \;\; \nu\cdot\nabla(\phi w)|_{\partial\Omega_3} = -\nu\cdot(\phi g)|_{\partial\Omega_3}$$

which leads to

$$\|\nabla^2(\phi w)\|_{q,\Omega_3} \le C\|g\|_{W^{1,q}(\Omega_3)} + C\|w\|_{W^{1,q}(\Omega_3)}, \tag{69}$$

where ν denotes the outer unit normal to $\partial\Omega_3$. On the other hand, $(1-\phi)w$ obeys

$$-\Delta\{(1-\phi)w\} = (1-\phi)(\mathrm{div}\, g) + 2\nabla\phi\cdot\nabla w + (\Delta\phi)w =: h \quad \text{in } \mathbb{R}^n.$$

For Riesz transform $\mathcal{R}$, we know

$$\| |x|\mathcal{R}h\|_{q,\mathbb{R}^n} \le C\| |x|h\|_{q,\mathbb{R}^n}$$

from Muckenhoupt theory for singular integrals, as long as $n' < q < \infty$; in fact, for such q, weight $|x|^q$ belongs to Muckenhoupt class $\mathcal{A}_q(\mathbb{R}^n)$; see Farwig and Sohr ([35], Section 2), and Torchinsky ([36], Chapter IX) for details. We thus obtain

$$\begin{aligned} \| |x|\nabla^2\{(1-\phi)w\}\|_{q,\mathbb{R}^n} &= \| |x|(\mathcal{R}\otimes\mathcal{R})h\|_{q,\mathbb{R}^n} \le C\| |x|h\|_{q,\mathbb{R}^n} \\ &\le C\| |x|(\mathrm{div}\, g)\|_q + C\|w\|_{W^{1,q}(\Omega_3)} \end{aligned} \tag{70}$$

for $n' < q < \infty$. We collect (68)–(70) to conclude (67). □

Proposition 2. *Suppose* (A1)–(A8). *Let $1 < q < \infty$. Given $m \in (0,\infty)$, there is a constant $C = C(m,q,n,\Omega) > 0$ such that*

$$\|T(t,s)f\|_{W^{1,q}(\Omega_3)} \le C(t-s)^{-n/2q}\|f\|_q \tag{71}$$

$$\|T(t,s)f\|_{\infty,\Omega_3} \le C(t-s)^{-n/2q}\|f\|_q \tag{72}$$

$$\|\partial_t T(t,s)f\|_{W^{-1,q}(\Omega_3)} \le C(t-s)^{-n/2q}\|f\|_q \tag{73}$$

for all (t,s) with $t-s>1$ and $f \in L^q_\sigma(\Omega)$ whenever $\|(b,M)\| \le m$.

Proof. Given $q \in (1, \infty)$, let us take $\varepsilon > 0$ to be so small that

$$\varepsilon < \min\left\{\frac{n}{2} - \frac{n}{2q}, \frac{n}{2} - 1\right\},$$

and then choose pair (p_0, q_0), such that $(n/p_0 - n/q_0)/2 = n/2 - \varepsilon$ as well as $1 < p_0 < q < q_0 < \infty$. If, in particular, $f \in L^q(\Omega)^n$ fulfils $f(x) = 0$ a.e. $\mathbb{R}^n \setminus B_3$ (but does not necessarily satisfy the solenoidal condition), it follows from $(26)_{j=1}$ and $(43)_{j=0}$ that

$$\|T(t,s)Pf\|_{W^{1,q_0}(\Omega)} \le C\|T(t-1,s)Pf\|_{q_0} \le C(t-s-1)^{-n/2+\varepsilon}\|f\|_{p_0}$$

for $t - s > 2$, which implies

$$\|T(t,s)Pf\|_{W^{1,q}(\Omega_3)} \le C(t-s)^{-1/2}(1+t-s)^{-n/2+1/2+\varepsilon}\|f\|_q \tag{74}$$

for all $t > s \ge 0$.

For the proof of (71), it suffices to show it for $u(t) = T(t,s)f$ with $f \in C^\infty_{0,\sigma}(\Omega)$. For such f, function $\widetilde{u}(t)$ given by (52) satisfies (71) because of (36). Let us consider $v(t) = u(t) - \widetilde{u}(t)$ obeying (55). Since both $\widetilde{f}$ and $F(t)$ vanish outside Ω_3, one can apply (74) to those vector fields. Then, we immediately see that $T(t,s)\widetilde{f}$ satisfies the desired estimate. Combining (58) with the above observation, we also find

$$\int_s^t \|T(t,\tau)PF(\tau)\|_{W^{1,q}(\Omega_3)}\,d\tau \le C\|f\|_q\left(\int_s^{(s+t)/2} + \int_{(s+t)/2}^t\right)\beta(\tau)\,d\tau$$

with

$$\beta(\tau) = (t-\tau)^{-1/2}(1+t-\tau)^{-n/2+1/2+\varepsilon}(\tau-s)^{-1/2}(1+\tau-s)^{-n/2q+1/2},$$

for $\tau \in (s,t)$, which concludes (71).

By the Sobolev embedding relation, (72) follows directly from (71) when $q > n$. This, together with $(43)_{j=0}$, implies (72) for other case $q \le n$, too.

We next show (73) for case $q > n'$. From (28), along with (24), it follows in this case that

$$\|L(t)T(t,s)f\|_{W^{-1,q}(\Omega_3)} \le c_2(t-s)^{-\alpha}\|f\|_q \tag{75}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in Z_q(\Omega)$. Let us take ε and (p_0, q_0) as above. If $g \in W^{1,q}(\Omega)^n$ fulfills $g(x) = 0$ a.e. $\mathbb{R}^n \setminus B_3$ (so that $Pg \in Z_q(\Omega)$), then we see that $T(t-1,s)Pg \in Z_{q_0}(\Omega)$ by Lemma 3, which, together with $(43)_{j=0}$ and (75), leads to

$$\begin{aligned}
\|\partial_t T(t,s)Pg\|_{W^{-1,q_0}(\Omega_3)} &= \|L(t)T(t,t-1)T(t-1,s)Pg\|_{W^{-1,q_0}(\Omega_3)} \\
&\le C\|T(t-1,s)Pg\|_{q_0} \\
&\le C(t-s-1)^{-n/2+\varepsilon}\|g\|_{p_0}
\end{aligned}$$

for $t - s > 2$. This, combined with (28), implies that

$$\|\partial_t T(t,s)Pg\|_{W^{-1,q}(\Omega_3)} \le C(t-s)^{-\alpha}(1+t-s)^{-n/2+\alpha+\varepsilon}\|g\|_q \tag{76}$$

for all $t > s \ge 0$. Let $f \in C^\infty_{0,\sigma}(\Omega)$; then, it follows from (37) and (53) that the temporal derivative of function $\widetilde{u}(t)$ given by (52) enjoys (73). Consider $\partial_t v(t) = \partial_t u(t) - \partial_t \widetilde{u}(t)$ with $u(t) = T(t,s)f$, which obeys

$$\partial_t v(t) = \partial_t T(t,s)\widetilde{f} + PF(t) + \int_s^t \partial_t T(t,\tau)PF(\tau)\,d\tau$$

in $W^{-1,q}(\Omega_3)$ by the latter part of (A4). We know (58) for second term $PF(t)$, whereas (76) can be applied to $g = \widetilde{f}$ and $g = F(\tau)$ since $F(\tau) \in W^{1,q}(\Omega)^n$ follows from (A6)–(A7), (51) and (53) in view of (56). We combine (76) with (58) to furnish

$$\int_s^t \|\partial_t T(t,\tau)PF(\tau)\|_{W^{-1,q}(\Omega_3)}\, d\tau \le C\|f\|_q \left(\int_s^{(s+t)/2} + \int_{(s+t)/2}^t\right)\widetilde{\beta}(\tau)\, d\tau$$

with

$$\widetilde{\beta}(\tau) = (t-\tau)^{-\alpha}(1+t-\tau)^{-n/2+\alpha+\varepsilon}(\tau-s)^{-1/2}(1+\tau-s)^{-n/2q+1/2}$$

for $\tau \in (s,t)$, which yields (73) for $f \in C_{0,\sigma}^\infty(\Omega)$; therefore, for $f \in L_\sigma^q(\Omega)$ provided $q \in (n', \infty)$. If, in particular, $f \in Z_q(\Omega)$ with $q \in (n', \infty)$, one can rewrite (73) as

$$\|L(t)T(t,s)f\|_{W^{-1,q}(\Omega_3)} \le C(t-s)^{-n/2q}\|f\|_q \tag{77}$$

for $t - s > 1$. Other case $q \in (1, n']$ is discussed in the following way: Let $f \in C_{0,\sigma}^\infty(\Omega)$ and fix $r \in (n', \infty)$, then we have $T((t+s)/2, s)f \in Y_r(\Omega) \subset Z_r(\Omega)$ by (23). We thus utilize (77) with such r and $(43)_{j=0}$ to find

$$\begin{aligned}\|\partial_t T(t,s)f\|_{W^{-1,q}(\Omega_3)} &\le C\|L(t)T(t,s)f\|_{W^{-1,r}(\Omega_3)} \\ &\le C(t-s)^{-n/2r}\|T((t+s)/2,s)f\|_r \\ &\le C(t-s)^{-n/2q}\|f\|_q\end{aligned}$$

for $t - s > 2$. The proof is complete. □

Corollary 1. *Suppose* (A1)–(A8). *Let* $1 < q < \infty$. *Set* $u(t) = T(t,s)f$ *with* $f \in L_\sigma^q(\Omega)$ *and single out pressure* $p(t)$ *associated with* $u(t)$ *subject to* $\int_{\Omega_3} p\, dx = 0$. *Let* $\phi \in C_0^\infty(B_3)$ *be the cut-off function as in the proof of Proposition 1, and* $\mathbb{B}$ *the Bogovskii operator in domain* $G = B_3 \setminus \overline{B_1}$. *Then, for each* $m \in (0, \infty)$, *there is a constant* $C = C(m,q,n,\Omega) > 0$, *such that*

$$\|p(t)\|_{q,\Omega_3} + \|\mathbb{B}[\partial_t u(t)\cdot\nabla\phi]\|_{q,G} \le C\|f\|_q \begin{cases} (t-s)^{-\alpha}, & 0 < t-s \le 1, \\ (t-s)^{-n/2q}, & t-s > 1, \end{cases} \tag{78}$$

for all $f \in L_\sigma^q(\Omega)$, *whenever* $\|(b,M)\| \le m$.

Proof. By (1) together with (57) in which B_3 is replaced by Ω_3 and by (53) we find

$$\|p(t)\|_{q,\Omega_3} \le C\|\nabla p(t)\|_{W^{-1,q}(\Omega_3)} \le C\|\partial_t u(t)\|_{W^{-1,q}(\Omega_3)} + C\|u(t)\|_{W^{1,q}(\Omega_3)}$$

and

$$\|\mathbb{B}[\partial_t u(t)\cdot\nabla\phi]\|_{q,G} \le C\|\partial_t u(t)\cdot\nabla\phi\|_{W^{1,q/(q-1)}(G)^*} \le C\|\partial_t u(t)\|_{W^{-1,q}(\Omega_3)}.$$

Hence, we collect (26), (28), (71) and (73) to conclude (78). □

Let us complete the proof of Theorem 1 by showing estimates near spatial infinity.

Proof of Theorem 1. Let $1 < q < \infty$. We show

$$\|\nabla T(t,s)f\|_{q,\mathbb{R}^n\setminus B_3} \le C(t-s)^{-\min\{1/2, n/2q\}}\|f\|_q \tag{79}$$

$$\|T(t,s)f\|_{\infty,\mathbb{R}^n\setminus B_3} \le C(t-s)^{-n/2q}\|f\|_q \tag{80}$$

for all (t,s) with $t - s > 1$ and $f \in L_\sigma^q(\Omega)$, which, combined with (71) and (72), conclude (65) and $(43)_{j=0}$ with $r = \infty$, where $q \in (n/2, \infty)$ has to be assumed for (80); however, the other case, $q \in (1, n/2]$, is easily discussed at the very end. Given $f \in C_{0,\sigma}^\infty(\Omega)$, we set $u(t) = T(t,s)f$. Let us take cut-off function $\phi \in C_0^\infty(B_3)$ and Bogovskii operator $\mathbb{B}$, both of which are the same as in the proof of Proposition 1. By $p(t)$, we denote the pressure associated with $u(t)$, such that $\int_{\Omega_3} p\,dx = 0$, and consider

$$v(t) := (1-\phi)u(t) + \mathbb{B}[u(t)\cdot\nabla\phi], \qquad p_v(t) := (1-\phi)p(t) \tag{81}$$

which obeys the following equation in terms of evolution operator $T_{\mathbb{R}^n}(t,s)$:

$$v(t) = T_{\mathbb{R}^n}(t,s)\widetilde{f} + \int_s^t T_{\mathbb{R}^n}(t,\tau)P_{\mathbb{R}^n}H(\tau)\,d\tau, \tag{82}$$

where $\widetilde{f} = (1-\phi)f + \mathbb{B}[f\cdot\nabla\phi]$ and

$$\begin{aligned} H(x,t) = {} & 2\nabla\phi\cdot\nabla u(t) + (\Delta\phi + b\cdot\nabla\phi)u(t) - \Delta\mathbb{B}[u(t)\cdot\nabla\phi] \\ & - b\cdot\nabla\mathbb{B}[u(t)\cdot\nabla\phi] + M\mathbb{B}[u(t)\cdot\nabla\phi] \\ & + \mathbb{B}[\partial_t u(t)\cdot\nabla\phi] - (\nabla\phi)p(t). \end{aligned} \tag{83}$$

Since $H(t)$ vanishes outside B_3, one can employ (26), (71), and (78) with (53) to obtain

$$\|H(t)\|_{r,\mathbb{R}^n} \le C(t-s)^{-\alpha}(1+t-s)^{-n/2q+\alpha}\|f\|_q \tag{84}$$

for all $t > s \ge 0$ and $r \in (1,q]$. In view of $v(t) = u(t)$ in $\mathbb{R}^n \setminus B_3$ (see (81)), let us consider $\|\nabla v(t)\|_{q,\mathbb{R}^n}$ and $\|v(t)\|_{\infty,\mathbb{R}^n}$ by using (82). We immediately observe from (36) that

$$\|\nabla T_{\mathbb{R}^n}(t,s)\widetilde{f}\|_{q,\mathbb{R}^n} \le C(t-s)^{-1/2}\|f\|_q, \quad \|T_{\mathbb{R}^n}(t,s)\widetilde{f}\|_{\infty,\mathbb{R}^n} \le C(t-s)^{-n/2q}\|f\|_q$$

for $t > s \ge 0$, whereas (36) and (84) imply that

$$\begin{aligned} \int_s^t \|\nabla T_{\mathbb{R}^n}(t,\tau)P_{\mathbb{R}^n}H(\tau)\|_{q,\mathbb{R}^n}\,d\tau &\le C\|f\|_q\left(\int_s^{(s+t)/2} + \int_{(s+t)/2}^t\right)\gamma(\tau)\,d\tau, \\ \int_s^t \|T_{\mathbb{R}^n}(t,\tau)P_{\mathbb{R}^n}H(\tau)\|_{\infty,\mathbb{R}^n}\,d\tau &\le C\|f\|_q\left(\int_s^{(s+t)/2} + \int_{(s+t)/2}^t\right)\widetilde{\gamma}(\tau)\,d\tau, \end{aligned}$$

where

$$\begin{aligned} \gamma(\tau) &= (t-\tau)^{-1/2}(1+t-\tau)^{-(n/r-n/q)/2}(\tau-s)^{-\alpha}(1+\tau-s)^{-n/2q+\alpha}, \\ \widetilde{\gamma}(\tau) &= (t-\tau)^{-n/2q}(1+t-\tau)^{-(n/r-n/q)/2}(\tau-s)^{-\alpha}(1+\tau-s)^{-n/2q+\alpha}, \end{aligned}$$

for $\tau \in (s,t)$. Hence, a suitable choice of exponent $r \in (1,q]$ (depending on q) leads us to (79) for $q \in (1,\infty)$, which concludes $(43)_{j=1}$, and (80) for $q \in (n/2,\infty)$. We thus find $(43)_{j=0}$ with $r = \infty$ for such q, which, together with $(43)_{j=0}$ $(r < \infty)$, provides the desired L^∞-estimate for every $q \in (1,\infty)$. The proof of Theorem 1 is complete. □

4. Conclusions

The stability or attainability of physically relevant basic (Navier–Stokes) flow V is a significant issue in mathematical fluid dynamics. Estimate (7) for the associated linearized flow describes linearized stability with a definite decay rate of disturbance, and it is always a crucial step toward nonlinear stability, where nonlinearity is regarded as a small perturbation as long as the initial disturbance is small enough. The exterior problem is even more interesting since the rigid motion (translation or rotation) of the obstacle is involved in stability analysis. Spectral analysis of the linearized operator through resolvent problem is quite useful to deduce (7) when basic flow V is steady. In this paper, we considered the situation where both the motion of the obstacle and basic flow V were time-dependent (see (9) as a typical example), for which stability analysis is much less developed because spectral analysis does not work well. The novelty is to provide the essence of the new approach proposed by the present author [1,2] to show linearized stability (7) even for nonautonomous systems in exterior domains under reasonable assumptions on regularity of the evolution operator, including smoothing estimate (10) of the temporal derivative, linearized stability for the whole space problem, and energy Relations (12) and (13) with dissipation. Theorem 1, together with standard analysis of the nonlinear problem, tells us roughly that linearized stability for the whole space problem and the energy structure lead to nonlinear stability in exterior domains. Emphasis is on how we utilize the energy relation to find this conclusion. In addition to example (9) with $V = 0$ discussed in [1,2], we could apply Theorem 1 to Case (9), with V decaying faster than scale-critical rate $O(|x|^{-1})$ at spatial infinity. This is indeed the case when the better spatial decay structure of V with wake behind the translating obstacle is available. As mentioned at the end of the introduction, we still have to await further analysis to apply our theory to the more important case of (9) with V, which decays at the scale-critical rate uniformly in t. This is left as future work.

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